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ON ORTHOGONAL SYSTEMS IN HILBERT C^*-MODULES

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ABSTRACT. Analogues for Hilbert C^*-modules of classical results of Fourier series theory in Hilbert spaces are considered. Relations between different properties of orthogonal and orthonormal systems for Hilbert C^*-modules are studied with special attention paid to the differences with the well-known Hilbert space situation.

KEYWORDS: *Orthogonal systems, Fourier series, Hilbert C^*-modules.*

MSC (2000): 46L08, 46L05, 46L10.

1. INTRODUCTION

In this paper we study properties of orthogonal and orthonormal systems in Hilbert C^*-modules. Actually the theory of Hilbert C^*-modules is at an intermediate stage between the theory of Hilbert spaces and the theory of general Banach spaces and can be considered as a "quantization" of the Hilbert space theory. Roughly speaking by quantization here we mean the following: there are crucial notions and definitions of the theory that include commutative objects like functions or just scalars and one replaces them in some proper way by noncommutative objects like elements of an arbitrary C^*-algebra. From this point of view the definition of Hilbert C^*-modules can be obtained by replacing complex vector spaces with modules over a C^*-algebra and allowing the inner product to take values in this C^*-algebra. This concept originally arose in [3] for commutative C^*-algebras and it was studied in the general noncommutative context in [11], [15]. The theory of Hilbert C^*-modules has a number of effects, related to the operator nature of the "coefficients" of their elements, that make it much more complicated to handle with respect to the usual Hilbert space theory. For example, a closed submodule of a Hilbert C^*-module need not be orthogonally (or even topologically — in the sense of direct sums of closed Banach submodules) complemented, a bounded A-linear operator in a Hilbert module over a C^*-algebra A need not have and adjoint, a Hilbert C^*-module need not be self-dual, i.e. canonically isomorphic to its C^*-dual module (cf. [11], [5], [8]).

Any Hilbert space can be described as a space of sequences (or nets in the non-separable case) $\{c_i\}$ of complex numbers such that the series $\sum_i c_i^* c_i$ converge in norm. The reason that any vector is represented in a unique way by its coordinate sequence is explained via the Fourier series theory: any Hilbert space admits a complete orthonormal system which automatically has to be closed (this exactly means that the Parseval equality is valid for the system); consequently, it forms an orthonormal basis for the Hilbert space. Unfortunately, but not surprisingly, for Hilbert C^*-modules this scheme does not work and they do not admit orthonormal bases in general (e.g. [5], [8]). The reason is, as we will discuss more thoroughly below, that the Fourier series of a vector x of a Hilbert C^*-module M with respect to a certain orthonormal system of M need not converge in norm to x even when this orthonormal system is complete (Example 3.4).

An efficient way to cope with this difficulty is provided by the concept of frame that was introduced in [1], [2] for countably generated Hilbert modules. We remind that a sequence $\{x_i\}$ of vectors of a Hilbert module over a unital C^*-algebra is called a *frame* if for any vector x in the Hilbert module there are real constants $C, D > 0$ such that

$$C\langle x, x\rangle \leqslant \sum_i \langle x, x_i\rangle\langle x_i, x\rangle \leqslant D\langle x, x\rangle.$$

The frame is said to be *tight* if $C = D$, and to be *normalized* if $C = D = 1$. The frame is named *standard normalized tight* if one has that $\langle x, x\rangle = \sum_i \langle x, x_i\rangle\langle x_i, x\rangle$ with respect to the norm topology for any vector x in the Hilbert module, a condition which is the analogue of the Parseval equality. The following crucial result about frames describes the conditions such that the reconstruction formula holds.

THEOREM 1.1 (cf. Theorem 4.1 of [1]). *Let A be a unital C^*-algebra, M be a finitely or countably generated Hilbert A-module and $\{x_i\}$ be a normalized tight frame of M. Then the reconstruction formula*

$$x = \sum_i e_i\langle e_i, x\rangle$$

holds for every $x \in M$, in the sense of convergence in norm, if and only if the frame $\{x_i\}$ is standard.

Let us emphasize, it is important for the statement of Theorem 1.1 that in the definition of a standard (normalized tight) frame we understand the Parseval equality $\langle x, x\rangle = \sum_i \langle x, x_i\rangle\langle x_i, x\rangle$ with respect to the norm topology (for the details see [1], [2]).

