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SPECTRAL MATRIX FOR STURM–LIOUVILLE OPERATORS ON TWO-SIDED UNBOUNDED TIME SCALES

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ABSTRACT. We establish existence of a spectral matrix for Sturm–Liouville operators on two-sided unbounded time scales. A Parseval equality and an expansion in eigenfunctions formula are obtained in terms of the spectral matrix. Our results unify and extend the well-known results on existence of a spectral matrix for Sturm–Liouville operators on the whole real axis and their discrete analogs.

KEYWORDS: *Time scale, delta derivative, nabla derivative, spectral matrix, Parseval equality, expansion in eigenfunctions.*

MSC (2010): 34L10, 34N05.

INTRODUCTION

There has been an enormous amount of activity on the spectral theory of differential and difference operators, i.e., on the investigation of the spectra of differential and difference operators and the expansion of given functions in terms of the eigenfunctions of these operators, see [5], [21] and the references given therein. The interest in spectral problems has been particularly stimulated by the development of quantum mechanics. In particular, quantum mechanics demands a detailed study of "singular" differential and difference operators, of operators, for example, which are defined on some unbounded interval. Such operators may, in general, have a continuous spectrum as well as a discrete spectrum; and then an expansion in terms of eigenfunctions takes the form of a Stieltjes integral.

During the past twenty years a lot of effort has been put into the study of dynamic equations on time scales (see [7], [8] and references given therein). This field of research unifies and extends discrete and continuous dynamical systems (i.e., difference and differential equations). A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$, the natural numbers $\mathbb{N}$, the nonnegative integers $\mathbb{N}_0$, the set $h\mathbb{Z} = \{hk : k \in \mathbb{Z}\}$ with fixed $h > 0$, and the set $q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$ with fixed $q > 1$ are examples of time scales, as are

$$[0,1] \cup [2,3], \ [-1,0] \cup \mathbb{N}, \text{ and the Cantor set.}$$

The calculus of time scales was initiated by Stefan Hilger in [15] in order to create a theory that can unify discrete and continuous analysis. Indeed, below in Appendix A we will introduce the delta derivative (Δ-derivative) f^Δ for a function $f : \mathbb{T} \to \mathbb{C}$, and it turns out that:

(i) $f^\Delta(t) = f'(t)$ is the usual derivative if $\mathbb{T} = \mathbb{R}$,
(ii) $f^\Delta(t) = \frac{f(t+h)-f(t)}{h}$ is the usual forward difference quotient if $\mathbb{T} = h\mathbb{Z}$,
(iii) $f^\Delta(t) = \frac{f(qt)-f(t)}{(q-1)t}$ is the usual q-derivative (Jackson derivative) if $\mathbb{T} = q^{\mathbb{N}_0}$.

In [15], Hilger introduced also dynamic equations (or Δ-differential equations) on time scales in order to unify and extend the theories of ordinary differential equations and difference equations (including q-difference equations as well). The concept of a nabla derivative (∇-derivative) was introduced later in [4] in order to construct higher-order, in particular second-order, formally self-adjoint differential expressions on time scales by making use of both delta and nabla derivatives simultaneously. For a general introduction to the calculus of time scales we refer the reader to the original paper by Hilger [15] and to the textbooks by Bohner and Peterson [7], [8] (for terminology and notation related to the time scale calculus see Appendix A below).

There are several papers concerning spectral problems on a bounded time scale interval (the regular case) [1], [2], [3], [9], [11], [13], [18]. Spectral problems on semi-unbounded time scale intervals (the singular case) have started to be considered only quite recently [14], [16], [17], [22], [24], [26]. The present paper deals with the eigenvalue problem for Sturm–Liouville operators on two-sided unbounded time scales, of the form

$$-[p(t)y^\Delta(t)]^\nabla + q(t)y(t) = \lambda y(t), \quad t \in (-\infty, \infty)_\mathbb{T}. \tag{0.1}$$

We prove the existence of a spectral matrix and obtain an expansion formula in terms of the spectral matrix for equation (0.1) by considering this problem as a limit of the corresponding eigenvalue problems on bounded time scale intervals. Such a method in the case of classical differential equations was applied in 1950 by Levitan, Levinson, and Yosida independently from each other (see [10], [20], [27]).

This paper is organized as follows. In Section 1, the main result of the paper is presented whose proof is given in Section 2. In Section 3 we consider two special cases of time scales $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. For the convenience of the reader, at the end of the paper we placed an appendix in which the usual notions and notation connected to time scale calculus are presented. For easy reference, we presented in that appendix also the classical theorems of Helly.

1. SPECTRAL MATRIX, PARSEVAL EQUALITY, AND EIGENFUNCTION EXPANSION

Let $\mathbb{T}$ be a time scale (an arbitrary nonempty closed subset of $\mathbb{R}$) which is unbounded from both below and above so that

$$\inf \mathbb{T} = -\infty \quad \text{and} \quad \sup \mathbb{T} = \infty.$$

We will denote such $\mathbb{T}$ also as $(-\infty, \infty)_{\mathbb{T}}$ and call it a *two-sided unbounded time scale.* All the time scale intervals will be equiped with the subindex $\mathbb{T}$ while the real axis intervals will be written without any subindex. For standard notions and notation connected to time scales we refer to Appendix A at the end of this paper.

Consider equation (0.1), where λ is a spectral (complex) parameter. Often we will deal with the real values of λ. The function $p(t)$ is assumed to be a real-valued ∇-differentiable function on $(-\infty, \infty)_{\mathbb{T}}$ with the piecewise continuous (in the time scale topology) nabla derivative p^{∇} and such that $p(t) > 0$ for all $t \in (-\infty, \infty)_{\mathbb{T}}$, $q(t)$ is assumed to be a real-valued function piecewise continuous on $(-\infty, \infty)_{\mathbb{T}}$.

A function $y : \mathbb{T} \to \mathbb{C}$ is called a *solution* of equation (0.1) if y is Δ-differentiable on $(-\infty, \infty)_{\mathbb{T}}$, y^{Δ} is ∇-differentiable on $(-\infty, \infty)_{\mathbb{T}}$, and equation (0.1) is satisfied.

