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PROPER ACTIONS OF GROUPOIDS ON C^*-ALGEBRAS

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ABSTRACT. In 1990, Rieffel defined a notion of proper action of a group H on a C^*-algebra A. He then defined a generalized fixed point algebra A^α for this action and showed that A^α is Morita equivalent to an ideal of the reduced crossed product. We generalize Rieffel's notion to define proper groupoid dynamical systems and show that the generalized fixed point algebra for proper groupoid actions is Morita equivalent to a subalgebra of the reduced crossed product. We give some nontrivial examples of proper groupoid dynamical systems and show that if $(\mathscr{A}, G, \alpha)$ is a groupoid dynamical system such that G is principal and proper, then the action of G on $\mathscr{A}$ is saturated, that is the generalized fixed point algebra is Morita equivalent to the reduced crossed product.

KEYWORDS: *Proper actions, groupoid crossed products, generalized fixed point algebras, reduced groupoid crossed products, locally compact groupoids, Morita equivalence.*

MSC (2000): 46L05, 46L08.

1. INTRODUCTION

In an effort to study deformation quantization of Poisson manifolds, Rieffel introduced a notion of proper group actions on C^*-algebras [27]. These actions are meant to behave like proper actions of groups on spaces. To that end, he also defined a generalized fixed point algebra for proper dynamical systems which has some of the same properties as the generalized fixed point algebra, $C_0(G\backslash X)$, for a proper action of a group G on a space X. The main theorem of [27] (Theorem 1.5) shows that the generalized fixed point algebra for a proper dynamical system is Morita equivalent to an ideal of the reduced crossed product. This generalizes a theorem of Green's ([7], Corollary 15) which gives a Morita equivalence between $C_0(X) \rtimes G$ and $C_0(G\backslash X)$ whenever G acts freely and properly on X. Since Rieffel introduced proper actions they have been studied in [9], [10], [11], [13], [15], [8].

In [27], Rieffel also identifies a class of proper actions with the property that the generalized fixed point algebra is Morita equivalent to the reduced crossed product, Rieffel calls these actions *saturated*. Saturated actions not only more closely resemble the situation in Green's theorem, they have also proved to be the actions most useful in applications [9], [10], [11].

The study of generalized fixed point algebras for proper group dynamical systems has lead to a wide range of interesting results in operator theory. For example they have been used to prove results in nonabelian duality theory [13], graph algebras [10], [15] and the equivariant Brauer semigroup [9]. When one is interested in extending these results to the groupoid setting, one is naturally lead to seek an appropriate notion of a generalized fixed point algebra for groupoid dynamical systems and therefore a notion of proper groupoid dynamical systems.

In this paper we propose a definition of proper groupoid dynamical systems and define a generalized fixed point algebra for these systems. Our main theorem is as follows:

THEOREM. *If a groupoid dynamical system is proper, then the generalized fixed point algebra for the action is Morita equivalent to a subalgebra of the reduced crossed product.*

Note that this theorem generalizes both Theorem 1.5 of [27] and Corollary 15 of [7]. We also present some examples of proper groupoid dynamical systems and give conditions that guarantee that these examples are saturated. To prove saturation in our examples, we needed to use a new averaging argument to overcome the fact that translations of open sets in groupoids are not necessarily open. We believe this argument can be applied to prove other density results. Along the way we recover a result in [19] showing that $C^*(G)$ has continuous trace when G is principal and proper. Although our results are about reduced crossed products, the work of Corollary 2.1.17 and Proposition 6.1.10 in [1] shows that the groupoids in our examples are amenable, so in these examples our results apply to the full crossed product.

There has been considerable interest recently in groupoid crossed products and groupoid dynamical systems. To learn more about these objects the reader is encouraged to look at the excellent exposition in [21]. To our knowledge groupoid crossed products were first introduced by Renault in [24]. They have since been studied in terms of their ideal structure in [3], [20], [24], [25], [21], [4], [5], [6], implicitly in the study of the equivariant Brauer Group and groupoid cohomology in [12] and with relation to inverse semigroups [14], [22]. The present paper hopes to provided a tool to illuminate the study of these fascinating objects further.

We should note that there is some debate in the literature about the correct definition of proper *group* dynamical systems [2], [16], [28]. We have chosen to generalize Rieffel's original definition [27] because (while it is not intrinsic) it gives a Morita equivalence result with the generalized fixed point algebra and is thus the definition most widely used in applications.

We begin with a section on preliminaries which includes a brief introduction to groupoid dynamical systems, induced representations and the reduced groupoid crossed product. In Section 3 we define proper groupoid dynamical systems and the generalized fixed point algebra and prove our main theorem. Section 4 is devoted to fleshing out two examples and Section 5 is devoted showing freeness guarantees that these examples are saturated.

1.1. CONVENTIONS. Throughout this paper we will use the following conventions. If A is a C^*-algebra, then $M(A)$ will denote the multiplier algebra of A and $Z(A)$ will denote its center. If $\pi : A \to B$ is nondegenerate, then $\overline{\pi}$ will denote its extension to $M(A)$. If X_1 and X_2 are spaces equipped with maps $\tau_i : X_i \to T$, then $X_1 * X_2$ denotes the set $\{(x,y) \in X_1 \times X_2 : \tau_1(x) = \tau_2(y)\}$. Throughout, G will denote a second countable locally compact Hausdorff groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ ([23], Definition I.2.2). We will use the notational conventions for groupoids established in [17] which are the same as those in [23] except that we use s to denote the source map. If G acts on a topological space X (on the left), then X is fibred over $G^{(0)}$ by a map r_X ([17], Definition 2.13). Furthermore there exists a map $\Phi : G * X \to X \times X$ given by $(\gamma, x) \mapsto (\gamma \cdot x, x)$. We say that the action of G on X is *free* if this map is injective and we say the action is *proper* if Φ is a proper map. Note that if G acts properly on a locally compact Hausdorff space X, then the orbit space X/G is locally compact and Hausdorff ([1], Proposition 2.1.12). We say G is *principal* if the natural action of G on its unit space given by $\gamma \cdot s(\gamma) = r(\gamma)$ is free, we say G is *proper* if this action is proper. We will show in Proposition 4.1 that proper actions of groupoids on spaces give rise to proper groupoid dynamical systems as defined in Definition 3.1, so there should be no cause for confusion between the two uses of the word *proper*. Unless otherwise stated we will assume that all of our C^*-algebras are separable and all spaces X are locally compact and Hausdorff. We use χ_E to denote the characteristic function of the set E.

2. PRELIMINARIES

2.1. $C_0(X)$-ALGEBRAS. Groupoids must act on fibred objects, so to construct groupoid dynamical systems we need fibred C^*-algebras. To that end, for a locally compact Hausdorff space X, a $C_0(X)$-*algebra* is a C^*-algebra A together with a nondegenerate homomorphism of $C_0(X)$ into $Z(M(A))$. $C_0(X)$-algebras are well studied objects in their own right, but for our needs it is enough to know that they have an associated fibred structure. Specifically, if $C_{0,x}(X)$ is the set of functions in $C_0(X)$ vanishing at $x \in X$, then $I_x := \overline{C_{0,x}(X) \cdot A}$ is an ideal in A and $A(x) := A/I_x$ is called the fibre of A over x. The image of a in $A(x)$ is denoted by $a(x)$, and the set $\{A(x) : x \in X\}$ gives rise to an *upper semicontinuous C^*-bundle* $\mathscr{A}$ over X ([32], Theorem C.26).

DEFINITION 2.1. Let X be a locally compact Hausdorff space. An *upper semicontinuous C^*-bundle* over X is a topological space $\mathscr{A}$ together with a continuous open surjection $p = p_{\mathscr{A}} : \mathscr{A} \to X$ such that each fibre $A(x) := p^{-1}(\{x\})$ is a C^*-algebra and $\mathscr{A}$ satisfies the following axioms:

(i) the map $a \mapsto \|a\|$ is upper semicontinuous from $\mathscr{A}$ to $\mathbb{R}^+$ (that is, for all $\varepsilon > 0$, $\{a \in \mathscr{A} : \|a\| < \varepsilon\}$ is open);

(ii) the maps $(a, b) \mapsto a + b$ and $(a, b) \mapsto ab$ are continuous from $\mathscr{A} * \mathscr{A}$ to $\mathscr{A}$;

(iii) for each $k \in \mathbb{C}$, the maps $a \mapsto ka$ and $a \mapsto a^*$ are continuous from $\mathscr{A}$ to $\mathscr{A}$;

(iv) if $\{a_i\}$ is a net in $\mathscr{A}$ such that $p(a_i) \to x$ and $\|a_i\| \to 0$, then $a_i \to 0_x$ (where 0_x is the zero element of $A(x)$).

The point is, if we let $A = \Gamma_0(X, \mathscr{A})$ be the C^*-algebra of continuous sections of $\mathscr{A}$ vanishing at infinity, then A is a $C_0(X)$-algebra. The relationship $A = \Gamma_0(X, \mathscr{A})$ defines a one-to-one correspondence between $C_0(X)$-algebras and upper semicontinuous C^*-bundles. Throughout this paper we will denote bundles by script letters $\mathscr{A}$ and the corresponding section algebras by the corresponding Roman letter A. For a more detailed discussion of $C_0(X)$-algebras the reader is encouraged to see Appendix C of [32].

2.2. THE REDUCED CROSSED PRODUCT.

DEFINITION 2.2. Let G be a second countable locally compact groupoid with Haar system $\{\lambda^u\}_{u \in G^{(0)}}$ and $\mathscr{A}$ be an upper semicontinuous C^*-bundle over $G^{(0)}$. Suppose the associated $C_0(X)$-algebra, $A = \Gamma_0(G^{(0)}, \mathscr{A})$ is separable. An action α of G on A is a family of $*$-isomorphisms $\{\alpha_\gamma\}_{\gamma \in G}$ such that:

(i) for each $\gamma \in G$, $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$;

(ii) for all $(\gamma, \eta) \in G^{(2)}$, $\alpha_{\gamma\eta} = \alpha_\gamma \circ \alpha_\eta$;

(iii) the map $(\gamma, a) \mapsto \alpha_\gamma(a)$ is a continuous map from $G * \mathscr{A}$ to $\mathscr{A}$.

The triple $(\mathscr{A}, G, \alpha)$ is called a *(groupoid) dynamical system*.

Given a dynamical system, we can construct a convolution algebra which we then complete to obtain the reduced crossed product. The remainder of this section is devoted to a sketch of this construction. First we need the following definition.

DEFINITION 2.3. Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system, we define a bundle over G called the *pullback bundle* of $\mathscr{A}$ via r to be

$$r^*\mathscr{A} := \{(\gamma, a) : r(\gamma) = p_{\mathscr{A}}(a)\} \subset G \times \mathscr{A}, \tag{2.1}$$

with bundle map $p_{r^*\mathscr{A}} : (\gamma, a) \mapsto \gamma$.

First note that the fibre of $r^*\mathscr{A}$ over γ, $p_{r^*\mathscr{A}}^{-1}(\gamma)$, is naturally isomorphic to $A(r(\gamma))$, the fibre of $\mathscr{A}$ over $r(\gamma)$. Now the underlying function space for the convolution algebra in the construction of the reduced crossed product is given by $\Gamma_c(G, r^*\mathscr{A})$, the set of continuous compactly supported sections of $r^*\mathscr{A}$. Note that for $f \in \Gamma_c(G, r^*\mathscr{A})$, we can view $f(\gamma)$ as an element in $A(r(\gamma))$, and so for

$\eta \in G$ with $s(\eta) = r(\gamma)$, $\alpha_\eta(f(\gamma))$ makes sense as an element of $A(r(\eta))$. We now use the following proposition to define a $*$-algebra structure on $\Gamma_c(G, r^*\mathscr{A})$.

PROPOSITION 2.4 ([21], Proposition 4.4). *Let G be a groupoid with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, and $\Gamma_c(G, r^*\mathscr{A})$ be the set of continuous compactly supported sections of $r^*\mathscr{A}$. Then $\Gamma_c(G, r^*\mathscr{A})$ is a $*$-algebra with respect to the operations*

$$f * g(\gamma) := \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))\mathrm{d}\lambda^{r(\gamma)}(\eta) \quad \textit{and} \quad f^*(\gamma) = \alpha_\gamma(f(\gamma^{-1})^*).$$

The goal is to complete this convolution algebra in the norm induced by regular representations. Since we use regular representations extensively in the sequel we will sketch their construction here. To continue we need the notion of a Borel Hilbert bundle. For our purposes a Borel Hilbert bundle $X * \mathfrak{H}$ over X is bundle of Hilbert spaces, $X * \mathfrak{H} = \{\mathcal{H}(x)\}_{x\in X}$, along with a Borel structure satisfying some technical conditions (see Definition F.1 of [32]). Given a measure μ on X we can form the Hilbert space $L^2(X * \mathfrak{H}, \mu)$ in the obvious way. This Hilbert space is just the direct integral $\int_X^{\oplus} \mathcal{H}(x)\mathrm{d}\mu(x)$ and gives us the notion of a fibred Hilbert space that we need for groupoid representations.