Just to mention some applications, the frame approach has already shown its usefulness for the description of conditional expectations of finite index and for the analysis of some classes of C^*-algebras (see references in [1], [2]). It is also very useful to investigate finitely generated projective modules arising from vector bundles and in particular for finding bases for the space of sections of non-trivial vector bundles ([16], Proposition 7.2).

In the present paper we will not deal directly with frames, but nevertheless our considerations are very close to the frame approach. Also, as it will be discussed thoroughly in Section 3, some help to overcome the lack of orthonormal bases for a general Hilbert module sometimes comes from Kasparov's stabilization theorem [4]. Our aim with the present note is two-fold. On the one hand we seek to obtain natural analogues — for arbitrary (i.e. non necessarily countably generated) Hilbert modules over operator algebras — of well-known results about Fourier series and orthonormal systems in Hilbert spaces. On the other hand to highlight, mainly using examples, some of the differences between these two theories. We will show that any Hilbert C^*-modules have complete orthogonal systems (Proposition 2.3), but a complete orthogonal and even orthonormal system needs not be closed at the same time (Examples 2.1, 3.4). Also the completeness of an orthogonal system does not imply that it forms a basis even in some weak sense (Example 2.2); despite these results, there is an analogue of the Bessel inequality for Hilbert C^*-modules. Fourier series of vectors with respect to some orthogonal system need not converge in norm, but only with respect to the strong topology (Theorem 2.5). We also describe interrelations between different properties of orthonormal systems in Hilbert C^*-modules (Theorem 2.10, Corollary 2.11).

2. ORTHOGONAL SYSTEMS IN HILBERT C^*-MODULES

In the sequel, $(M, \langle\cdot,\cdot\rangle)$ is always a Hilbert module over a C^*-algebra A, unless otherwise explicitly stated. A collection $\{e_i\}_{i\in I}$, indexed by some set I, of vectors from M is called *orthogonal* if $\langle e_i, e_j\rangle = 0$ whenever $i \neq j$. The orthogonal system $\{e_i\}_{i\in I}$ is said to be *quasi-orthonormal* if there are (self-adjoint) projections p_i in A such that $\langle e_i, e_i\rangle = p_i$ for all $i \in I$ and it is said to be *orthonormal* provided A is unital (where this not the case we would be able to join the unit, but there will be no need for such complications in the following) and for the inner squares it happens that $\langle e_i, e_i\rangle = 1$ for all $i \in I$.

Let $\{e_i\}_{i\in I}$ be an orthogonal system of M, x be an arbitrary vector in M and $F \subset I$ be any finite subset. Then

$$S_F = \sum_{i\in F} e_i\langle e_i, x\rangle$$

stands for the corresponding partial sum of the Fourier series with respect to $\{e_i\}_{i\in I}$ and a straightforward computation provides the formula:

$$\langle x - S_F, x - S_F\rangle = \langle x, x\rangle - 2\sum_{i\in F}\langle x, e_i\rangle\langle e_i, x\rangle + \sum_{i\in F}\langle x, e_i\rangle\langle e_i, e_i\rangle\langle e_i, x\rangle. \tag{2.1}$$

Next, given an orthogonal system $\{e_i\}_{i\in I}$ of M, in the sequel we will explore consequences and relations among them of the following conditions:

(c1) The system $\{e_i\}_{i\in I}$ generates M over A,

$$M = \overline{\operatorname{span}_A\{e_i : i \in I\}},$$

that is to say, the norm-closure of its A-linear span coincides with M.

(c2) For any x of M there are elements a_i of A such that

$$x = \sum_{i\in I} e_i a_i,$$

where convergence in norm is meant and

$$\sum_{i\in I} e_i a_i = \lim_{F\in\mathcal{F}} \sum_{i\in F} e_i a_i$$

indicates the limit over the set $\mathcal{F}$ of all finite subsets of I, directed by inclusions.

(c3) The system $\{e_i\}_{i\in I}$ is said to be *closed* if it happens that for any $x \in M$ the series

$$\sum_{i\in I} (2\langle x, e_i\rangle\langle e_i, x\rangle - \langle x, e_i\rangle\langle e_i, e_i\rangle\langle e_i, x\rangle)$$

converges in norm to $\langle x, x\rangle$. Using (2.1) this exactly means that any vector of M is the limit in norm of its Fourier series.

