J. OPERATOR THEORY
70:1(2013), 3–31
doi: 10.7900/jot.2011apr01.1921

THE LEBESGUE DECOMPOSITION OF REPRESENTABLE FORMS OVER ALGEBRAS

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Communicated by Şerban Strătilă

ABSTRACT. This paper contains new results on Lebesgue type decompositions of nonnegative forms on complex algebras. We introduce the concept of representable forms, and we discuss the representability of the regular and the singular parts. A result on topologically irreducible representations in the context of the Lebesgue decomposition is included.

We prove a general Lebesgue decomposition theorem for representable positive functionals on $*$-algebras by the above-mentioned results on representable forms. This theory was studied by other authors in very different ways. We also clarify the correspondence of this kind of decompositions.

The Lebesgue decomposition theorem for measures follows from our results.

A completion of the proof of a theorem due to S. Hassi, Z. Sebestyén and H. de Snoo is included.

KEYWORDS: *Forms, parallel sums, Lebesgue type decompositions, representations, irreducible representations, positive functionals, positive measures.*

MSC (2010): Primary 46L51, 47A67; Secondary 46C50, 46L30, 47A07, 28A50.

INTRODUCTION

Recently S. Hassi, Z. Sebestyén and H. de Snoo presented a general Lebesgue type decomposition theorem on complex vector spaces [6]. A given semi-inner product $\mathfrak{t}$ (form, for short) on a complex vector space $\mathcal{D}$ is decomposed to a sum of forms

$$\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}} \tag{0.1}$$

with respect to another form $\mathfrak{w}$ on $\mathcal{D}$. The main point of this decomposition is that the so-called regular part $\mathfrak{t}_{\mathrm{reg}}$ *is closable with respect to* $\mathfrak{w}$, i.e. for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$ the property

$$\mathfrak{w}(x_n, x_n) \to 0 \wedge \mathfrak{t}_{\mathrm{reg}}(x_n - x_m, x_n - x_m) \to 0$$

implies that $\mathfrak{t}_{\text{reg}}(x_n, x_n) \to 0$, while the so-called singular part $\mathfrak{t}_{\text{sing}}$ *and* $\mathfrak{w}$ *are singular*, i.e. for any form $\mathfrak{p}$ on $\mathcal{D}$ the property

$$(\mathfrak{p}(x,x) \leqslant \mathfrak{w}(x,x)) \wedge (\mathfrak{p}(x,x) \leqslant \mathfrak{t}_{\text{sing}}(x,x)) \quad (\forall x \in \mathcal{D})$$

implies that $\mathfrak{p} = 0$.

The main purpose of this paper is to gain new results on this type of Lebesgue decomposition on complex algebras. The interesting case is that when the algebra's multiplication is involved to the investigations, namely we study decomposition of forms $\mathfrak{t}$ such that the left multiplication operators are well-defined and bounded with respect to the form $\mathfrak{t}$. For this we introduce the concept of representable forms. These forms induce representations of the algebra. It will turn out that the regular part of a representable form $\mathfrak{t}$ is always representable. Furthermore, we give a characterization theorem for the representability of the singular part. In this result closed invariant subspaces of the generated representation appear. From this we can study the Lebesgue decomposition of forms such that the induced representation is topologically irreducible. It is important to note that we do not make any assumptions for the algebra, neither the commutativity, nor the existence of unit element.

The case of involutive algebras (throughout the paper $*$-algebra) and representable positive functionals is also interesting. We introduce a general Lebesgue decomposition theorem for representable positive functionals. We will prove that for a representable positive functional the Lebesgue decomposition related to forms always yields regular and singular parts that were derived from representable positive functionals.

There are other authors who studied the $*$-algebra case previously, in various approaches. H. Kosaki in [8] presented a Lebesgue decomposition for normal states on σ-finite von Neumann algebras. S. Gudder in [5] proved a Lebesgue decomposition theorem for positive functionals on unital Banach $*$-algebras. We will show that our result coincides with Kosaki's decomposition on σ-finite von Neumann algebras and with Gudder's on unital Banach $*$-algebras, respectively. Hence our theorem generalizes these two results. According to non-uniqueness of Kosaki's decomposition, this fact allows us to show that our decomposition is not unique, as well as Gudder's.

It is also a natural question what is the connection between the decomposition (0.1) and the famous Lebesgue decomposition related to measures. We will present a theorem on the existence of the Lebesgue decomposition related to positive finite measures, directly from the decomposition that we claimed for representable positive functonals on $*$-algebras. This result will imply a theorem of Hassi, Sebestyén and de Snoo. They proved that if we take forms induced by measures, then $\mathfrak{t}_{\text{reg}}$ (respectively $\mathfrak{t}_{\text{sing}}$) in the decomposition (0.1) coincides with the form generated by the absolutely (respectively singular) part of the decomposed measure ([6], Theorem 5.5). However, there is a gap in their proof. We will introduce a counterexample to illustrate this problem, moreover we will fill the gap in the original proof.

We have to mention other papers concerning with Lebesgue decomposition. We mentioned before H. Kosaki's [8] and S. Gudder's [5] results. T. Ando [1] decomposed a bounded positive linear operator into a sum of positive linear operators (almost dominated and singular part, respectively) with respect to another bounded positive operator. In [15] B. Simon claimed a decomposition similar to (0.1) for a densely defined form on a Hilbert space. A. Inoue [7] proved a Lebesgue decomposition theorem for positive invariant sesquilinear forms on $*$-algebras.

The set-up of the paper is the following. Section 1 includes essential definitions and achievements on the Lebesgue decomposition of forms (all of these results were taken from the work of Hassi, Sebestyén and de Snoo [6]). In Section 2 we present new results for representable forms on complex algebras. We do not make any assumptions for the algebra. We prove that the regular part of a representable form is always representable, and we give a characterization for the representability of the singular part by means of closed invariant subspaces. With the aid of this result we discuss the Lebesgue decomposition of representable forms which generate topologically irreducible representations. Section 3 deals with the $*$-algebra case. With the help of the involution and the characterization theorem for the singular part we show that the regular part and the singular part in the decomposition of a representable positive functional with respect to a representable form are always derived from representable positive functionals. This is a general Lebesgue type decomposition for representable positive functionals. In the second part of Section 3 we prove that this decomposition coincides with Kosaki's decomposition on σ-finite von Neumann algebras [8] and with Gudder's decomposition on unital Banach $*$-algebras [5]. We also give an answer to the question of uniqueness. Section 4 contains a new proof to the well-known Lebesgue theorem for positive measures, directly from our results. Section 5 introduces a completion for the proof of a theorem due to Hassi, Sebestyén and de Snoo. A counterexample to the original proof's deficiency is also given in the last section of the paper.

NOTATIONS

If $\mathfrak{t}$ is a form on the complex vector space $\mathcal{D}$, then for $x \in \mathcal{D}$ let $\mathfrak{t}[x] := \mathfrak{t}(x,x)$, i.e. the quadratic form generated by $\mathfrak{t}$. The kernel of $\mathfrak{t}$ will be denoted by $\ker \mathfrak{t}$, i.e. $\ker \mathfrak{t} := \{x \in \mathcal{D} : \mathfrak{t}[x] = 0\}$. If $\mathfrak{w}$ is another form on $\mathcal{D}$, then $\mathfrak{w} \leqslant \mathfrak{t}$ means that $\mathfrak{w}[x] \leqslant \mathfrak{t}[x]$ holds true for all $x \in \mathcal{D}$.

A form $\mathfrak{t}$ induces an inner product on the quotient space $\mathcal{D}/\ker \mathfrak{t}$, that is for any $x, y \in \mathcal{D}$, $\langle x + \ker \mathfrak{t}, y + \ker \mathfrak{t}\rangle_{\mathfrak{t}} := \mathfrak{t}(x,y)$. Then $(\mathcal{H}_{\mathfrak{t}}, \langle\cdot,\cdot\rangle_{\mathfrak{t}})$ stands for the Hilbert space associated to the form $\mathfrak{t}$, i.e. the completion of the pre-Hilbert space

$(\mathcal{D}/\ker \mathfrak{t}, \langle\cdot,\cdot\rangle_{\mathfrak{t}})$. If $\mathfrak{w}$ is another form on $\mathcal{D}$, then $\mathcal{H}_{\mathfrak{t}} \oplus \mathcal{H}_{\mathfrak{w}}$ denotes the orthogonal sum of the Hilbert space $\mathcal{H}_{\mathfrak{t}}$ and $\mathcal{H}_{\mathfrak{w}}$, with scalar product

$$\langle (x_1, y_1), (x_2, y_2)\rangle_{\mathfrak{t}\oplus\mathfrak{w}} := \langle x_1, x_2\rangle_{\mathfrak{t}} + \langle y_1, y_2\rangle_{\mathfrak{w}} \quad (x_1, x_2 \in \mathcal{H}_{\mathfrak{t}}, y_1, y_2 \in \mathcal{H}_{\mathfrak{w}}).$$

For a Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ stands for the space of bounded linear operators.

If $\mathscr{A}$ is a $*$-algebra and f is positive linear functional on $\mathscr{A}$ (i.e. $f(a^*a) \geqslant 0$ for all $a \in \mathscr{A}$), then $\widetilde{f}$ stands for the semi inner product generated by f on $\mathscr{A}$, namely $\widetilde{f}(a, b) := f(b^*a)$ for all $a, b \in \mathscr{A}$. For positive functionals f and g, $f \leqslant g$ denotes the natural ordering, i.e. $\widetilde{f} \leqslant \widetilde{g}$. A positive functional f is said to be *representable*, if there exists a cyclic representation $\pi : \mathscr{A} \to B(\mathcal{H})$ on a Hilbert space $\mathcal{H}$ with cyclic vector $\xi \in \mathcal{H}$ (i.e. $\overline{\pi(\mathscr{A})\xi} = \mathcal{H}$) such that for any $a \in \mathscr{A}$

$$f(a) = \langle \pi(a)\xi, \xi\rangle$$

holds true. By the aid of the remarkable Gelfand–Naimark–Segal (GNS) construction there are simple equivalent conditions to this property (see [9], [14]). The usual notations $(\mathcal{H}_f, \langle\cdot,\cdot\rangle_f)$, π_f and ξ_f stand for the associated Hilbert space, representation and the distinguished cyclic vector from the GNS-construction, respectively. If $\mathscr{A}$ is unital, then the unit element of $\mathscr{A}$ will be denoted by $\mathbf{1}$.

For $*$-algebras, positive functionals and the GNS-construction the reader is also referred to [2], [10], [13].

1. A GENERAL LEBESGUE TYPE DECOMPOSITION THEOREM FOR FORMS

This part of the paper contains the main tools and results that we need to gain new results in the latter sections. All of the definitions and theorems are adopted from article [6].

DEFINITION 1.1. Let $\mathcal{D}$ be a complex vector space, $\mathfrak{t}$ and $\mathfrak{w}$ forms on $\mathcal{D}$.
(i) $\mathfrak{w}$ is *closable* with respect to $\mathfrak{t}$ if for any sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}$:

$$(\lim_{n\to+\infty} \mathfrak{t}[x_n] = 0 \wedge \lim_{n,m\to+\infty} \mathfrak{w}[x_n - x_m] = 0) \Rightarrow \lim_{n\to+\infty} \mathfrak{w}[x_n] = 0.$$

(i) $\mathfrak{w}$ and $\mathfrak{t}$ are *singular* if for any form $\mathfrak{p}$ on $\mathcal{D}$:

$$(\mathfrak{p} \leqslant \mathfrak{w} \wedge \mathfrak{p} \leqslant \mathfrak{t}) \Rightarrow \mathfrak{p} = 0.$$

For positive linear functionals f and g on a $*$-algebra we say that f is *closable* with respect to g, if the induced form $\widetilde{f}$ is closable with respect to the other induced form $\widetilde{g}$. We say that f and g are *singular*, if the induced forms $\widetilde{f}$ and $\widetilde{g}$ are singular.

The following theorem ([6], Proposition 2.2) presents a concept of fundamental importance in this theory.

THEOREM 1.2. *Let $\mathcal{D}$ be a complex vector space, and let $\mathfrak{t}, \mathfrak{w}$ be forms on $\mathcal{D}$. Then the formula*

$$(\mathfrak{t} : \mathfrak{w})[x] := \inf\{\mathfrak{t}[x-z] + \mathfrak{w}[z] : z \in \mathcal{D}\} \quad (x \in \mathcal{D}).$$

defines a quadratic form on $\mathcal{D}$, hence uniquely determines a form $(\mathfrak{t} : \mathfrak{w})$ on $\mathcal{D}$.

DEFINITION 1.3. The form $(\mathfrak{t} : \mathfrak{w})$ is the *parallel sum* of the forms $\mathfrak{t}$ and $\mathfrak{w}$.

REMARK 1.4. It is easy to see that $(\mathfrak{t} : \mathfrak{w}) \leqslant \mathfrak{t}$ and $(\mathfrak{t} : \mathfrak{w}) \leqslant \mathfrak{w}$. For other properties of the parallel sum, the reader is referred to Lemma 2.3 of [6].

The next proposition shows the connection between singularity and the parallel sum ([6], Proposition 2.10).

THEOREM 1.5. *If $\mathfrak{t}$ and $\mathfrak{w}$ are two forms on the complex vector space $\mathcal{D}$, then the following statements are equivalent:*
(i) $\mathfrak{t}$ *and* $\mathfrak{w}$ *are singular.*
(ii) $(\mathfrak{t} : \mathfrak{w}) = 0$.

According to the previous theorem, the following statement ([6], Corollary 3.2) introduces a geometric characterization of singularity. This property plays a key role in our paper, therefore we give the sketch of the proof.

THEOREM 1.6. *Let $\mathcal{D}$ be a complex vector space and let $\mathfrak{t}$, $\mathfrak{w}$ be forms on $\mathcal{D}$. Then for the densely defined linear isometry*

$$\mathbb{U} : \mathcal{H}_{\mathfrak{t}+\mathfrak{w}} \to \mathcal{H}_{\mathfrak{t}} \oplus \mathcal{H}_{\mathfrak{w}}; \quad \mathbb{U}(x + \ker(\mathfrak{t}+\mathfrak{w})) := (x + \ker \mathfrak{t}, x + \ker \mathfrak{w}) \tag{1.1}$$

the following statements are equivalent:
(i) $(\mathfrak{t} : \mathfrak{w}) = 0$.
(ii) $\overline{\operatorname{ran}}\mathbb{U} = \mathcal{H}_{\mathfrak{t}} \oplus \mathcal{H}_{w}$.