Suppose π is a (separable) $C_0(G^{(0)})$-linear representation of A on $\mathcal{H}_\pi$. Then by Proposition F.26 of [32] there exists a Borel Hilbert bundle $G^{(0)} * \mathfrak{H}$, a finite measure $\mu_\pi = \mu$ on $G^{(0)}$ (note: μ need not be quasi invariant) and a Borel family of representations $\{\pi_u\}_{u\in G^{(0)}}$ of A on $\mathcal{H}(u)$ such that π is unitarily equivalent to the representation

$$\rho = \int_{G^{(0)}}^{\oplus} \pi_u \mathrm{d}\mu(u) \quad \text{given by } (\rho(a)h)(u) = \pi_u(a)(h(u)), \quad \text{for } h \in L^2(X * \mathfrak{H}, \mu). \tag{2.2}$$

Using the proof of Proposition F.26 of [32] we see that $I_u \subset \ker(\pi_u)$ μ-almost everywhere so that π_u descends to a well defined representation on $A(u)$. Therefore

$$\pi_u(a)h(u) = \pi_u(a(u))h(u) \quad \mu\text{-almost everywhere.} \tag{2.3}$$

We can then form the pull-back Hilbert bundle $s^*(G^{(0)} * \mathfrak{H}) =: G *_s \mathfrak{H}$ and define the measure $\nu^{-1} = \int_{G^{(0)}} \lambda_u \mathrm{d}\mu$ (where $\lambda_u(E) = \lambda^u(E^{-1})$) to form a new Hilbert space $L^2(G *_s \mathfrak{H}, \nu^{-1})$. Now the functions $h \in L^2(G *_s \mathfrak{H}, \nu^{-1})$ have the property that $h(\gamma) \in \mathcal{H}(s(\gamma))$. So that,

$$\operatorname{Ind}\pi(f)h(\gamma) = \int_G \pi_{s(\gamma)}(\alpha_\gamma^{-1}(f(\eta)))h(\eta^{-1}\gamma)\mathrm{d}\lambda^{r(\gamma)}(\eta) \tag{2.4}$$

defines a representation of $\Gamma_c(G, r^*\mathscr{A})$ induced by π on $L^2(G *_s \mathfrak{H}, \nu^{-1})$. We call these representations *regular* and define the reduced norm on $\Gamma_c(G, r^*\mathscr{A})$ to be

$$\|f\|_r := \sup\{\|\operatorname{Ind}\pi(f)\| : \pi \text{ is a } C_0(G^{(0)})\text{-linear representation of } A\}. \tag{2.5}$$

REMARK 2.5. This definition is consistent with those given in Definition 2.45 of [17] and p. 82 of [23], but is *a priori* different from that given in p. 146 of [1]. We suspect that all of these definitions agree, but have yet to prove it. However, the set of regular representations used in p. 146 of [1] is the subset of the regular representations defined above such that μ_π is a point mass measure. Thus $\|\cdot\|_{\mathrm{r}}$ is greater than or equal to the norm $\|\cdot\|_{\mathrm{red}}$ considered in [1] which is enough for our purposes.

As usual we can define the reduced crossed product of a dynamical system $(\mathscr{A}, G, \alpha)$, denoted $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$, to be the completion of $\Gamma_{\mathrm{c}}(G, r^*\mathscr{A})$ in the norm $\|\cdot\|_{\mathrm{r}}$.

In this paper we will also use the I-norm on $\Gamma_{\mathrm{c}}(G, r^*\mathscr{A})$ given by

$$\|f\|_I := \max\Big\{ \sup\Big\{ \int \|f\| \mathrm{d}\lambda_u \Big\}, \sup\Big\{ \int \|f\| \mathrm{d}\lambda^u \Big\} \Big\}.$$

We denote the completion of $\Gamma_{\mathrm{c}}(G, r^*\mathscr{A})$ in this norm by $L^I(G, r^*\mathscr{A})$.

We should note that $\operatorname{Ind}\pi(f)$ makes sense for $f \in L^I(G, r^*\mathscr{A})$, so $A \rtimes_{\alpha,\mathrm{r}} G$ is also the completion of $L^I(G, r^*\mathscr{A})$ in $\|\cdot\|_{\mathrm{r}}$.

3. PROPER ACTIONS

3.1. DEFINING PROPER DYNAMICAL SYSYTEMS.

The following definition is modeled after Definition 1.2 of [27].

DEFINITION 3.1. Suppose $(\mathscr{A}, G, \alpha)$ is a groupoid dynamical system and let $A = \Gamma_0(G^{(0)}, \mathscr{A})$ be the associated $C_0(G^{(0)})$-algebra. We say that the dynamical system $(\mathscr{A}, G, \alpha)$ is *proper* if there exists a dense $*$-subalgebra $A_0 \subset A$, such that the following two conditions hold:

(i) For all $a, b \in A_0$, the function ${}_E\langle a, b\rangle : \gamma \mapsto a(r(\gamma))\alpha_\gamma(b(s(\gamma))^*)$ is integrable. That is, the function $\gamma \mapsto {}_E\langle a, b\rangle(\gamma)$ is in $L^I(G, r^*\mathscr{A})$.

(ii) Let

$$M(A_0)^\alpha := \{d \in M(A) : A_0 d \subset A_0,\ \overline{\alpha}_\gamma(d(s(\gamma))) = d(r(\gamma))\}. \tag{3.1}$$

Then for all $a, b \in A_0$, there exists a unique element $\langle a, b\rangle_D \in M(A_0)^\alpha$ such that for all $c \in A_0$

$$(c \cdot \langle a, b\rangle_D)(u) = \int_G c(r(\gamma))\alpha_\gamma(a^* b(s(\gamma)))\mathrm{d}\lambda^u(\gamma). \tag{3.2}$$

For a proper dynamical system, $(\mathscr{A}, G, \alpha)$, we denote $\operatorname{span}\{{}_E\langle a, b\rangle : a, b \in A_0\}$ by E_0. Now since the functions ${}_E\langle a, b\rangle$ are integrable, $E_0 \subset \mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$ and we denote $E = \overline{E}_0$ in $A \rtimes_{\alpha,\mathrm{r}} G$.

REMARK 3.2. One may wonder at first why we chose $A_0 \subset A$ instead of $A_0 \subset \mathscr{A}$. But since condition (i) is a condition about integrability of sections, A_0 had to be a subset of the section algebra instead of a subset of the bundle.

REMARK 3.3. In Section 1 of [27], for a group dynamical system (B, H, β), Rieffel defines $M(B_0)^{\beta} := \{d \in M(B) : B_0 d \subset B_0,\ \overline{\beta}_{\gamma}(d) = d\}$. That is $M(B_0)^{\beta}$ is the set of β-invariant elements of $M(B)$ that map B_0 to itself. Now, in a groupoid dynamical system $(\mathscr{A}, G, \alpha)$, $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$, thus if $s(\gamma) \neq r(\gamma)$ then $\overline{\alpha}_\gamma(c)$ can not equal c for $c \in M(A(s(\gamma)))$. However, if $d \in M(A)$, then d fibres over $G^{(0)}$ and $\overline{\alpha}_\gamma(d(s(\gamma)))$ acts on $A(r(\gamma))$. So we will call $d \in M(A)$ α-invariant if $\overline{\alpha}_\gamma(d(s(\gamma))) = d(r(\gamma))$ for all $\gamma \in G$. This is how we define it in (3.1).

To see that this is a reasonable definition, first note that if G is a group then $r(\gamma) = s(\gamma) = e$ for all $\gamma \in G$. Thus $\overline{\alpha}_\gamma(d(s(\gamma))) = d(r(\gamma))$ reduces to $\overline{\alpha}_\gamma(d) = d$, which is the definition of α-invariant in the group case. Also compare to Lemma 3.1.11 of [1] and consider the following example.

EXAMPLE 3.4. Let $A = C_0(G^{(0)})$, then A is a $C_0(G^{(0)})$-algebra and the associated upper semicontinuous C^*-bundle is $\mathscr{T} := G^{(0)} \times \mathbb{C}$ (i.e. $C_0(G^{(0)}) \cong \Gamma_0(G^{(0)}, \mathscr{T})$). Let G act on $\mathscr{T}$ by left translation, that is $\mathrm{lt}_\gamma(s(\gamma), \xi) = (r(\gamma), \xi)$. Now $M(A) \cong \Gamma^{\mathrm{b}}(G^{(0)}, \mathscr{T}) \cong C^{\mathrm{b}}(G^{(0)})$, where $\Gamma^{\mathrm{b}}(G^{(0)}, \mathscr{T})$ (respectively $C^{\mathrm{b}}(G^{(0)})$) denotes the continuous bounded sections (respectively functions) on $G^{(0)}$, so if $d \in M(A)$ is lt-invariant, then

$$(r(\gamma), d(r(\gamma))) = \overline{\mathrm{lt}}_\gamma(s(\gamma), d(s(\gamma))) = (r(\gamma), d(s(\gamma))).$$

That is d is constant on orbits and we can view d as a function in $C^{\mathrm{b}}(G \backslash G^{(0)})$. We should note that a little work shows $C_0(G^{(0)}) \rtimes_{\mathrm{lt,r}} G \cong C^*_{\mathrm{r}}(G)$.

EXAMPLE 3.5. Suppose (A, H, β) is a proper group dynamical system with respect to the subalgebra A_0 as in Definition 1.2 of [27]. Then (A, H, β) is a proper groupoid dynamical system with respect to Definition 3.1 once we make the standard allowances for the lack of modular function in the groupoid definition.

REMARK 3.6. Definition 1.2 of [27] has an extra condition that we do not assume in Definition 3.1. He assumes that $\beta_s(A_0) \subset A_0$ for all s in the group H where (A, H, β) is a group dynamical system. This assumption allows him to show that E is an ideal in the reduced crossed product. Unfortunately, we have not yet been able to find a well defined analogous condition for groupoid dynamical systems. This means a group dynamical system can be a proper *groupoid* dynamical system without being a proper *group* dynamical system under Definition 1.2 of [27].

LEMMA 3.7. *If $(\mathscr{A}, G, \alpha)$ is a proper dynamical system then the action*

$$(f \cdot c)(u) := \int_G f(\gamma)\alpha_\gamma(c(s(\gamma)))\mathrm{d}\lambda^u(\gamma), \tag{3.3}$$

for $f \in L^I(G, r^*\mathscr{A})$ and $c \in A$, and inner product in condition (i) *of Definition* 3.1 *define a pre-Hilbert module structure on ${}_{E_0}A_0$.*

Proof. First note that $\|\gamma \mapsto f(\gamma)\alpha_\gamma(c(s(\gamma)))\| \leqslant \|f\|_I\|c\|$ so that the action is bounded. For $a, b, c \in A_0$, equations (3.2) and (3.3) imply that

$$ {}_E\langle a, b\rangle \cdot c = a\langle b, c\rangle_D. $$

Furthermore, since $\langle b, c\rangle_D \in M(A_0)^\alpha$ by Definition 3.1, the right hand side is in A_0, therefore $E_0 \cdot A_0 \subset A_0$. The linear and adjoint relations are routine. To show E_0 is a subalgebra of $A \rtimes_{\alpha,\mathrm{r}} G$ and the action of E_0 commutes with the inner product, we perform the following computation. For $a, b, c, d \in A_0$,

$$
\begin{aligned}
{}_E\langle a, b\rangle * {}_E\langle c, d\rangle(\gamma) &= \int_G {}_E\langle a, b\rangle(\eta)\alpha_\eta({}_E\langle c, d\rangle(\eta^{-1}\gamma))\mathrm{d}\lambda^{r(\gamma)}(\eta) \\
&= \int_G {}_E\langle a, b\rangle(\eta)\alpha_\eta(c(s(\eta))\alpha_{\eta^{-1}\gamma}(d(s(\gamma))^*))\mathrm{d}\lambda^{r(\gamma)}(\eta) \\
&= \Big(\int_G {}_E\langle a, b\rangle(\eta)\alpha_\eta(c(s(\eta)))\mathrm{d}\lambda^{r(\gamma)}(\eta)\Big)\alpha_\gamma(d(s(\gamma))^*) \\
&= {}_E\langle {}_E\langle a, b\rangle \cdot c, d\rangle(\gamma).
\end{aligned}
$$

To show the inner product is positive we will use the following lemma.