(c4) The system $\{e_i\}_{i\in I}$ is said to be *complete* provided there is no non-zero vector x of M such that $\langle e_i, x\rangle = 0$ for all $i \in I$.

It is clear that condition (c2) implies (c1). The next example shows that the converse is not true: condition (c1) does not imply (c2) in general.

EXAMPLE 2.1. Let $A = C_0(0,1] = \{f \in C[0,1] : f(0) = 0\}$ and $M = A$ be the Hilbert A-module with respect to the inner product:

$$\langle a, b\rangle = a^* b, \quad a, b \in A.$$

Then, the one-point-set $\varepsilon = \{f\}$, where $f(x) = x$ for $x \in [0,1]$, is an orthogonal and, clearly, complete system of M. Suppose B is the closure of the $*$-algebra

$$\{fg : g \in C_0(0,1]\}.$$

Then, with $\{g_i\}$ standing for the approximative identity of $C_0(0,1]$, the C^*-algebra B contains f as the limit $f = \lim_i fg_i$. As a consequence, B separates points of the interval $[0,1]$ and, consequently, coincides with $C_0(0,1]$ by the Stone–Weierstrass theorem (cf. Theorem IV.10 of [14]). Thus, the system ε satisfies (c1). But at the same time it does not satisfy (c2) since, for instance, the function f cannot be represented as a product fg for any $g \in C_0(0,1]$.

The next example shows that there are complete orthogonal systems not satisfying (c1).

EXAMPLE 2.2. We will slightly modify Example 2.1. Let $A = C_0(0,1]$ and $M = A$ again, but now take $\varepsilon = \{g\}$, where

$$g(x) = \begin{cases} x & \text{if } 0 \leqslant x \leqslant 1/2, \\ 1-x & \text{if } 1/2 < x \leqslant 1. \end{cases}$$

Clearly, ε is a complete orthogonal system for M. However it cannot satisfy (c1) since the closure of the set $\{gh : h \in C_0(0,1]\}$ rather then to A itself, it belongs to the suspension $SA = \{f \in C[0,1] : f(0) = f(1) = 0\}$ of A.

A use of the Zorn lemma directly ensures that any pre-Hilbert C^*-module admits a complete orthogonal system, more precisely the next statement is true.

PROPOSITION 2.3. *Every orthogonal system of a pre-Hilbert C^*-module can be enlarged to a complete orthogonal system; also, every orthogonal system of norm one vectors can be enlarged to a complete orthogonal system of norm one vectors.*

This observation may be strengthen for von Neumann modules (see [17]). Indeed, let $B(G)$ denotes the set of all linear bounded operators in a Hilbert space G, $A \subset B(G)$ be a von Neumann algebra acting non-degenerately on G, and M be a Hilbert A-module. Then the algebraic tensor product $M \otimes G$ becomes a pre-Hilbert space with respect to the inner product $\langle x \otimes g, x' \otimes g' \rangle = \langle g, \langle x, x' \rangle g' \rangle$. Let $H = \overline{M \otimes G}$ stands for the Hilbert space completion of $M \otimes G$. We can consider in a natural way the module M as a linear subspace of the space $B(G, H)$ of all bounded linear operators from G to H. Then M is said to be a *von Neumann module* if it is strongly closed in $B(G, H)$. These modules behave themselves like Hilbert spaces, mostly because they are necessarily self-dual. As for the fact that any von Neumann module admits a complete quasi-orthonormal system one has the following ([17], Theorem 4.11).

Now, the analogue of the Bessel inequality for an orthogonal system $\{e_i\}_{i \in I}$ of norm one vectors in a pre-Hilbert C^*-module exists only as a finite version, i.e.

$$\sum_{i \in F} \langle x, e_i \rangle \langle e_i, x \rangle \leqslant \langle x, x \rangle$$

holds for every finite subset $F \subset I$ and any vector x. But provided $\{e_i\}_{i \in I}$ is made of norm one vectors and fulfils (c3) the restriction on finiteness may be omitted, i.e. under these additional conditions

$$\sum_{i \in I} \langle x, e_i \rangle \langle e_i, x \rangle \leqslant \langle x, x \rangle.$$

To make certain of the last inequality we need just a direct use of the following auxiliary result.