Proof. (i) $\Rightarrow$ (ii) It is enough to prove that $\{(x + \ker \mathfrak{t}, y + \ker \mathfrak{w}) : x, y \in \mathcal{D}\} \subseteq \overline{\operatorname{ran}}\mathbb{U}$, since it is a dense subspace of $\mathcal{H}_{\mathfrak{t}} \oplus \mathcal{H}_{\mathfrak{w}}$. Let $x, y \in \mathcal{D}$ be arbitrary vectors. Since $(\mathfrak{t} : \mathfrak{w}) = 0$, we have that $0 = (\mathfrak{t} : \mathfrak{w})[x] = \inf\{\mathfrak{t}[x-z] + \mathfrak{w}[z] : z \in \mathcal{D}\}$. This implies the existence of a sequence $(x_n)_{n\in\mathbb{N}}$ such that $\mathfrak{t}[x - x_n] + \mathfrak{w}[x_n] \to 0$. Similarly, there exists a sequence $(y_n)_{n\in\mathbb{N}}$ such that $\mathfrak{t}[y_n] + \mathfrak{w}[y - y_n] \to 0$. Then for the sequence $(x_n + y_n)_{n\in\mathbb{N}}$ we conclude that

$$\mathbb{U}(x_n+y_n+\ker(\mathfrak{t}+\mathfrak{w})) = (x_n+y_n+\ker \mathfrak{t}, x_n+y_n+\ker \mathfrak{w}) \to (x+\ker \mathfrak{t}, y+\ker \mathfrak{w}).$$

(ii) $\Rightarrow$ (i) Fix an arbitrary $x \in \mathcal{D}$. By (ii) there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $(x_n + \ker \mathfrak{t}, x_n + \ker \mathfrak{w}) \to (x + \ker \mathfrak{t}, 0 + \ker \mathfrak{w})$, hence

$$(\mathfrak{t} : \mathfrak{w})[x] = \inf\{\mathfrak{t}[x-z] + \mathfrak{w}[z] : z \in \mathcal{D}\} \leqslant \mathfrak{t}[x - x_n] + \mathfrak{w}[x_n] \to 0. \qquad \blacksquare$$

The following remarkable theorem is the main achievement in this decomposition theory of forms ([6], Theorems 2.11, 3.6, 3.8 and 3.9). This statement is the basis of our results.

THEOREM 1.7 (Lebesgue decomposition of forms). *Let* $\mathfrak{t}, \mathfrak{w}$ *be forms on the complex vector space* $\mathcal{D}$. *Define a form* $\mathfrak{t}_{\mathrm{reg}}$ *by the equation*

$$\mathfrak{t}_{\mathrm{reg}}[x] := \sup_{n \in \mathbb{N}} (\mathfrak{t} : n\mathfrak{w})[x] \quad (x \in \mathcal{D}) \tag{1.2}$$

and let $\mathfrak{t}_{\mathrm{sing}} := \mathfrak{t} - \mathfrak{t}_{\mathrm{reg}}$. *Then in the decomposition* $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$ *the form* $\mathfrak{t}_{\mathrm{reg}}$ *is closable with respect to* $\mathfrak{w}$, *and the forms* $\mathfrak{t}_{\mathrm{sing}}$ *and* $\mathfrak{w}$ *are singular. Moreover, the form* $\mathfrak{t}_{\mathrm{reg}}$ *is the greatest among all of the forms* $\mathfrak{p}$ *such that* $\mathfrak{p} \leqslant \mathfrak{t}$ *and* $\mathfrak{p}$ *is closable with respect to* $\mathfrak{w}$.

The forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$ are called the regular and singular part of $\mathfrak{t}$, respectively.

REMARK 1.8. In Theorem 2.11 of [6] for a form the property *almost dominated by* $\mathfrak{w}$ appears, but this attribute is equivalent to that the form is *closable with respect to* $\mathfrak{w}$ ([6], Theorem 3.8). In our results only the latter property plays role, that is why the previous theorem was stated in that form.

We note that the terminology *almost dominated* has also appeared in [8].

The last statement shows a connection between the regular part and the singular part ([6], Proposition 3.13):

PROPOSITION 1.9. *Let* $\mathfrak{t}, \mathfrak{w}$ *be forms on the complex vector space* $\mathcal{D}$. *Then*

$$(\mathfrak{t}_{\mathrm{sing}} : (\mathfrak{w} + \mathfrak{t}_{\mathrm{reg}})) = 0, \tag{1.3}$$

i.e. the forms $\mathfrak{t}_{\mathrm{sing}}$ *and* $\mathfrak{w} + \mathfrak{t}_{\mathrm{reg}}$ *are singular.*

REMARK 1.10. The previous proposition implies that $(\mathfrak{t}_{\mathrm{reg}} : \mathfrak{t}_{\mathrm{sing}}) = 0$. An easy consequence of this fact is that if a form $\mathfrak{t}$ simultaneously singular and closable with respect to a non-zero form $\mathfrak{w}$, then $\mathfrak{t} = 0$. Indeed, singularity implies in the Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$ that $\mathfrak{t} = \mathfrak{t}_{\mathrm{sing}}$, meanwhile closability yields that $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}$. Thus $\mathfrak{t} = 2(\mathfrak{t} : \mathfrak{t}) = 2(\mathfrak{t}_{\mathrm{reg}} : \mathfrak{t}_{\mathrm{sing}}) = 0$.

2. THE DECOMPOSITION OF REPRESENTABLE FORMS ON ALGEBRAS

We introduce new results on the Lebesgue decomposition of forms over complex algebras. The interesting case is that when the multiplication of the algebra is involved to the investigations, hence let us begin this section with a definition. As we mentioned before, we do not assume that the algebra has a unit element.

DEFINITION 2.1. Let $\mathscr{A}$ be a complex algebra. Then a form $\mathfrak{t}$ on $\mathscr{A}$ is *representable*, if

$$(\forall a \in \mathscr{A})(\exists \lambda_a \in [0, +\infty)) \quad (\forall b \in \mathscr{A}) : \mathfrak{t}[ab] \leqslant \lambda_a \mathfrak{t}[b], \tag{2.1}$$

i.e. for any $a \in \mathscr{A}$ the left multiplication operators

$$L_a : \mathscr{A} / \ker \mathfrak{t} \to \mathscr{A} / \ker \mathfrak{t}; L_a(b + \ker \mathfrak{t}) := ab + \ker \mathfrak{t} \quad (b \in \mathscr{A})$$

are well-defined and bounded on the pre-Hilbert space $(\mathscr{A} / \ker \mathfrak{t}, \langle \cdot, \cdot \rangle_{\mathfrak{t}})$.

Denote by $\pi_{\mathfrak{t}}(a)$ the unique bounded linear extension of L_a to $\mathcal{H}_{\mathfrak{t}}$.

It is easy to prove the following proposition. This result justifies the use of the term *representable*.

PROPOSITION 2.2. *The map $\pi_{\mathfrak{t}} : \mathscr{A} \to B(\mathcal{H}_{\mathfrak{t}})$ is a representation of $\mathscr{A}$ on the Hilbert space $(\mathcal{H}_{\mathfrak{t}}, \langle \cdot, \cdot \rangle_{\mathfrak{t}})$.*

Let us call $\pi_{\mathfrak{t}}$ the representation generated (or induced) by the representable form $\mathfrak{t}$.

The importance of representable forms lies on the following simple fact: every cyclic representation $\pi : \mathscr{A} \to B(\mathcal{H})$ of a complex algebra $\mathscr{A}$ on a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ is unitarily equivalent to a representation induced by a representable form. Indeed, let $x_0 \in \mathcal{H}$ be a cyclic vector and define a form on $\mathscr{A}$ by

$$\mathfrak{t} : \mathscr{A} \times \mathscr{A} \to \mathbb{C}; \quad \mathfrak{t}(a, b) := \langle \pi(a)x_0, \pi(b)x_0 \rangle.$$

This is a representable form, since for any $a, b \in \mathscr{A}$ we have

$$\begin{aligned} \mathfrak{t}[ab] &= \langle \pi(ab)x_0, \pi(ab)x_0 \rangle = \langle \pi(a)\pi(b)x_0, \pi(a)\pi(b)x_0 \rangle \\ &\leqslant \|\pi(a)\|^2 \langle \pi(b)x_0, \pi(b)x_0 \rangle = \|\pi(a)\|^2 \mathfrak{t}[b]. \end{aligned}$$

To see that π and $\pi_{\mathfrak{t}}$ are unitarily equivalent, let $\mathbb{V}$ be the densely defined operator

$$\mathbb{V} : \pi(\mathscr{A})x_0 \to \mathcal{H}_{\mathfrak{t}}; \quad \mathbb{V}(\pi(a)x_0) := a + \ker \mathfrak{t}.$$

It is easy to check that $\mathbb{V}$ is a linear isometry, hence the unique linear extension V of $\mathbb{V}$ to $\mathcal{H}$ is unitary. Moreover, for every $a \in \mathscr{A}$ and arbitrary element $\pi(b)x_0$ of the dense subspace $\pi(\mathscr{A})x_0$ we obtain

$$(V\pi(a))(\pi(b)x_0) = V(\pi(ab)x_0) = ab + \ker \mathfrak{t} = \pi_{\mathfrak{t}}(a)(b + \ker \mathfrak{t}) = (\pi_{\mathfrak{t}}(a)V)(\pi(b)x_0),$$

that is for any $a \in \mathscr{A}$

$$V\pi(a) = \pi_{\mathfrak{t}}(a)V$$

holds true, namely π and $\pi_{\mathfrak{t}}$ are unitarily equivalent representations (and obviously $\pi_{\mathfrak{t}}$ is also cyclic with the vector Vx_0).

REMARK 2.3. We note here that when $\mathscr{A}$ is commutative, then

$$\mathfrak{t}^*(a, b) := \langle \pi(b)^* x_0, \pi(a)^* x_0 \rangle$$

is also a representable form on $\mathscr{A}$. It is easy to see that $\mathfrak{t}^*$ is a form, and the representability follows from the validity of the next inequality for any $a, b \in \mathscr{A}$:

$$\begin{aligned} \mathfrak{t}^*[ab] &= \langle \pi(ab)^* x_0, \pi(ab)^* x_0 \rangle = \langle \pi(b)^* \pi(a)^* x_0, \pi(b)^* \pi(a)^* x_0 \rangle \\ &= \langle \pi(a)^* \pi(b)^* x_0, \pi(a)^* \pi(b)^* x_0 \rangle \leqslant \|\pi(a)^*\|^2 \langle \pi(b)^* x_0, \pi(b)^* x_0 \rangle = \|\pi(a)\|^2 \mathfrak{t}^*[b]. \end{aligned}$$

The term *representable* is also motivated by the well-known representable positive functionals on $*$-algebras. This kind of functionals induce a form and a cyclic representation on a Hilbert space via the GNS-construction. This induced form is obviously representable in the sense of Definition 2.1. The difference in our terminology is that we do not assume the representations to be cyclic. For instance, 9.4.25 in [9] gives an example to a positive linear functional ω such that is not representable in the classical sense, while the induced form $\widetilde{\omega}$ is representable in the sense of Definition 2.1. But we note that if the algebra has unit element $\mathbf{1}$, then a representable form $\mathfrak{t}$ induces a cyclic representation with cyclic vector $\mathbf{1} + \ker \mathfrak{t}$.

The primary question in terms of the Lebesgue decomposition for representable forms is whether or not the regular part and the singular part are representable. The purpose of this section is to answer this.

With the aid of our results we will be able to study the case of forms such that the generated representation is topologically irreducible. Moreover, Theorem 2.6, our main theorem gives the opportunity to get a general Lebesgue decomposition theorem for representable positive functionals on $*$-algebras (Corollary 3.2). The latter will be discussed in Section 3.

First we prove a lemma on the parallel sum of representable forms.

LEMMA 2.4. *Let $\mathscr{A}$ be a complex algebra and let $\mathfrak{t}, \mathfrak{w}$ be representable forms on $\mathscr{A}$. Then their parallel sum $(\mathfrak{t} : \mathfrak{w})$ is also a representable form on $\mathscr{A}$:*

$$\|\pi_{(\mathfrak{t}:\mathfrak{w})}(a)\| \leqslant \max\{\|\pi_{\mathfrak{t}}(a)\|, \|\pi_{\mathfrak{w}}(a)\|\} \quad (a \in \mathscr{A}). \tag{2.2}$$

Proof. We prove that for every $a, b \in \mathscr{A}$

$$(\mathfrak{t} : \mathfrak{w})[ab] \leqslant \max\{\|\pi_{\mathfrak{t}}(a)\|^2, \|\pi_{\mathfrak{w}}(a)\|^2\}(\mathfrak{t} : \mathfrak{w})[b],$$

holds, and this implies the desired result. By definition we have

$$\begin{aligned}
(\mathfrak{t} : \mathfrak{w})[ab] &= \inf\{\mathfrak{t}[ab - c] + \mathfrak{w}[c] : c \in \mathscr{A}\} \leqslant \inf\{\mathfrak{t}[ab - ac] + \mathfrak{w}[ac]\} : c \in \mathscr{A}\} \\
&\leqslant \inf\{\|\pi_{\mathfrak{t}}(a)\|^2 \mathfrak{t}[b - c] + \|\pi_{\mathfrak{w}}(a)\|^2 \mathfrak{w}[c]\} : c \in \mathscr{A}\} \\
&\leqslant \max\{\|\pi_{\mathfrak{t}}(a)\|^2, \|\pi_{\mathfrak{w}}(a)\|^2\} \inf\{\mathfrak{t}[b - c] + \mathfrak{w}[c] : c \in \mathscr{A}\} \\
&= \max\{\|\pi_{\mathfrak{t}}(a)\|^2, \|\pi_{\mathfrak{w}}(a)\|^2\}(\mathfrak{t} : \mathfrak{w})[b],
\end{aligned}$$

thus (2.2) follows. ∎

The regular part was defined by supremum of parallel sums, hence the previous lemma provides that the regular part is always representable:

