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MODULE WEAK BANACH–SAKS AND MODULE SCHUR PROPERTIES OF HILBERT C^*-MODULES

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ABSTRACT. Continuing research on Banach–Saks and Schur properties started by C.-H. Chu, M. Kusuda, and the authors, we investigate analogous properties in the Banach C^*-module context. As an environment serves the class of Hilbert C^*-modules. Some properties of weak module topologies on Hilbert C^*-modules are described. Natural module analogues of the classical weak Banach–Saks and Schur properties are defined and studied. A number of useful characterizations of properties of Hilbert C^*-modules is obtained. In particular, some interrelations of these properties with the self-duality property of countably generated Hilbert C^*-modules are established.

KEYWORDS: *C^*-algebras, Hilbert C^*-modules, module Banach–Saks properties, module Schur property, self-duality.*

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1. INTRODUCTION

Originally, the Banach–Saks property was introduced and studied for the particular case of $L^p([0,1])$ with $1 < p < \infty$ by S. Banach and S. Saks in [1]. In the sequel the Banach–Saks property has been considered for various classes of Banach spaces (see, for example, [3], [6], [11], [14], [16], [24]).

In the present paper we extend the considerations of [16], [11] on the Banach–Saks and Schur properties from the situation of Banach spaces to the situation of certain Banach C^*-modules. For Hilbert C^*-modules we introduce some natural module analogues of the weak Banach–Saks and Schur properties and study their potential. The main idea of these constructions is to replace the weak topology of Banach spaces by a weak topology of Hilbert C^*-modules. But there are at least two suitable candidates for such a replacement — the weak topology generated by the inner product and the weak topology generated by all C^*-linear bounded functionals on the Hilbert C^*-module. We considered both of them, but mostly we are interested in the first variant.

In the second section we briefly recall some facts of the general Hilbert C^*-module theory. In the third section we obtain some properties of the mentioned weak module topologies (see Definition 3.1).

The forth section is dedicated to the study of the module Schur property (see Definition 4.2). In particular, we prove that a C^*-algebra with a strictly positive element (considered as a Hilbert module over itself) has the module Schur property if and only if it is unital (Theorem 4.6), and this requirement concerning strictly positive elements is essential (Example 4.7). Finitely generated projective modules over σ-unital C^*-algebras with Schur property have the module Schur property (Corollary 4.10), whereas standard Hilbert modules $l_2(A)$ do not have the module Schur property for any C^*-algebras A (Proposition 4.11).

In the fifth section we consider two possible candidates for the definition of the module Banach–Saks property (see properties (mBS1) and (mBS2) in the beginning of the fifth section). But it occurs, the first of them is fulfilled for all Hilbert C^*-modules (Theorem 5.3). So we choose the second of them as the definition of the module Banach–Saks property (Definition 5.4). We prove that the module Schur property implies the module Banach–Saks property (Proposition 5.5), that a C^*-algebra with a strictly positive element has the module Banach–Saks property if and only if it is unital (Theorem 5.6), and that finitely generated projective Hilbert modules have the module Banach–Saks property (Corollary 5.8).

In the last section some interrelations between both the module Schur and the module weak Banach–Saks properties and self-duality of Hilbert C^*-modules are considered.

2. PRELIMINARIES

A (right) *pre-Hilbert C^*-module* over a C^*-algebra A (cf. [25]) is a right A-module V equipped with an A-valued inner product $\langle \cdot, \cdot \rangle : V \times V \to A$, which is A-linear in the second variable, fulfils $\langle x, y \rangle = \langle y, x \rangle^*$; is *positive*, i.e. $\langle x, x \rangle \geqslant 0$, and is *definite*, i.e. $\langle x, x \rangle = 0$ if and only if $x = 0$. A pre-Hilbert A-module is a *Hilbert A-module* provided it is a Banach space with respect to the norm $\|x\| = \|\langle x, x \rangle\|^{1/2}$.