Let t_0 be a fixed point in $\mathbb{T}$. Denote by $\varphi_1(t, \lambda)$, $\varphi_2(t, \lambda)$ the solutions of equation (0.1), real for real λ, satisfying the initial conditions

$$\varphi_1(t_0, \lambda) = 1, \quad \varphi_1^{[\Delta]}(t_0, \lambda) = 0, \tag{1.1}$$

$$\varphi_2(t_0, \lambda) = 0, \quad \varphi_2^{[\Delta]}(t_0, \lambda) = 1, \tag{1.2}$$

where by $y^{[\Delta]}(t)$ we denote the quasi Δ-derivative of $y(t)$ defined by $y^{[\Delta]}(t) = p(t)y^{\Delta}(t)$. We have, for the Wronskian of the solutions φ_1 and φ_2,

$$W_t(\varphi_1, \varphi_2) = \varphi_1(t)\varphi_2^{[\Delta]}(t) - \varphi_1^{[\Delta]}(t)\varphi_2(t) = 1 \neq 0$$

and therefore φ_1 and φ_2 are linearly independent solutions of equation (0.1).

Next, denote by $\mathcal{H}$ the Hilbert space of all ∇-measurable (see [12]) functions $f : (-\infty, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that

$$\int_{-\infty}^{\infty} f^2(t) \nabla t < \infty,$$

with the inner product

$$\langle f_1, f_2 \rangle_{\nabla} = \int_{-\infty}^{\infty} f_1(t) f_2(t) \nabla t.$$

Let

$$\omega(\lambda) = [\omega_{jk}(\lambda)]_{j,k=1}^{2}, \quad -\infty < \lambda < \infty,$$

be a fixed 2×2 matrix-function with the following properties:

(a) All functions $\omega_{jk}(\lambda)$ are real-valued and continuous on the right, so that $\omega_{jk}(\lambda+0) = \omega_{jk}(\lambda)$.

(b) For every $\lambda \in (-\infty,\infty)_{\mathbb{R}}$, $\omega(\lambda)$ is a symmetric matrix, that is, $\omega_{12}(\lambda) = \omega_{21}(\lambda)$.

(c) For $\lambda < \mu$, the difference $\omega(\mu) - \omega(\lambda)$ is a positive semidefinite matrix, that is, for any $\xi_1, \xi_2 \in \mathbb{R}$,

$$\sum_{j,k=1}^{2} [\omega_{jk}(\mu) - \omega_{jk}(\lambda)]\xi_j\xi_k \geqslant 0. \tag{1.3}$$

Since the diagonal elements of a positive semidefinite matrix are nonnegative, it follows that each diagonal element $\omega_{jj}(\lambda)$ is a nondecreasing function. Moreover, the function $\omega_{12}(\lambda) = \omega_{21}(\lambda)$ is of finite variation on every bounded interval; this follows from the inequality

$$2|\omega_{12}(\mu) - \omega_{12}(\lambda)| \leqslant |\omega_{11}(\mu) - \omega_{11}(\lambda)| + |\omega_{22}(\mu) - \omega_{22}(\lambda)|, \tag{1.4}$$

which is an immediate consequence of (1.3). Indeed, (1.3) implies

$$[\omega_{12}(\mu) - \omega_{12}(\lambda)]^2 \leqslant [\omega_{11}(\mu) - \omega_{11}(\lambda)][\omega_{22}(\mu) - \omega_{22}(\lambda)]$$

and hence (1.4) follows.

Because of (c), ω is said to be nondecreasing.

Denote by $L^2_\omega(-\infty,\infty)$ the Hilbert space of all real vector-functions $h(\lambda) = [h_1(\lambda), h_2(\lambda)]$ such that

$$\int_{-\infty}^{\infty} \sum_{j,k=1}^{2} h_j(\lambda)h_k(\lambda)\mathrm{d}\omega_{jk}(\lambda) < \infty, \tag{1.5}$$

with the inner product

$$\langle h, \widetilde{h}\rangle_\omega = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} h_j(\lambda)\widetilde{h}_k(\lambda)\mathrm{d}\omega_{jk}(\lambda).$$

The nondecreasing nature of ω guarantees that the integral in (1.5) is nonnegative.

Take any finite points a and b in $\mathbb{T}$ such that $a < t_0 < b$ and denote $\delta = (a,b]_{\mathbb{T}}$ for brevity. The main result of this paper is the following statement.

THEOREM 1.1. *For equation* (0.1) *there is a matrix-function* $\omega(\lambda) = [\omega_{jk}(\lambda)]_{j,k=1}^2$, $-\infty < \lambda < \infty$, *satisfying the above conditions* (a)–(c) *and having the following properties:*

(i) *If* $f \in \mathcal{H}$ *there exists a vector-function* $F = [F_1, F_2] \in L^2_\omega(-\infty,\infty)$ *such that*

$$\lim_{\delta \to (-\infty,\infty)_{\mathbb{T}}} \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} \{F_j(\lambda) - F_{\delta j}(\lambda)\}\{F_k(\lambda) - F_{\delta k}(\lambda)\}\mathrm{d}\omega_{jk}(\lambda) = 0 \tag{1.6}$$

and the Parseval equality

$$\int_{-\infty}^{\infty} f^2(t)\nabla t = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{jk}(\lambda) \tag{1.7}$$

holds, where

$$F_{\delta j}(\lambda) = \int_{\delta} f(t)\varphi_j(t,\lambda)\nabla t = \int_a^b f(t)\varphi_j(t,\lambda)\nabla t. \tag{1.8}$$

(ii) *The integral*

$$\int_{-\infty}^{\infty} \sum_{j,k=1}^{2} F_j(\lambda)\varphi_k(t,\lambda)\mathrm{d}\omega_{jk}(\lambda)$$

converges in $\mathcal{H}$ to f, that is,

$$\lim_{\Lambda\to\infty} \int_{-\infty}^{\infty} \left\{ f(t) - \int_{-\Lambda}^{\Lambda} \sum_{j,k=1}^{2} F_j(\lambda)\varphi_k(t,\lambda)\mathrm{d}\omega_{jk}(\lambda) \right\}^2 \nabla t = 0. \tag{1.9}$$

The 2×2 matrix-function $\omega(\lambda) = [\omega_{jk}(\lambda)]_{j,k=1}^2$ we call a *spectral matrix* for equation (0.1). By

$$\int_{-\infty}^{\infty} f(t)\varphi_j(t,\lambda)\nabla t$$

will be meant the value at λ of the function F_j defined by (1.6). Thus

$$F_j(\lambda) = \int_{-\infty}^{\infty} f(t)\varphi_j(t,\lambda)\nabla t \quad (j=1,2) \tag{1.10}$$

and $[F_1, F_2]$ may be regarded as a generalized Fourier transform of f by means of the functions φ_1 and φ_2. It follows that

$$f(t) = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} F_j(\lambda)\varphi_k(t,\lambda)\mathrm{d}\omega_{jk}(\lambda), \tag{1.11}$$

where the equality is meant in the sense of (1.9), and (1.11) (it can be regarded as the inverse Fourier transform of $[F_1, F_2]$) we call the *expansion formula*. If f, for example, is continuous and vanishes for all $t \in \mathbb{T}$ with large $|t|$ then the integral in (1.10) exists in the ordinary sense.