LEMMA 3.8. *Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system and π a (separable) $C_0(G^{(0)})$-linear representation of $A = \Gamma_0(G^{(0)}, \mathscr{A})$. Decompose π as in* (2.2)*. If $a \in A_0$ and $h \in L^2(G *_s \mathfrak{H}, \nu^{-1})$ then*

$$
\begin{aligned}
\langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h, h\rangle = \int_{G^{(0)}} \Big\langle &\int_G \pi_u(\alpha^{-1}_{\eta^{-1}}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^u(\eta), \\
&\int_G \pi_u(\alpha^{-1}_{\gamma^{-1}}(a(s(\gamma)^*)))h(\gamma^{-1})\mathrm{d}\lambda^u(\gamma)\Big\rangle_{\mathcal{H}(u)}(\gamma)\mathrm{d}\mu(u). \qquad (3.4)
\end{aligned}
$$

Proof. We now compute:

$$
\begin{aligned}
&\langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h, h\rangle \\
&= \int_G \langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h(\gamma), h(\gamma)\rangle_{\mathcal{H}(s(\gamma))}\mathrm{d}\nu^{-1}(\gamma) \\
&= \int_G\int_G \langle \pi_{s(\gamma)}(\alpha_\gamma^{-1}(a(r(\eta))\alpha_\eta(a(s(\eta))^*)))h(\eta^{-1}\gamma), h(\gamma)\rangle_{\mathcal{H}(s(\gamma))}\mathrm{d}\lambda^{r(\gamma)}(\eta)\mathrm{d}\nu^{-1}(\gamma) \\
&= \int_G\int_G \langle \pi_{s(\gamma)}(\alpha^{-1}_{\eta^{-1}\gamma}(a(s(\eta))^*))h(\eta^{-1}\gamma), \\
&\qquad\qquad \pi_{s(\gamma)}(\alpha_\gamma^{-1}(a(r(\gamma)^*)))h(\gamma)\rangle_{\mathcal{H}(s(\gamma))}\mathrm{d}\lambda^{r(\gamma)}(\eta)\mathrm{d}\nu^{-1}(\gamma).
\end{aligned}
$$

Using the left invariance of the Haar system to replace η with $\gamma\eta$ the above becomes

$$= \int_G \Big\langle \int_G \pi_{s(\gamma)}(\alpha_{\eta^{-1}}^{-1}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^{s(\gamma)}(\eta),$$
$$\pi_{s(\gamma)}(\alpha_\gamma^{-1}(a(r(\gamma)^*)))h(\gamma)\Big\rangle_{\mathcal{H}(s(\gamma))}\mathrm{d}\nu^{-1}(\gamma).$$

But $s(\gamma) = r(\eta)$ so the above becomes

$$= \int_G \Big\langle \int_G \pi_{r(\eta)}(\alpha_{\eta^{-1}}^{-1}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^{s(\gamma)}(\eta),$$
$$\pi_{s(\gamma)}(\alpha_\gamma^{-1}(a(r(\gamma)^*)))h(\gamma)\Big\rangle_{\mathcal{H}(s(\gamma))}\mathrm{d}\nu^{-1}(\gamma)$$
$$= \int_G \Big\langle \int_G \pi_{r(\eta)}(\alpha_{\eta^{-1}}^{-1}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^{r(\gamma)}(\eta),$$
$$\pi_{r(\gamma)}(\alpha_{\gamma^{-1}}^{-1}(a(s(\gamma)^*)))h(\gamma^{-1})\Big\rangle_{\mathcal{H}(r(\gamma))}\mathrm{d}\nu(\gamma).$$

Where ν is the image of ν^{-1} (see p. 441) under inversion and is hence defined by $\nu = \int \lambda^u \mathrm{d}\mu$. By decomposing ν and noticing $r(\eta)=r(\gamma)=u$ the above is equal to

$$\int_{G^{(0)}} \Big\langle \int_G \pi_u(\alpha_{\eta^{-1}}^{-1}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^u(\eta),$$
$$\int_G \pi_u(\alpha_{\gamma^{-1}}^{-1}(a(s(\gamma)^*)))h(\gamma^{-1})\mathrm{d}\lambda^u(\gamma)\Big\rangle_{\mathcal{H}(u)}(\gamma)\mathrm{d}\mu(u). \quad \blacksquare$$

Now, since μ is a positive measure, Lemma 3.8 gives that $\operatorname{Ind}\pi({}_E\langle a,a\rangle)$ is positive. This holds for all induced representations, so ${}_E\langle a,a\rangle$ is positive as an element of $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$ and hence as an element of E, so that ${}_{E_0}A_0$ is a pre-Hilbert module and thus completes to a Hilbert E-module. ∎

3.2. MORITA EQUIVALENCE.

THEOREM 3.9. *Let $(\mathscr{A}, G, \alpha)$ be a proper dynamical system with respect to A_0, $D_0 = \operatorname{span}\{\langle a,b\rangle_D : a,b \in A_0\}$, $E_0 = \operatorname{span}\{{}_E\langle a,b\rangle : a,b \in A_0\}$, and E, A^α be the closures of E_0, D_0 in $\mathscr{A} \rtimes_\alpha G$ and $M(A)$ respectively. Then A_0 equipped with the E_0-action defined in equation* (3.3) *and inner products defined in Definition* 3.1 *is a $E_0 - D_0$ pre-imprimitivity bimodule.*

REMARK 3.10. We call A^α the generalized fixed point algebra for the dynamical system $(\mathscr{A}, G, \alpha)$. So that Theorem 3.9 gives that the generalized fixed point algebra is Morita equivalent to a subalgebra of the reduced crossed product.

Proof of Theorem 3.9. The proof of this theorem follows from Section 1 of [27] fairly closely. From Lemma 3.7, A_0 is a pre-Hilbert E_0-module. The goal is to show that A^α is the imprimitivity algebra for the resulting Hilbert module. From the definition of the D_0-valued inner product and the definition of the E_0-action, it is easy to see that D_0 satisfies the algebraic conditions.

It remains to show that the D_0-action is bounded and adjointable, so that $D_0 \subset \mathcal{L}({}_E\overline{A}_0)$ and furthermore, that the norm of $d \in D_0$ as an element of $\mathcal{L}({}_E\overline{A}_0)$ coincides with its norm as an element of $M(A)$. The last statement ensures that A_0 completes to an $E - A^\alpha$-imprimitivity bimodule.

First, we show that the action of $M(A_0)^\alpha$ on A_0 is bounded. Let π be a $C_0(G^{(0)})$-linear representation of A, and $\operatorname{Ind}\pi$ be the corresponding representation of the reduced crossed product. Pick $a \in A_0$ and $d \in M(A_0)^\alpha$. Using ad in Lemma 3.8 we get

$$\begin{aligned}
&\langle \operatorname{Ind}\pi({}_E\langle ad, ad\rangle)h, h\rangle \\
&\quad = \int_{G^{(0)}} \Big\langle \int_G \pi_u(\alpha^{-1}_{\eta^{-1}}((ad(s(\eta)))^*))h(\eta^{-1})\mathrm{d}\lambda^u(\eta), \\
&\qquad\qquad \int_G \pi_u(\alpha^{-1}_{\gamma^{-1}}((ad(s(\gamma))^*)))h(\gamma^{-1})\mathrm{d}\lambda^u(\gamma)\Big\rangle_{\mathscr{H}(u)} \mathrm{d}\mu(u).
\end{aligned} \tag{3.5}$$

Using the fact that $r(\gamma) = r(\eta) = u$ and $\overline{\alpha}_\gamma(d(s(\gamma))) = d(r(\gamma))$, (3.5) is equal to

$$\begin{aligned}
&= \int_{G^{(0)}} \Big\langle \int_G \overline{\pi}_u(d(u)^*)\pi_{r(\eta)}(\alpha^{-1}_{\eta^{-1}}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^u(\eta), \\
&\qquad\qquad \int_G \overline{\pi}_u(d(u)^*)\pi_{r(\gamma)}(\alpha^{-1}_{\gamma^{-1}}(a(s(\gamma))^*))h(\gamma^{-1})\mathrm{d}\lambda^u(\gamma)\Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u) \\
&\leqslant \int_{G^{(0)}} \Big\langle \|d\|^2_{M(A)} \int_G \pi_{r(\eta)}(\alpha^{-1}_{\eta^{-1}}(a(s(\eta))^*))h(\eta^{-1})\mathrm{d}\lambda^u(\eta), \\
&\qquad\qquad \int_G \pi_{r(\gamma)}(\alpha^{-1}_{\gamma^{-1}}(a(s(\gamma))^*))h(\gamma^{-1})\mathrm{d}\lambda^u(\gamma)\Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u) \\
&= \|d\|^2_{M(A)}\langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h, h\rangle.
\end{aligned}$$

Here the inequality follows from the fact that $d^*d \leqslant \|d\|^2$, μ is a positive measure and the integrand is positive. Since this holds for all induced representations of $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$ we have $\|ad\|_{A_0} \leqslant \|a\|_{A_0}\|d\|_{M(A)}$.

It is not hard to see that d^* is the adjoint for d as an element of $L({}_E\overline{A}_0)$, so that d extends to an adjointable operator on ${}_E\overline{A}_0$.

It remains to show the norm of d as an element of $M(A)$ is the same as the norm of d as an element of $\mathcal{L}({}_E\overline{A}_0)$. We do this by taking a faithful a representation π of A and constructing an $a \in A_0$ and $h \in L^2(G *_s \mathfrak{H})$ such that $\langle \operatorname{Ind}\pi({}_E\langle ad, ad\rangle)h, h\rangle$ is close to $\|d\|^2_{M(A)}$ and $\langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h, h\rangle$ is close to 1. It follows that $\langle \operatorname{Ind}\pi({}_E\langle ad, ad\rangle h), h\rangle$ is close to $\|d\|^2_{M(A)}\langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h, h\rangle$.

For a faithful representation $\pi : A \to L^2(G^{(0)} * \mathfrak{H})$ of A, the idea due to Rieffel ([27], p. 151) is the following. We first pick $v \in L^2(G^{(0)} * \mathfrak{H})$ such that $\overline{\pi}(d)v$ is close to $\|d\|$, and find an $a \in A_0$ such that $\pi(a)v$ is close to v. We then let h be a vector in $L^2(G *_s \mathfrak{H})$ that extends v and is supported on a small neighborhood of $G^{(0)}$. The calculation used to show d is bounded will then also show that $\langle \operatorname{Ind}\pi({}_E\langle ad, ad\rangle)h, h\rangle$ is close to $\|d\|^2_{M(A)}$ and $\langle \operatorname{Ind}\pi({}_E\langle a, a\rangle)h, h\rangle$ is close to 1. We are left with checking the technical details.

Let $d \in M(A_0)^\alpha$ be given and let $\varepsilon > 0$ be small. Suppose that π is a faithful nondegenerate representation of A. Then $\overline{\pi}$ is a faithful nondegenerate representation of $M(A)$. Thus there exists $v \in \mathcal{H}_\pi$ such that

$$\|\overline{\pi}(d)v\|^2 + \frac{\varepsilon}{6} > \|d\|^2_{M(A)}. \tag{3.6}$$

Now there exists a Borel Hilbert bundle $G^{(0)} * \mathfrak{H}$ and a finite measure μ on $G^{(0)}$ such that $\pi \cong \int^{\oplus}_{G^{(0)}} \pi_u \,\mathrm{d}\mu(u)$. We identify π with its direct integral and $\mathcal{H}_\pi$ with $L^2(G^{(0)} * \mathfrak{H})$ and furthermore we can assume that v under this identification has compact support K_v.

To justify the above assumption on v, note that since μ is a finite measure on a second countable locally compact Hausdorff space, it is regular by Theorem 2.18 of [29]. Thus, there exists a compact set K such that $\mu(X \setminus K)$ is small. Now suppose $v' \in L^2(G^{(0)} * \mathfrak{H})$, then $\chi_K v'$ is also in $L^2(G^{(0)} * \mathfrak{H})$, and has compact support. Furthermore, it is not hard to see that $\|v' - \chi_K v'\|$ is small. Lastly, an easy computation shows that if $\|v'\| = 1$ and v'' is close to v' then the normalization of v'' is also close to v'. So if v does not have compact support we can replace v with $\chi_K v / \|\chi_K v\|$ for some compact set $K \subset X$ sufficiently large.

Now pick a_0 close to an approximate unit of A such that

$$\|\pi((a_0 d)^*)v\|^2 + \frac{\varepsilon}{6} > \|\overline{\pi}(d^*)v\|^2, \tag{3.7}$$

$$\|\pi(a_0)v\|^2 + \frac{\varepsilon}{6\|d\|^2} > \|v\|^2 = 1, \text{ and} \tag{3.8}$$

$$\|a_0\| < 1. \tag{3.9}$$

We will use a constant multiple of a_0 as our a. Before we state the next lemma we need a definition. A subset L of a topological groupoid G is called *s-relatively compact*, if $L \cap s^{-1}(K)$ is relatively compact for every compact subset $K \subset G^{(0)}$. r-relatively compact subsets are defined similarly. A compactness argument shows the next lemma which we will use to find an appropriate small neighborhood of $G^{(0)}$.

LEMMA 3.11. *Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system, and suppose $a \in \Gamma_0(G^{(0)}, \mathscr{A})$. Fix $\varepsilon > 0$, then there exists an open neighborhood V of $G^{(0)}$ in G such that V is both r- and s-relatively compact and $\|\alpha_\gamma(a(s(\gamma))) - a(r(\gamma))\| < \varepsilon$ for all $\gamma \in V$.*

Using Lemma 3.11 pick a symmetric r, s-relatively compact open neighborhood V_ε of $G^{(0)}$ such that for all $\gamma \in V_\varepsilon$

$$\|\alpha_\gamma(a_0^*(s(\gamma))) - a_0^*(r(\gamma))\| < \frac{\varepsilon}{12\|d\|^2}. \tag{3.10}$$

Since V_ε is s-relatively compact, $s^{-1}(u) \cap V_\varepsilon$ is relatively compact. Hence,

$$\lambda_u(V_\varepsilon) = \lambda_u(s^{-1}(u) \cap V_\varepsilon) \leqslant \lambda_u(\overline{s^{-1}(u) \cap V_\varepsilon}) < \infty.$$

Furthermore, since V_ε is open and $u \in V_\varepsilon$, we have $\lambda_u(V_\varepsilon) \neq 0$. Thus

$$\widetilde{h}(\gamma) := \frac{\chi_{V_\varepsilon}(\gamma)v(s(\gamma))}{\lambda_{s(\gamma)}(V_\varepsilon)} \tag{3.11}$$

is defined and less than infinity for all $\gamma \in G$.