THEOREM 2.5. *Let $\mathscr{A}$ be a complex algebra and let $\mathfrak{t}, \mathfrak{w}$ be representable forms on $\mathscr{A}$. Let $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$ be the Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Then the form $\mathfrak{t}_{\mathrm{reg}}$ is a representable form on $\mathscr{A}$, moreover for every $a \in \mathscr{A}$ we infer that*

$$\|\pi_{\mathfrak{t}_{\mathrm{reg}}}(a)\| \leqslant \max\{\|\pi_{\mathfrak{t}}(a)\|, \|\pi_{\mathfrak{w}}(a)\|\}. \tag{2.3}$$

Proof. The regular part $\mathfrak{t}_{\mathrm{reg}}$ was defined in Theorem 1.7 by the formula

$$\mathfrak{t}_{\mathrm{reg}}[a] := \sup_{n\in\mathbb{N}}(\mathfrak{t} : n\mathfrak{w})[a] \quad (a \in \mathscr{A}),$$

hence it is enough to prove the next inequality for every $n \in \mathbb{N}$ and $a, b \in \mathscr{A}$:

$$(\mathfrak{t} : n\mathfrak{w})[ab] \leqslant \max\{\|\pi_{\mathfrak{t}}(a)\|^2, \|\pi_{\mathfrak{w}}(a)\|^2\}(\mathfrak{t} : n\mathfrak{w})[b].$$

Fix an arbitrary $n \in \mathbb{N}$ and two elements $a, b \in \mathscr{A}$. For the form $\mathfrak{w}$ we have that

$$n\mathfrak{w}[ab] \leqslant n\|\pi_{\mathfrak{w}}(a)\|^2\mathfrak{w}[b],$$

hence $\|\pi_{\mathfrak{w}}(a)\| = \|\pi_{n\mathfrak{w}}(a)\|$. From Lemma 2.4 it follows that

$$(\mathfrak{t} : n\mathfrak{w})[ab] \leqslant \max\{\|\pi_{\mathfrak{t}}(a)\|^2, \|\pi_{\mathfrak{w}}(a)\|^2\}(\mathfrak{t} : n\mathfrak{w})[b],$$

and taking supremum in n on both sides of this inequality gives (2.3). ∎

The following theorem is our main result in the context of the Lebesgue decomposition of representable forms on algebras. It turns out that the regular part and the singular part are closely connected to closed invariant subspaces of the representation induced by the decomposition of forms.

In Section 3 by means of involution, this theorem allows us to gain a stronger statement for $*$-algebras (Theorem 3.1).

THEOREM 2.6. *Let $\mathscr{A}$ be a complex algebra and let $\mathfrak{t}, \mathfrak{w}$ be representable forms on $\mathscr{A}$, and let $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$ be the Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Then the densely defined linear isometry*

$$\begin{aligned} &\mathbb{U} : \mathscr{A} / \ker \mathfrak{t} \to \mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}, \\ &\mathbb{U}(a + \ker \mathfrak{t}) := (a + \ker \mathfrak{t}_{\mathrm{reg}}, a + \ker \mathfrak{t}_{\mathrm{sing}}) \quad (a \in \mathscr{A}), \end{aligned} \tag{2.4}$$

admits an unitary extension U to $\mathcal{H}_{\mathfrak{t}}$, and:

(i) *The closed linear subspace $U^{-1}(\{0 + \ker \mathfrak{t}_{\mathrm{reg}}\} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}})$ is $\pi_{\mathfrak{t}}$-invariant.*

(ii) *For the $\pi_{\mathfrak{t}}$-invariance of the closed linear subspace $U^{-1}(\mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \{0 + \ker \mathfrak{t}_{\mathrm{sing}}\})$ it is necessary and sufficient that the form $\mathfrak{t}_{\mathrm{sing}}$ is representable.*

If $\mathfrak{t}_{\mathrm{sing}}$ is representable, then the following estimates are valid for every $a \in \mathscr{A}$:

$$\|\pi_{\mathfrak{t}_{\mathrm{reg}}}(a)\| \leqslant \|\pi_{\mathfrak{t}}(a)\|, \quad \|\pi_{\mathfrak{t}_{\mathrm{sing}}}(a)\| \leqslant \|\pi_{\mathfrak{t}}(a)\|.$$

Moreover, for all $a \in \mathscr{A}$

$$U\pi_{\mathfrak{t}}(a)U^{-1} = \pi_{\mathfrak{t}_{\mathrm{reg}}}(a) \oplus \pi_{\mathfrak{t}_{\mathrm{sing}}}(a). \tag{2.5}$$

Proof. The forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$ are singular (Remark 1.10), and by Theorem 1.6 this is equivalent to $\overline{\mathrm{ran}}\mathbb{U} = \mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}$, hence $\mathbb{U}$ has an unitary extension U. Define a representation of $\mathscr{A}$ by

$$\psi_{\mathfrak{t}} : \mathscr{A} \to B(\mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}), \quad \psi_{\mathfrak{t}}(a) := U\pi_{\mathfrak{t}}(a)U^{-1}, \tag{2.6}$$

which is a representation unitarily equivalent to $\pi_{\mathfrak{t}}$.

(i) The closed linear subspace $U^{-1}(\{0+\ker \mathrm{t_{reg}}\} \oplus \mathcal{H}_{\mathrm{t_{sing}}})$ is π_{t}-invariant if and only if $\{0+\ker \mathrm{t_{reg}}\} \oplus \mathcal{H}_{\mathrm{t_{sing}}}$ is ψ_{t}-invariant. Fix an $a \in \mathscr{A}$. We show that for the dense linear subspace

$$\mathcal{Y} = \{(0+\ker \mathrm{t_{reg}}, b+\ker \mathrm{t_{sing}}) : b \in \mathscr{A}\}$$

of $\{0+\ker \mathrm{t_{reg}}\} \oplus \mathcal{H}_{\mathrm{t_{sing}}}$ the relation $\psi_{\mathrm{t}}(a)(\mathcal{Y}) \subseteq \{0+\ker \mathrm{t_{reg}}\} \oplus \mathcal{H}_{\mathrm{t_{sing}}}$ holds true, thus the closure of $\mathcal{Y}$ is also an invariant subspace, since the operator $\psi_{\mathrm{t}}(a)$ is continuous.

Let $b \in \mathscr{A}$ be an arbitrary element. Since $\overline{\mathrm{ran}}\mathbb{U} = \mathcal{H}_{\mathrm{t_{reg}}} \oplus \mathcal{H}_{\mathrm{t_{sing}}}$, there exists a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that

$$(b_n+\ker \mathrm{t_{reg}}, b_n+\ker \mathrm{t_{sing}}) \to (0+\ker \mathrm{t_{reg}}, b+\ker \mathrm{t_{sing}}).$$

The sequence $(b_n+\ker \mathrm{t})_{n\in\mathbb{N}} = (U^{-1}(b_n+\ker \mathrm{t_{reg}}, b_n+\ker \mathrm{t_{sing}}))_{n\in\mathbb{N}}$ is convergent in $\mathcal{H}_{\mathrm{t}}$, so by the continuity of the operator $\pi_{\mathrm{t}}(a)$ the sequence $(\pi_{\mathrm{t}}(a)(b_n+\ker \mathrm{t}))_{n\in\mathbb{N}} = (ab_n+\ker \mathrm{t})_{n\in\mathbb{N}}$ is also convergent. This implies that the sequence $(ab_n+\ker \mathrm{t_{sing}})_{n\in\mathbb{N}}$ is Cauchy, since $\mathrm{t_{sing}} \leqslant \mathrm{t}$. From this it follows that $(ab_n+\ker \mathrm{t_{sing}})_{n\in\mathbb{N}}$ converges to a vector $\xi_{ab} \in \mathcal{H}_{\mathrm{t_{sing}}}$. Furthermore, $\mathrm{t_{reg}}$ is representable according to Theorem 2.5, hence

$$ab_n+\ker \mathrm{t_{reg}} = \pi_{\mathrm{t_{reg}}}(a)(b_n+\ker \mathrm{t_{reg}}) \to 0+\ker \mathrm{t_{reg}}.$$

Then for the continuous operator $\psi_{\mathrm{t}}(a)$ we infer that

$$\begin{aligned}\psi_{\mathrm{t}}(a)(0+\ker \mathrm{t_{reg}}, b+\ker \mathrm{t_{sing}}) &= \lim_{n\to+\infty} \psi_{\mathrm{t}}(a)(b_n+\ker \mathrm{t_{reg}}, b_n+\ker \mathrm{t_{sing}})\\ &= \lim_{n\to+\infty} U(\pi_{\mathrm{t}}(a)(b_n+\ker \mathrm{t})) = \lim_{n\to+\infty} U(ab_n+\ker \mathrm{t})\\ &= \lim_{n\to+\infty}(ab_n+\ker \mathrm{t_{reg}}, ab_n+\ker \mathrm{t_{sing}})\\ &= (0+\ker \mathrm{t_{reg}}, \xi_{ab}+\ker \mathrm{t_{sing}}),\end{aligned}$$

that is $\psi_{\mathrm{t}}(a)(\mathcal{Y}) \subseteq \{0+\ker \mathrm{t_{reg}}\} \oplus \mathcal{H}_{\mathrm{t_{sing}}}$.

(ii) For the necessity assume that $U^{-1}(\mathcal{H}_{\mathrm{t_{reg}}} \oplus \{0+\ker \mathrm{t_{sing}}\})$ is π_{t}-invariant, which is equivalent to the ψ_{t}-invariance of $\mathcal{H}_{\mathrm{t_{reg}}} \oplus \{0+\ker \mathrm{t_{sing}}\}$. Let $a, c \in \mathscr{A}$ be arbitrary elements. By the invariance for the vector $(c+\ker \mathrm{t_{reg}}, 0+\ker \mathrm{t_{sing}})$ we have that

$$\psi_{\mathrm{t}}(a)(c+\ker \mathrm{t_{reg}}, 0+\ker \mathrm{t_{sing}}) \in \mathcal{H}_{\mathrm{t_{reg}}} \oplus \{0+\ker \mathrm{t_{sing}}\},$$

hence there exists a $\xi_{ac} \in \mathcal{H}_{\mathrm{t_{reg}}}$ such that

$$\psi_{\mathrm{t}}(a)(c+\ker \mathrm{t_{reg}}, 0+\ker \mathrm{t_{sing}}) = (\xi_{ac}, 0+\ker \mathrm{t_{sing}}).$$

Since $\overline{\mathrm{ran}}\mathbb{U} = \mathcal{H}_{\mathrm{t_{reg}}} \oplus \mathcal{H}_{\mathrm{t_{sing}}}$, there exists a sequence $(c_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that

$$(c_n+\ker \mathrm{t_{reg}}, c_n+\ker \mathrm{t_{sing}}) \to (0+\ker \mathrm{t_{reg}}, c+\ker \mathrm{t_{sing}}).$$

By Theorem 2.5 the form $\mathrm{t_{reg}}$ is representable, hence $\mathrm{t_{reg}}[ac_n] \to 0$, so

$$ac-ac_n+\ker \mathrm{t_{reg}} = \pi_{\mathrm{t_{reg}}}(a)(c-c_n+\ker \mathrm{t_{reg}}) \to ac+\ker \mathrm{t_{reg}}.$$

Then we have that

$$
\begin{aligned}
(\xi_{ac}, 0+\ker \mathfrak{t}_{\mathrm{sing}}) &= \psi_{\mathfrak{t}}(a)(c+\ker \mathfrak{t}_{\mathrm{reg}}, 0+\ker \mathfrak{t}_{\mathrm{sing}}) \\
&= \psi_{\mathfrak{t}}(a)(\lim_{n\to+\infty}(c-c_n+\ker \mathfrak{t}_{\mathrm{reg}}, c-c_n+\ker \mathfrak{t}_{\mathrm{sing}})) \\
&= \lim_{n\to+\infty} \psi_{\mathfrak{t}}(a)(c-c_n+\ker \mathfrak{t}_{\mathrm{reg}}, c-c_n+\ker \mathfrak{t}_{\mathrm{sing}}) \\
&= \lim_{n\to+\infty} (ac-ac_n+\ker \mathfrak{t}_{\mathrm{reg}}, ac-ac_n+\ker \mathfrak{t}_{\mathrm{sing}}),
\end{aligned}
$$

hence $ac - ac_n + \ker \mathfrak{t}_{\mathrm{sing}} \to 0 + \ker \mathfrak{t}_{\mathrm{sing}}$. As a consequence we conclude that $\mathfrak{t}_{\mathrm{sing}}[ac_n] \to \mathfrak{t}_{\mathrm{sing}}[ac]$. From this and $\mathfrak{t}_{\mathrm{reg}}[ac_n] \to 0$ it follows that

$$
\begin{aligned}
\mathfrak{t}_{\mathrm{sing}}[ac] &= \lim_{n\to+\infty} \mathfrak{t}_{\mathrm{sing}}[ac_n] = \lim_{n\to+\infty} (\mathfrak{t}_{\mathrm{reg}}[ac_n] + \mathfrak{t}_{\mathrm{sing}}[ac_n]) \\
&= \lim_{n\to+\infty} \mathfrak{t}[ac_n] \leqslant \|\pi_{\mathfrak{t}}(a)\|^2 \lim_{n\to+\infty} \mathfrak{t}[c_n] \\
&= \|\pi_{\mathfrak{t}}(a)\|^2 \lim_{n\to+\infty} (\mathfrak{t}_{\mathrm{reg}}[c_n] + \mathfrak{t}_{\mathrm{sing}}[c_n]) = \|\pi_{\mathfrak{t}}(a)\|^2 \mathfrak{t}_{\mathrm{sing}}[c],
\end{aligned} \tag{2.7}
$$

that is the form $\mathfrak{t}_{\mathrm{sing}}$ is representable, and

$$
\|\pi_{\mathfrak{t}_{\mathrm{sing}}}(a)\| \leqslant \|\pi_{\mathfrak{t}}(a)\|.
$$

Similarly, since $\mathfrak{t}_{\mathrm{reg}}[a(c-c_n)] \to \mathfrak{t}_{\mathrm{reg}}[ac]$ and $\mathfrak{t}_{\mathrm{sing}}[a(c-c_n)] \to 0$, then

$$
\begin{aligned}
\mathfrak{t}_{\mathrm{reg}}[ac] &= \lim_{n\to+\infty} \mathfrak{t}_{\mathrm{reg}}[a(c-c_n)] = \lim_{n\to+\infty} (\mathfrak{t}_{\mathrm{reg}}[a(c-c_n)] + \mathfrak{t}_{\mathrm{sing}}[a(c-c_n)]) \\
&= \lim_{n\to+\infty} \mathfrak{t}[a(c-c_n)] \leqslant \|\pi_{\mathfrak{t}}(a)\|^2 \lim_{n\to+\infty} \mathfrak{t}[c-c_n] \\
&= \|\pi_{\mathfrak{t}}(a)\|^2 \lim_{n\to+\infty} (\mathfrak{t}_{\mathrm{reg}}[c-c_n] + \mathfrak{t}_{\mathrm{sing}}[c-c_n]) = \|\pi_{\mathfrak{t}}(a)\|^2 \mathfrak{t}_{\mathrm{reg}}[c],
\end{aligned} \tag{2.8}
$$

that is

$$
\|\pi_{\mathfrak{t}_{\mathrm{reg}}}(a)\| \leqslant \|\pi_{\mathfrak{t}}(a)\|.
$$

The argument for the sufficiency will be similar to the proof of (i). Assume that $\mathfrak{t}_{\mathrm{sing}}$ is representable, and let $a \in \mathscr{A}$ be an arbitrary element. It is enough to prove that the dense linear subspace

$$
\mathcal{Z} = \{(d+\ker \mathfrak{t}_{\mathrm{reg}}, 0+\ker \mathfrak{t}_{\mathrm{sing}}) : d \in \mathscr{A}\}
$$

of $\mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \{0+\ker \mathfrak{t}_{\mathrm{sing}}\}$ is a $\psi_{\mathfrak{t}}(a)$ invariant subspace.