2. PROOF OF THEOREM 1.1

Let a and b be any (finite) points in $\mathbb{T}$ such that $a < t_0 < b$. For convenience we put $\delta = (a, b]_{\mathbb{T}}$ and together with equation (0.1) consider the eigenvalue problem:

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = \lambda y(t), \quad t \in \delta = (a, b]_{\mathbb{T}}, \tag{2.1}$$

$$y(a)\sin\alpha + y^{[\Delta]}(a)\cos\alpha = 0, \quad y(b)\sin\beta + y^{[\Delta]}(b)\cos\beta = 0, \tag{2.2}$$

where $0 \leqslant \alpha, \beta < \pi$. This problem is investigated in [18].

Denote by $\lambda_{\delta n}$ $(n = 1, 2, \ldots, N_\delta;\ N_\delta \leqslant \infty)$ the eigenvalues of (2.1), (2.2). Each eigenvalue $\lambda_{\delta n}$ is real and simple. We arrange $\lambda_{\delta n}$'s as

$$\lambda_{\delta 1} < \lambda_{\delta 2} < \lambda_{\delta 3} < \cdots .$$

Let $y_{\delta 1}(t), y_{\delta 2}(t), y_{\delta 3}(t), \ldots$ be the corresponding to these eigenvalues orthonormal and real eigenfunctions of (2.1), (2.2). Since $\varphi_1(t, \lambda)$ and $\varphi_2(t, \lambda)$ are linearly independent solutions of equation (0.1), we have

$$y_{\delta n}(t) = \alpha_{\delta n1}\varphi_1(t, \lambda_{\delta n}) + \alpha_{\delta n2}\varphi_2(t, \lambda_{\delta n}) = \sum_{j=1}^{2} \alpha_{\delta nj}\varphi_j(t, \lambda_{\delta n}) \tag{2.3}$$

with some real numbers $\alpha_{\delta n1}$ and $\alpha_{\delta n2}$.

Denote by $\mathcal{H}_\delta$ the Hilbert space of all ∇-measurable functions $f : \delta \to \mathbb{R}$ such that

$$\int_\delta f^2(t)\nabla t = \int_a^b f^2(t)\nabla t < \infty$$

and that $f(b) = 0$ in the case b is left-scattered and $\cos\beta = 0$. For functions f, g in the space $\mathcal{H}_\delta$ the inner product and norm are defined by

$$\langle f, g\rangle_\nabla = \int_a^b f(t)g(t)\nabla t, \quad \|f\|_\nabla = \langle f, f\rangle_\nabla^{1/2}.$$

The Parseval equality for problem (2.1), (2.2) is (see [18])

$$\int_a^b f^2(t)\nabla t = \sum_{n=1}^{N_\delta} \left\{ \int_a^b f(t)y_{\delta n}(t) \right\}^2 \nabla t, \tag{2.4}$$

where $f \in \mathcal{H}_\delta$. Using (2.3) in (2.4), we get

$$\int_a^b f^2(t)\nabla t = \sum_{n=1}^{N_\delta} \sum_{j,k=1}^{2} \alpha_{\delta nj}\alpha_{\delta nk} F_{\delta j}(\lambda_{\delta n}) F_{\delta k}(\lambda_{\delta n}), \tag{2.5}$$

where

$$F_{\delta j}(\lambda) = \int_{\delta} f(t)\varphi_j(t,\lambda)\nabla t = \int_a^b f(t)\varphi_j(t,\lambda)\nabla t \quad (j=1,2). \tag{2.6}$$

Now we introduce the jump functions $\omega_{\delta jk}(\lambda)$ $(j,k=1,2)$ with the jumps at the eigenvalues of (2.1), (2.2) by

$$\omega_{\delta jk}(\lambda) = \begin{cases} -\sum\limits_{\lambda<\lambda_{\delta n}\leqslant 0} \alpha_{\delta nj}\alpha_{\delta nk} & \text{for } \lambda \leqslant 0, \\ \sum\limits_{0<\lambda_{\delta n}\leqslant \lambda} \alpha_{\delta nj}\alpha_{\delta nk} & \text{for } \lambda > 0. \end{cases} \tag{2.7}$$

Then equality (2.5) can be written as

$$\int_a^b f^2(t)\nabla t = \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} F_{\delta j}(\lambda)F_{\delta k}(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda) \tag{2.8}$$

in terms of a Stieltjes integral.

It follows from (2.7) that the matrix $\omega_\delta(\lambda) = [\omega_{\delta jk}(\lambda)]_{j,k=1}^2$ has the following properties:

(i) $\omega_\delta(\lambda)$ is a symmetric matrix $(\omega_{\delta jk} = \omega_{\delta kj})$, $\omega_\delta(0)$ is the zero matrix, and $\omega_\delta(\lambda+0) = \omega_\delta(\lambda)$.

(ii) If $\lambda < \mu$, then $\omega_\delta(\mu) - \omega_\delta(\lambda)$ is positive semidefinite.

(iii) The total variation of $\omega_{\delta jk}$ is finite on every finite λ interval.

Note that the property (ii) holds as follows:

$$\begin{aligned}\sum_{j,k=1}^{2}[\omega_{jk}(\mu)-\omega_{jk}(\lambda)]\xi_j\xi_k &= \sum_{j,k=1}^{2}\Big\{\sum_{\lambda_{\delta n}\in(\lambda,\mu]}\alpha_{\delta nj}\alpha_{\delta nk}\Big\}\xi_j\xi_k = \sum_{\lambda_{\delta n}\in(\lambda,\mu]}\Big\{\sum_{j,k=1}^{2}\alpha_{\delta nj}\alpha_{\delta nk}\xi_j\xi_k\Big\} \\ &= \sum_{\lambda_{\delta n}\in(\lambda,\mu]}(\alpha_{\delta n1}\xi_1 + \alpha_{\delta n2}\xi_2)^2 \geqslant 0\end{aligned}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$.

We will show that there is at least one limiting matrix ω as $\delta \to (-\infty,\infty)_{\mathbb{T}}$ and that the Parseval equality (2.8) is valid with $\delta = (a,b]_{\mathbb{T}}$ replaced by $(-\infty,\infty)_{\mathbb{T}}$ and ω_δ replaced by ω.