CLAIM 3.12. $\widetilde{h} \in L^2(G *_s \mathfrak{H}, \nu^{-1})$.

Proof. Now

$$\begin{aligned}\|\widetilde{h}\|_2^2 &= \int_{G^{(0)}} \int_G \Big\langle \frac{\chi_{V_\varepsilon}(\gamma)v(s(\gamma))}{\lambda_{s(\gamma)}(V_\varepsilon)}, \frac{\chi_{V_\varepsilon}(\gamma)v(s(\gamma))}{\lambda_{s(\gamma)}(V_\varepsilon)} \Big\rangle_{\mathcal{H}(s(\gamma))} \mathrm{d}\lambda_u(\gamma)\mathrm{d}\mu(u) \\ &= \int_{G^{(0)}} \frac{\chi_{K_v}(u)}{(\lambda_u(V_\varepsilon))} \langle v(u), v(u)\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u).\end{aligned}$$

So to show that $\widetilde{h} \in L^2(G *_s \mathfrak{H}, \nu^{-1})$ it suffices to show that $\chi_{K_v}/\lambda_u(V_\varepsilon) \in L^\infty(G^{(0)}, \mu)$. Pick $\psi \in C_c(G)$ such that $\psi|_{K_v} \equiv 1$, $0 \leqslant \psi \leqslant 1$, and $\operatorname{supp}(\psi) \subset V_\varepsilon$. Then by the properties of the Haar system the function

$$\lambda(\psi) : u \mapsto \int_G \psi \mathrm{d}\lambda_u$$

is continuous. So $\lambda(\psi)|_{K_v}$ has a minimum m. Since $\psi|_{K_v} \equiv 1$, $\int \psi \mathrm{d}\lambda_u > 0$ for $u \in K_v$ so that $m > 0$. But $\psi \leqslant \chi_{V_\varepsilon}$, so for $u \in K_v$, we have $m \leqslant \lambda(\psi)(u) \leqslant \lambda_u(V_\varepsilon)$. Thus $\chi_{K_v}/(\lambda_u(V_\varepsilon)) \in L^\infty(G^{(0)}, \mu)$, giving $\widetilde{h} \in L^2(G *_s \mathfrak{H}, \nu^{-1})$. ∎

For $k = \|\widetilde{h}\|$, define:

$$h(\gamma) = \frac{\widetilde{h}(\gamma)}{k} = \frac{\chi_{V_\varepsilon}(\gamma)v(s(\gamma))}{k\lambda_{s(\gamma)}(V_\varepsilon)} \quad \text{and} \quad a = ka_0.$$

The next claim uses this h and a to get the estimates we need to complete the proof.

CLAIM 3.13. *For a, d, π, h, v, and ε chosen as above,*

$$|\langle \operatorname{Ind} \pi({}_E\langle ad, ad\rangle)h, h\rangle - \langle \pi((a_0d)^*)v, \pi((a_0d)^*)v\rangle_{\mathcal{H}_\pi}| < \frac{\varepsilon}{6} \text{ and} \tag{3.12}$$

$$|\langle \operatorname{Ind} \pi({}_E\langle a, a\rangle)h, h\rangle - \langle \pi((a_0)^*)v, \pi((a_0)^*)v\rangle_{\mathcal{H}_\pi}| < \frac{\varepsilon}{6\|d\|^2}. \tag{3.13}$$

Proof. We will compute the estimate for (3.12), the computation for (3.13) is exactly the same. First note that

$$\langle \pi((a_0d)^*)v, \pi((a_0d)^*)v\rangle_{\mathcal{H}_\pi} = \int_{G^{(0)}} \Big\langle \pi_u((a_0d)(u))\lambda_u(V_\varepsilon)\pi_u((a_0d)^*(u)) \lambda_u(V_\varepsilon)\frac{v(u)}{\lambda_u(V_\varepsilon)}, \frac{v(u)}{\lambda_u(V_\varepsilon)}\Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u). \tag{3.14}$$

Using Lemma 3.8 with ad and h we compute:

$$\begin{aligned}
&\langle \mathrm{Ind}\pi({}_E\langle ad, ad\rangle)h, h\rangle \\
&= \int_{G^{(0)}} \Big\langle \int_G \pi_u(\alpha_{\eta^{-1}}((ad(r(\eta)))^*))h(\eta)\mathrm{d}\lambda_u(\eta), \\
&\qquad \int_G \pi_u(\alpha_{\gamma^{-1}}((ad(r(\gamma))^*)))h(\gamma)\mathrm{d}\lambda_u(\gamma)\Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u) \\
&= \int_{G^{(0)}} \Big\langle \overline{\pi}_u(d^*(u)) \int_{V_\varepsilon} \pi_u(\alpha_{\eta^{-1}}(a_0(r(\eta))^*))\mathrm{d}\lambda_u(\eta)\frac{v(u)}{\lambda_u(V_\varepsilon)}, \\
&\qquad \overline{\pi}_u(d^*(u)) \int_{V_\varepsilon} \pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*)))\mathrm{d}\lambda_u(\gamma)\frac{v(u)}{\lambda_u(V_\varepsilon)}\Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u).
\end{aligned}$$

Since $\pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*))) \in B(\mathcal{H}(u))$ for all γ, the map

$$\gamma \mapsto \pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*))) \tag{3.15}$$

is continuous. This along with the compactness of $\overline{s^{-1}(u) \cap V_\varepsilon}$, implies there exists an operator $L(u) \in B(\mathcal{H}(u))$ such that

$$L(u) = \int_{V_\varepsilon} \pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*)))\mathrm{d}\lambda_u(\gamma), \text{ giving}$$

$$\langle \mathrm{Ind}\,\pi({}_E\langle ad, ad\rangle)h, h\rangle = \int_{G^{(0)}} \Big\langle L(u)^*\overline{\pi}_u(dd^*(u))L(u)\frac{v(u)}{\lambda_u(V_\varepsilon)}, \frac{v(u)}{\lambda_u(V_\varepsilon)}\Big\rangle \mathrm{d}\mu(u).$$

Thus from (3.14),

$$\begin{aligned}
&|\langle \mathrm{Ind}\,\pi({}_E\langle ad, ad\rangle)h, h\rangle - \langle \pi((a_0d)^*)v, \pi((a_0d)^*)v\rangle| \\
&\qquad \leqslant \int_{G^{(0)}} \|L(u)^*\overline{\pi}_u(d(u))\overline{\pi}_u(d^*(u))L(u) - \pi_u(a_0dd^*a_0^*(u))(\lambda_u(V_\varepsilon))^2\| \\
&\qquad\qquad \cdot \Big\langle \frac{v(u)}{\lambda_u(V_\varepsilon)}, \frac{v(u)}{\lambda_u(V_\varepsilon)}\Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u).
\end{aligned} \tag{3.16}$$

CLAIM 3.14. *With L, π_u, a_0, d, ε, and V_ε as above*

$$\|L(u)^*\overline{\pi}_u(d(u))\overline{\pi}_u(d^*(u))L(u) - \pi_u(a_0dd^*a_0^*(u))(\lambda_u(V_\varepsilon))^2\| < \frac{\varepsilon}{6}(\lambda_u(V_\varepsilon))^2.$$

Proof. First note that

$$\|L(u)\| \leqslant \int\limits_{V_\varepsilon} \|\pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*)))\| \mathrm{d}\lambda_u(\gamma) < \lambda_u(V_\varepsilon)$$

since $\|a_0\| < 1$ from equation (3.9). An unenlightening computation now shows

$$\begin{aligned} &\|L(u)^*\overline{\pi}_u(d(u))\overline{\pi}_u(d^*(u))L(u) - \pi_u(a_0 dd^* a_0^*(u))(\lambda_u(V_\varepsilon))^2\| \\ &\qquad\qquad \leqslant 2\|d\|^2\lambda_u(V_\varepsilon)\|L(u) - \pi_u(a_0^*(u))\lambda_u(V_\varepsilon)\|. \end{aligned} \tag{3.17}$$

Now,

$$\begin{aligned} &\|L(u) - \pi_u(a_0^*(u))\lambda_u(V_\varepsilon)\| \\ &\quad = \Big\| \int\limits_{V_\varepsilon} \pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*)))\mathrm{d}\lambda_u(\gamma) - \pi_u(a_0^*(u))\lambda_u(V_\varepsilon) \Big\| \\ &\quad \leqslant \int\limits_{V_\varepsilon} \|\pi_u(\alpha_{\gamma^{-1}}(a_0(r(\gamma)^*))) - \pi_u(a_0^*(u))\| \mathrm{d}\lambda_u(\gamma) < \frac{\varepsilon}{12\|d\|^2}\lambda_u(V_\varepsilon) \end{aligned}$$

by equation (3.10). Thus using equation (3.17), we have

$$\begin{aligned} &\|L(u)^*\overline{\pi}_u(d(u))\overline{\pi}_u(d(u)^*)L(u) \\ &\qquad - \pi_u(a_0 dd^* a_0^*(u))(\lambda_u(V_\varepsilon))^2\| < \frac{\varepsilon}{6}(\lambda_u(V_\varepsilon))^2. \end{aligned}$$ ∎

Combining Claim 3.14 with (3.16) we get

$$\begin{aligned} &|\langle \mathrm{Ind}\,\pi({}_E\langle ad, ad\rangle)h, h\rangle - \langle \pi((a_0 d)^*)v, \pi((a_0 d)^*)v\rangle| \\ &\qquad < \int\limits_{G^{(0)}} \frac{\varepsilon}{6}(\lambda_u(V_\varepsilon))^2 \Big\langle \frac{v(u)}{\lambda_u(V_\varepsilon)}, \frac{v(u)}{\lambda_u(V_\varepsilon)} \Big\rangle_{\mathcal{H}(u)} \mathrm{d}\mu(u) = \frac{\varepsilon}{6}, \end{aligned} \tag{3.18}$$

giving equation (3.12). ∎

Thus by combining equations (3.6), (3.7) and (3.12) we get

$$|\langle \mathrm{Ind}\,\pi({}_E\langle ad, ad\rangle)h, h\rangle - \|d\|^2_{M(A)}| < \frac{\varepsilon}{2}. \tag{3.19}$$

Similarly by combining equations (3.8) and (3.13) we get

$$|\langle \mathrm{Ind}\,\pi({}_E\langle a, a\rangle)h, h\rangle - 1| < \frac{\varepsilon}{2\|d\|^2_{M(A)}}. \tag{3.20}$$

Now equations (3.19) and (3.20) give

$$|\langle \mathrm{Ind}\,\pi({}_E\langle ad, ad\rangle)h, h\rangle - \|d\|^2_{M(A)}\langle \mathrm{Ind}\,\pi({}_E\langle a, a\rangle)h, h\rangle| < \varepsilon.$$

Thus $\|d\|_{\mathcal{L}({}_E\overline{A}_0)} = \|d\|_{M(A)}$ as desired. This completes the proof of Theorem 3.9. ∎

4. FUNDAMENTAL EXAMPLES

PROPOSITION 4.1. *Suppose G is a groupoid acting properly on space X, then $(C_0(X), G, \mathrm{lt})$ is a proper groupoid dynamical system with respect to the dense subalgebra $C_c(X)$. Furthermore, $C_0(X)^{\mathrm{lt}} \cong C_0(G\backslash X)$.*

Before proceeding we should note that if $\mathscr{C}$ is the upper semicontinuous C^*-bundle associated to $C_0(X)$, it is not hard to see that the fibres of $\mathscr{C}$ are given by $\{C_0(r_X^{-1}(u))\}_{u\in G^{(0)}}$. The action lt is then given by $\mathrm{lt}_\gamma(f)(x) = f(\gamma^{-1}\cdot x)$ for $x \in r_X^{-1}(r(\gamma))$ and $f \in C_0(r_X^{-1}(s(\gamma)))$. Furthermore, the bundle $\mathscr{C}$ is actually a *continuous* C^*-bundle by Theorem 3.26 of [32], that is the map from $\mathscr{C} \to \mathbb{C}$ given by $c \mapsto \|c\|$ is continuous.

Now to show Proposition 4.1 we need to show:

(i) For all $f, g \in C_c(X)$, the function

$${}_E\langle f, g\rangle(\gamma) := f|_{r_X^{-1}(r(\gamma))}\,\mathrm{lt}_\gamma(g^*|_{r_X^{-1}(s(\gamma))}) = (x \in r_X^{-1}(r(\gamma)) \mapsto f(x)\overline{g(\gamma^{-1}\cdot x)})$$

is integrable;

(ii) If $f, g \in C_c(X)$, there exists a function $\langle f, g\rangle_D \in C^{\mathrm{b}}(G\backslash X) \subset M(C_c(X))^{\mathrm{lt}}$, such that for all $h \in C_c(X)$

$$h\langle f, g\rangle_D|_{r_X^{-1}(u)} = \int_G h|_{r_X^{-1}(u)}\,\mathrm{lt}_\gamma(f^*g|_{r_X^{-1}(s(\gamma))})\mathrm{d}\lambda^u(\gamma). \tag{4.1}$$

First we will show that ${}_E\langle f, g\rangle$ is integrable for $f, g \in C_c(X)$. Consider the continuous function

$$G * X \to \mathbb{C}$$
$$(\gamma, x) \mapsto f(x)\overline{g(\gamma^{-1}\cdot x)}.$$

Using the properness of the G-action, it is not hard to see that this function has compact support and is hence integrable. This gives that ${}_E\langle f, g\rangle$ is integrable.