LEMMA 2.4. *Let $\{e_i\}_{i \in I}$ be an orthogonal system of norm one vectors in a pre-Hilbert C^*-module. Then the following conditions are equivalent:*

(i) *the net*

$$\Big\{A_F = \sum_{i\in F}(2\langle x,e_i\rangle\langle e_i,x\rangle - \langle x,e_i\rangle\langle e_i,e_i\rangle\langle e_i,x\rangle) : F \text{ is a finite subset of } I\Big\}$$

converges in norm;

(ii) *the net* $\Big\{B_F = \sum_{i\in F}\langle x,e_i\rangle\langle e_i,x\rangle : F \text{ is a finite subset of } I\Big\}$ *converges in norm.*

Proof. Clearly, (ii) implies (i), so we just need verify the inverse implication. Assume (i) is true and denote $C_F = \sum_{i\in F}\langle x,e_i\rangle\langle e_i,e_i\rangle\langle e_i,x\rangle$. As a consequence $B_F = A_F - B_F + C_F$, whence B_F converges if and only if $B_F - C_F$ does. To finish the argument it only remains to observe that $0 \leqslant B_F - C_F \leqslant A_F$. ∎

Although the Parseval equality does not hold for arbitrary Hilbert C^*-modules, there is a weakened version in the W^*-case.

PROPOSITION 2.5. *Suppose $\{e_i\}_{i\in I}$ is an orthogonal system of norm one vectors in a Hilbert module M over a von Neumann algebra. Then for any vector $x \in M$ the net*

$$a_F = 2\sum_{i\in F}\langle x,e_i\rangle\langle e_i,x\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,e_i\rangle\langle e_i,x\rangle,$$

indexed by finite subsets F of I, converges with respect to the strong topology.

Proof. Clearly, the elements a_F are positive, and the equality (2.1) implies $a_F \leqslant \langle x,x\rangle$ for all finite subsets F of I. It only remains to check that the net $\{a_F\}$ is not decreasing, and the required result will follow from Theorem 4.1.1 of [9]. So, let G, F be finite subsets of I and $F \subset G$; one gets:

$$\begin{aligned} a_G - a_F &= 2\sum_{i\in G\setminus F}\langle x,e_i\rangle\langle e_i,x\rangle - \sum_{i\in G\setminus F}\langle x,e_i\rangle\langle e_i,e_i\rangle\langle e_i,x\rangle \\ &= \sum_{i\in G\setminus F}\langle x,e_i\rangle\langle e_i,x\rangle + \sum_{i\in G\setminus F}\langle e_i,x\rangle^*(1-\langle e_i,e_i\rangle)\langle e_i,x\rangle \geqslant 0, \end{aligned}$$

under our assumption that the e_i's are norm one vectors. ∎

LEMMA 2.6 (The optimality property of Fourier series). *Suppose $\{e_i\}_{i\in I}$ is an orthonormal system of M, $x \in M$ is an arbitrary vector. Then*

$$\langle x - S_F, x - S_F\rangle \leqslant \Big\langle x - \sum_{i\in F} e_i a_i\, , x - \sum_{i\in F} e_i a_i\Big\rangle$$

for any elements $a_i \in A$ and for any finite subset $F \subset I$. Moreover, in the above expression the equality occurs if and only if $a_i = \langle e_i, x\rangle$ for any $i \in F$.

Proof. The above inequality follows from the following sequence of transformations:

$$\begin{aligned}\Big\langle x - \sum_{i\in F} e_i a_i\,, x - \sum_{i\in F} e_i a_i \Big\rangle &= \langle x,x\rangle - \sum_{i\in F}\langle x,e_i\rangle a_i - \sum_{i\in F} a_i^*\langle e_i,x\rangle + \sum_{i\in F} a_i^* a_i \\ &= \langle x,x\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,x\rangle + \sum_{i\in F}(a_i-\langle e_i,x\rangle)^*(a_i-\langle e_i,x\rangle) \\ &= \langle x-S_F, x-S_F\rangle + \sum_{i\in F}(a_i-\langle e_i,x\rangle)^*(a_i-\langle e_i,x\rangle),\end{aligned}$$

using identity (2.1):

$$\langle x-S_F, x-S_F\rangle = \langle x,x\rangle - \sum_{i\in F}\langle x,e_i\rangle\langle e_i,x\rangle,$$

for orthonormal systems. ∎

The next example emphasizes that for bases which are orthogonal but not orthonormal there is no uniqueness of the decomposition (c2).