LEMMA 2.1. *For any positive real number Λ there is a positive constant $M = M(\Lambda)$ not depending on δ such that*

$$V_{-\Lambda}^{\Lambda}(\omega_{\delta jk}) = \int_{-\Lambda}^{\Lambda} |\mathrm{d}\omega_{\delta jk}(\lambda)| \leqslant M, \tag{2.9}$$

that is, the total variations of the functions $\omega_{\delta jk}(\lambda)$ on the interval $[-\Lambda,\Lambda]$ are bounded uniformly relative to $\delta = (a,b]_{\mathbb{T}}$. In particular, using $\omega_{\delta jk}(0) = 0$,

$$|\omega_{\delta jk}(\lambda)| \leqslant M \quad \textit{for all } \lambda \in [-\Lambda,\Lambda] \textit{ and all } \delta = (a,b]_{\mathbb{T}}. \tag{2.10}$$

Proof. In order to prove inequality (2.9) it is sufficient to prove this inequality only for $j=k$ because, by virtue of (1.4) with ω_{jk} replaced by $\omega_{\delta jk}$,

$$\int_{-\Lambda}^{\Lambda} |\mathrm{d}\omega_{\delta 12}(\lambda)| \leqslant \frac{1}{2}\Big\{ \int_{-\Lambda}^{\Lambda} \mathrm{d}\omega_{\delta 11}(\lambda) + \int_{-\Lambda}^{\Lambda} \mathrm{d}\omega_{\delta 22}(\lambda) \Big\}. \tag{2.11}$$

Further, the functions $\varphi_j^{[\Delta^{m-1}]}(t,\lambda)$ $(j,m=1,2)$ are continuous in (t,λ) and at $t=t_0$ are equal to δ_{jm} by (1.1), (1.2). Therefore, given Λ and $0<\varepsilon<1$ (the number ε will be chosen later) there are points t' and t'' in $\mathbb{T}$ with $t'<t_0<t''$ and close to t_0 such that

$$|\varphi_j^{[\Delta^{m-1}]}(t,\lambda) - \delta_{jm}| < \varepsilon \quad \text{for } t \in (t',t'')_{\mathbb{T}},\ \lambda \in [-\Lambda,\Lambda]. \tag{2.12}$$

Let $g(t)$ be a nonnegative continuously ∇-differentiable function on $(a,b]_{\mathbb{T}}$ vanishing outside of $(t',t'')_{\mathbb{T}}$ and normalized so that

$$\int_{t'}^{t''} g(t)\Delta t = 1. \tag{2.13}$$

The Parseval equality (2.8) applied to the function $(-1)^{m-1}g^{[\nabla^{m-1}]}(t)$, for fixed m, $(m=1,2)$, gives

$$\begin{aligned}\int_{t'}^{t''} \{g^{[\nabla^{m-1}]}(t)\}^2 \nabla t &= \int_{-\infty}^{\infty} \sum_{j,k=1}^{2} G_{jm}(\lambda)G_{km}(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda) \\ &\geqslant \int_{-\Lambda}^{\Lambda} \sum_{j,m=1}^{2} G_{jm}(\lambda)G_{km}(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda),\end{aligned} \tag{2.14}$$

where

$$G_{jm}(\lambda) = (-1)^{m-1}\int_{t'}^{t''} g^{[\nabla^{m-1}]}(t)\varphi_j(t,\lambda)\nabla t = \int_{t'}^{t''} g(t)\varphi_j^{[\Delta^{m-1}]}(t,\lambda)\Delta t,$$

$$g^{[\nabla^0]}(t) = g(t), \quad g^{[\nabla^1]}(t) = g^{[\nabla]}(t) = p(t)g^{\nabla}(t).$$

In the latter equality we have used the integration by parts formula (A.5) given below in Appendix. Hence, by (2.12), (2.13),

$$\begin{aligned}|G_{jm}(\lambda) - \delta_{jm}| &= \Big| \int_{t'}^{t''} g(t)\varphi_j^{[\Delta^{m-1}]}(t,\lambda)\Delta t - \delta_{jm}\int_{t'}^{t''} g(t)\Delta t \Big| \\ &= \Big| \int_{t'}^{t''} g(t)[\varphi_j^{[\Delta^{m-1}]}(t,\lambda) - \delta_{jm}]\Delta t \Big| \leqslant \varepsilon \int_{t'}^{t''} g(t)\Delta t = \varepsilon,\end{aligned}$$

that is,

$$|G_{jm}(\lambda)-\delta_{jm}|\leqslant\varepsilon\quad\text{for }\lambda\in[-\Lambda,\Lambda]\text{ and }j,m=1,2.$$

Therefore

$$|G_{jm}(\lambda)|\leqslant\delta_{jm}+\varepsilon,\ |G_{jj}(\lambda)|\geqslant 1-\varepsilon\quad\text{for }\lambda\in[-\Lambda,\Lambda]\text{ and }j,m=1,2. \tag{2.15}$$

Then we have

$$\begin{aligned}
I&=\int_{-\Lambda}^{\Lambda}\sum_{j,m=1}^{2}G_{jm}(\lambda)G_{km}(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)\\
&=\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}|G_{jj}(\lambda)|^2\mathrm{d}\omega_{\delta jj}(\lambda)+\int_{-\Lambda}^{\Lambda}\sum_{j\neq k}G_{jm}(\lambda)G_{km}(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)\\
&\geqslant\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}|G_{jj}(\lambda)|^2\mathrm{d}\omega_{\delta jj}(\lambda)-\int_{-\Lambda}^{\Lambda}\sum_{j\neq k}|G_{jm}(\lambda)||G_{km}(\lambda)||\mathrm{d}\omega_{\delta jk}(\lambda)|.
\end{aligned}$$

Using inequality (2.11) gives

$$\begin{aligned}
I&\geqslant\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}|G_{jj}(\lambda)|^2\mathrm{d}\omega_{\delta jj}(\lambda)-\frac{1}{2}\int_{-\Lambda}^{\Lambda}\sum_{j\neq k}|G_{jm}(\lambda)||G_{km}(\lambda)|\{\mathrm{d}\omega_{\delta jj}(\lambda)+\mathrm{d}\omega_{\delta kk}(\lambda)\}\\
&=2\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}|G_{jj}(\lambda)|^2\mathrm{d}\omega_{\delta jj}(\lambda)-\int_{-\Lambda}^{\Lambda}\sum_{j,k=1}^{2}|G_{jm}(\lambda)||G_{km}(\lambda)|\mathrm{d}\omega_{\delta jj}(\lambda).
\end{aligned}$$