For the convenience of the reader we also give a sketch of the dual module theory that will be essentially used in the sequel. Suppose V denotes a Hilbert A-module again. Let $V' = \mathrm{Hom}_A(V, A)$ be the set of all A-linear bounded maps from V to A, the *dual* (left Banach) *module* for V. Apparently, there is an isometric module embedding

$$\wedge : V \to V', \quad x^\wedge(\cdot) = \langle x, \cdot \rangle. \tag{2.1}$$

A Hilbert A-module is said to be *self-dual* if this map is surjective. Generally speaking, V' is far not always a Hilbert C^*-module provided A is just an arbitrary C^*-algebra. The inner product can be extended from V to V' (and even in such way that the dual module will be self-dual) if and only if A is monotone complete (cf. Theorem 4.7 of [8], and [12], [19], although in the more important case of W^*-algebras such an extension was described in [25]). As usual, the *bidual module* V'' should be defined by the second iteration, i.e. $V'' = \mathrm{Hom}_A(V', A)$. For a functional $F \in V''$ one can define a functional $\widetilde{F} \in V'$ by the formula

$$\widetilde{F}(x) = F(\widehat{x}), \quad x \in V .$$

Then the map $F \mapsto \widetilde{F}$ is an A-module isometry from V'' into V', and it allows to extend the inner product from V to V'' in the following way

$$\langle F, G \rangle = F(\widetilde{G}), \quad F, G \in V'' ,$$

where the norm arising from this inner product coincides with the operator norm of V''. Now, let $x \in V$, $f \in V'$ and put

$$\dot{x}(f) = f(x)^*. \tag{2.2}$$

Then the map (2.2) is an isometric module embedding of V into V'' (see [21], [26] for details). In opposition to the situation of Banach spaces, the third dual V''' for V is isomorphic to V', whereas the Banach modules V, V' and V'' may be pairwise non-isomorphic in particular situations [26].

More information about Hilbert C^*-modules can be found in [17], [20], [21], [30], [31], for example.

3. WEAK MODULE TOPOLOGIES ON HILBERT C^*-MODULES

We are going to establish some properties of the topologies $\tau_{\widehat{V}}$ and $\tau_{V'}$ for arbitrary Hilbert C^*-modules which are defined in the sequel. In particular, we demonstrate that whenever both these topologies coincide on a certain countably generated Hilbert C^*-module over a unital C^*-algebra then this module has to be self-dual.

DEFINITION 3.1. Let $(V, \langle \cdot, \cdot \rangle)$ be a pre-Hilbert module over a C^*-algebra A, and let V' be its dual module. The V'-weak module topology $\tau_{V'}$ on V is generated by the family of semi-norms

$$\{\nu_f\}_{f \in V'}, \quad \text{where } \nu_f(x) = \|f(x)\|,\ x \in V , \tag{3.1}$$

and the $\widehat{V}$-weak module topology $\tau_{\widehat{V}}$ on V is generated by the family of semi-norms

$$\{\mu_z\}_{z \in V}, \quad \text{where } \mu_z(x) = \|\langle z, x \rangle\|,\ x \in V. \tag{3.2}$$

Obviously, $\tau_{\widehat{V}}$ is not stronger than $\tau_{V'}$, and these topologies coincide whenever V is self-dual. What about the opposite interrelation? Do they always coincide? The next example demonstrates that for arbitrary V the topologies $\tau_{\widehat{V}}$ and $\tau_{V'}$ are distinct, in general.

EXAMPLE 3.2. Let (X, ρ) be a locally compact metric space, $A = C_0(X)$ be the corresponding C^*-algebra and $V = A$ be the Hilbert A-module under consideration. Let $\widetilde{X} = X \cup \{\widetilde{x}\}$ denote the one-point compactification of X. By the commutativity of A both the two-sided and the left multiplier algebras $M(A)$ and $LM(A)$ coincide. Moreover, the A-dual Banach A-module V' of V can be identified with the Banach space $\mathrm{End}_A(A)$ of A-linear bounded operators from A to itself which is isometrically isomorphic to the left multiplier algebra $LM(A)$ of A. So V' exactly coincides with the algebra $C_{\mathrm{b}}(X)$ of all bounded continuous functions on X, and $V' = C_{\mathrm{b}}(X)$ can be considered to act by left multiplication on $V = C_0(X)$ (see [23], [31]). Consider a sequence $\{x_i\}$ in X which converges to $\widetilde{x}$ with respect to the topology of $\widetilde{X}$. For this sequence choose a sequence of open neighborhoods

$$O_{\delta_i}(x_i) = \{y \in X : \rho(y, x_i) < \delta_i\},$$

under the additional supposition that the sequence of positive numbers $\{\delta_i\}$ has to be a null sequence. Let $f_i \in C_0(X)$ be elements such that $f_i(x_i) = 1$, $f_i = 0$ outside of $O_{\delta_i}(x_i)$ and $0 \leqslant f_i(x) \leqslant 1$ for any $x \in X$. Then for any $g \in C_0(X)$ one has