It remains to show property (ii). Given $F \in C_c(X)$, it suffices to show there exists a function $d \in C^{\mathrm{b}}(G\backslash X)$ such that

$$h(x)d(x) = \int_G h(x)F(\gamma^{-1}\cdot x)\mathrm{d}\lambda^{r_X(x)}(\gamma) \quad \forall\, h \in C_c(X),\ x \in X. \tag{4.2}$$

Using the properness of the G-action, a compactness argument shows the set $L := \{\gamma \in G : F(\gamma^{-1}\cdot x) \neq 0\}$ is relatively compact for a fixed $x \in X$, and hence the function $\gamma \mapsto F(\gamma^{-1}\cdot x)$ is $\lambda^{r_X(x)}$-integrable. So we can define

$$d(x) := \int_G F(\gamma^{-1}\cdot x)\mathrm{d}\lambda^{r_X(x)}(\gamma).$$

This d certainly satisfies equation (4.2). It remains to show that $d(x) \in C^{\mathrm{b}}(G\backslash X)$. For this, we will use the following stronger lemma from [18] which we will restate here for convenience.

LEMMA 4.2 ([18], Lemma 2.9). *Let G act properly on the left of a locally compact Hausdorff space X, if $f \in C_c(X)$, then*

$$\lambda(f)([x]) = \int_G f(\gamma^{-1} \cdot x) \mathrm{d}\lambda^{r_X(x)}(\gamma)$$

defines a map of $C_c(X)$ onto $C_c(G\backslash X)$.

Lemma 4.2 now guarantees that $d(x) = \lambda(F)(x)$ is in $C_c(G\backslash X)$. So condition (ii) is satisfied and the action of G on $C_0(X)$ by left translation is proper. Furthermore, the onto assertion of Lemma 4.2 gives that the generalized fixed point algebra, $C_0(X)^{\mathrm{lt}}$, is $C_0(G\backslash X)$.

REMARK 4.3. Suppose $X = G^{(0)}$ in Proposition 4.1, since $r_{G^{(0)}}$ is the identity map, the associated bundle $\mathscr{C} = \mathscr{T} = G^{(0)} \times \mathbb{C}$. Furthermore, by Theorem 3.9, $C_0(G\backslash G^{(0)})$ is Morita equivalent to a subalgebra of $C_0(G^{(0)}) \rtimes_{\mathrm{lt,r}} G \cong C^*_{\mathrm{r}}(G)$. In particular if $G = H \times X$ is the transformation group groupoid then $C_0(H\backslash X)$ is Morita equivalent to a subalgebra of $C_0(X) \rtimes_{\mathrm{lt,r}} H$.

PROPOSITION 4.4. *Let G be a proper groupoid, $(\mathscr{A}, G, \alpha)$ a groupoid dynamical system, and A be the $C_0(G^{(0)})$-algebra corresponding to $\mathscr{A}$. Then $(\mathscr{A}, G, \alpha)$ is proper with respect to the subalgebra $A_0 = C_c(G^{(0)}) \cdot A$.*

REMARK 4.5. Notice that the hypotheses and conclusions of Proposition 4.4 are similar to those of Theorem 5.7 in [28].

REMARK 4.6. If $G = H \times X$ is a transformation group groupoid, then G acts properly on its unit space if and only if H acts properly on X.

Proof of Proposition 4.4. First note that $C_c(G^{(0)}) \cdot A$ is dense in A. To show that the dynamical system $(\mathscr{A}, G, \alpha)$ is proper, we first need to show that the functions

$$\begin{aligned} {}_E\langle f \cdot a, g \cdot b\rangle : \gamma \mapsto & f(r(\gamma))a(r(\gamma))\alpha_\gamma(\overline{g(s(\gamma))}b^*(s(\gamma))) \\ & (= f(r(\gamma))\overline{g(s(\gamma))}a(r(\gamma))\alpha_\gamma(b^*(s(\gamma)))) \end{aligned}$$

are integrable for $a, b \in A$ and $f, g \in C_c(G^{(0)})$. Using the properness of the G it is not hard to see that these functions have compact support. To finish showing that $\|{}_E\langle f \cdot a, g \cdot b\rangle\|_I < \infty$, we use the following lemma.

LEMMA 4.7. *Let G be a groupoid, $\mathscr{B}$ be an upper semicontinuous C^*-bundle over G and suppose $f \in \Gamma_c(G, \mathscr{B})$. Then $\|f\|_I < \infty$.*

The proof of Lemma 4.7 relies on the following proposition. The proof of this proposition is a standard compactness argument which we omit.

PROPOSITION 4.8. *Let X be a locally compact Hausdorff space, and $f : X \to \mathbb{R}_{\geqslant 0}$ be an upper semicontinuous function with compact support, then $\|f\|_\infty < \infty$.*

Proof of Lemma 4.7. Let K be the support of f. By Proposition 4.8 we know that $\|f\|_\infty < \infty$. So

$$\int_G \|f(\gamma)\| \mathrm{d}\lambda^u(\gamma) \leqslant \int_G \chi_K(\gamma)\|f\|_\infty \mathrm{d}\lambda^u(\gamma) = \|f\|_\infty \lambda^u(K).$$

Similarly $\int_G \|f(\gamma)\|\mathrm{d}\lambda_u(\gamma) \leqslant \|f\|_\infty \lambda_u(K)$. Since $\sup\{\lambda^u(K), \lambda_u(K)\} < \infty$ we have $\|f\|_I < \infty$. ∎

It remains to show that for $f \cdot a, g \cdot b \in C_c(G^{(0)}) \cdot A$ that there exists an element $\langle f \cdot a, g \cdot b\rangle_D \in M(C_c(G^{(0)}) \cdot A)^\alpha$ such that for all $h \cdot c \in A_0$,

$$((h \cdot c)\langle f \cdot a, g \cdot b\rangle_D)([u]) = \int_G (h \cdot c)(r(\gamma))\alpha_\gamma((f \cdot a)^*(s(\gamma))(g \cdot b)(s(\gamma)))\mathrm{d}\lambda^u(\gamma).$$

For this we will follow Lemma 6.17 of [31] and Lemma 3.5 of [9].

REMARK 4.9. Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system, and suppose G acts on the left of a locally compact Hausdorff space X. We can define the pull back bundle

$$r_X^*\mathscr{A} := \{(x, a) : r_X(x) = p_{\mathscr{A}}(a)\}$$

and a continuous action of G on $r_X^*\mathscr{A}$ via

$$\alpha_\gamma^{r_X}(x, a) = (\gamma \cdot x, \alpha_\gamma(a)).$$

DEFINITION 4.10. Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system and suppose that G acts on the left of a locally compact Hausdorff space X. Define

$$\begin{aligned}\mathrm{Ind}_G^{G^{(0)}}(\mathscr{A}, \alpha) := \{f \in \Gamma^{\mathrm{b}}(X, r_X^*\mathscr{A}) : f(x) = (\alpha_\gamma^{r_X})^{-1}(f(\gamma \cdot x))\\ \text{and } ([u] \mapsto \|f(u)\|) \text{ vanishes at } \infty\}.\end{aligned}$$

To finish the proof of Proposition 4.4, we will show that $\mathrm{Ind}_G^X(\mathscr{A}, \alpha) \subset M(A)^\alpha$ and that for $a, b \in A_0$, there exists a $d \in \mathrm{Ind}_G^X(\mathscr{A}, \alpha)$ satisfying the required properties in Definition 3.1.

REMARK 4.11. If H is a group and A is a C^*-algebra, $\mathrm{Ind}_H^{G^{(0)}}(A, \alpha)$ is normally defined as:

$$\begin{aligned}\mathrm{Ind}_H^X(A, \alpha) := \{f \in C^{\mathrm{b}}(X, A) : f(x) = \alpha_s^{-1}(f(s \cdot x))\\ \text{and } ([u] \mapsto \|f(u)\|) \in C_0(H\backslash X)\}.\end{aligned}$$

This definition does not make sense for an upper semicontinuous C^*-bundle, since the norm is upper semicontinuous. But in the group case, the continuity of $[u] \mapsto \|f(u)\|$ is implied by the condition that $f \in C^{\mathrm{b}}(X, A)$. So the important part of the condition $([u] \mapsto \|f(u)\|) \in C_0(H\backslash X)$ is that the function $[u] \mapsto \|f(u)\|$ vanishes at infinity.

LEMMA 4.12. *Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system. For $f \in C_c(G^{(0)})$ and $a \in A = \Gamma_0(G^{(0)}, \mathscr{A})$,*

$$\lambda(f \cdot a)(u) := \int_G \alpha_\gamma(f(s(\gamma)a(s(\gamma))))\mathrm{d}\lambda^u(\gamma) \tag{4.3}$$

gives a well-defined element of $\mathrm{Ind}_G^{G^{(0)}}(\mathscr{A}, \alpha)$.

The proof of Lemma 4.12 is essentially the same as that of Lemma 6.17 in [31] with some minor modifications, so we omit it.

CLAIM 4.13. *If $f \in \mathrm{Ind}_G^{G^{(0)}}(\mathscr{A}, \alpha)$ then*

$$m_f : a \mapsto (v \mapsto f(v)a(v))$$

is a multiplier of $A = \Gamma_0(G^{(0)}, \mathscr{A})$.

Proof. Note that $f(v) \in A(v) \subset M(A(v))$ and for $f \in \Gamma^{\mathrm{b}}(G^{(0)}, \mathscr{A})$ (continuous bounded sections), $a \in \Gamma_0(G^{(0)}, \mathscr{A})$ we have $(v \mapsto f(v)a(v)) \in \Gamma_0(G^{(0)}, \mathscr{A})$. Similarly, $v \mapsto f(v)^* a(v) \in \Gamma_0(G^{(0)}, \mathscr{A})$. So Lemma C.11 of [32] implies that $f \in M(A)$. ∎

Claim 4.13 and Lemma 4.12 give that $\lambda(f \cdot a) \in M(A)$ for all $f \in C_c(G^{(0)})$ and $a \in A$. Furthermore, since A is a $C_0(G^{(0)})$-algebra, $C_c(G^{(0)}) \subset Z(M(A))$, so if $m \in M(A)$, $g \in C_c(G^{(0)})$ and $b \in A$ then $m(g \cdot a) = g \cdot (ma) \in C_c(G^{(0)}) \cdot A$. Thus, $\lambda(f \cdot a)(C_c(G^{(0)}) \cdot A) \subset C_c(G^{(0)}) \cdot A$ and so $\lambda(f \cdot a) \in M(C_c(G^{(0)}) \cdot A)$.

Notice

$$\alpha_\eta(\lambda(f \cdot a)(s(\eta))) = \alpha_\eta\Big(\int_G \alpha_\gamma(f \cdot a(s(\gamma)))\mathrm{d}\lambda^{s(\eta)}(\gamma)\Big) = \int_G \alpha_{\eta\gamma}(f \cdot a(s(\eta\gamma)))\mathrm{d}\lambda^{s(\eta)}(\gamma)$$
$$= \int_G \alpha_\gamma(f \cdot a(s(\gamma)))\mathrm{d}\lambda^{r(\eta)}(\gamma) = \lambda(f \cdot a)(r(\eta)).$$

Thus $\lambda(f \cdot a) \in M(A_0)^\alpha$.

Finally, we need to show that for $g \in C_c(G^{(0)})$ and $b \in A$, then

$$((g \cdot b)\lambda(f \cdot a))(u) = \int_G g(r(\gamma))b(r(\gamma))\alpha_\gamma(f \cdot a(s(\gamma)))\mathrm{d}\lambda^u(\gamma).$$

But this is just a straight forward calculation.

Notice that $(f \cdot a)(g \cdot b) = fg \cdot ab \in C_c(G^{(0)}) \cdot A$. Thus we can define $\langle f \cdot a, g \cdot b\rangle_D := \lambda((f \cdot a)^*(g \cdot b))$ and from the above argument this $\langle f \cdot a, g \cdot b\rangle_D$ has the desired properties, making $(\mathscr{A}, G, \alpha)$ a proper dynamical system with respect to the subalgebra $C_c(G^{(0)}) \cdot A$. ∎

REMARK 4.14. The subalgebra $E \subset \mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$ guaranteed by Proposition 4.4 and Theorem 3.9 is actually an ideal in $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$. To see this, suppose $f \in \Gamma_c(G, r^*\mathscr{A})$, and $a, b \in A_0$ then by a similar calculation to the one given in Lemma 3.8 we get

$$(f *_E\langle a, b\rangle)(\gamma) = {}_E\langle f \cdot a, b\rangle(\gamma).$$

So to show that E is an ideal in $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$ it suffices to show that $f \cdot A_0 \subset A_0$. Now suppose G acts properly on its unit space and $A_0 = C_{\mathrm{c}}(G^{(0)}) \cdot A$ as in Proposition 4.4, if $a \in A_0$ then

$$f \cdot a : u \mapsto \int_G f(\eta)\alpha_\eta(a(s(\eta)))\mathrm{d}\lambda^u(\eta).$$

But the integrand $f(\eta)\alpha_\eta(a(s(\eta)))$ is continuous since f, a and the G action are. It has compact support since f does. Thus $f \cdot a$ is a continuous compactly supported section, giving $f \cdot a \in A_0$, so $f \cdot A_0 \subset A_0$ and hence E is an ideal.