Hence, by virtue of the frst inequality in (2.15),

$$I\geqslant 2\int_{-\Lambda}^{\Lambda}|G_{mm}(\lambda)|^2\mathrm{d}\omega_{\delta mm}(\lambda)-\int_{-\Lambda}^{\Lambda}\sum_{j,k=1}^{2}(\delta_{jm}+\varepsilon)(\delta_{km}+\varepsilon)\mathrm{d}\omega_{\delta jj}(\lambda).$$

Finaly, using in the last inequality the second inequality in (2.15), we get

$$\begin{aligned}
I&\geqslant 2(1-\varepsilon)^2\int_{-\Lambda}^{\Lambda}\mathrm{d}\omega_{\delta mm}(\lambda)-(1+2\varepsilon)\int_{-\Lambda}^{\Lambda}\mathrm{d}\omega_{\delta mm}(\lambda)-(1+2\varepsilon)\varepsilon\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}\mathrm{d}\omega_{\delta jj}(\lambda)\\
&=(1-6\varepsilon+2\varepsilon^2)\int_{-\Lambda}^{\Lambda}\mathrm{d}\omega_{\delta mm}(\lambda)-(1+2\varepsilon)\varepsilon\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}\mathrm{d}\omega_{\delta jj}(\lambda).
\end{aligned}$$

Consequently, it follows from (2.14) that

$$\int_{t'}^{t''}\{g^{[\nabla^{m-1}]}(t)\}^2\nabla t\geqslant(1-6\varepsilon+2\varepsilon^2)\int_{-\Lambda}^{\Lambda}\mathrm{d}\omega_{\delta mm}(\lambda)-(1+2\varepsilon)\varepsilon\int_{-\Lambda}^{\Lambda}\sum_{j=1}^{2}\mathrm{d}\omega_{\delta jj}(\lambda).$$

Summing this inequality for $m=1,2$ we get

$$\sum_{m=1}^{2}\int_{t'}^{t''}\{g^{[\nabla^{m-1}]}(t)\}^2\nabla t \geqslant (1-8\varepsilon-2\varepsilon^2)\int_{-\Lambda}^{\Lambda}\sum_{m=1}^{2}\mathrm{d}\omega_{\delta mm}(\lambda). \tag{2.16}$$

If we choose ε so that

$$1-8\varepsilon-2\varepsilon^2>\frac{1}{2},$$

then we get from (2.16) the statement of the lemma. ∎

Now we show that making use of Lemma 2.1 and the Helly theorems it is possible to get a spectral matrix and the Parseval equality for equation (0.1).

Let $\Lambda_1<\Lambda_2<\cdots$ be positive numbers such that $\Lambda_n\to\infty$ as $n\to\infty$. By Lemma 2.1, the nondecreasing functions $\omega_{\delta 11}$ and $\omega_{\delta 22}$ are uniformly (with respect to δ) bounded on $[-\Lambda_1,\Lambda_1]$, and the function $\omega_{\delta 12}=\omega_{\delta 21}$ is uniformly bounded and has bounded total variation on $[-\Lambda_1,\Lambda_1]$. Then, by Theorem A.8 we can choose a sequence of matrix-functions $\omega_{\delta_n^{(1)}}=[\omega_{\delta_n^{(1)}jk}]_{j,k=1}^{2}$ $(n=1,2,3,\ldots)$ that converges to a nondecreasing matrix-function $\omega^{(1)}$ on $-\Lambda_1\leqslant\lambda\leqslant\Lambda_1$. Similarly, the sequence $(\omega_{\delta_n^{(1)}})_{n\in\mathbb{N}}$ contains a subsequence $(\omega_{\delta_n^{(2)}})_{n\in\mathbb{N}}$ that converges to a nondecreasing matrix-function $\omega^{(2)}$ on $[-\Lambda_2,\Lambda_2]$ and obviously $\omega^{(1)}(\lambda)=\omega^{(2)}(\lambda)$ for $\lambda\in[-\Lambda_1,\Lambda_1]$. If we continue this process, then the "diagonal sequence" $(\omega_{\delta_n^{(n)}})_{n\in\mathbb{N}}$ will converge to a nondecreasing matrix-function ω at every point of $(-\infty,\infty)$.

To get the Parseval equality, first assume that the function f satisfies the conditions:

(i) f vanishes outside the interval $\delta'=(a',b']_{\mathbb{T}}$, where $a',b'\in\mathbb{T}$ and $a<a'<t_0<b'<b$.
(ii) f is Δ-differentiable on $(a',b']_{\mathbb{T}}$ with the continuously ∇-differentiable f^{Δ}.

Applying to this function the Parseval equality (2.8) we get

$$\int_{a'}^{b'}f^2(t)\nabla t=\int_{-\infty}^{\infty}\sum_{j,k=1}^{2}F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda), \tag{2.17}$$

where

$$F_j(\lambda)=\int_a^b f(t)\varphi_j(t,\lambda)\nabla t\quad (j=1,2).$$

Since the function f vanishes in some neighborhoods of the points $t=a$ and $t=b$, we obtain by using Theorem A.7 (the integration by parts formulae (A.4) and (A.5)) that

$$F_j(\lambda)=\frac{1}{\lambda}\int_a^b f(t)\{-[p(t)\varphi_j^{\Delta}(t,\lambda)]^{\nabla}+q(t)\varphi_j(t,\lambda)\}\nabla t$$
$$=-\frac{1}{\lambda}f(t)p(t)\varphi_j^{\Delta}(t,\lambda)\ |_a^b+\frac{1}{\lambda}\int_a^b f^{\Delta}(t)p(t)\varphi_j^{\Delta}(t,\lambda)\Delta t+\frac{1}{\lambda}\int_a^b f(t)q(t)\varphi_j(t,\lambda)\nabla t$$
$$=-\frac{1}{\lambda}f(t)p(t)\varphi_j^{\Delta}(t,\lambda)\ |_a^b+\frac{1}{\lambda}f^{\Delta}(t)p(t)\varphi_j(t,\lambda)\ |_a^b$$
$$+\frac{1}{\lambda}\int_a^b\{-[p(t)f^{\Delta}(t)]^{\nabla}+q(t)f(t)\}\varphi_j(t,\lambda)\nabla t$$
$$=\frac{1}{\lambda}\int_a^b\{-[p(t)f^{\Delta}(t)]^{\nabla}+q(t)f(t)\}\varphi_j(t,\lambda)\nabla t=\frac{1}{\lambda}X_j(\lambda),$$

where

$$X_j(\lambda)=\int_a^b\{-[p(t)f^{\Delta}(t)]^{\nabla}+q(t)f(t)\}\varphi_j(t,\lambda)\nabla t.$$