Let $(d+\ker \mathfrak{t}_{\mathrm{reg}}, 0+\ker \mathfrak{t}_{\mathrm{sing}}) \in \mathcal{Z}$ arbitrary. Since $\operatorname{ran} \mathbb{U}$ is dense in $\mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}$, there exists a sequence $(d_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that

$$
(d_n+\ker \mathfrak{t}_{\mathrm{reg}}, d_n+\ker \mathfrak{t}_{\mathrm{sing}}) \to (d+\ker \mathfrak{t}_{\mathrm{reg}}, 0+\ker \mathfrak{t}_{\mathrm{sing}}).
$$

Both of the forms $\mathfrak{t}_{\mathrm{reg}}$ and $\mathfrak{t}_{\mathrm{sing}}$ are representable, hence it follows that

$$
\psi_{\mathfrak{t}}(a)(d+\ker \mathfrak{t}_{\mathrm{reg}}, 0+\ker \mathfrak{t}_{\mathrm{sing}}) = \lim_{n\to+\infty} \psi_{\mathfrak{t}}(a)(d_n+\ker \mathfrak{t}_{\mathrm{reg}}, d_n+\ker \mathfrak{t}_{\mathrm{sing}})
$$

$$
\begin{aligned}
&= \lim_{n\to+\infty} U\pi_{\mathfrak{t}}(a)(d_n + \ker \mathfrak{t})\\
&= \lim_{n\to+\infty} U(ad_n + \ker \mathfrak{t})\\
&= \lim_{n\to+\infty} (\pi_{\mathfrak{t}_{\mathrm{reg}}}(a)(d_n + \ker \mathfrak{t}_{\mathrm{reg}}), \pi_{\mathfrak{t}_{\mathrm{sing}}}(a)(d_n + \ker \mathfrak{t}_{\mathrm{sing}}))\\
&= (ad + \ker \mathfrak{t}_{\mathrm{reg}}, 0 + \ker \mathfrak{t}_{\mathrm{sing}}),
\end{aligned}
$$

that is $\psi_{\mathfrak{t}}(a)(\mathcal{Z}) \subseteq \mathcal{Z}$.

Finally let $a, x, y \in \mathscr{A}$ arbitrary elements, and let $(z_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{A}$ such that

$$(z_n + \ker \mathfrak{t}_{\mathrm{reg}}, z_n + \ker \mathfrak{t}_{\mathrm{sing}}) \to (x + \ker \mathfrak{t}_{\mathrm{reg}}, y + \ker \mathfrak{t}_{\mathrm{sing}}).$$

If $\mathfrak{t}_{\mathrm{sing}}$ is representable, then

$$
\begin{aligned}
&(U\pi_{\mathfrak{t}}(a)U^{-1})(x + \ker \mathfrak{t}_{\mathrm{reg}}, y + \ker \mathfrak{t}_{\mathrm{sing}})\\
&= \psi_{\mathfrak{t}}(a)(x + \ker \mathfrak{t}_{\mathrm{reg}}, y + \ker \mathfrak{t}_{\mathrm{sing}}) = \lim_{n\to+\infty} \psi_{\mathfrak{t}}(a)(z_n + \ker \mathfrak{t}_{\mathrm{reg}}, z_n + \ker \mathfrak{t}_{\mathrm{sing}})\\
&= \lim_{n\to+\infty} U(\pi_{\mathfrak{t}}(a)(z_n + \ker \mathfrak{t})) = \lim_{n\to+\infty} U(az_n + \ker \mathfrak{t})\\
&= \lim_{n\to+\infty} (\pi_{\mathfrak{t}_{\mathrm{reg}}}(a)(z_n + \ker \mathfrak{t}_{\mathrm{reg}}), \pi_{\mathfrak{t}_{\mathrm{sing}}}(a)(z_n + \ker \mathfrak{t}_{\mathrm{sing}}))\\
&= (ax + \ker \mathfrak{t}_{\mathrm{reg}}, ay + \ker \mathfrak{t}_{\mathrm{sing}}) = (\pi_{\mathfrak{t}_{\mathrm{reg}}}(a) \oplus \pi_{\mathfrak{t}_{\mathrm{sing}}}(a))(x + \ker \mathfrak{t}_{\mathrm{reg}}, y + \ker \mathfrak{t}_{\mathrm{sing}}),
\end{aligned}
$$

hence the bounded operators $U\pi_{\mathfrak{t}}(a)U^{-1}$ and $\pi_{\mathfrak{t}_{\mathrm{reg}}}(a) \oplus \pi_{\mathfrak{t}_{\mathrm{sing}}}(a)$ are equal on a dense subspace of $\mathcal{H}_{\mathfrak{t}_{\mathrm{reg}}} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}$, thus (2.5) follows. The proof is complete. ∎

In Section 3 we will prove that the singular part of a form generated by a representable positive functional on a $*$-algebra is always a representable form. However, the following example demonstrates that the singular part is not necessarily representable in general, moreover, this can occur over unital commutative algebras. Hence for an arbitrary algebra the statement in part (ii) cannot be strengthened.

EXAMPLE 2.7. Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be an infinite dimensional separable Hilbert space and let $(e_n)_{n\in\mathbb{N}}$ be a complete orthonormal system in $\mathcal{H}$. Denote by S the shift operator associated to $(e_n)_{n\in\mathbb{N}}$, i.e. the bounded linear operator such that $Se_n = e_{n+1}$ for all $n \in \mathbb{N}$, and let $\mathscr{A}$ be the subalgebra of $B(\mathcal{H})$ generated by S and the identity I. Furthermore, let P be the one-dimensional orthogonal projection onto the subspace spanned by e_1. Now we define two forms on $\mathscr{A}$:

$$\mathfrak{v}(A, B) = \langle PAe_1, PBe_1\rangle, \quad \mathfrak{w}(A, B) = \langle (I - P)Ae_1, (I - P)Be_1\rangle \quad (A, B \in \mathscr{A}).$$

It is clear that $\mathfrak{v}$ and $\mathfrak{w}$ are truly forms on $\mathscr{A}$. Moreover,

$$\mathfrak{t}(A, B) := \mathfrak{v}(A, B) + \mathfrak{w}(A, B) = \langle Ae_1, Be_1\rangle$$

is also a form, which is representable ($\mathfrak{t}[AB] \leqslant \|A\|^2\mathfrak{t}[B]$). We will prove the following three statements:

(a) The form $\mathfrak{v}$ is representable.

(b) The form $\mathfrak{w}$ is not representable.

(c) The Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{v}$ is $\mathfrak{t} = \mathfrak{v} + \mathfrak{w}$, where the regular part is $\mathfrak{v} = \mathfrak{t}_{\mathrm{reg}}$, the singular part is $\mathfrak{w} = \mathfrak{t}_{\mathrm{sing}}$, respectively.

(a) We have to show that for every $A \in \mathscr{A}$ there exists a positive number λ_A such that for every $B \in \mathscr{A}$ the inequality $\mathfrak{v}[AB] \leqslant \lambda_A \mathfrak{v}[B]$ holds. Fix an $A \in \mathscr{A}$; then A has the form $A = \sum\limits_{j=0}^{n} \lambda_j S^j$ for some $n \in \mathbb{N}$ and $\lambda_j \in \mathbb{C}$ $(j = 0, \dots, n)$. Since every positive power of S has its range in $\ker P$, therefore $PA = \lambda_0 P$. Hence it follows that

$$\mathfrak{v}[AB] = \langle PABe_1, PABe_1 \rangle = |\lambda_0|^2 \langle PBe_1, PBe_1 \rangle = |\lambda_0|^2 \mathfrak{v}[B],$$

that is $|\lambda_0|^2$ is suitable for λ_A.

(b) Assume that $\mathfrak{w}$ is representable. Then for S there exists $\lambda_S \geqslant 0$ such that for all $B \in \mathscr{A}$ we have $\mathfrak{w}[SB] \leqslant \lambda_S \mathfrak{w}[B]$. If we take $B = I$, then we obtain

$$1 = \langle e_2, e_2 \rangle = \langle (I-P)Se_1, (I-P)Se_1 \rangle = \mathfrak{w}[SI] \leqslant \lambda_S \mathfrak{w}[I] = \lambda_S \langle (I-P)e_1, (I-P)e_1 \rangle = 0,$$

which is a contradiction.

(c) The form $\mathfrak{v}$ is closable with respect to itself and smaller than $\mathfrak{t}$, thus by the extremal property of $\mathfrak{t}_{\mathrm{reg}}$ the inequality $\mathfrak{v} \leqslant \mathfrak{t}_{\mathrm{reg}}$ holds (Theorem 1.7). Consequently it is enough to prove that the inequality $(\mathfrak{t} : n\mathfrak{v})[A] \leqslant \mathfrak{v}[A]$ is true for every $A \in \mathscr{A}$ and $n \in \mathbb{N}$, since the regular part of $\mathfrak{t}$ is $\mathfrak{t}_{\mathrm{reg}} = \sup\limits_{n \in \mathbb{N}} (\mathfrak{t} : n\mathfrak{v})$ (and in this case $\mathfrak{w} = \mathfrak{t}_{\mathrm{sing}}$ also follows).

Let us fix an arbitrary $A \in \mathscr{A}$ and $n \in \mathbb{N}$. The operator A has the form $A = \sum\limits_{j=0}^{m} \lambda_j S^j$ for some $m \in \mathbb{N}$ and $\lambda_j \in \mathbb{C}$ $(j = 0, \dots, m)$. Let $C \in \mathscr{A}$ be the operator $\sum\limits_{j=1}^{m} \lambda_j S^j$. Then

$$\begin{aligned}
(\mathfrak{t} : n\mathfrak{v})[A] &= \inf\{\mathfrak{v}[A-B] + \mathfrak{w}[A-B] + n\mathfrak{v}[B] : B \in \mathscr{A}\} \\
&\leqslant \mathfrak{v}[A-C] + \mathfrak{w}[A-C] + n\mathfrak{v}[C] = \mathfrak{v}[\lambda_0 I] + \mathfrak{w}[\lambda_0 I] + n\mathfrak{v}\Big[\sum_{j=1}^{m} \lambda_j S^j\Big] \\
&= \|P\lambda_0 I e_1\|^2 + \|(I-P)\lambda_0 I e_1\|^2 + n\Big\|P\Big(\sum_{j=1}^{m} \lambda_j S^j\Big)e_1\Big\|^2 \\
&= \Big\|\Big(P\lambda_0 I + P\Big(\sum_{j=1}^{m} \lambda_j S^j\Big)\Big)e_1\Big\|^2 + 0 + 0 = \mathfrak{v}[A],
\end{aligned}$$

and whereby follows that $\mathfrak{t}_{\mathrm{reg}} = \mathfrak{v}$.

Therefore $\mathfrak{t}$ and $\mathfrak{v}$ are representable forms on the unital commutative algebra $\mathscr{A}$ such that the singular part in the Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{v}$ is not a representable form. ∎

We have seen at the beginning of this section that every cyclic representation of an algebra on a Hilbert space is unitarily equivalent to a representation induced by a representable form. A non-zero topologically irreducible representation is cyclic, so it is an interesting question what can we assert about the decomposition of forms $\mathfrak{t}$ such that $\pi_{\mathfrak{t}}$ is a topologically irreducible representation of the algebra. The last theorem of the section gives the answer to this.

THEOREM 2.8. *Let* $\mathfrak{t}$ *and* $\mathfrak{w}$ *be non-zero representable forms on a complex algebra* $\mathscr{A}$*, where* $\pi_{\mathfrak{t}}$ *is topologically irreducible. Then either*

(i) $\mathfrak{t}$ *is closable with respect to* $\mathfrak{w}$*,*

or

(ii) $\mathfrak{t}$ *and* $\mathfrak{w}$ *are singular.*

If $\pi_{\mathfrak{w}}$ *is also topologically irreducible, then the occurence of* (i) *implies that* $\mathfrak{w}$ *is closable with respect to* $\mathfrak{t}$ *as well.*

Proof. Let $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$ be the Lebesgue decomposition of $\mathfrak{t}$ with respect to $\mathfrak{w}$. Using the notations of (2.4) in Theorem 2.6, we conclude by Theorem 2.6(i) that the closed linear subspace $U^{-1}(\{0 + \ker \mathfrak{t}_{\mathrm{reg}}\} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}})$ of $\mathcal{H}_{\mathfrak{t}}$ is $\pi_{\mathfrak{t}}$-invariant. But $\pi_{\mathfrak{t}}$ is topologically irreducible, hence this subspace must be trivial:

(i) If $U^{-1}(\{0 + \ker \mathfrak{t}_{\mathrm{reg}}\} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}) = \{0 + \ker \mathfrak{t}\}$, then $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}}$, namely $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$.