A similar argument shows that the subalgebra $E \subset C_0(X) \rtimes_{\mathrm{lt},\mathrm{r}} G$ (see Proposition 4.1), guaranteed by Theorem 3.9 is also an ideal of the reduced crossed product.

5. SATURATION

Theorem 3.9 guarantees that if $(\mathscr{A}, G, \alpha)$ is a proper dynamical system, then the generalized fixed point algebra is Morita equivalent to a subalgebra of the reduced crossed product, $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$. This theorem is most useful when this subalgebra is itself an object we would like to study. In particular, we are interested in when this subalgebra is actually the algebra $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G$. So following [27] we make the following definition.

DEFINITION 5.1. We call a proper dynamical system $(\mathscr{A}, G, \alpha)$, *saturated* if ${}_{E_0}A_{0D_0}$ completes to a $\mathscr{A} \rtimes_{\alpha,\mathrm{r}} G - A^\alpha$ imprimitivity bimodule in Theorem 3.9.

Saturated dynamical systems are the proper dynamical systems primarily studied in applications [9], [10], [11], [13]. So it is important to find some conditions which guarantee that a given proper action is saturated.

The goal of this section is to show the following theorem.

THEOREM 5.2. *Suppose $(\mathscr{A}, G, \alpha)$ is a groupoid dynamical system and let $A = \Gamma_0(G^{(0)}, \mathscr{A})$ be the associated $C_0(G^{(0)})$-algebra. Suppose further that G is principal and proper. Then the action of G on $\mathscr{A}$ is saturated with respect to the dense subalgebra $C_{\mathrm{c}}(G^{(0)}) \cdot A$.*

In particular, Theorem 5.2 shows that if $\mathscr{A} = \mathscr{T}$ and $\alpha = \mathrm{lt}$, then $C_0(G\backslash G^{(0)})$ is Morita equivalent to $C^*_{\mathrm{r}}(G)$. This is the content of Theorem 5.9.

5.1. THE SCALAR CASE.

In order to prove Theorem 5.2, we will first show that if G is principal and proper, then the action of G on $C_0(G^{(0)})$ is saturated with respect to the subalgebra $C_{\mathrm{c}}(G^{(0)})$. Let $\mathscr{T} = G^{(0)} \times \mathbb{C}$, then $\Gamma_0(G^{(0)}, \mathscr{T}) \cong C_0(G^{(0)})$. Recall from Proposition 4.1, that the dynamical system $(C_0(G^{(0)}), G, \mathrm{lt})$ is proper.

To show that the action is saturated we need to show spans of elements of the form

$$(5.1)\qquad {}_E\langle f, g\rangle(\gamma) := f(r(\gamma))\overline{g(\gamma^{-1}\cdot r(\gamma))} = f(r(\gamma))\overline{g(s(\gamma))}$$

are dense in $C_0(G^{(0)}) \rtimes_{\mathrm{lt,r}} G$. For this it suffices to show that they are dense in $\Gamma_{\mathrm{c}}(G, r^*\mathscr{T}) \cong C_{\mathrm{c}}(G)$ in the inductive limit topology. To see this note that regular representations (see (2.4) or p. 82 of [23]) are clearly continuous with respect to the inductive limit topology so that density in the inductive limit topology implies density in norm. We will follow the proof in [26] and construct a special approximate identity. To construct this approximate identity, we need the following key lemma which is the groupoid analogue of Lemma p. 306 of [26]. The proof follows that given in [26].

LEMMA 5.3. *Let G be principal and proper. Then for each $u \in G^{(0)}$ and open neighborhood $N \subset G$ of u, there exists an open neighborhood $U \subset G^{(0)}$ of u such that $\{\gamma : \gamma\cdot U \cap U \neq \varnothing\} \subset N$.*

Proof. By way of contradiction assume there exists an open neighborhood $N \subset G$ of u such that Lemma 5.3 does not hold. Then given an open neighborhood $W \subset G^{(0)}$ of u, there exists $\gamma_W \in G$ and $v_W \in W$ such that $\gamma_W \notin N$ and $\gamma_W \cdot v_W \in W$. For each open neighborhood $W \subset G^{(0)}$ pick such a $\gamma_W \in G$ and $v_W \in W$ and order the nets $\{\gamma_W\}$ and $\{v_W\}$ by reverse inclusion.

Let K be compact neighborhood of u in $G^{(0)}$. Since $(\gamma_W \cdot v_W, v_W)$ is eventually in $K \times K$, it has a convergent subnet. By the properness of G, $\{\gamma_W\}$ has a convergent subnet $\gamma_{W_i} \to \gamma$. Note that since $\{\gamma_{W_i}\}$ is a subnet, W_i is a fundamental system for u, thus $(\gamma_{W_i}\cdot v_{W_i}, v_{W_i}) \to (u,u)$. Hence $\gamma\cdot u = u$, but by assumption $\{\gamma_W\}$ is never in the open neighborhood N of u so $\gamma \neq u$. This contradicts the freeness of the action. ∎

As in p. 307 of [26], we will use Lemma 5.3 to construct an approximate unit for A, but first we need another lemma.

LEMMA 5.4. *Functions of the form $\gamma \mapsto g(r(\gamma)) \int_G g(s(\eta))\mathrm{d}\lambda^{r(\gamma)}(\eta)$ are dense in $C_{\mathrm{c}}^{+}(G^{(0)})$ (the positive, continuous compactly supported functions) for the inductive limit topology.*

Proof. Let $f \in C_{\mathrm{c}}^{+}(G^{(0)})$ and $\delta > 0$ be given. Define F on $G\backslash G^{(0)}$ by

$$(5.2)\qquad F([u]) := \int_G f(s(\gamma))\mathrm{d}\lambda^{u}(\gamma).$$

Let $C = \{u \in G^{(0)} :\ f(u) \geqslant \delta\}$ and let $[C]$ be the image of C in $G\backslash G^{(0)}$.

Let $m = \inf\{F([u]) : [u] \in [C]\}$, where $[u]$ denotes the image of u in $G\backslash G^{(0)}$. Since $[C]$ is compact and F is continuous, F attains its minimum on $[C]$. Furthermore, since f is continuous, positive and bounded away from zero on C we have that $m > 0$.

Let $U = \{u : F([u]) > m/2\}$. By the above argument $C \subset U$. Construct $Q \in C_c(G\backslash G^{(0)})$ such that $0 \leqslant Q \leqslant 1$, $Q([v]) = 1$ for $[v] \in [C]$, and $Q([v]) = 0$ for $v \notin U$.

Thus $Q/\sqrt{F} \in C_c(G\backslash G^{(0)})$. Define $g := fQ/\sqrt{F}$. Then $g \in C_c^+(G^{(0)})$ and $\operatorname{supp}(g) \subset \operatorname{supp}(f)$. Furthermore, a simple calculation shows

$$\Big| f(u) - g(u) \int_G g(s(\gamma)) \mathrm{d}\lambda^u(\gamma) \Big| < \delta. \quad \blacksquare$$

We are now ready to construct a special approximate unit for A.

LEMMA 5.5 (Approximate Identity). *Let G be principal and proper. Then there exists an approximate identity for $C_c(G)$ in the inductive limit topology given by the net $\Phi_{N,D,\varepsilon}$ indexed by decreasing neighborhoods N of $G^{(0)}$, increasing compact subsets D of $G^{(0)}$, and decreasing $\varepsilon > 0$ which satisfies:*

(i) $\Phi_{N,D,\varepsilon}(\gamma) = 0$ *if* $\gamma \notin N$ *and* $\geqslant 0$ *otherwise;*

(ii) $\Big| \int_G \Phi_{N,D,\varepsilon}(\gamma) \mathrm{d}\lambda^u(\gamma) - 1 \Big| < \varepsilon$ *for* $u \in D$*;*

(iii) $\Phi_{N,D,\varepsilon}(\gamma) = \sum_E \langle g_i^{N,D,\varepsilon}, g_i^{N,D,\varepsilon} \rangle(\gamma) = \sum g_i^{N,D,\varepsilon}(r(\gamma)) \overline{g_i^{N,D,\varepsilon}(s(\gamma))}$ *for some* $g_i^{N,D,\varepsilon} \in C_c(G^{(0)})$.

Proof. Let N be a neighborhood of $G^{(0)}$, D be a compact subset of $G^{(0)}$, and $\varepsilon > 0$ be given. Note that D is also compact in G, so we can choose an open set $V \subset G$ such that $D \subset V \subset \overline{V} \subset N$. Then using Lemma 5.3 there exists a finite open covering $\{U_i\}_{i=0}^n$ of D such that for each i,

$$\{\gamma \in G : \gamma \cdot U_i \cap U_i \neq \emptyset\} \subset V. \tag{5.3}$$

For each i, pick $h_i \in C_c(U_i)$ such that $h(u) := \sum h_i(u)$ is strictly positive on D. Now let

$$m = \frac{\inf(h|_D)}{2} \quad \text{and} \quad g(u) := \max\{h(u), m\}.$$

Note that $m \neq 0$ since h actually attains a minimum on the compact set D. Furthermore, since h is strictly positive on D, this minimum must be bigger than 0. Thus $g > 0$ on $G^{(0)}$ and g is continuous. Therefore the following is in $C_c(U_i)$:

$$f_i(u) := \frac{h_i(u)}{g(u)}. \tag{5.4}$$

CLAIM 5.6. *If* $f(u) := \sum f_i(u)$ $(= h(u)/g(u))$ *then* $0 \leqslant f \leqslant 1$ *and* $f \equiv 1$ *on* D.

Proof. We begin by showing $0 \leqslant f \leqslant 1$. Now if $h(u) \geqslant m$ then $g(u) = h(u)$ and thus $f(u) = 1$. If $h(u) \leqslant m$ then $g(u) = m$ and so $f(u) = h(u)/m \leqslant m/m =$

1. Thus $f \leqslant 1$ and since we know each f_i is positive and by definition, $m \leqslant h$ on D, we get the result. ∎

So to finish the proof of Lemma 5.5, let $M = \operatorname{card}\{U_i\}$. For each f_i defined in (5.4), use Lemma 5.4 to pick $g_i^{N,D,\varepsilon}$ so that

$$\left| f_i(u) - g_i^{N,D,\varepsilon}(u) \int_G g_i^{N,D,\varepsilon}(s(\gamma)) \mathrm{d}\lambda^u(\gamma) \right| < \frac{\varepsilon}{M} \tag{5.5}$$

with $\operatorname{supp}(g_i^{N,D,\varepsilon}) \subset \operatorname{supp}(f_i)$. Now define

$$\Phi_{N,D,\varepsilon} := \sum {}_E\langle\, g_i^{N,D,\varepsilon},\, g_i^{N,D,\varepsilon}\,\rangle. \tag{5.6}$$

We need to show this $\Phi_{N,D,\varepsilon}$ satisfies conditions (i) and (ii) from Lemma 5.5.

For condition (i), notice $\operatorname{supp}(g_i^{N,D,\varepsilon}) \subset \operatorname{supp}(f_i) \subset U_i$, and by the definition of U_i (equation (5.3))

$$\operatorname{supp}({}_E\langle\, g_i^{N,D,\varepsilon},\, g_i^{N,D,\varepsilon}\,\rangle) \subset \{\gamma \in G : \gamma \cdot U_i \cap U_i \neq \emptyset\} \subset V \subset N.$$

Since i was arbitrary we have $\operatorname{supp}(\Phi_{N,D,\varepsilon}) \subset N$ as desired.

For property (ii), let $u \in D$, then

$$\begin{aligned}\left|\int_G \Phi_{N,D,\varepsilon}(\gamma)\mathrm{d}\lambda^u(\gamma) - 1\right| &= \left| \sum g_i^{N,D,\varepsilon}(r(u)) \int_G \overline{g_i^{N,D,\varepsilon}(s(\gamma))} \mathrm{d}\lambda^u(\gamma) - 1\right| \\ &\leqslant \left|\sum f_i(u) - g_i^{N,D,\varepsilon}(r(u)) \int_G \overline{g_i^{N,D,\varepsilon}(s(\gamma))}\mathrm{d}\lambda^u(\gamma)\right| + \left|\sum f_i(u) - 1\right|\end{aligned}$$

which is less than ε by our assumptions on g_i, f_i and D.

It is left to show that $\{\Phi_{N,D,\varepsilon}\}$ is actually an approximate identity for $C_c(G)$ in the inductive limit topology.

Let $F \in C_c(G)$ be arbitrary. First we will show that $\operatorname{supp}(\Phi_{N,D,\varepsilon}) * F$ is eventually in some compact set. Now

$$(\Phi_{N,D,\varepsilon} * F)(\gamma) = \int_G \Phi_{N,D,\varepsilon}(\eta) F(\eta^{-1}\gamma) \mathrm{d}\lambda^u(\eta).$$

So for $(\Phi_{N,D,\varepsilon} * F)(\gamma) \neq 0$ there is an η such that $\eta \in \operatorname{supp}(\Phi_{N,D,\varepsilon}) \subset N$ and $\eta^{-1}\gamma \in \operatorname{supp}(F)$. That is $\gamma \in N \cdot \operatorname{supp}(F)$, thus

$$\operatorname{supp}(\Phi_{N,D,\varepsilon} * F) \subset \overline{N \cdot \operatorname{supp}(F)}. \tag{5.7}$$

To continue we need a definition: a neighborhood W of $G^{(0)}$ is called *diagonally compact* (respectively *conditionally compact*) if VW and WV are compact (respectively relatively compact) for every compact (respectively compact) set V in G.