Therefore, for any $\Lambda>0$,

$$\int_{|\lambda|>\Lambda}\sum_{j,k=1}^{2}F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)$$
$$=\int_{|\lambda|>\Lambda}\frac{1}{\lambda^2}\sum_{j,k=1}^{2}X_j(\lambda)X_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)\leqslant\frac{1}{\Lambda^2}\int_{|\lambda|>\Lambda}\sum_{j,k=1}^{2}X_j(\lambda)X_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)$$
$$\leqslant\frac{1}{\Lambda^2}\int_{-\infty}^{\infty}\sum_{j,k=1}^{2}X_j(\lambda)X_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)=\frac{1}{\Lambda^2}\int_a^b\{-[p(t)f^{\Delta}(t)]^{\nabla}+q(t)f(t)\}^2\nabla t.$$

In the last equality we have used the Parseval equality (2.8) accordingly. Hence, taking into account equality (2.17),

$$\Big|\int_{a'}^{b'}f^2(t)\nabla t-\int_{-\Lambda}^{\Lambda}\sum_{j,k=1}^{2}F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)\Big|=\int_{|\lambda|>\Lambda}\sum_{j,k=1}^{2}F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{\delta jk}(\lambda)$$
$$\leqslant\frac{1}{\Lambda^2}\int_a^b\{-[p(t)f^{\Delta}(t)]^{\nabla}+q(t)f(t)\}^2\nabla t. \tag{2.18}$$

Passing to the limit in (2.18) through the sequence $(\omega_{\delta_n^{(n)}})_{n\in\mathbb{N}}$ and using Theorem A.9, we get

$$\Big|\int_{a'}^{b'} f^2(t)\nabla t - \int_{-\Lambda}^{\Lambda}\sum_{j,k=1}^{2} F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{jk}(\lambda)\Big| \leqslant \frac{1}{\Lambda^2}\int_{-\infty}^{\infty}\{-[p(t)f^{\Delta}(t)]^{\nabla}+q(t)f(t)\}^2\nabla t.$$

Letting $\Lambda\to\infty$ we obtain

$$\int_{a'}^{b'} f^2(t)\nabla t = \int_{-\infty}^{\infty}\sum_{j,k=1}^{2} F_j(\lambda)F_k(\lambda)\mathrm{d}\omega_{jk}(\lambda),$$

that is, the Parseval equality for any function f satisfying the above conditions (i) and (ii).

Extension of the Parseval equality to arbitrary functions of space $\mathcal{H}$ and derivation of the expansion formula (1.11) from the Parseval equality (1.7) are realized in a standard way (see [16]).

3. EXAMPLES

1. In the case $\mathbb{T}=\mathbb{R}=(-\infty,\infty)$, the real axis, for functions $y:\mathbb{T}\to\mathbb{C}$ we have

$$y^{\Delta}(t)=y^{\nabla}(t)=y'(t),\quad t\in\mathbb{R},$$

and equation (0.1) becomes the ordinary Sturm–Liouville equation on the whole real axis

$$-[p(t)y'(t)]'+q(t)y(t)=\lambda y(t),\quad -\infty<t<\infty, \tag{3.1}$$

where $p(t)$ is a real-valued differentiable function with the piecewise continuous derivative $p'(t)$ on $(-\infty,\infty)$, $p(t)>0$, and $q(t)$ is a piecewise continuous real-valued function on $(-\infty,\infty)$. Theorem 1.1 gives a spectral matrix and an expansion result for equation (3.1). Such a result for equation (3.1) was for the first time established earlier by H. Weyl [25]. Its various proofs were later offered by Levitan, Levinson, Yosida, and Titchmarsh (see [10], [20], [23], [27]).

2. In the case $\mathbb{T}=\mathbb{Z}=\{\dots,-2,-1,0,1,2,\dots\}$, for functions $y:\mathbb{T}\to\mathbb{C}$ we have

$$y^{\Delta}(t)=y(t+1)-y(t),\quad y^{\nabla}(t)=y(t)-y(t-1),\quad t\in\mathbb{Z},$$

and equation (0.1) can be written in the form

$$-p(t-1)y(t-1)+q_1(t)y(t)-p(t)y(t+1)=\lambda y(t),\quad t\in\mathbb{Z}, \tag{3.2}$$

where

$$q_1(t)=p(t-1)+p(t)+q(t).$$

Setting

$$a_n=-p(n),\quad b_n=q_1(n),\quad y_n=y(n),\quad n\in\mathbb{Z},$$

we can rewrite equation (3.2) in the form

$$a_{n-1}y_{n-1} + b_n y_n + a_n y_{n+1} = \lambda y_n, \quad n \in \mathbb{Z}. \tag{3.3}$$

Theorem 1.1 gives a spectral matrix and an expansion result for the discrete equation (difference equation) (3.3). Such a result for equation (3.3) was obtained earlier by Berezanskii in [6].

Appendix A. PRELIMINARIES ON TIME SCALES CALCULUS

In this section, for the convenience of the reader, we introduce and describe, following [7], [8], some basic concepts concerning the calculus on time scales. At the end of this section, for easy reference, we present also the classical theorems of Helly.

Any time scale $\mathbb{T}$, being a closed subset of the real numbers $\mathbb{R}$, is a complete metric space with the metric (distance)

$$d(t,s) = |t-s| \quad \text{for } t,s \in \mathbb{T}. \tag{A.1}$$

Consequently, according to the well-known theory of general metric spaces we have for $\mathbb{T}$ the fundamental concepts such as open balls (intervals), neighborhoods of points, open sets, closed sets, compact sets and so on. In particular, for a given $\delta > 0$, the δ-neighborhood $U_\delta(t)$ of a given point $t \in \mathbb{T}$ is the set of all points $s \in \mathbb{T}$ such that $d(t,s) < \delta$. By a neighborhood of a point $t \in \mathbb{T}$ is meant an arbitrary set in $\mathbb{T}$ containing a δ-neighborhood of the point t. Also we have for functions $f : \mathbb{T} \to \mathbb{C}$ the concepts of limit, continuity, and the properties of continuous functions on general complete metric spaces (note that, in particular, any function $f : \mathbb{Z} \to \mathbb{C}$ is continuous at each point of $\mathbb{Z}$). It turns out that, unlike for functions on general metric spaces, we can introduce and investigate a concept of derivative for functions on time scales. This proves to be possible due to the special structure of the time scale.