EXAMPLE 2.7. Consider $A = L^\infty[0,1]$, $M = A$ with the usual inner product and let an orthogonal system of M be given by $\varepsilon = \{f_1, f_2\}$, where

$$f_1(x) = \begin{cases} 1 & \text{if } 0 \leqslant x \leqslant 1/2, \\ 0 & \text{if } 1/2 < x \leqslant 1, \end{cases} \quad \text{and} \quad f_2(x) = \begin{cases} 0 & \text{if } 0 \leqslant x \leqslant 1/2, \\ 1 & \text{if } 1/2 < x \leqslant 1. \end{cases}$$

Then $g = f_1 g + f_2 g$ for any $g \in A$, so ε forms the basis in the sense of the condition (c2). But now uniqueness does not hold since, for instance, the unit function $\mathbf{1}$ of A complies:

$$\mathbf{1} = f_1 \cdot \mathbf{1} + f_2 \cdot \mathbf{1} = f_1 \cdot f_1 + f_2 \cdot f_2.$$

Using a Banach space like terminology (cf. [7]) we say that an orthogonal system $\{e_i\}_{i\in I}$ of M forms an orthogonal *Schauder basis* for M (over A) if $\{e_i\}_{i\in I}$ satisfies (c2) and the coefficients in the decomposition (c2) are unique for any vector x of M.

Let us remind (cf. [8]) that an element x of M is called *non-singular* if its inner square $\langle x,x\rangle$ is invertible in A. Clearly, an orthogonal system $\{e_i\}_{i\in I}$ of M satisfying the condition (c2) is an orthogonal Schauder basis provided it consists of non-singular vectors. Indeed, in this case the coefficients a_i of the decomposition (c2) take the form

$$a_i = \langle e_i,e_i\rangle^{-1}\langle e_i,x\rangle,$$

from which one infers their uniqueness. The next theorem gives additional properties.

THEOREM 2.8. *Assume an orthogonal system $\{e_i\}_{i\in I}$ of a Hilbert module M over an unital C^*-algebra A, satisfying the condition* (c2), *contains at least one singular vector, e_t say. Then the system $\{e_i\}_{i\in I}$ does not form a Schauder basis if at least one of the following conditions holds:*

(i) *zero is an isolated point of the spectrum of $\langle e_t,e_t\rangle$;*

(ii) *for any element a of A which is both non-invertible and non-zero, there is a non-zero element b of A such that $ab = 0$.*

Proof. Firstly, suppose that (i) is true and consider the following continuous function

$$f(x) = \begin{cases} 1 & \text{if } \; x = 0, \\ 0 & \text{otherwise}, \end{cases}$$

on the spectrum of $\langle e_t, e_t \rangle$. Then, the element $b = f(\langle e_t, e_t \rangle)$ is not zero ([14], VII.3), belongs to A and

$$\langle e_t, e_t \rangle \, b = 0. \tag{2.2}$$

Therefore $e_t(b + \mathbf{1}) = e_t\mathbf{1}$ meaning that $\{e_i\}_{i \in I}$ does not form a Schauder basis. The same argument is valid under the assumption (ii) as well, because it directly yields the equality (2.2). ∎

We give examples of C^*-algebras with and without the property (ii) of Theorem 2.8.

EXAMPLE 2.9. Any unital commutative C^*-algebra, for instance $C[0, 1]$, does not satisfy the condition (ii) in Theorem 2.8. Condition (ii) holds for finitely dimensional C^*-algebras. But the algebra $B(H)$ of bounded linear operators on a separable Hilbert space H does not enjoy (ii). To see this it suffices to take the operator $a = \mathrm{diag}(1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots)$; it is compact and not invertible, but there is no non-zero b in $B(H)$ such that $ab = 0$.

THEOREM 2.10. *Let $\{e_i\}_{i \in I}$ be an orthonormal system in a Hilbert module M over a unital C^*-algebra A. Then the conditions* (c1)–(c3) *are equivalent and each of them is strictly stronger than* (c4).