Let N_0 be some open neighborhood of $G^{(0)}$; then by Lemma 2.7 of [19] there exists an open symmetric conditionally compact set W_0 with $\overline{W}_0$ diagonally compact, such that $G^{(0)} \subset W_0 \subset \overline{W}_0 \subset N_0$. Thus $\overline{W}_0 \cdot \operatorname{supp}(F)$ is compact and $\operatorname{supp}(\Phi_{N,D,\varepsilon} * F) \subset \overline{W}_0 \cdot \operatorname{supp}(F)$ for $N \subset W_0$.

We will use W_0 to show $\{\Phi_{N,D,\varepsilon} * F\} \to F$ uniformly. Let $\delta > 0$ be given. Now compute

$$
\begin{aligned}
|\Phi_{N,D,\varepsilon}F(\gamma) - F(\gamma)| &= \Big| \int_G \Phi_{N,D,\varepsilon}(\eta)F(\eta^{-1}\gamma)\mathrm{d}\lambda^{r(\gamma)}(\eta) - F(\gamma)\Big| \\
&\leqslant \int_G |\Phi_{N,D,\varepsilon}(\eta)|\,|F(\eta^{-1}\gamma) - F(\gamma)|\mathrm{d}\lambda^{r(\gamma)}(\eta) \qquad (5.8)\\
&\quad + \|F\|_\infty \Big|\chi_{\mathrm{supp}(F)}(\gamma)\int_G \Phi_{N,D,\varepsilon}(\eta)\mathrm{d}\lambda^{r(\gamma)}(\eta) - 1\Big|.
\end{aligned}
$$

Notice if $r(\mathrm{supp}(F)) \subset D$ then by property (ii) the second term of (5.8) is less than $\|F\|_\infty \cdot \varepsilon$. So if we choose $\varepsilon < \delta/(2\|F\|_\infty)$ the second term of (5.8) is less than $\delta/2$. It remains to show that the first term is eventually less than $\delta/2$. By way of contradiction assume

$$\int_G |\Phi_{N,D,\varepsilon}(\eta)|\,|F(\eta^{-1}\gamma) - F(\gamma)|\mathrm{d}\lambda^{r(\gamma)}(\eta) \geqslant \frac{\delta}{2} \quad \forall\,(N,D,\varepsilon).$$

So, if we choose W_0 as above, and if $N \subset W_0$ then for $\gamma \notin (\overline{W}_0 \cdot \mathrm{supp}(F) \cup \mathrm{supp}(F))$ the first term of (5.8) is 0. Thus we can restrict our attention to when $\gamma \in (\overline{W}_0 \cdot \mathrm{supp}(F) \cup \mathrm{supp}(F))$, which is compact since it is the union of two compact sets.

CLAIM 5.7. *There exists an open neighborhood N of $G^{(0)}$ such that for $\gamma \in (\overline{W}_0 \cdot \mathrm{supp}(F) \cup \mathrm{supp}(F))$, $\eta \in N$ we have $|F(\eta^{-1}\gamma) - F(\gamma)| < \delta/4$.*

Proof. By way of contradiction assume the claim is false. Then for each neighborhood N of $G^{(0)}$ we can choose $\gamma_N \in (\overline{W}_0 \cdot \mathrm{supp}(F) \cup \mathrm{supp}(F))$ and $\eta_N \in N$ such that $|F(\eta_N^{-1}\gamma_N) - F(\gamma_N)| \geqslant \delta/4$. Since γ_N is a net in a compact set it has a convergent subnet which by relabeling we can assume $\gamma_N \to \gamma$. Also take the corresponding subnet of η_N.

Pick an r-relatively compact neighborhood (see p. 447) $N_0 \subset W_0$ of $G^{(0)}$, and set $K = r(\overline{W}_0 \cdot \mathrm{supp}(F) \cup \mathrm{supp}(F))$. Then $r^{-1}(K) \cap N_0$ is relatively compact and for $N \subset N_0$, $r(\eta_N) = r(\gamma_N) \in K$. Thus $\eta_N \in r^{-1}(K) \cap N_0$ which is relatively compact by assumption. Thus η_N must have a convergent subnet $\eta_{N_i} \to \eta$. By our choice of η_{N_i} we must have $\eta \in G^{(0)}$. Choose this subnet of η_N and the corresponding subnet of γ_N and relabel. Thus $\eta_N^{-1}\gamma_N \to \eta^{-1}\gamma = \gamma$, hence $|F(\eta_N^{-1}\gamma_N) - F(\gamma_N)| \to |F(\gamma) - F(\gamma)| = 0$ a contradiction. ∎

Next a simple computation using (5.5) shows that

CLAIM 5.8. *For $\varepsilon < 1$, the integral $\int_G |\Phi_{N,D,\varepsilon}(\eta)|\mathrm{d}\lambda^u(\eta) < 2$.*

Thus if we pick N_0 as in Claim 5.7, $D = \operatorname{supp}(F)$ and $\varepsilon = \delta/(2\|F\|_\infty)$, then by the discussion after (5.8),

$$\begin{aligned}|\Phi_{N,D,\varepsilon} * F(\gamma) - F(\gamma)| &< \int_G |\Phi_{N,D,\varepsilon}(\eta)|\, |F(\eta^{-1}\gamma) - F(\gamma)| \mathrm{d}\lambda^{r(\gamma)}(\eta) + \frac{\delta}{2} \\ &< \int_G |\Phi_{N,D,\varepsilon}(\eta)| \frac{\delta}{4} \mathrm{d}\lambda^{r(\gamma)}(\eta) + \frac{\delta}{2} \leqslant \delta\end{aligned}$$

by Claims 5.7 and 5.8 and property (i). Hence $\Phi_{N,D,\varepsilon}$ is an approximate identity for A in the inductive limit topology so Lemma 5.5 is proved. ∎

Lemma 5.5 shows $\operatorname{span}\{{}_E\langle f, g\rangle\}$ is dense in $C_c(G)$ in the inductive limit topology and thus dense in $C^*_r(G) = C_0(G^{(0)}) \rtimes_{\mathrm{lt,r}} G$, so combined with Proposition 4.1 and Theorem 3.9 we have:

THEOREM 5.9. *Suppose G is principal and proper. Then the dynamical system $(C_0(G^{(0)}), G, \mathrm{lt})$ is saturated with respect to the dense subalgebra $C_c(G^{(0)})$, that is $C_0(G\backslash G^{(0)}) \cong C_0(G^{(0)})^{\mathrm{lt}}$ is Morita equivalent to $C^*_r(G)$.*

Note that this theorem implies that the spectrum of $C^*_r(G)$ is $G\backslash G^{(0)}$ and furthermore that $C^*_r(G)$ is *globally* Morita equivalent to $C_0(G\backslash G^{(0)})$. So by applying Proposition 5.15 of [31] we get the following result.

COROLLARY 5.10. *If G is a second countable groupoid acting freely and properly on its unit space, then $C^*_r(G)$ has continuous trace with trivial Dixmier–Douady invariant.*

REMARK 5.11. Since G is principal and proper, Corollary 2.1.17 of [1] implies G is properly amenable. Thus Definition 2.1.13 of [1] and Definition 2.2.2 of [1] show that it is topologically amenable. So by Proposition 3.35 of [1] we have that G is measure-wise amenable. Thus Proposition 6.1.8 of [1] gives that $C^*(G) = C^*_{\mathrm{red}}(G)$. We use the notation $C^*_{\mathrm{red}}(G)$ here because it is *a priori* different from $C^*_r(G)$ defined in Section 2.2 and Definition II.2.8 of [23]. In [1], $\|\cdot\|_{\mathrm{red}}$ is defined using only those representations induced by point mass measures on $G^{(0)}$, therefore

$$\|\cdot\|_{\mathrm{red}} \leqslant \|\cdot\|_{\mathrm{r}} \leqslant \|\cdot\|_{\mathrm{universal}}.$$

Now, Proposition 6.1.8 of [1] implies that $\|\cdot\|_{\mathrm{red}}$ is the same as the universal norm, thus $\|\cdot\|_{\mathrm{r}}$ must be the same as the universal norm as well and hence $C^*_r(G) = C^*(G)$. Thus Corollary 5.10 recovers Proposition 2.2 of [19].

REMARK 5.12. If $G = H \times X$ is a transformation group groupoid the condition G acts freely and properly on its unit space means that H acts freely and properly on X. Therefore, Corollary 5.10 and Remark 5.11 give Corollary 15 of [7].

5.2. PROOF OF THEOREM 5.2. We now prove Theorem 5.2, which states that if a groupoid G acts freely and properly on its unit space then the action of G on any upper semicontinuous C^*-bundle is saturated, that is finite linear combinations of the inner product

$$_E\langle f\cdot a, g\cdot b\rangle := \gamma \mapsto f(r(\gamma))a(r(\gamma))\alpha_\gamma((g(s(\gamma))b(s\gamma))^*)$$

$f, g \in C_c(G^{(0)}),\ a, b \in A = \Gamma_0(G^{(0)}, \mathscr{A})$ are dense in $\mathscr{A} \rtimes_{\alpha,r} G$. This proof follows Appendix C of [11] fairly closely. We proceed in several steps. The first two steps show that we can consider functions of compact support. Then we cover the support of the function we want to approximate by small enough neighborhoods, so that the action on these neighborhoods is almost trivial, and finally we use a partition of unity to complete the approximation.

Step 1. Show that the span of sections of the form

$$F(\gamma) = \phi(\gamma)f(r(\gamma))a(r(\gamma))g(r(\gamma))b(r(\gamma))^* \tag{5.9}$$

are dense in $\Gamma_c(G, r^*\mathscr{A})$ in the inductive limit topology, where $\phi \in C_c(G),\ f, g \in C_c(G^{(0)})$, and $a, b \in A = \Gamma_0(G^{(0)}, \mathscr{A})$.

To see this first note that A_0^2 is dense in A. Now from Proposition 1.3 of [30], we know that

$$C_0(G) \otimes_{C_0(G^{(0)})} \Gamma_0(G^{(0)}, \mathscr{A}) \cong \Gamma_0(G, r^*\mathscr{A})$$

where the isomorphism is given on elementary tensors by

$$\Phi : f \otimes a \mapsto (\gamma \mapsto f(\gamma)a(r(\gamma))).$$

Note that $\Phi(C_c(G) \odot A_0^2)$ is a $C_0(G)$-module. Furthermore, since A_0^2 is dense in A we have $A_0^2(r(\gamma))$ is dense in $A(r(\gamma))$. Thus by Proposition C.24 of [32], $\Phi(C_c(G) \odot A_0^2)$ is dense in $\Gamma_0(G, r^*\mathscr{A})$ and hence in $\Gamma_c(G, r^*\mathscr{A})$ in the uniform topology.

Given $\psi \in \Gamma_c(G, r^*\mathscr{A})$ pick a net $\psi'_j \to \psi$ uniformly with $\psi'_j \in \Phi(C_c(G) \odot A_0^2)$. Pick $\omega \in C_c(G)$ such that $0 \leqslant \omega \leqslant 1$ and $\omega \equiv 1$ on $\operatorname{supp}(\psi)$. Then $\psi = \omega\psi = \lim \omega\psi'_j$. Let $\psi_j = \omega\psi'_j$ then $\psi_j \to \psi$ uniformly and $\operatorname{supp}(\psi_j) \subset \operatorname{supp}(\omega)$ which is compact. Thus $\psi_j \to \psi$ in the inductive limit topology.

Note that every element of $\Phi(C_c(G) \odot A_0^2)$ is of the form (5.9). Thus it suffices to show that elements of the form (5.9) can be approximated by elements of E in the inductive limit topology.

Step 2. Show that elements of the form

$$\gamma \mapsto \phi(\gamma)f(r(\gamma))a(r(\gamma))\alpha_\gamma(\overline{g(s(\gamma))}b^*(s(\gamma))) \tag{5.10}$$

are in E with $\phi \in C_c(G),\ f, g \in C_c(G^{(0)}),\ a, b \in A$.