(ii) If $U^{-1}(\{0 + \ker \mathfrak{t}_{\mathrm{reg}}\} \oplus \mathcal{H}_{\mathfrak{t}_{\mathrm{sing}}}) = \mathcal{H}_{\mathfrak{t}}$, then $\mathfrak{t} = \mathfrak{t}_{\mathrm{sing}}$, i.e. $\mathfrak{t}$ and $\mathfrak{w}$ are singular.

Now suppose that $\pi_{\mathfrak{w}}$ is also topologically irreducible and case (i) holds, that is $\mathfrak{t}$ is closable with respect to $\mathfrak{w}$. The above proved results imply that $\mathfrak{w}$ is closable with respect to $\mathfrak{t}$, or $\mathfrak{w}$ and $\mathfrak{t}$ are singular, but by Remark 1.10 we conlude that the latter case never occurs for non-zero forms. ∎

3. THE $*$-ALGEBRA CASE

In this section we apply the previously claimed results on $*$-algebras. The especially interesting case is that when the representable forms are derived from positive functionals. Other authors dealt with this case before. H. Kosaki in [8] studied the Lebesgue decomposition of normal positive functionals on σ-finite von Neumann algebras. S. Gudder in [5] proved a decomposition theorem on unital Banach $*$-algebras. Our results give the opportunity to gain a more general theorem, wherein no assumptions needed for the $*$-algebra, even the unitality. The first part contains this general Lebesgue decomposition theorem for representable positive linear functionals. In the second part we show that our decomposition corresponds with Kosaki's result on σ-finite von Neumann algebras and with Gudder's result on Banach $*$-algebras, as well. Since on an unital Banach $*$-algebra (and of course, on a von Neumann algebra) every positive linear functional is representable (see Theorem 11.3.7 of [9]), therefore our result, Corollary 3.2, generalizes these decomposition theorems to arbitrary $*$-algebras. We also discuss the question of the decompositions' uniqueness.

As we mentioned in the Introduction, for a positive functional f we will use the term *representable* in the classical sense, just like the notations $(\mathcal{H}_f, \langle\cdot,\cdot\rangle_f)$, $\xi_f \in \mathcal{H}_f$ and $\pi_f : \mathscr{A} \to B(\mathcal{H}_f)$ for the associated Hilbert space, distinguished cyclic vector and $*$-representation given by the GNS-construction.

3.1. THE DECOMPOSITION OF POSITIVE LINEAR FUNCTIONALS. We have seen in Example 2.7 that the singular part of a representable form is not necessarily representable. But it will turn out that if we proceed from forms generated by representable positive functionals, then the singular part is representable. Moreover, both of the regular and singular parts is derived from positive functionals. Hence the following result stengthens Theorem 2.6 for positive functionals on $*$-algebras. We note that we do not assume any conditions for the $*$-algebra, neither the existence of unit element.

THEOREM 3.1. *Let $\mathscr{A}$ be a $*$-algebra, let t be a representable positive linear functional on $\mathscr{A}$ and let $\mathfrak{w}$ be a representable form on $\mathscr{A}$. Let $\widetilde{t} = \mathrm{t}_{\mathrm{reg}} + \mathrm{t}_{\mathrm{sing}}$ be the Lebesgue decomposition of $\widetilde{t}$ with respect to $\mathfrak{w}$. Then the forms $\mathrm{t}_{\mathrm{reg}}$ and $\mathrm{t}_{\mathrm{sing}}$ are representable, and in addition, there exist positive functionals t_{r} and t_{s} on $\mathscr{A}$ such that $\widetilde{t_{\mathrm{r}}} = \mathrm{t}_{\mathrm{reg}}$, $\widetilde{t_{\mathrm{s}}} = \mathrm{t}_{\mathrm{sing}}$. Moreover, the following two estimates hold true for every $a \in \mathscr{A}$:*

$$\|\pi_{t_{\mathrm{r}}}(a)\| \leqslant \|\pi_t(a)\|, \quad \|\pi_{t_{\mathrm{s}}}(a)\| \leqslant \|\pi_t(a)\|.$$

Proof. We use the notations of Theorem 2.6. It follows from Theorem 2.5 that $\mathrm{t}_{\mathrm{reg}}$ is representable. In Theorem 2.6 it was proved that $U^{-1}(\{0 + \ker \mathrm{t}_{\mathrm{reg}}\} \oplus \mathcal{H}_{\mathrm{t}_{\mathrm{sing}}})$ is a π_t-invariant subspace. But π_t is a $*$-representation, thus we obtain that the orthogonal complementary subspace $U^{-1}(\mathcal{H}_{\mathrm{t}_{\mathrm{reg}}} \oplus \{0 + \ker \mathrm{t}_{\mathrm{sing}}\})$ is also a π_t-invariant subspace. By Theorem 2.6 this implies that $\mathrm{t}_{\mathrm{sing}}$ is representable, and for any $a \in \mathscr{A}$ we have

$$\|\pi_{\mathrm{t}_{\mathrm{reg}}}(a)\| \leqslant \|\pi_t(a)\|, \quad \|\pi_{\mathrm{t}_{\mathrm{sing}}}(a)\| \leqslant \|\pi_t(a)\|.$$

We only have to prove that the forms $\mathrm{t}_{\mathrm{reg}}$ and $\mathrm{t}_{\mathrm{sing}}$ are derived from positive functionals. The positive functional t is representable, i.e. there is a cyclic vector $\xi_t \in \mathcal{H}_t$ such that $\pi_t(a)\xi_t = a + \ker \widetilde{t}$, $t(a) = \langle \pi_t(a)\xi_t, \xi_t\rangle_t$ hold true for all $a \in \mathscr{A}$. Since $\mathcal{H}_{\mathrm{t}_{\mathrm{reg}}} \oplus \mathcal{H}_{\mathrm{t}_{\mathrm{sing}}}$ is an orthogonal sum, there exist unique vectors $\xi_{\mathrm{r}} \in \mathcal{H}_{\mathrm{t}_{\mathrm{reg}}}$, $\xi_{\mathrm{s}} \in \mathcal{H}_{\mathrm{t}_{\mathrm{sing}}}$ such that $U(\xi_t) = (\xi_{\mathrm{r}}, \xi_{\mathrm{s}})$. Then for all $a \in \mathscr{A}$ we obtain that

$$(U\pi_t(a)U^{-1})(\xi_{\mathrm{r}}, \xi_{\mathrm{s}}) = U(\pi_t(a)\xi_t) = U(a + \ker \widetilde{t}) = (a + \ker \mathrm{t}_{\mathrm{reg}}, a + \ker \mathrm{t}_{\mathrm{sing}}). \tag{3.1}$$

We have seen in (2.5) of Theorem 2.6 such that for every $a \in \mathscr{A}$

$$U\pi_t(a)U^{-1} = \pi_{\mathrm{t}_{\mathrm{reg}}}(a) \oplus \pi_{\mathrm{t}_{\mathrm{sing}}}(a),$$

hence combining this and (3.1)

$$(\pi_{\mathrm{t}_{\mathrm{reg}}}(a) \oplus \pi_{\mathrm{t}_{\mathrm{sing}}}(a))(\xi_{\mathrm{r}}, \xi_{\mathrm{s}}) = (\pi_{\mathrm{t}_{\mathrm{reg}}}(a)\xi_{\mathrm{r}}, \pi_{\mathrm{t}_{\mathrm{sing}}}(a)\xi_{\mathrm{s}}) = (a + \ker \mathrm{t}_{\mathrm{reg}}, a + \ker \mathrm{t}_{\mathrm{sing}})$$

follows. As a consequence we have that ξ_{r} and ξ_{s} are cylic vectors for the representations $\pi_{\mathfrak{t}_{\mathrm{reg}}}$ and $\pi_{\mathfrak{t}_{\mathrm{sing}}}$, respectively.

Define the linear functionals t_{r} and t_{s} on $\mathscr{A}$ by the following equations:

$$t_{\mathrm{r}}(a) := \langle \pi_{\mathfrak{t}_{\mathrm{reg}}}(a)\xi_{\mathrm{r}}, \xi_{\mathrm{r}}\rangle_{\mathfrak{t}_{\mathrm{reg}}}, \quad t_{\mathrm{s}}(a) := \langle \pi_{\mathfrak{t}_{\mathrm{sing}}}(a)\xi_{\mathrm{s}}, \xi_{\mathrm{s}}\rangle_{\mathfrak{t}_{\mathrm{sing}}}$$

We show that for any $a, b \in \mathscr{A}$

$$t_{\mathrm{r}}(b^*a) = \mathfrak{t}_{\mathrm{reg}}(a,b), \quad t_{\mathrm{s}}(b^*a) = \mathfrak{t}_{\mathrm{sing}}(a,b)$$

holds true, hence t_{r} and t_{s} are positive, and the regular and singular parts are derived from the positive functionals t_{r} and t_{s}, respectively.

Let $a, b \in \mathscr{A}$ be arbitrary elements. Then we conclude that

$$\begin{aligned}
t_{\mathrm{r}}(b^*a) &= \langle \pi_{\mathfrak{t}_{\mathrm{reg}}}(b^*a)\xi_{\mathrm{r}}, \xi_{\mathrm{r}}\rangle_{\mathfrak{t}_{\mathrm{reg}}} = \langle (\pi_{\mathfrak{t}_{\mathrm{reg}}}(b^*a)\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}}), (\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle U\pi_t(b^*a)U^{-1}(\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}}), (\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle U\pi_t(b^*)U^{-1}U\pi_t(a)U^{-1}(\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}}), (\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle U\pi_t(a)U^{-1}(\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}}), U\pi_t(b)U^{-1}(\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle (\pi_{\mathfrak{t}_{\mathrm{reg}}}(a)\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}}), (\pi_{\mathfrak{t}_{\mathrm{reg}}}(b)\xi_{\mathrm{r}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle (a+\ker\mathfrak{t}_{\mathrm{reg}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}}), (b+\ker\mathfrak{t}_{\mathrm{sing}}, 0+\ker_{\mathfrak{t}_{\mathrm{sing}}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} = \mathfrak{t}_{\mathrm{reg}}(a,b).
\end{aligned}$$

The proof is analogous for the singular part:

$$\begin{aligned}
t_{\mathrm{s}}(b^*a) &= \langle \pi_{\mathfrak{t}_{\mathrm{sing}}}(b^*a)\xi_{\mathrm{s}}, \xi_{\mathrm{s}}\rangle_{\mathfrak{t}_{\mathrm{sing}}} = \langle (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \pi_{\mathfrak{t}_{\mathrm{sing}}}(b^*a)\xi_{\mathrm{s}}), (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \xi_{\mathrm{s}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle U\pi_t(b^*a)U^{-1}(0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \xi_{\mathrm{s}}), (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \xi_{\mathrm{s}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle U\pi_t(b^*)U^{-1}U\pi_t(a)U^{-1}(0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \xi_{\mathrm{s}}), (0+\ker_{\mathfrak{t}_{\mathrm{sing}}}, \xi_{\mathrm{s}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle U\pi_t(a)U^{-1}(0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \xi_{\mathrm{s}}), U\pi_t(b)U^{-1}(0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \xi_{\mathrm{s}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \pi_{\mathfrak{t}_{\mathrm{sing}}}(a)\xi_{\mathrm{s}}), (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, \pi_{\mathfrak{t}_{\mathrm{sing}}}(b)\xi_{\mathrm{s}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} \\
&= \langle (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, a+\ker\mathfrak{t}_{\mathrm{sing}}), (0+\ker_{\mathfrak{t}_{\mathrm{reg}}}, b+\ker\mathfrak{t}_{\mathrm{sing}})\rangle_{\mathfrak{t}_{\mathrm{reg}}\oplus\mathfrak{t}_{\mathrm{sing}}} = \mathfrak{t}_{\mathrm{sing}}(a,b).
\end{aligned}$$

This finishes the proof. ∎

COROLLARY 3.2 (Lebesgue decomposition of representable positive functionals). *Let $\mathscr{A}$ be a $*$-algebra and let f, g be representable positive functionals on $\mathscr{A}$. Then f admits a Lebesgue decomposition $f = f_{\mathrm{reg}} + f_{\mathrm{sing}}$ to a sum of representable positive functionals such that f_{sing} and g are singular, while f_{reg} is closable with respect to g. The form induced by f_{reg} is*

$$\widetilde{f}_{\mathrm{reg}}[a] = \sup_{n\in\mathbb{N}}(\widetilde{f} : n\widetilde{g})[a] \quad (a \in \mathscr{A}).$$

Moreover, the form $\widetilde{f}_{\mathrm{reg}}$ is the greatest among all of the forms $\mathfrak{p}$ such that $\mathfrak{p} \leqslant \widetilde{f}$ and $\mathfrak{p}$ is closable with respect to $\widetilde{g}$.

The proof is clear by Theorem 1.7 and Theorem 3.1.

REMARK 3.3. We note that the decomposition is not necessarily unique, see Proposition 3.21.

3.2. THE CORRESPONDENCE OF THE DECOMPOSITIONS FOR POSITIVE FUNCTIONALS. In this part we prove that our result, Corollary 3.2 yields the same decomposition that H. Kosaki obtained on σ-finite von Neumann algebras ([8], Theorem 3.5) and that S. Gudder found on unital Banach $*$-algebras ([5], Corollary 3). These facts and an example due to Kosaki ([8], 10.6) imply that our decomposition is not unique, as well as Gudder's.

Kosaki's and Gudder's definitions for singularity differs from ours, but the coincidence of the decompositions implies that all of these definitions result the same concept.

We note that for positive functionals f and g the terminology *f is closable with respect g* appears as *f is strongly g-absolutely continuous* in Gudder's paper ([5], (ii) p. 142), while in Kosaki's article it appears as *f is g-absolutely continuous* ([8], Theorem 2.2, Definition 2.5).

THE DECOMPOSITION OF KOSAKI. Now we recall Kosaki's result from [8]. He dealt with the case of σ-finite von Neumann algebras, and normal positive functionals appeared in his decomposition theorem. For von Neumann algebras, the reader is referred to [3], [9] and [13].

DEFINITION 3.4. Let $\mathscr{A}$ be a von Neumann algebra. Then a positive linear functional f on $\mathscr{A}$ is called

(i) *normal*, if $f(a) = \sup_{\beta\in B} f(a_\beta)$ holds true whenever $(a_\beta)_{\beta\in B}$ is a norm-bounded increasing net of self-adjoint elements with supremum a;

(ii) *faithful*, if for any $a \in \mathscr{A}$, $f(a^*a) = 0$ implies that $a = 0$;

(iii) *state*, if $f(\mathbf{1}) = 1$.