$$\begin{aligned} \lim_{i\to\infty} \|gf_i\| &= \lim_{i\to\infty} \sup_{x\in X} |g(x)f_i(x)| = \lim_{i\to\infty} \sup_{x\in O_{\delta_i}(x_i)} |g(x)f_i(x)| \\ &\leqslant \lim_{i\to\infty} \sup_{x\in O_{\delta_i}(x_i)} |g(x)| = 0 \end{aligned}$$

what means $\{f_i\}$ is a null sequence with respect to the $\tau_{\widehat{V}}$-topology. On the other hand consider the constant function h on X, which always equals to 1. Then h belongs to $C_{\mathrm{b}}(X) = V'$ and

$$\|hf_i\| = \sup_{x\in X} |f_i(x)| = 1$$

for any $i \in \mathbb{N}$. Consequently, $\{f_i\}$ does not converge to zero with respect to the topology $\tau_{V'}$.

PROPOSITION 3.3. *Let $\{x_n\}$ be a $\tau_{\widehat{V}}$-converging sequence of a Hilbert C^*-module V. Then the set $\{x_n\}$ is bounded in norm.*

Proof. Consider the sets

$$A_{k,n} = \{y \in V : \|\langle y, x_n\rangle\| \leqslant k\}$$

and $A_k = \bigcap_n A_{k,n}$. These sets are closed in norm, because the function $y \mapsto \|\langle y, x_n\rangle\|$ is continuous for any fixed x_n. The $\tau_{\widehat{V}}$-convergence of $\{x_n\}$ implies that $V = \bigcup_k A_k$. By the Baire's theorem there are a number k_0 and a ball $B(y, \varepsilon) = \{z \in V : \|y - z\| < \varepsilon\}$ such that $B(y, \varepsilon) \subset A_{k_0}$. Thus the sequence $\{x_n\}$, understood as a subset of V', is bounded on the ball $B(y, \varepsilon)$ and, consequently, on any ball of V, in particular on $B(0,1)$. That exactly means

$$\begin{aligned} C \geqslant \sup_{\|y\|\leqslant 1} \|\langle y, x_n\rangle\| &= \sup_{\|y\|\leqslant 1} \|\widehat{y}(x_n)\| \\ = \sup_{\|y\|\leqslant 1} \|\dot{x}_n(\widehat{y})\| &= \sup_{\|\widehat{y}\|\leqslant 1} \|\dot{x}_n(\widehat{y})\| = \|\dot{x}_n\| = \|x_n\| \end{aligned}$$

for some constant C and for all $n \in \mathbb{N}$, where the properties of the maps (2.1) and (2.2) have been applied. ∎

PROPOSITION 3.4. *Let $\{x_n\}$ be a $\tau_{V'}$-converging sequence of a pre-Hilbert C^*-module V. Then the set $\{x_n\}$ is bounded in norm.*

Proof. We should only remark that V' is a Banach module even if V is just a pre-Hilbert one. Further arguments are similar to those applied in the proof of Proposition 3.3. ∎

Let us remind that a subset M of a topological vector space X is said to be *bounded* if for any open neighborhood U of zero in X there is real number $s > 0$ such that $M \subset tU$ for any $t > s$.

PROPOSITION 3.5. *The following conditions for a subset Q of a Hilbert C^*-module V are equivalent:*

(i) *Q is bounded with respect to the topology $\tau_{\widehat{V}}$;*

(ii) *Q is norm-bounded.*

Proof. Because the norm topology is stronger than the topology $\tau_{\widehat{V}}$ one has only to check the implication (i) $\Rightarrow$ (ii).

Let Q be bounded with respect to the topology $\tau_{\widehat{V}}$. Let $\{x_n\} \in Q$ be a $\tau_{\widehat{V}}$-bounded sequence. So, for any $y \in V$ one can find a real number $t > 0$ satisfying

$$\{x_n\} \subset t\{x : \|\langle y, x\rangle\| < 1\}\,.$$

Consequently,

$$\|\langle y, x_n\rangle\| < t \quad \text{for all } n \in \mathbb{N}.$$

In particular, the sets $\{\|\langle y, x_n\rangle\|\}_{n=1}^{\infty}$ are bounded for any fixed $y \in V$. Consider the sets

$$A_{k,n} = \{y \in V : \|\langle y, x_n\rangle\| \leqslant k\}$$

and $A_k = \bigcap_n A_{k,n}$. These sets are closed in norm, because the function $y \mapsto \|\langle y, x_n\rangle\|$ is continuous for any fixed x_n. Moreover, $V = \bigcup_k A_k$. By the Baer's theorem there are a number k_0 and a ball $B(y,\varepsilon) = \{z \in V : \|y - z\| < \varepsilon\}$ such that $B(y,\varepsilon) \subset A_{k_0}$. Therefore, the sequence $\{x_n\}$ is module weakly bounded on the ball $B(y,\varepsilon)$ and, as a consequence, on any ball of this type. Thus the sequence