By Theorem 5.9, the action of G on $C_0(G^{(0)})$ is saturated with respect to $C_c(G^{(0)})$. Thus, given $\varepsilon > 0$, we can find $g_i, h_i \in C_c(G^{(0)})$ such that

$$\Big\| \phi(\gamma) - \sum_i g_i(r(\gamma))\overline{h_i(s(\gamma))} \Big\| < \frac{\varepsilon}{\|f\|_\infty \|a\| \|b\| \|g\|_\infty}. \tag{5.11}$$

Furthermore, we can arrange it so that if W is a compact neighborhood of the support of ϕ, then $\operatorname{supp}(\gamma \mapsto \sum_i g_i(r(\gamma))\overline{h_i(s(\gamma))}) \subset W$. Let $a_i = g_i f \cdot a$ and $b_i = h_i g \cdot b$ then

$$\Big\| \sum_i {}_E\langle a_i, b_i\rangle(\gamma) - \phi(\gamma) f(r(\gamma)) a(r(\gamma)) \alpha_\gamma(\overline{g(s(\gamma))} b^*(s(\gamma))) \Big\|$$
$$= \Big\| \sum_i g_i(r(\gamma))\overline{h_i(s(\gamma))}\, {}_E\langle f \cdot a, g \cdot b\rangle(\gamma) - \phi(\gamma)\, {}_E\langle f \cdot a, g \cdot b\rangle \Big\|$$
$$\leqslant \Big\| \sum_i g_i(r(\gamma))\overline{h_i(s(\gamma))} - \phi(\gamma) \Big\| \|{}_E\langle f \cdot a, g \cdot b\rangle\| < \frac{\|f\|_\infty \|a\| \|b\| \|g\|_\infty \varepsilon}{\|f\|_\infty \|a\| \|b\| \|g\|_\infty} = \varepsilon.$$

Since W does not depend on ε and ε is arbitrary, we must have

$$\gamma \mapsto \phi(\gamma) f(r(\gamma)) a(r(\gamma)) \alpha_\gamma(\overline{g(s(\gamma))} b^*(s(\gamma))) \in E.$$

Step 3. Show that the functions of the form (5.10) can be used to approximate the functions of the form (5.9) in the inductive limit topology.

REMARK 5.13. At this point in Appendix C of [11], the authors find a neighborhood N of the identity in the group such that $\|b^* - \alpha_s(b^*)\|$ is small for $s \in N$. They then translate this neighborhood to find a finite collection of open sets Nr_i such that $\operatorname{supp}(\phi) \subset \bigcup Nr_i$. They use this open cover to construct a partition of unity, $\{\phi_i\}$, and define

$$F_i(s) := \phi(s)\phi_i(s)(f \cdot a)\alpha_{sr_i^{-1}}((g \cdot b)^*). \tag{5.12}$$

This is a fairly standard approximation argument in group crossed products. Unfortunately, this argument does not work for groupoids, since the translation of an open set in a groupoid by a groupoid element is not necessarily open. However, the translation UV_i of an open set $U \subset G$ by an open set $V_i \subset G$ is open in a groupoid. But now we do not have an element r_i to plug into an analogous equation to (5.12). The idea which motivates what follows is to average $\alpha_{\gamma\eta^{-1}}(g(r(\eta)) \cdot b(r(\eta))^*)$ over $\eta \in V_i$.

Fix

$$F(\gamma) = \phi(\gamma) f(r(\gamma)) a(r(\gamma)) g(r(\gamma)) b(r(\gamma))^*$$

as in (5.9) with $\phi \in C_c(G)$, $f, g \in C_c(G^{(0)})$, $a, b \in A = \Gamma_0(G^{(0)}, \mathscr{A})$ and let $\varepsilon > 0$ be given. Define

$$K := \operatorname{supp}(\phi). \tag{5.13}$$

Note that since the norm is upper semicontinuous, the set

$$N_\varepsilon := \Big\{ \gamma : \|b^*(r(\gamma))\overline{g(r(\gamma))} - \alpha_\gamma(b^*(s(\gamma))\overline{g(s(\gamma))})\| < \frac{\varepsilon}{\|\phi\| \|f\| \|a\|} \Big\} \tag{5.14}$$

is open. Furthermore, it is nonempty since $G^{(0)} \subset N_\varepsilon$. Now we need a lemma whose proof follows easily from the continuity of multiplication in G, so we omit it.

LEMMA 5.14. *For every $\eta \in G$, there exists an open neighborhood U_η of η such that $U_\eta \cdot U_\eta^{-1} \subset N_\varepsilon$.*

Now for $\eta \in K = \operatorname{supp}(\phi)$, let U_η be an open neighborhood of η as in Lemma 5.14. Then $\{U_\eta\}_{\eta\in K}$ is an open cover of the compact set K, therefore there is a finite subcover $\{U_i\}_{i=1}^n$. Furthermore, since G is locally compact and Hausdorff, G is regular. Thus, for all $\eta \in K$ there exists a neighborhood V_η of η with compact closure such that $\eta \in V_\eta \subset \overline{V}_\eta \subset U_i$ for some i. Now $\{V_\eta\}_{\eta\in K}$ is an open cover of the compact set K, therefore there is a finite subcover $\{V_j\}_{j=1}^m$. We have arranged it so that for each j there exists i such that $\overline{V_j} \subset U_i$. For each $j = 1, \dots, m$ pick such an i and define

$$\sigma : \{1, \dots, m\} \to \{1, \dots, n\} \quad \text{so that } \overline{V_j} \subset U_{\sigma(j)}.$$

For each V_j pick a function $\psi_j \in C_c^+(G)$ such that $0 \leqslant \psi_j \leqslant 1$, $\psi_j|_{\overline{V_j}} \equiv 1$ and $\operatorname{supp}(\psi_j) \subset U_{\sigma(j)}$. Also, pick a partition of unity $\{\phi_j\} \subset C_c^+(G)$ subordinate to the subcover $\{V_j\}_{j=1}^m$. That is, $\operatorname{supp}(\phi_i) \subset V_j$, $0 \leqslant \phi_j \leqslant 1$, $\sum \phi_j \equiv 1$ on K and $\sum \phi_j \equiv 0$ off of $\bigcup V_j$. We will use these functions to ensure that groupoid elements lie in N_ε.

Define

$$\omega_j(u) := \int_G \psi_j(\gamma) \mathrm{d}\lambda_u(\gamma). \tag{5.15}$$

Now ω_j is continuous since $\psi_j \in C_c(G)$ and λ_u is a (right) Haar system. Furthermore,

$$\gamma \mapsto \frac{\phi_j(\gamma)}{\omega_j(s(\gamma))}$$

is continuous, since

$$\operatorname{supp}(\phi_j) \subset V_j \subset \overline{V_j} \subset \operatorname{supp}(\psi_j) \subset \operatorname{supp}(w_j \circ s).$$

Define

$$\begin{aligned} f_j(\gamma) := \phi(\gamma) \frac{\phi_j(\gamma)}{\omega_j(s(\gamma))} f(r(\gamma)) a(r(\gamma)) \\ \alpha_\gamma\Big(\int_G \psi_j(\eta)\, \alpha_{\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))}) \mathrm{d}\lambda_{s(\gamma)}(\eta)\Big). \end{aligned} \tag{5.16}$$

REMARK 5.15. The functions f_j are of the form of equation (5.10). To see this notice $\gamma \mapsto \phi(\gamma)(\phi_j(\gamma))/(\omega_j(s(\gamma))) \in C_c(G)$ and that $\alpha_{\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))}) \in A(s(\eta))$. Thus

$$\int_G \psi_j(\eta)\, \alpha_{\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))}) \mathrm{d}\lambda_u(\eta) \in A(u).$$

But $\psi_j(\eta)\ \alpha_{\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})$ has compact support since ψ_j does. Now from the properties of Haar systems we have that

$$\Big(u \mapsto \int_G \psi_j(\eta)\ \alpha_{\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})\mathrm{d}\lambda_u(\eta)\Big) \in \Gamma_{\mathrm{c}}(G^{(0)}, \mathscr{A})$$

and thus is of the form $\overline{h(u)}c^*(u)$ for $c \in A$ and $h \in C_{\mathrm{c}}(G^{(0)})$.

REMARK 5.16. If $\gamma \in \operatorname{supp}(f_j)$ then $\gamma \in \operatorname{supp}(\phi_j)$, so that $\gamma \in U_{\sigma(j)}$. Now if the integrand in (5.16) is nonzero, then $\eta \in \operatorname{supp}(\psi_j)$ so that $\eta \in U_{\sigma(j)}$. That is $\gamma\eta^{-1} \in N_\varepsilon$.

REMARK 5.17. Now $\operatorname{supp}(f_j) \subset \operatorname{supp}(\phi) = K$, so that $\operatorname{supp}(\sum f_j) \subset K$.

To finish the proof we compute:

$$\Big\|F(\gamma) - \sum(f_j(\gamma))\Big\| = \Big\|\sum(\phi_j(\gamma)F(\gamma) - f_j(\gamma))\Big\|$$

(since $\sum \phi_j(\gamma) \equiv 1$ on $K \supset \operatorname{supp}(F) \cap \operatorname{supp}(\sum f_j)$)

$$= \Big\|\sum \phi(\gamma)\phi_j(\gamma)f(r(\gamma))a(r(\gamma))\Big(b^*(r(\gamma))\overline{g(r(\gamma))} \cdots$$

$$- \int_G \frac{\psi_j(\eta)}{\omega_j(s(\gamma))}\ \alpha_{\gamma\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})\mathrm{d}\lambda_{s(\gamma)}(\eta)\Big)\Big\|$$

$$\leqslant \|\phi\|_\infty\|f\|_\infty\|a\|\chi_K(\gamma)\Big\|\sum \phi_j(\gamma)\Big(b^*(r(\gamma))\overline{g(r(\gamma))} \cdots$$

$$- \int_G \frac{\psi_j(\eta)}{\omega_j(s(\gamma))}\ \alpha_{\gamma\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})\mathrm{d}\lambda_{s(\gamma)}(\eta)\Big)\Big\|$$

$$\leqslant \|\phi\|_\infty\|f\|_\infty\|a\|\chi_K(\gamma)\sum\Big(\phi_j(\gamma)\Big\|\Big(\int_G \frac{\psi_j(\eta)}{\omega_j(s(\gamma))}\ b^*(r(\gamma)) \cdots$$

$$\overline{g(r(\gamma))}\mathrm{d}\lambda_{s(\gamma)}(\eta) - \int_G \frac{\psi_j(\eta)}{\omega_j(s(\gamma))}\ \alpha_{\gamma\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})\mathrm{d}\lambda_{s(\gamma)}(\eta)\Big)\Big\|\Big)$$

(since $\int_G (\psi_j(\eta))/(0\omega_j(u))\mathrm{d}\lambda_u(\eta) \equiv 1$ on $\operatorname{supp}(\phi_j)$ and $b^*(r(\gamma))\overline{g(r(\gamma))}$ does not depend on η)

$$\leqslant \|\phi\|_\infty\|f\|_\infty\|a\|\chi_K(\gamma)\sum\Big(\phi_j(\gamma)\int_G \frac{\psi_j(\eta)}{\omega_j(s(\gamma))}$$

$$\| b^*(r(\gamma))\overline{g(r(\gamma))} - \alpha_{\gamma\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})\|\mathrm{d}\lambda_{s(\gamma)}(\eta)\Big). \tag{5.17}$$

But by Remark 5.16 and equation (5.14), we know that

$$\| b^*(r(\gamma))\overline{g(r(\gamma))} - \alpha_{\gamma\eta^{-1}}(b^*(r(\eta))\overline{g(r(\eta))})\| < \frac{\varepsilon}{\|\phi\|_\infty\|f\|_\infty\|a\|}.$$

So that (5.17) is less than

$$\|\phi\|_\infty\|f\|_\infty\|a\|\chi_K(\gamma)\sum\Big(\frac{\varepsilon}{\|\phi\|_\infty\|f\|_\infty\|a\|}\phi_j(\gamma)\int\limits_G\frac{\psi_j(\eta)}{\omega_j(s(\gamma))}\mathrm{d}\lambda_{s(\gamma)}(\eta)\Big)$$
$$=\|\phi\|_\infty\|f\|_\infty\|a\|\frac{\varepsilon}{\|\phi\|_\infty\|f\|_\infty\|a\|}=\varepsilon$$

since $\int\limits_G(\psi_j(\eta))/(\omega_j(u))\mathrm{d}\lambda_u(\eta)\equiv 1$ on $\operatorname{supp}(\phi_j)$ and $\sum\phi_j(\gamma)\equiv 1$ on K.

Thus we can approximate F by $\sum f_j$ in the inductive limit topology. Now Steps 1 and 2 along with Remark 5.15 gives the density of $\operatorname{span}\{_E\langle f\cdot a, g\cdot b\rangle : f,g\in C_c(G^{(0)}), a,b\in A\}$ in $\Gamma_c(G,r^*\mathscr{A})$ in the inductive limit topology and hence $\operatorname{span}\{_E\langle f\cdot a, g\cdot b\rangle : f,g\in C_c(G^{(0)}), a,b\in A\}$ is dense in $\mathscr{A}\rtimes_{\alpha,\mathrm{r}} G$. Thus the dynamical system $(\mathscr{A},G,\alpha)$ is saturated and we obtain Theorem 5.2.

REMARK 5.18. As in Remark 5.11 for G acting freely and properly on its unit space, $\mathscr{A}\rtimes_{\alpha,\mathrm{r}} G=\mathscr{A}\rtimes_\alpha G$ using Proposition 6.1.10 of [1] or Theorem 3.6 of [25]. So that A^α is Morita equivalent to $\mathscr{A}\rtimes_\alpha G$.

COMMENT. After this paper was accepted we were able to show that if a proper groupoid dynamical system $(\mathscr{A},G,\alpha)$ with respect to A_0 satisfies $C_c(G)\cdot A_0\subset A_0$, where the action of $C_c(G)$ on A_0 is given by $f\cdot a=\int\limits_G f(\gamma)\alpha_\gamma(a(s(\gamma)))\mathrm{d}\lambda^u(\gamma)$, then the subalgebra E guaranteed by Theorem 3.9 is an ideal. The details will appear elsewhere.

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