DEFINITION 3.5. Let $\mathscr{A}$ be a von Neumann algebra, f a normal positive functional, g a normal faithful state on $\mathscr{A}$. Then f is said to be *g-singular*, if $p = 0$ is true for every normal positive functional p which simultaneously satisfies $p \leqslant f$ and $p \leqslant g$.

Before the next theorem we recall that a von Neumann algebra is σ-finite, if for every family of non-zero pairwise orthogonal projections of $\mathscr{A}$ is countable. Below is in our terms Kosaki's result on normal positive functionals ([8], Theorem 3.3 and Theorem 3.5).

THEOREM 3.6 (Kosaki). *Let $\mathscr{A}$ be a σ-finite von Neumann algebra, let f be a normal positive linear functional on $\mathscr{A}$ and let g be a normal faithful state on $\mathscr{A}$. Then f admits a decomposition $f = f_{\mathrm{r}} + f_{\mathrm{s}}$ to a sum of normal positive functionals, where f_{r} is closable with respect to g while f_{s} is g-singular. Moreover, f_{r} is the greatest among all of the normal positive functionals p such that $p \leqslant f$ and p is closable with respect to g.*

The following simple fact allows us to prove the coincidence of Kosaki's and our decomposition: for positive functionals f and p, where f is normal, the assumption $p \leqslant f$ implies that p is normal. Namely, let $(a_\beta)_{\beta\in B}$ be a norm-bounded increasing net of self-adjoint elements with supremum a. Then from the positivity of p we obtain that $\sup\limits_{\beta\in B} p(a_\beta) \leqslant p(a)$. Furthermore, by the normality of f for any $\varepsilon > 0$ there exists $\beta_0 \in B$ such that $0 \leqslant f(a) - f(a_{\beta_0}) \leqslant \varepsilon$, consequently

$$0 \leqslant p(a) - p(a_{\beta_0}) = p(a - a_{\beta_0}) \leqslant f(a - a_{\beta_0}) = f(a) - f(a_{\beta_0}) \leqslant \varepsilon,$$

hence $p(a) = \sup\limits_{\beta\in B} p(a_\beta)$ follows.

THEOREM 3.7. *Let $\mathscr{A}$ be a σ-finite von Neumann algebra, let f be a normal positive linear functional on $\mathscr{A}$ and let g be a normal faithful state on $\mathscr{A}$. Let $f = f_{\mathrm{reg}} + f_{\mathrm{sing}}$ be the Lebesgue decomposition of f with respect to g by Corollary* 3.2 *and $f = f_{\mathrm{r}} + f_{\mathrm{s}}$ by Theorem* 3.6*, respectively. Then $f_{\mathrm{reg}} = f_{\mathrm{r}}$ and $f_{\mathrm{sing}} = f_{\mathrm{s}}$, i.e. the two decompositions correspond.*

Proof. Since f_{r} is closable with respect to g and $f_{\mathrm{r}} \leqslant f$ holds true, thus the maximality of f_{reg} implies that $f_{\mathrm{r}} \leqslant f_{\mathrm{reg}}$. On the other hand, f_{reg} is also closable with respect to g and $f_{\mathrm{reg}} \leqslant f$, moreover the normality of f implies that f_{reg} is normal. So the maximality of f_{r} among the normal positive functionals infers that $f_{\mathrm{reg}} \leqslant f_{\mathrm{r}}$, whereby $f_{\mathrm{reg}} = f_{\mathrm{r}}$ and $f_{\mathrm{sing}} = f_{\mathrm{s}}$ follows. ∎

REMARK 3.8. For a von Neumann algebra the property σ-finite is equivalent to the existence of a normal faithful state (that is why the assumption *σ-finite* was needed in Theorem 3.6). Our decomposition does not need faithfulness, it is valid for an arbitrary $*$-algebra.

COROLLARY 3.9. *Let $\mathscr{A}$ be a σ-finite von Neumann algebra, let f be a normal positive functional on $\mathscr{A}$ and let g be a normal faithful state on $\mathscr{A}$. Then f is g-singular if and only if f and g are singular.*

Proof. The functional f is g-singular if and only if $f_{\mathrm{r}} = 0$ holds true in Theorem 3.6 ([8], p. 706). By Theorem 3.7 this is equivalent to $f_{\mathrm{reg}} = 0$ in Corollary 3.2, namely $f = f_{\mathrm{sing}}$, and this case occurs if and only if f and g are singular ([6], Corollary 3.11). ∎

REMARK 3.10. Kosaki also obtained a remarkable result on g-singularity ([8], Theorem 8.1).

THE DECOMPOSITION OF GUDDER. In [5] Gudder proved a remarkable theorem on the closability of positive functionals (Theorem 1). This result is very general, since it is valid on arbitrary $*$-algebra. However, for his Lebesgue type decomposition he assumed that the algebra is unital Banach $*$-algebra.

We have seen that the regular part in our and Kosaki's decomposition has a maximal property with respect to the natural ordering. We will prove that Gudder's result also yields this property (and this fact will imply the correspondence of the decompositions).

First of all we take two definitions from Gudder's paper (pages 141 and 146).

DEFINITION 3.11. Let $\mathscr{A}$ be a $*$-algebra and let f, g be positive functionals on $\mathscr{A}$. A sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ with the property

$$(\lim_{n\to+\infty} \widetilde{g}[a_n] = 0 \wedge \lim_{n,m\to+\infty} \widetilde{f}[a_n - a_m] = 0) \tag{3.2}$$

is called (g, f) *sequence*.

DEFINITION 3.12. Let $\mathscr{A}$ be a $*$-algebra and let f, g be positive functionals on $\mathscr{A}$. Then f is called *g-semisingular* if there exists a (g, f) sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that for all $a \in \mathscr{A}$ the following holds true:

$$\lim_{n\to+\infty} f(a_n^* a) = f(a). \tag{3.3}$$

Now we recall Gudder's decomposition in our terms ([5], Corollary 3).

THEOREM 3.13 (Gudder). *Let f and g be positive linear functionals on a Banach $*$-algebra $\mathscr{A}$ with unit. Then f admits a decomposition $f = f_{\mathrm{r}} + f_{\mathrm{s}}$ to a sum of positive functionals, where f_{r} is closable with respect to g and f_{s} is g-semisingular, moreover there exists a (g, f) sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ which shows the g-semisingularity of f_{s}.*

For the correspondence, we will need the following characterization of semisingularity. It can be found in another paper of the author ([16]), but we introduce the proof for the readers convenience.

PROPOSITION 3.14. *Let $\mathscr{A}$ be a $*$-algebra with unit and let f, g be positive functionals on $\mathscr{A}$, and let $(a_n)_{n\in\mathbb{N}}$ be a (g, f) sequence in $\mathscr{A}$. Then*

(i) (3.3) *holds if and only if $\widetilde{f}[\mathbf{1} - a_n] \to 0$.*

(ii) *f is g-semisingular if and only if g is f-semisingular.*

Proof. (i) Assume that first (3.3) holds. We have to prove that $\widetilde{f}[\mathbf{1} - a_n] \to 0$. The sequence $(a_n + \ker \widetilde{f})_{n\in\mathbb{N}}$ is Cauchy in the Hilbert space $\mathcal{H}_f$, therefore it converges to a vector $\xi \in \mathcal{H}_f$. Hence for every $a \in \mathscr{A}$ we have that

$$\langle a + \ker \widetilde{f}, a_n + \ker \widetilde{f} \rangle_f \to \langle a + \ker \widetilde{f}, \xi \rangle_f.$$

At the same time (3.3) implies that

$$\langle a + \ker \widetilde{f}, a_n + \ker \widetilde{f} \rangle_f = f(a_n^* a) \to f(a) = \langle a + \ker \widetilde{f}, \mathbf{1} + \ker \widetilde{f} \rangle_f,$$

thus for every $a \in \mathscr{A}$ we obtain

$$\langle a + \ker \widetilde{f}, \xi - (\mathbf{1} + \ker \widetilde{f}) \rangle_f = 0.$$

This means that the vector $\xi - (\mathbf{1} + \ker \widetilde{f})$ is orthogonal to the elements of the dense linear subspace $\mathscr{A}/\ker \widetilde{f}$, hence $\xi - (\mathbf{1} + \ker \widetilde{f}) = 0$, that is $\xi = \mathbf{1} + \ker \widetilde{f}$. Consequently $\widetilde{f}[\mathbf{1} - a_n] \to 0$.

Conversely assume for $(a_n)_{n\in\mathbb{N}}$ that $\widetilde{g}[a_n] \to 0$ and $\widetilde{f}[\mathbf{1} - a_n] \to 0$, and fix an arbitrary $a \in \mathscr{A}$. By the Cauchy–Schwarz inequality we have

$$|f(a) - f(a_n^* a)|^2 = |f((\mathbf{1} - a_n)^* a)|^2 \leqslant f((\mathbf{1} - a_n)^*(\mathbf{1} - a_n)) f(a^* a) = \widetilde{f}[\mathbf{1} - a_n]\widetilde{f}[a] \to 0,$$

hence $f(a_n^* a) \to f(a)$, i.e. (3.3) holds.

(ii) It is enough to prove that if f is g-semisingular then g is f-semisingular. By (i) there is a sequence $(a_n)_{n\in\mathbb{N}}$ such that $\widetilde{g}[a_n] \to 0$ and $\widetilde{f}[\mathbf{1} - a_n] \to 0$. Then for the sequence $(b_n)_{n\in\mathbb{N}} = (\mathbf{1} - a_n)_{n\in\mathbb{N}}$ we have that

$$\widetilde{f}[b_n] = \widetilde{f}[\mathbf{1} - a_n] \to 0, \quad \widetilde{g}[\mathbf{1} - b_n] = \widetilde{g}[\mathbf{1} - (\mathbf{1} - a_n)] = \widetilde{g}[a_n] \to 0,$$

thus g is f-semisingular according to (a). ∎

The correspondence of Kosaki's and our result depended on the fact that the regular parts are maximal in a sort of sense in both decomposition. In the next theorem we prove an analogous statement for Gudder's result.

THEOREM 3.15. *Let f and g be positive linear functionals on a Banach $*$-algebra $\mathscr{A}$ with unit, and let $f = f_{\mathrm{r}} + f_{\mathrm{s}}$ be the Lebesgue decomposition of f by Theorem* 3.13. *Then f_{r} is the greatest among all of the positive functionals p such that $p \leqslant f$ and p is closable with respect to g.*

Proof. Let us fix a positive functional p closable with respect to g and let $p \leqslant f$. Denote the positive functional $f - p$ by q, and let $(a_n)_{n\in\mathbb{N}}$ be a (g, f) sequence which shows the g-semisingularity of f_{s}. Then we have that

$$f = f_{\mathrm{r}} + f_{\mathrm{s}} = p + q,$$

and by Proposition 3.14(i) the following follows:

$$\lim_{n\to+\infty} (a_n + \ker \widetilde{f_{\mathrm{s}}}) = \mathbf{1} + \ker \widetilde{f_{\mathrm{s}}}. \tag{3.4}$$

The functional f dominates all of the functionals $f_{\mathrm{r}}, f_{\mathrm{s}}, p$ and q, hence the sequence $(a_n)_{n\in\mathbb{N}}$ is a (g, f_{r}), (g, f_{s}), (g, p) and (g, q) sequence as well. From the closability of f_{r} and p it follows that

$$\langle a_n + \ker \widetilde{f_{\mathrm{r}}}, a_n + \ker \widetilde{f_{\mathrm{r}}} \rangle_{f_{\mathrm{r}}} = f_{\mathrm{r}}[a_n] \to 0, \quad \langle a_n + \ker \widetilde{p}, a_n + \ker \widetilde{p} \rangle_p = p[a_n] \to 0. \tag{3.5}$$

As a consequence, from the Cauchy–Schwarz inequality for any $a \in \mathscr{A}$ we have

$$|f_{\mathrm{r}}(a_n^* a)|^2 \leqslant f_{\mathrm{r}}[a_n] f_{\mathrm{r}}[a] \to 0, \quad |p(a_n^* a)|^2 \leqslant p[a_n] p[a] \to 0.$$

Thus for every $a \in \mathscr{A}$ we obtain that

$$\begin{aligned} f_{\mathrm{s}}(a) &= \lim_{n\to+\infty} f_{\mathrm{s}}(a_n^* a) = \lim_{n\to+\infty} (f_{\mathrm{r}}(a_n^* a) + f_{\mathrm{s}}(a_n^* a)) \\ &= \lim_{n\to+\infty} f(a_n^* a) = \lim_{n\to+\infty} (p(a_n^* a) + q(a_n^* a)) = \lim_{n\to+\infty} q(a_n^* a). \end{aligned}$$

The sequence $(a_n + \ker \widetilde{q})_{n\in\mathbb{N}}$ is Cauchy in $\mathcal{H}_q$, hence there exists a vector $\zeta \in \mathcal{H}_q$ such that

$$\zeta = \lim_{n\to+\infty} (a_n + \ker \widetilde{q}). \tag{3.6}$$

Thus for every $a \in \mathscr{A}$ we have

$$f_{\mathrm{s}}(a) = \lim_{n\to+\infty} q(a_n^* a) = \lim_{n\to+\infty} \langle a + \ker \widetilde{q}, a_n + \ker \widetilde{q}\rangle_q = \langle \pi_q(a)(\mathbf{1} + \ker \widetilde{q}), \zeta\rangle_q. \tag{3.7}$$

Since a positive functional on a unital Banach $*$-algebra is representable, all of the operators $\pi_{f_{\mathrm{r}}}(a), \pi_{f_{\mathrm{s}}}(a), \pi_p(a)$ and $\pi_q(a)$ are continuous. Then from (3.5) we infer that

$$\begin{aligned} f_{\mathrm{r}}(a_n^* a a_n) &= \langle \pi_{f_{\mathrm{r}}}(a)(a_n + \ker \widetilde{f}_{\mathrm{r}}), a_n + \ker \widetilde{f}_{\mathrm{r}}\rangle_{f_{\mathrm{r}}} \to 0, \\ p(a_n^* a a_n) &= \langle \pi_p(a)(a_n + \ker \widetilde{p}), a_n + \ker \widetilde{p}\rangle_p \to 0. \end{aligned}$$