Proof. It is clear that (c3) implies the completeness of $\{e_i\}_{i \in I}$. On the other hand Example 3.4 below ensures that (c4) does not imply (c3). To show that (c1) implies $(c2)$ let us consider an arbitrary vector $x \in M$. Then for any $\delta > 0$ one can find a finite subset $G \subset I$ and elements $a_i \in A$ such that

$$\Big\| x - \sum_{i \in G} e_i a_i \Big\| < \delta.$$

Now for any finite set $F \subset I$ containing G put

$$b_i = \begin{cases} a_i & \text{if } \; i \in G, \\ 0 & \text{if } \; i \in F \setminus G. \end{cases}$$

Applying Lemma 2.6 we conclude that

$$\delta > \Big\| x - \sum_{i \in F} e_i b_i \Big\| \geqslant \Big\| x - \sum_{i \in F} e_i \langle e_i, x \rangle \Big\|.$$

This just means that $\lim\limits_{F} \sum\limits_{i \in F} e_i \langle e_i, x \rangle = x$.

It is clear that the decomposition of x in (c2) is unique for any $x \in M$, besides the coefficients $a_i = \langle e_i, x\rangle$, so (c2) implies (c3). Besides this, obviously, (c3) implies (c1). This finishes the proof. ∎

According to [2] we call an orthogonal system $\{e_i\}_{i\in I}$ of M an orthogonal *standard Riesz basis* if it is a standard frame satisfying (c1) and endowed with the additional property that A-linear combinations $\sum\limits_{j\in S} e_j a_j$ with coefficients $a_j \in A$ and $S \subset I$ are equal to zero if and only if $e_j a_j$ equals zero for any $j \in S$.

COROLLARY 2.11. *Let $\{e_i\}_{i\in I}$ be an orthonormal system in a Hilbert module M over a unital C^*-algebra A. Then the following conditions are equivalent:*

(i) $\{e_i\}_{i\in I}$ *is a Schauder basis;*

(ii) $\{e_i\}_{i\in I}$ *is a standard Riesz basis;*

(iii) $\{e_i\}_{i\in I}$ *satisfies any of the conditions* (c1)–(c3).

Proof. Since the decomposition of x in (c2) is unique for any $x \in M$, (c2) holds for $\{e_i\}_{i\in I}$ if and only if $\{e_i\}_{i\in I}$ forms a Schauder basis. Moreover, clearly, (i) $\Rightarrow$ (ii) $\Rightarrow$ (c2). ∎

3. ORTHONORMAL BASES AND STANDARD HILBERT C^*-MODULES

The *standard Hilbert module* over a C^*-algebra A, which is denoted by $l_2(A)$ or H_A, consists of all sequences (a_i) of elements of A such that the series $\sum\limits_{i=1}^{\infty} a_i^* a_i$ converges in norm. The inner product of elements $x = (a_i)$ and $y = (b_i)$ of $l_2(A)$ is given by $\langle x, y\rangle = \sum\limits_{i=1}^{\infty} a_i^* b_i$. According to Kasparov's stabilization theorem [4] every countably generated Hilbert C^*-module is a direct summand of $l_2(A)$. The notion of a standard Hilbert C^*-module can be naturally generalized for any cardinality in the following way. Let I be an arbitrary set and $(a_i)_{i\in I}$ be a collection of elements from A indexed by I. Given that the collection $\mathcal{F}$ of finite subsets of I is partially ordered by inclusions we form a net $\Big\{ \sum\limits_{i\in F} a_i^* a_i : F \in \mathcal{F} \Big\}$ of finite sums. If this net converges in norm we will declare by definition that the series $\sum\limits_{i\in I} a_i^* a_i$ converges in norm. Then the Hilbert A-module $H_{A,I}$ is made of all collections $(a_i)_{i\in I}$ of elements of A such that the series $\sum\limits_{i\in I} a_i^* a_i$ converges in norm with the inner product of elements $x = (a_i)$ and $y = (b_i)$ of $H_{A,I}$ given by $\langle x, y\rangle = \sum\limits_{i\in I} a_i^* b_i$.

It is a well-known fact that a Hilbert module M over a unital C^*-algebra A possesses an orthonormal system $\{e_i\}_{i\in I}$ that satisfies the condition (c2) (such collection of vectors is said to be an *orthonormal basis*) if and only if M is isomorphic to the standard A-module $H_{A,I}$. Let us recall just the sketch of the proof of this assertion. The orthonormal basis $\{e_i\}_{i\in I}$ of M is closed by Theorem 2.10, consequently both $x = \sum_{i \in I} e_i \langle e_i, x\rangle$ for any x of M and the series $\sum_{i \in I} \langle e_i, x\rangle^* \langle e_i, x\rangle$ converges in norm. In particular, the Fourier coefficients $\{\langle e_i, x\rangle\}_{i\in I}$ of x belong to $H_{A,I}$. Thus, one has a well defined A-linear map from M to $H_{A,I}$ given by:

$$x \mapsto \{\langle e_i, x\rangle\}_{i\in I}.$$

The straightforward verification shows that this map is actually an isomorphism. This result was extended for the frame context in Theorem 4.1 of [2].