In the definition of the derivative the so-called forward and backward jump operators play an important role.

DEFINITION A.1. Let $\mathbb{T}$ be a time scale. We define the *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\} \quad \text{for } t \in \mathbb{T},$$

while the *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{for } t \in \mathbb{T}.$$

In this definition we put in addition $\sigma(\max \mathbb{T}) = \max \mathbb{T}$ if there exists a finite $\max \mathbb{T}$, and $\rho(\min \mathbb{T}) = \min \mathbb{T}$ if there exists a finite $\min \mathbb{T}$.

Obviously both $\sigma(t)$ and $\rho(t)$ are in $\mathbb{T}$ when $t \in \mathbb{T}$. This is because of our assumption that $\mathbb{T}$ is closed subset of $\mathbb{R}$.

Let $t \in \mathbb{T}$. If $\sigma(t) > t$, we say that t is *right-scattered*, while if $\rho(t) < t$ we say that t is *left-scattered*. Also, if $t < \max \mathbb{T}$ and $\sigma(t) = t$, then t is called *right-dense*, if $t > \min \mathbb{T}$ and $\rho(t) = t$, then t is called *left-dense*. Points that are right-dense or left-dense are called *dense* and points that are right-scattered and left scattered at the same time are called *isolated*.

EXAMPLE A.2. If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$. If $\mathbb{T} = h\mathbb{Z}$, then $\sigma(t) = t + h$ and $\rho(t) = t - h$. On the other hand, if $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, then $\sigma(t) = qt$ and $\rho(1) = 1$, $\rho(t) = q^{-1}t$ for $t > 1$.

Let $\mathbb{T}^\kappa$ denote Hilger's truncated ("kappen" = lop off) above set consisting of $\mathbb{T}$ except for a possible left-scattered maximal point. Similarly, $\mathbb{T}_\kappa$ denotes the below truncated set obtained from $\mathbb{T}$ by deleting a possible right-scattered minimal point. Now we consider a function $f : \mathbb{T} \to \mathbb{C}$ and define the so-called delta (or Hilger) derivative of f at a point $t \in \mathbb{T}^\kappa$ and the nabla derivative of f at a point $t \in \mathbb{T}_\kappa$.

DEFINITION A.3. (Delta and Nabla Derivatives) Assume $f : \mathbb{T} \to \mathbb{C}$ is a function and $t \in \mathbb{T}^\kappa$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U (in $\mathbb{T}$) of t such that

$$|f(\sigma(t)) - f(s) - f^\Delta(t)[\sigma(t) - s]| \leqslant \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U. \tag{A.2}$$

We call $f^\Delta(t)$ the *delta* (or *Hilger*) *derivative* of f at t. Similarly, the *nabla derivative* $f^\nabla(t)$ of f at a point $t \in \mathbb{T}_\kappa$ is defined to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U (in $\mathbb{T}$) of t such that

$$|f(\rho(t)) - f(s) - f^\nabla(t)[\rho(t) - s]| \leqslant \varepsilon|\rho(t) - s| \quad \text{for all } s \in U.$$

If $t \in \mathbb{T} \setminus \mathbb{T}^\kappa$, then $f^\Delta(t)$ is not uniquely defined, since for such a point t small neighborhoods U of t consist only of t, and besides, we have $\sigma(t) = t$. Therefore (A.2) holds for an arbitrary number $f^\Delta(t)$. This is why we omit a maximal left-scattered point. Similarly, $f^\nabla(t)$ is not defined uniquely if $t \in \mathbb{T} \setminus \mathbb{T}_\kappa$.

Note that for calculations of the delta and nabla derivatives it is convenient in practice to use their definition in the limit form

$$f^\Delta(t) = \lim_{s \to t} \frac{f(\sigma(t)) - f(s)}{\sigma(t) - s} \quad \text{and} \quad f^\nabla(t) = \lim_{s \to t} \frac{f(\rho(t)) - f(s)}{\rho(t) - s}, \tag{A.3}$$

where the limits are taken in $\mathbb{T}$ with the metric (A.1) and in the first limit we assume $s \neq \sigma(t)$, while in the second limit we assume $s \neq \rho(t)$.

Using (A.3) and taking into account Example A.2 yield the following result.

EXAMPLE A.4. If $\mathbb{T} = \mathbb{R}$, then $f^\Delta(t) = f^\nabla(t) = f'(t)$, the ordinary derivative of f at t. If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$, then

$$f^\Delta(t) = \frac{f(t+h) - f(t)}{h} \quad \text{and} \quad f^\nabla(t) = \frac{f(t) - f(t-h)}{h}.$$

In particular, in the case $\mathbb{T} = \mathbb{Z}$, we have

$$f^{\Delta}(t) = f(t+1) - f(t) \quad \text{and} \quad f^{\nabla}(t) = f(t) - f(t-1).$$

If $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$, then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{and} \quad f^{\nabla}(t) = \frac{f(q^{-1}t) - f(t)}{(q^{-1}-1)t} \quad (\text{for } t > 1).$$

The Δ-integration is defined as the inverse operation to the Δ-differentiation, while the ∇-integration is defined as the inverse to the ∇-differentiation.

DEFINITION A.5. (Delta and Nabla Integrals) A function $F : \mathbb{T} \to \mathbb{C}$ is called a *Δ-antiderivative* of $f : \mathbb{T} \to \mathbb{C}$ provided $F^{\Delta}(t) = f(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. Then we define the *$\Delta$-integral* from a to b of f by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all } a, b \in \mathbb{T}.$$

A function $\Phi : \mathbb{T} \to \mathbb{C}$ is called a *∇-antiderivative* of $f : \mathbb{T} \to \mathbb{C}$ provided $\Phi^{\nabla}(t) = f(t)$ holds for all $t \in \mathbb{T}_{\kappa}$. Then we define the *$\nabla$-integral* from a to b of f by

$$\int_a^b f(t)\nabla t = \Phi(b) - \Phi(a) \quad \text{for all } a, b \in \mathbb{T}.$$

EXAMPLE A.6. Let $f : \mathbb{T} \to \mathbb{C}$ be a function and $a, b \in \mathbb{T}$ with $a < b$. Then we have the following:

(i) If $\mathbb{T} = \mathbb{R}$ and f is continuous, then

$$\int_a^b f(t)\Delta t = \int_a^b f(t)\nabla t = \int_a^b f(t)\mathrm{d}t,$$

where the last integral is the ordinary integral.