On the other hand, (3.4) and (3.6) implies that

$$\begin{aligned} \lim_{n\to+\infty} f_{\mathrm{s}}(a_n^* a a_n) &= \lim_{n\to+\infty} \langle \pi_{f_{\mathrm{s}}}(a)(a_n + \ker \widetilde{f}_{\mathrm{s}}), a_n + \ker \widetilde{f}_{\mathrm{s}}\rangle_{f_{\mathrm{s}}} \\ &= \langle \pi_{f_{\mathrm{s}}}(a)(\mathbf{1} + \ker \widetilde{f}_{\mathrm{s}}), \mathbf{1} + \ker \widetilde{f}_{\mathrm{s}}\rangle_{f_{\mathrm{s}}} \\ &= \langle a + \ker \widetilde{f}_{\mathrm{s}}, \mathbf{1} + \ker \widetilde{f}_{\mathrm{s}}\rangle_{f_{\mathrm{s}}} = f_{\mathrm{s}}(a), \end{aligned} \tag{3.8}$$

and

$$\lim_{n\to+\infty} q(a_n^* a a_n) = \lim_{n\to+\infty} \langle \pi_q(a)(a_n + \ker \widetilde{q}), a_n + \ker \widetilde{q}\rangle_q = \lim_{n\to+\infty} \langle \pi_q(a)\zeta, \zeta\rangle_q.$$

Hence for any $a \in \mathscr{A}$ we conclude that

$$\begin{aligned} f_{\mathrm{s}}(a) &= \lim_{n\to+\infty} f_{\mathrm{s}}(a_n^* a a_n) = \lim_{n\to+\infty} (f_{\mathrm{r}}(a_n^* a a_n) + f_{\mathrm{s}}(a_n^* a a_n)) = \lim_{n\to+\infty} f(a_n^* a a_n) \\ &= \lim_{n\to+\infty} (p(a_n^* a a_n) + q(a_n^* a a_n)) = \lim_{n\to+\infty} q(a_n^* a a_n) = \langle \pi_q(a)\zeta, \zeta\rangle_q. \end{aligned} \tag{3.9}$$

If we combine (3.7) and (3.9), then we get the following equation:

$$\langle \pi_q(a)(\mathbf{1} + \ker \widetilde{q}), \zeta\rangle_q = f_{\mathrm{s}}(a) = \langle \pi_q(a)\zeta, \zeta\rangle_q.$$

Rearranging this we have that

$$\langle \pi_q(a)((\mathbf{1} + \ker \widetilde{q}) - \zeta), \zeta\rangle_q = 0$$

holds true for every $a \in \mathscr{A}$. Thus for $b \in \mathscr{A}$ replace a with $b^* a$ in the previous equality, and then we have

$$0 = \langle \pi_q(b^* a)((\mathbf{1} + \ker \widetilde{q}) - \zeta), \zeta\rangle_q = \langle \pi_q(a)((\mathbf{1} + \ker \widetilde{q}) - \zeta), \pi_q(b)\zeta\rangle_q,$$

which implies that the closed π_q-invariant subspaces $\overline{\pi_q(\mathscr{A})((\mathbf{1} + \ker \widetilde{q}) - \zeta)}$ and $\overline{\pi_q(\mathscr{A})\zeta}$ are orthogonal.

To complete the proof, let $P \in B(\mathcal{H}_q)$ be the orthogonal projection onto $\overline{\pi_q(\mathscr{A})\zeta}$. Since $\operatorname{ran} P$ is a π_q-invariant subspace, then for any $a \in \mathscr{A}$ the equality $P\pi_q(a) = \pi_q(a)P$ holds true. Furthermore,

$$P(\mathbf{1} + \ker \widetilde{q}) = P((\mathbf{1} + \ker \widetilde{q}) - \zeta + \zeta) = P((\mathbf{1} + \ker \widetilde{q}) - \zeta) + P\zeta = P\zeta = \zeta,$$

hence from (3.9) for every $a \in \mathscr{A}$ we obtain that

$$\begin{aligned} f_{\mathrm{s}}(a^*a) &= \langle \pi_q(a^*a)\zeta, \zeta\rangle_q = \langle \pi_q(a)\zeta, \pi_q(a)\zeta\rangle_q \\ &= \langle \pi_q(a)P(\mathbf{1}+\ker\widetilde{q}), \pi_q(a)P(\mathbf{1}+\ker\widetilde{q})\rangle_q \\ &= \langle P\pi_q(a)(\mathbf{1}+\ker\widetilde{q}), P\pi_q(a)(\mathbf{1}+\ker\widetilde{q})\rangle_q \\ &\leqslant \|P\|^2\langle \pi_q(a)(\mathbf{1}+\ker\widetilde{q}), \pi_q(a)(\mathbf{1}+\ker\widetilde{q})\rangle_q \\ &\leqslant \langle \pi_q(a)(\mathbf{1}+\ker\widetilde{q}), \pi_q(a)(\mathbf{1}+\ker\widetilde{q})\rangle_q \\ &= \langle \pi_q(a^*a)(\mathbf{1}+\ker\widetilde{q}), \mathbf{1}+\ker\widetilde{q}\rangle_q = q(a^*a), \end{aligned}$$

namely $f_{\mathrm{s}} \leqslant q$, and this implies that $0 \leqslant q - f_{\mathrm{s}} = f_{\mathrm{r}} - p$. The proof is complete. ∎

REMARK 3.16. The equality $f_{\mathrm{s}}(a) = \lim\limits_{n\to+\infty} f_{\mathrm{s}}(a_n^*aa_n)$ in (3.8) was noticed by Gudder in [5], but the arguments were different than ours.

Now it is easy to prove the coincidence of Gudder's and our decomposition.

COROLLARY 3.17. *Let f and g be positive linear functionals on a Banach $*$-algebra $\mathscr{A}$ with unit. Let $f = f_{\mathrm{reg}} + f_{\mathrm{sing}}$ be the Lebesgue decomposition of f with respect to g by Corollary* 3.2 *and $f = f_{\mathrm{r}} + f_{\mathrm{s}}$ by Theorem* 3.13*, respectively. Then $f_{\mathrm{reg}} = f_{\mathrm{r}}$ and $f_{\mathrm{sing}} = f_{\mathrm{s}}$, i.e. the two decompositions correspond.*

Proof. Straightforward, since Corollary 3.2 yields that $f_{\mathrm{r}} \leqslant f_{\mathrm{reg}}$, meanwhile Theorem 3.15 implies that $f_{\mathrm{reg}} \leqslant f_{\mathrm{r}}$. Hence $f_{\mathrm{reg}} = f_{\mathrm{r}}$ and $f_{\mathrm{sing}} = f_{\mathrm{s}}$ hold true. ∎

REMARK 3.18. In [16] the author proved by techniques different from the above that for positive functionals on unital Banach $*$-algebras Theorem 1.7 and Theorem 3.13 yields the same decomposition.

As a consequence, we may conclude that the definitions of semisingularity and singularity are equivalent. This result also can be found in [16].

COROLLARY 3.19. *Let f and g be positive linear functionals on a Banach $*$-algebra $\mathscr{A}$ with unit. Then f is g-semisingular if and only if f and g are singular.*

For the proof see Corollary 2 in [16].

We state the following theorem wherein all of the three discussed decompositions are involved. This leads to non-uniqueness of Gudder's and our decomposition, as well.

THEOREM 3.20. *Let $\mathscr{A}$ be a σ-finite von Neumann algebra. Then for a normal positive functional f and a normal faithful state g on $\mathscr{A}$ the decomposition of Kosaki (Theorem* 3.6*), the decomposition of Gudder (Theorem* 3.13*) and the decomposition by Corollary* 3.2 *correspond.*

The proof is clear by Theorem 3.7 and Corollary 3.17.

PROPOSITION 3.21. *There exist a σ-finite von Neumann algebra $\mathscr{A}$ and normal positive functionals f and g on $\mathscr{A}$ such that g is a faithful state, and the above decompositions of f with respect to g is not unique.*

Proof. According to the previous theorem all of the decompositions correspond, hence Example 10.6 due to Kosaki ([8]) shows the non-uniqueness. ∎

4. APPLICATION FOR POSITIVE MEASURES

In this section we prove the existence of the classical Lebesgue decomposition for positive measures with the aid of our results.

Henceforth let $(X, \mathcal{M})$ be a measurable space (i.e. X is a non-void set, $\mathcal{M}$ is a σ-algebra in X), and denote by $\mathscr{A}$ the unital, commutative $*$-algebra generated by the characteristic functions of the measurable sets. We recall that if μ and ν are positive measures on $(X, \mathcal{M})$, then μ is absolutely continuous with respect to ν if and only if for any $A \in \mathcal{M}$, $\nu(A) = 0$ implies $\mu(A) = 0$; furthermore μ and ν are singular if and only if there exists $S \in \mathcal{M}$ such that $\nu(S) = 0$ and $\mu(X \setminus S) = 0$ holds true.

We will need the following lemma, which shows the connection between the absolute continuity (respectively singularity) of positive measures and the closability (respectively singularity) of the positive functionals induced by the measures. The decomposition will be a consequence of this lemma and Corollary 3.2.

The key arguments of the proof are due to F. Riesz, and the second implication of part (a) appears also in Lemma 5.1 of [6]. We include a proof for completeness.

LEMMA 4.1. *Let μ and ν be finite positive measures on $(X, \mathcal{M})$. Then the linear functionals defined on $\mathscr{A}$ by the formulas*

$$f(a) := \int_X a \, \mathrm{d}\mu, \quad g(a) := \int_X a \, \mathrm{d}\nu$$

are positive and representable. Furthermore

(i) *f is closable with respect to g $\Leftrightarrow$ μ is absolutely continuous with respect to ν.*

(ii) *f and g are singular $\Leftrightarrow$ μ and ν are singular.*

Proof. The functionals are well-defined and positive linear, since they are integrals by finite measures. The representability of a positive linear functional v is equivalent to the following two properties (see [9] and [14]):

(1) There exists $K \geqslant 0$ such that for any $a \in \mathscr{A}$: $|v(a)|^2 \leqslant Kv(a^*a)$.

(2) For all $a \in \mathscr{A}$ there exists $\lambda_a \geqslant 0$ such that for every $b \in \mathscr{A}$:

$$v(b^*a^*ab) \leqslant \lambda_a v(b^*b).$$

Since $\mathscr{A}$ is unital, then for all $a \in \mathscr{A}$ the Cauchy–Schwarz inequality yields

$$|f(a)|^2 \leqslant f(\mathbf{1})f(a^*a),$$

thus $f(\mathbf{1})$ is suitable for K in (i). Furthermore,

$$f(b^*a^*ab) = \int_X |a|^2|b|^2 \,\mathrm{d}\mu \leqslant \max_X |a|^2 \int_X |b|^2 \,\mathrm{d}\mu = \max_X |a|^2 f(b^*b) \quad (a, b \in \mathscr{A}),$$

thus for the element $a \in \mathscr{A}$, $\max_X |a|^2$ is suitable for λ_a in (ii). The very same argument shows that g is representable.

(i) "$\Rightarrow$" Let $A \in \mathcal{M}$ be a set such that $\nu(A) = 0$. Then for the constant sequence $(a_n)_{n\in\mathbb{N}} = (\chi_A)_{n\in\mathbb{N}}$ we conclude that

$$\widetilde{g}[a_n] = g(\overline{\chi}_A\chi_A) = g(\chi_A) = \int_X \chi_A \,\mathrm{d}\nu = 0,$$

meanwhile

$$\widetilde{f}[a_n - a_m] = f((\overline{\chi_A - \chi_A})(\chi_A - \chi_A)) = f(0) = 0,$$

hence the closability of f implies that

$$0 = \lim_{n\to+\infty} \widetilde{f}[a_n] = f(\overline{\chi}_A\chi_A) = f(\chi_A) = \int_X \chi_A \,\mathrm{d}\mu = \mu(A),$$

that is, μ is absolutely continuous with respect to ν.

"$\Leftarrow$" Let $(a_n)_{n\in\mathbb{N}}$ be a sequence in $\mathscr{A}$ such that

$$\int_X |a_n|^2 \,\mathrm{d}\nu = \lim_{n\to+\infty} \widetilde{g}[a_n] = \lim_{n,m\to+\infty} \widetilde{f}[a_n - a_m] = \int_X |a_n - a_m|^2 \,\mathrm{d}\mu = 0.$$

We will prove that $\widetilde{f}[a_n] \to 0$. A standard result of Riesz on $\mathscr{L}_2$ convergence yields that there exists a subsequence $(a_{k_n})_{n\in\mathbb{N}}$ of $(a_n)_{n\in\mathbb{N}}$ converging to zero ν-almost everywhere. By the absolute continuity we obtain that $\mu(\{x \in X : a_{k_n}(x) \nrightarrow 0\}) = 0$, namely $a_{k_n} \to 0$ μ-almost everywhere. But $(a_{k_n})_{n\in\mathbb{N}}$ is a subsequence of the Cauchy sequence $(a_n)_{n\in\mathbb{N}}$ in $\mathscr{L}_2(\mu)$, hence it follows that $\widetilde{f}[a_n] \to 0$.

(ii) "$\Rightarrow$" The singularity of f and g is equivalent to $(\widetilde{f} : \widetilde{g}) = 0$ according to Theorem 1.5. From this it follows that $(\widetilde{f} : \widetilde{g})[\mathbf{1}] = 0$, so there exists a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathscr{A}$ such that

$$\int_X |\mathbf{1} - b_n|^2 \,\mathrm{d}\mu = \widetilde{f}[\mathbf{1} - b_n] \to 0, \qquad \int_X |b_n|^2 \,\mathrm{d}\nu = \widetilde{g}[b_n] \to 0.$$

Using the result of Riesz, there exists a subsequence $(b_{k_n})_{n\in\mathbb{N}}$ of $(b_n)_{n\in\mathbb{N}}$ converging to $\mathbf{1}$ μ-almost everywhere. Applying this result again, there exists a subsequence of $(b_{k_n})_{n\in\mathbb{N}}$ such that converges to zero ν-almost everywhere. We may assume that $(b_n)_{n\in\mathbb{N}}$ already has these properties.

Denote by S the set $\{x \in X : b_n(x) \nrightarrow 0\}$. Then we infer that $X = S \cup (X \setminus S)$, $\nu(S) = 0$ and $\mu(X \setminus S) \leqslant \mu(\{x \in X : b_n(x) \nrightarrow \mathbf{1}\}) = 0$, thus μ and ν are singular measures.