Moreover the cardinality of an orthonormal basis in a Hilbert module is unique like it happens for a Hilbert space. Indeed, as it happens for a Hilbert space, the norm convergence of the series $\sum_{i \in I} a_i^* a_i$ implies that the number of its non-zero entries are at most countable. And then it remains only to apply well-known arguments similar to the ones for the Hilbert space case (cf. I. Section 5.4 of [10]). Thus we have proved the following statement.

PROPOSITION 3.1. *Any two closed orthonormal systems of a Hilbert module over a unital C^*-algebra have the same cardinality.*

Let us remark that the cardinality of a complete quasi-orthonormal system in a von Neumann module is not unique ([17], Remark 4.15). It is easy to see that the same is true for closed quasi-orthonormal systems (for instance, we can consider functions f_1, f_2 and 1 of Example 2.7).

The next example shows that there are orthonormal systems in standard Hilbert C^*-modules that cannot be extended to complete orthonormal systems, a situation that differs from the cases of orthogonal systems described in Proposition 2.3.

EXAMPLE 3.2. Assume $A = L^\infty[0,1]$, $M = l_2(A)$ and choose the functions f_1, f_2 as in Example 2.7. Let $\{e_i\}_{i=1}^\infty$ be the standard basis of $l_2(A)$ meaning that all entries of e_i are zero except the i-th, which is the identity of A. Then the vectors $\{x_i\}_{i=1}^\infty$ of M, where $x_i = f_1 e_i + f_2 e_{i+1}$, form an orthonormal system. It is not complete; indeed, suppose a vector $y = (g_1, g_2, \dots)$ of M is orthogonal to x_i for any i, that is its entries are such that:

$$g_1|_{[0,1/2]} = 0, \quad g_i = 0 \text{ for } i \geqslant 2; \tag{3.1}$$

this holds, for instance, for the non-zero vector $x = f_2 e_1$. On the other hand the family $\{x_i\}_{i=1}^\infty$ cannot be enlarged to a complete orthonormal system, because the inner square of any vector satisfying (3.1) cannot give the identity. Let us remark, by the way, that the vector x extends the set $\{x_i\}_{i=1}^\infty$ to a complete orthogonal system.

In a private communication Hanfeng Li has extended the above to every non-simple unital C^*-algebra A; the following example is due to him.

EXAMPLE 3.3. Let I be a non-trivial closed two-sided ideal of a non-simple unital C^*-algebra A. Take a nonzero element $a \in I$. Denote by J the closed right ideal of A generated by a; then J is contained in I. Thus as a right Hilbert C^*-module over A, one can not find any element $x \in J$ with $\langle x, x\rangle = 1$. On the other hand, J is countably generated as a Hilbert C^*-module over A. By Kasparov's theorem, the direct sum of J and $l_2(A)$ is isomorphic to $l_2(A)$. Thus, the standard basis in $l_2(A)$ cannot be extended to a complete orthonormal system for the direct sum of J and $l_2(A)$ (which is just $l_2(A)$).

The next example shows that there are complete orthonormal systems in standard Hilbert C^*-modules which are not closed. This is one of the crucial differences between general Hilbert C^*-modules and Hilbert spaces.

EXAMPLE 3.4 (This example was refined with crucial suggestions from M. Skeide). Suppose $A = L^\infty[0,1]$ and $M = l_2(A)$ is the standard countably generated module over A. The desired system $\{e_i\}_{i=1}^\infty$ of M, where $e_i = (f_{i1}, f_{i2}, f_{i3}, \dots)$ is constructed as follows. Let us denote by $\varphi_{[a,b]}$ the characteristic function of the interval $[a,b]$, i.e.

$$\varphi_{[a,b]}(x) = \begin{cases} 1 & \text{if } x \in [a,b], \\ 0 & \text{otherwise.} \end{cases}$$

Consider $c_i = 1 - (1/2^i)$ for any non-negative integer i. Then

$$f_{i1} = \varphi_{[c_{i-1},c_i]}, \quad i \geqslant 1; \quad f_{i(i+1)} = \varphi_{[c_i,1]}, \quad i \geqslant 1; \quad f_{ii} = \varphi_{[0,c_{i-1}]}, \quad i > 1;$$

and $f_{ij} = 0$ for all other positive integer values of i and j.