(ii) If $\mathbb{T} = h\mathbb{Z}$ $(h > 0)$ and $a = hm$, $b = hn$ with $m < n$, then

$$\int_a^b f(t)\Delta t = h\sum_{k=m}^{n-1} f(hk) \quad \text{and} \quad \int_a^b f(t)\nabla t = h\sum_{k=m+1}^{n} f(hk).$$

In particular, in the case $\mathbb{T} = \mathbb{Z}$, we have

$$\int_a^b f(t)\Delta t = \sum_{k=a}^{b-1} f(k) \quad \text{and} \quad \int_a^b f(t)\nabla t = \sum_{k=a+1}^{b} f(k).$$

(iii) If $\mathbb{T} = q^{\mathbb{N}_0}$ $(q > 1)$ and $a = q^m$, $b = q^n$ with $m < n$, then

$$\int_a^b f(t)\Delta t = (q-1)\sum_{k=m}^{n-1} q^k f(q^k) \quad \text{and} \quad \int_a^b f(t)\nabla t = (1-q^{-1})\sum_{k=m+1}^{n} q^k f(q^k).$$

The following integration by parts formulas on time scales hold:

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t)\mid_a^b - \int_a^b f(\sigma(t))g^{\Delta}(t)\Delta t,$$

$$\int_a^b f^{\nabla}(t)g(t)\nabla t = f(t)g(t)\mid_a^b - \int_a^b f(\rho(t))g^{\nabla}(t)\nabla t.$$

For more general treatment of the delta and nabla integrals on time scales (Riemann and Lebesgue delta and nabla integrals on time scales) see [12] and Chapter 5 in [8].

THEOREM A.7 (see Theorem 2.4 of [13]). *Let f and g be continuous (in the time scale topology) functions on $[a,b]_{\mathbb{T}}$. Suppose that f is Δ-differentiable on $[a,b)_{\mathbb{T}}$ with continuous and bounded f^{Δ} and g is ∇-differentiable on $(a,b]_{\mathbb{T}}$ with continuous and bounded g^{∇}. Then the following integration by parts formulae hold:*

$$\int_a^b f^{\Delta}(t)g(t)\Delta t = f(t)g(t)\mid_a^b - \int_a^b f(t)g^{\nabla}(t)\nabla t, \tag{A.4}$$

$$\int_a^b g^{\nabla}(t)f(t)\nabla t = g(t)f(t)\mid_a^b - \int_a^b g(t)f^{\Delta}(t)\Delta t. \tag{A.5}$$

Now following [4] we present some facts about the second-order linear dynamic equations on time scales.

Let $\mathbb{T}$ be a time scale. Consider the following second-order linear homogeneous dynamic equation

$$-[p(t)y^{\Delta}(t)]^{\nabla} + q(t)y(t) = 0, \quad t \in \mathbb{T}^* = \mathbb{T}^{\kappa} \cap \mathbb{T}_{\kappa}, \tag{A.6}$$

where $q : \mathbb{T} \to \mathbb{C}$ is a piecewise continuous function, $p : \mathbb{T} \to \mathbb{C}$ is ∇-differentiable on $\mathbb{T}_{\kappa}$, $p(t) \neq 0$ for all $t \in \mathbb{T}$, and $p^{\nabla} : \mathbb{T}_{\kappa} \to \mathbb{C}$ is piecewise continuous.

We define the *quasi Δ-derivative* $y^{[\Delta]}(t)$ of y at t by

$$y^{[\Delta]}(t) = p(t)y^{\Delta}(t).$$

For any point $t_0 \in \mathbb{T}^*$ and any complex constants c_0, c_1, equation (A.6) has a unique solution y satisfying the initial conditions

$$y(t_0) = c_0, \quad y^{[\Delta]}(t_0) = c_1.$$

If $y_1, y_2 : \mathbb{T} \to \mathbb{C}$ are two Δ-differentiable on $\mathbb{T}^{\kappa}$ functions, then their Wronskian is defined for $t \in \mathbb{T}^k$ by

$$W_t(y_1,y_2) = y_1(t)y_2^{[\Delta]}(t) - y_1^{[\Delta]}(t)y_2(t) = p(t)[y_1(t)y_2^{\Delta}(t) - y_1^{\Delta}(t)y_2(t)].$$

The Wronskian of any two solutions of equation (A.6) is independent of t. Two solutions of equation (A.6) are linearly independent if and only if their

Wronskian is nonzero. Equation (A.6) has two linearly independent solutions and every solution of equation (A.6) is a linear combination of these solutions.

Finally, let us present two classical theorems of Helly. A sequence $(h_n)_{n\in\mathbb{N}}$ of functions h_n defined on a set S is called *uniformly bounded* on S if there is a finite constant $C > 0$ such that

$$|h_n(\lambda)| \leqslant C \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda \in S.$$

THEOREM A.8 (Helly's Selection Theorem). *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of functions defined on a finite interval $\alpha \leqslant \lambda \leqslant \beta$ and satisfying the conditions*

$$\sup_{\alpha\leqslant\lambda\leqslant\beta} |h_n(\lambda)| \leqslant C, \quad V_\alpha^\beta(h_n) \leqslant M$$

for suitable constants C and M, so that the functions h_n are uniformly bounded and have bounded total variations on the interval $[\alpha,\beta]$. Then there exists a subsequence $(h_{n_k})_{k\in\mathbb{N}}$ and a function h of finite variation on $[\alpha,\beta]$ such that

$$\lim_{k\to\infty} h_{n_k}(\lambda) = h(\lambda), \quad \alpha \leqslant \lambda \leqslant \beta.$$

The following theorem gives conditions under which we can take the limit of a sequence $(h_n)_{n\in\mathbb{N}}$ of functions inside a Stieltjes integral.

THEOREM A.9 (Helly's Convergence Theorem for Integrals). *Let $(h_n)_{n\in\mathbb{N}}$ be a sequence of functions of finite variation on $[\alpha,\beta]$, converging to a function h at every point of $[\alpha,\beta]$. Suppose the sequence of total variations $\{V_\alpha^\beta(h_n)\}$ is bounded, so that*

$$V_\alpha^\beta(h_n) \leqslant M \quad (n = 1, 2, 3, \ldots)$$

for some constant $M > 0$. Then h is also of finite variation on $[\alpha,\beta]$ and

$$\lim_{n\to\infty} \int_\alpha^\beta f(\lambda)\mathrm{d}h_n(\lambda) = \int_\alpha^\beta f(\lambda)\mathrm{d}h(\lambda)$$

for every function f continuous on $[\alpha,\beta]$.

Proofs of the Helly theorems can be found in Section 36.5 of [19].

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