"$\Leftarrow$" Assume that there exists $S \in \mathcal{M}$ such that $\nu(S) = 0$ and $\mu(X \setminus S) = 0$. Then for any $a \in \mathscr{A}$ we obtain that

$$\begin{aligned}(\widetilde{f} : \widetilde{g})[a] &= \inf\{\widetilde{f}[a-b] + \widetilde{g}[b] : b \in \mathscr{A}\} \leqslant \widetilde{f}[a - a\chi_S] + \widetilde{g}[a\chi_S] \\ &= \widetilde{f}[a\chi_{X\setminus S}] + \widetilde{g}[a\chi_S] = \int\limits_{X\setminus S} |a|^2 \,\mathrm{d}\mu + \int\limits_S |a|^2 \,\mathrm{d}\nu = 0,\end{aligned}$$

hence f and g are singular. ∎

Now we turn to the existence of the Lebesgue decomposition. In our result, Corollary 3.2 not just forms but representable positive functionals appear, and this fact allows us to prove the existence directly. We only deal with positive finite measures, the other cases are consequences of the following result (see [4] and [12]).

THEOREM 4.2 (Lebesgue decomposition of measures). *Let μ and ν be finite positive measures on $(X, \mathcal{M})$. Then there exist μ_{reg} and μ_{sing} positive measures on $(X, \mathcal{M})$ such that $\mu = \mu_{\mathrm{reg}} + \mu_{\mathrm{sing}}$, where μ_{reg} is absolutely continuous with respect to ν, as well as μ_{sing} and ν are singular.*

Proof. Let f and g be positive functionals on $\mathscr{A}$ defined by

$$f(a) := \int\limits_X a \,\mathrm{d}\mu, \quad g(a) := \int\limits_X a \,\mathrm{d}\nu.$$

We have seen in Lemma 4.1 that these functionals are representable. Then by Corollary 3.2 we conclude that there exist representable positive linear functionals on $\mathscr{A}$ such that $f = f_{\mathrm{reg}} + f_{\mathrm{sing}}$, where f_{reg} is closable with respect to g, while f_{sing} and g are singular. Define the set-functions $\mu_{\mathrm{reg}}, \mu_{\mathrm{sing}} : \mathcal{M} \to \mathbb{C}$ by the formulas

$$\mu_{\mathrm{reg}}(A) := f_{\mathrm{reg}}(\chi_A), \quad \mu_{\mathrm{sing}}(A) := f_{\mathrm{sing}}(\chi_A).$$

The functionals f_{reg} and f_{sing} are positive linear, thus it follows that μ_{reg} and μ_{sing} are positive finitely additive set-functions dominated by the finite measure μ. It is well-known that these properties imply that μ_{reg} and μ_{sing} are σ-additive, namely μ_{reg} and μ_{sing} are positive measures (see [4]). Furthermore, for any $A \in \mathcal{M}$ we have

$$\mu(A) = \int\limits_X \chi_A \,\mathrm{d}\mu = f(\chi_A) = f_{\mathrm{reg}}(\chi_A) + f_{\mathrm{sing}}(\chi_A) = \mu_{\mathrm{reg}}(A) + \mu_{\mathrm{sing}}(A),$$

as well as for every $a \in \mathscr{A}$ the linearity of f_{reg} and f_{sing} implies that

$$f_{\mathrm{reg}}(a) = \int\limits_X a \,\mathrm{d}\mu_{\mathrm{reg}}, \quad f_{\mathrm{sing}}(a) = \int\limits_X a \,\mathrm{d}\mu_{\mathrm{sing}}.$$

Now from Lemma 4.1 we obtain that μ_{reg} is absolutely continuous with respect to ν, moreover μ_{sing} and ν are singular. ∎

5. A COMPLETION TO THE PROOF OF THE MEASURE AND FORM DECOMPOSITIONS CORRESPONDENCE IN THE PAPER OF HASSI, SEBESTYÉN AND DE SNOO

Hassi, Sebestyén and de Snoo presented the following noteworthy result in Theorem 5.5 of [6]:

THEOREM 5.1. *Let μ and ν be finite positive measures on the measurable space $(X, \mathcal{M})$, and let $\mathscr{A}$ be the vector space generated by the characteristic functions of the measurable sets. Define the forms $\mathfrak{t}, \mathfrak{w} : \mathscr{A} \times \mathscr{A} \to \mathbb{C}$ by the formulas*

$$\mathfrak{t}(a,b) := \int_X a\bar{b}\,\mathrm{d}\mu, \quad \mathfrak{w}(a,b) := \int_X a\bar{b}\,\mathrm{d}\nu.$$

Let $\mathfrak{t} = \mathfrak{t}_{\mathrm{reg}} + \mathfrak{t}_{\mathrm{sing}}$ be the Lebesgue decomposition of $\mathfrak{t}$ with respect to the form $\mathfrak{w}$ by Theorem 1.7. Let $\mu = \mu_{\mathrm{reg}} + \mu_{\mathrm{sing}}$ be the Lebesgue decomposition of μ with respect to ν, where μ_{reg} is absolutely continuous with respect to ν, while μ_{sing} and ν are singular. Then for any $a \in \mathscr{A}$ the followig hold true:

$$\mathfrak{t}_{\mathrm{reg}}[a] = \int_X |a|^2\,\mathrm{d}\mu_{\mathrm{reg}}, \quad \mathfrak{t}_{\mathrm{sing}}[a] = \int_X |a|^2\,\mathrm{d}\mu_{\mathrm{sing}}.$$

In other words, this theorem states that the regular part (respectively singular part) in the Lebesgue decomposition of forms induced by measures coincides with the form induced by the absolutely continuous part (respectively singular part) of the Lebesgue decomposition for measures.

First of all we note that this theorem is a consequence of our result, Theorem 4.2. Indeed, we proved the existence of the measure decomposition directly from the decomposition related to representable positive functionals (Corollary 3.2), hence the uniqueness of the decomposition related to measures implies Theorem 5.1.

Hassi, Sebestyén and de Snoo did not prove directly the Lebesgue decomposition related to measures from their decomposition. For the existence they used the result of von Neumann on the Radon–Nikodym derivative (see [11]), and then they proved Theorem 5.1. However, there is a gap in their proof. They pointed out that for any measurable set $M \in \mathcal{M}$ the inequality

$$\mathfrak{t}_{\mathrm{reg}}[\chi_M] \leqslant \int_X |\chi_M|^2\,\mathrm{d}\mu_{\mathrm{reg}} \tag{5.1}$$

holds true for the form $\mathfrak{t}_{\mathrm{reg}}$. They stated that this implies

$$\mathfrak{t}_{\mathrm{reg}}[a] \leqslant \int_X |a|^2\,\mathrm{d}\mu_{\mathrm{reg}} \tag{5.2}$$

for every $a \in \mathscr{A}$ (see (5.5) in [6]). Unfortunately, this implication is false in general, namely if a form generated by a measure is greater on the characteristic functions than another form, then we cannot conclude the inequality on the whole space of simple functions. The following example shows this fact.

EXAMPLE 5.2. Let $X = \{x_1, x_2\}$ be an arbitrary two-element set and let $\mathcal{M}$ be the power-set of X. Denote by ν the counting measure on the measurable space $(X, \mathcal{M})$, moreover let $\mathscr{A}$ be the two dimensional vector space of the simple functions on $(X, \mathcal{M})$. Denote by $\mathfrak{w}$ the form generated by ν on $\mathscr{A}$, namely

$$\mathfrak{w} : \mathscr{A} \times \mathscr{A} \to \mathbb{C}, \quad \mathfrak{w}(a,b) = \int_X a\overline{b}\, \mathrm{d}\nu.$$

Let $e, f \in \mathscr{A}$ be the vectors

$$e = \frac{3}{2}\chi_{\{x_1\}}, \quad f = \frac{3}{2}\chi_{\{x_1\}} + \frac{3}{2}\chi_{\{x_2\}} = \frac{3}{2}\chi_X.$$

These vectors are linearly independent, therefore the system $\{e, f\}$ is a basis for the linear space $\mathscr{A}$. Hence for $a, b \in \mathscr{A}$

$$a = \alpha_e e + \alpha_f f, \quad b = \beta_e e + \beta_f f \quad (\alpha_e, \alpha_f, \beta_e, \beta_f \in \mathbb{C})$$

the following formula defines a form on $\mathscr{A}$:

$$\mathfrak{t} : \mathscr{A} \times \mathscr{A} \to \mathbb{C}, \quad \mathfrak{t}(a,b) := \alpha_e\overline{\beta}_e + \alpha_f\overline{\beta}_f.$$

It is easy to calculate that for every $M \in \mathcal{M}$ the inequality $\mathfrak{t}[\chi_M] \leqslant \mathfrak{w}[\chi_M]$ holds true, moreover for the simple function $a := -\chi_{\{x_1\}} + \chi_{\{x_2\}}$ the inequality $\mathfrak{t}[a] > \mathfrak{w}[a]$ holds. ∎

On the other hand, the proof of Theorem 5.1 in [6] can be corrected. To see this, we will need the following simple lemma.

LEMMA 5.3. *Let $\mathcal{D}$ be a complex vector space, and let $\mathfrak{t}, \mathfrak{w}$ be forms on $\mathcal{D}$ such that $\mathfrak{w} \leqslant \mathfrak{t}$ holds true. Assume that $x, y \in \mathcal{D}$ are vectors such that $\mathfrak{w}(x,y) = 0$. If $\mathfrak{t}[x] \leqslant \mathfrak{w}[x]$ and $\mathfrak{t}[y] \leqslant \mathfrak{w}[y]$ hold true, then $\mathfrak{t}(x,y) = 0$.*

Proof. Suppose, by way of contradiction that $\mathfrak{t}(x,y) \neq 0$, and let $\lambda \in \mathbb{C}$ an arbitrary number. Then by $\mathfrak{t}[x] \leqslant \mathfrak{w}[x]$, $\mathfrak{t}[y] \leqslant \mathfrak{w}[y]$, $\mathfrak{w}(x,y) = 0$ and $\mathfrak{w} \leqslant \mathfrak{t}$ we infer that

$$\begin{aligned} |\lambda|^2\mathfrak{t}[x] + \mathfrak{t}[y] &\leqslant |\lambda|^2\mathfrak{w}[x] + \mathfrak{w}[y] = |\lambda|^2\mathfrak{w}[x] + \mathfrak{w}[y] + 2\Re(\lambda\mathfrak{w}(x,y)) \\ &= \mathfrak{w}[\lambda x + y] \leqslant \mathfrak{t}[\lambda x + y] = |\lambda|^2\mathfrak{t}[x] + \mathfrak{t}[y] + 2\Re(\lambda\mathfrak{t}(x,y)), \end{aligned}$$

namely for any $\lambda \in \mathbb{C}$

$$0 \leqslant 2\Re(\lambda\mathfrak{t}(x,y))$$

holds true. Now if we take $\lambda = -1/\mathfrak{t}(x,y)$ in this inequality, then we have that $0 \leqslant -2$, which is absurd. ∎

Completion to the proof of Theorem 5.1. Denote by $\mathfrak{w}$ the form on $\mathscr{A}$ generated by μ_{reg}, i.e.

$$\mathfrak{w}(a,b) = \int_X a\overline{b}\, \mathrm{d}\mu_{\mathrm{reg}} \quad (a, b \in \mathscr{A}).$$

The maximality of $\mathfrak{t}_{\mathrm{reg}}$ (see (5.6) in [6]) implies that

$$\mathfrak{w} \leqslant \mathfrak{t}_{\mathrm{reg}}. \tag{5.3}$$

We have to prove that the inequality $\mathfrak{t}_{\mathrm{reg}}[a] \leqslant \mathfrak{w}[a]$ holds true for any $a \in \mathscr{A}$. As we mentioned previously, Hassi, Sebestyén and de Snoo showed the inequality

$$\mathfrak{t}_{\mathrm{reg}}[\chi_M] \leqslant \mathfrak{w}[\chi_M] \tag{5.4}$$

for every $M \in \mathcal{M}$.

Let us fix an $a \in \mathscr{A}$. Then there exists a finite disjoint system of measurable sets $\{M_j\}_{j\in J}$ and a system of complex numbers $\{\lambda_j\}_{j\in J}$ such that

$$a = \sum_{j\in J} \lambda_j \chi_{M_j}.$$

Since the sets are disjoint, then for $j,k \in J$, $j \neq k$ we infer that

$$\mathfrak{w}(\chi_{M_j}, \chi_{M_k}) = 0. \tag{5.5}$$

From the properties in (5.3), (5.4) and (5.5) Lemma 5.3 implies

$$\mathfrak{t}_{\mathrm{reg}}(\chi_{M_j}, \chi_{M_k}) = 0$$

for any $j,k \in J$, $j \neq k$. Hence we have

$$\begin{aligned}
\mathfrak{t}_{\mathrm{reg}}[a] &= \mathfrak{t}_{\mathrm{reg}}\Big[\sum_{j\in J}\lambda_j\chi_{M_j}\Big] = \sum_{j\in J}|\lambda_j|^2\mathfrak{t}_{\mathrm{reg}}[\chi_{M_j}] + \sum_{j,k\in J,\, j\neq k} 2\Re(\lambda_j\overline{\lambda}_k \mathfrak{t}_{\mathrm{reg}}(\chi_{M_j},\chi_{M_k}))\\
&= \sum_{j\in J}|\lambda_j|^2\mathfrak{t}_{\mathrm{reg}}[\chi_{M_j}] \leqslant \sum_{j\in J}|\lambda_j|^2\mathfrak{w}[\chi_{M_j}]\\
&= \sum_{j\in J}|\lambda_j|^2\mathfrak{w}[\chi_{M_j}] + \sum_{j,k\in J,\, j\neq k} 2\Re(\lambda_j\overline{\lambda}_k \mathfrak{w}(\chi_{M_j},\chi_{M_k})) = \mathfrak{w}\Big[\sum_{j\in J}\lambda_j\chi_{M_j}\Big] = \mathfrak{w}[a],
\end{aligned}$$

and this completes the proof. ∎

Acknowledgements. I am grateful to Professor János Kristóf for several helpful conversations and valuable suggestions, and to Tamás Titkos for his comments to Section 5.

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Received April 01, 2011.