For such a construction we have the following properties:

(i) Only a finite number of functions $\{f_{ij}\}_{j=1}^\infty$ is non-zero for any i, apart from this, the sum $\sum\limits_{j=1}^\infty f_{ij} = 1$ everywhere on the interval $[0,1]$ (except either the points c_{i-1} and c_i if $i \geqslant 2$ or the point c_1 if $i = 1$, but subsets of zero measure are not significant). This implies: $\langle e_i, e_i\rangle = 1$ for any i.

(ii) Whenever $i \neq k$ the supports of the functions f_{ij} and f_{kj} do not intersect each other for any j. This implies: $\langle e_i, e_k\rangle = 0$ for $i \neq k$.

(iii) For any i the union over j of the supports of the functions f_{ij} coincides with the interval $[0,1]$; this means that the system $\{e_i\}$ is complete.

But at the same time the system $\{e_i\}$ cannot be closed since, for example, for the vector $x = (1,0,0,\dots)$ the series $\sum\limits_{i=1}^\infty \langle x, e_i\rangle\langle e_i, x\rangle$ does not converge in norm.

We remark that a situation similar to the one of the previous example cannot occur for finite orthonormal systems since, clearly, any finite complete orthonormal system of a Hilbert C^*-module is closed. Now we would like to describe an example of a countably, but not finitely generated Hilbert C^*-module possessing a finite complete orthogonal system. In fact the idea of the next example may be used for constructing families of such modules, corresponding to the branched coverings over compact Hausdorff spaces.

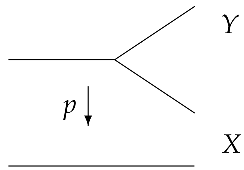

FIGURE 1. Example 3.5

EXAMPLE 3.5. Let us consider the map $p : Y \to X$ from Figure 1, where X is an interval, say $[-1, 1]$, and Y is the topological union of one interval with two copies of another half-interval with a branch point at 0. Then $C(Y)$ is a Banach $C(X)$-module for the action:

$$(f\xi)(y) = f(y)\,\xi(p(y)), \quad \text{for } f \in C(Y),\, \xi \in C(X).$$

Let us define the $C(X)$-valued inner product on $C(Y)$ by the formula

$$\langle f, g\rangle(x) = \frac{1}{\# p^{-1}(x)} \sum_{y \in p^{-1}(x)} \overline{f(y)} g(y), \tag{3.2}$$

where $\# p^{-1}(x)$ is the cardinality of $p^{-1}(x)$. It was shown in [13] that $C(Y)$ is a countably, but not finitely generated Hilbert $C(X)$-module with respect to the inner product (3.2). The space Y consists of the three intervals with the common boundary point; interval that we number in some arbitrary way. Then, for $i = 1, 2, 3$, let us consider all continuous on Y functions f_i that are not zero at all points of the i-th interval except the boundary point and are zero at the others points of Y. Clearly, the functions f_1, f_2, f_3 form a finite orthogonal complete system for $C(Y)$.

We finish the section by describing one family of non-standard bases for $l_2(L^\infty[0, 1])$.

EXAMPLE 3.6. Let $A = L^\infty[0, 1]$ and $M = l_2(A)$. For any positive integer n, consider the functions

$$f_i(x) = \begin{cases} 1 & \text{if } \ (i-1)/n \leqslant x < i/n, \\ 0 & \text{otherwise,} \end{cases}$$

and the matrix

$$F_n = \begin{pmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_2 & f_3 & \cdots & f_n & f_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_n & f_1 & \cdots & f_{n-2} & f_{n-1} \end{pmatrix}$$

We form a new matrix with an infinite number of rows and columns in the following manner:

$$B = \begin{pmatrix} F_n & 0 & \cdots & 0 & \cdots \\ 0 & F_n & \cdots & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & F_n & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix},$$

and introduce vectors e_i as the i-th rows of B. The system $\{e_i\}$ forms an orthonormal basis of the Hilbert module $l_2(A)$.

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