$\{x_n\}$, understood as a subset of V', is bounded on the ball $B(y,\varepsilon)$ and, consequently, on any ball of V, in particular on $B(0,1)$. That exactly means

$$\begin{aligned} C &\geqslant \sup_{\|y\|\leqslant 1} \|\langle y, x_n\rangle\| = \sup_{\|y\|\leqslant 1} \|\widehat{y}(x_n)\| \\ &= \sup_{\|y\|\leqslant 1} \|\dot{x}_n(\widehat{y})\| = \sup_{\|\widehat{y}\|\leqslant 1} \|\dot{x}_n(\widehat{y})\| = \|\dot{x}_n\| = \|x_n\| \end{aligned}$$

for some constant C and for all $n \in \mathbb{N}$, where the properties of the maps (2.1) and (2.2) have been applied. So the sequence $\{\|x_n\|\}$ turns out to be bounded. ∎

PROPOSITION 3.6. *The following conditions for a subset Q of a pre-Hilbert C^*-module V are equivalent:*

(i) *Q is bounded with respect to the topology $\tau_{V'}$;*

(ii) *Q is norm-bounded.*

Proof. The proof follows the arguments of the proofs of Propositions 3.3 and 3.5 with obvious changes in details. ∎

THEOREM 3.7. *Let A be a unital C^*-algebra, $V = l_2(A)$ be the standard countably generated Hilbert A-module. Suppose, the topologies $\tau_{\widehat{V}}$ and $\tau_{V'}$ coincide on V. Then V is self-dual and A has to be finite-dimensional.*

Proof. Consider a module functional $\beta \in V'$. By Proposition 2.5.5 of [21] there are elements b_i from A such that $\beta(x) = \sum_{i=1}^{\infty} b_i^* a_i$, where $x = (a_i) \in V$ and

$$\Big\| \sum_{i=1}^{N} b_i^* b_i \Big\| \leqslant C$$

for some constant C depending only on β and for all integers N. By the suppositions of the theorem there are vectors $y_1, \ldots, y_n$ of V and a constant $K > 0$ such that

$$\nu_\beta(x) \leqslant K \cdot \max\{\mu_{y_1}(x), \ldots, \mu_{y_n}(x)\} \tag{3.3}$$

for any $x \in V$, where we have used the notations (3.1) and (3.2). Let us fix the notations $y_i = (y_j^{(i)})$, $x = (a_i)$, where $y_j^{(i)}, a_i \in A$ and rewrite the inequality (3.3) in the form

$$\Big\| \sum_{i=1}^{\infty} b_i^* a_i \Big\| \leqslant K \cdot \max\left\{ \Big\| \sum_{j=1}^{\infty} \left(y_j^{(1)}\right)^* a_j \Big\|, \ldots, \Big\| \sum_{j=1}^{\infty} \left(y_j^{(n)}\right)^* a_j \Big\| \right\}. \tag{3.4}$$

For arbitrary positive integers k and N set $a_i = b_i$ if $k \leqslant i \leqslant k+N$, and $a_i = 0$ otherwise. Then the formula (3.4) gives

$$\begin{aligned}\Big\| \sum_{i=k}^{k+N} b_i^* b_i \Big\| &\leqslant K \cdot \max \Big\{ \Big\| \sum_{j=k}^{k+N} \big(y_j^{(1)}\big)^* b_j \Big\|, \ldots, \Big\| \sum_{j=k}^{k+N} \big(y_j^{(n)}\big)^* b_j \Big\| \Big\} \\ &\leqslant K \cdot \max \Big\{ \Big\| \sum_{j=k}^{k+N} \big(y_j^{(1)}\big)^* y_j^{(1)} \Big\|^{1/2} \Big\| \sum_{j=k}^{k+N} b_j^* b_j \Big\|^{1/2}, \ldots, \\ &\qquad \Big\| \sum_{j=k}^{k+N} \big(y_j^{(n)}\big)^* y_j^{(n)} \Big\|^{1/2} \Big\| \sum_{j=k}^{k+N} b_j^* b_j \Big\|^{1/2} \Big\} \\ &\leqslant C^{1/2} \cdot \max \Big\{ \Big\| \sum_{j=k}^{k+N} \big(y_j^{(1)}\big)^* y_j^{(1)} \Big\|^{1/2}, \ldots, \Big\| \sum_{j=k}^{k+N} \big(y_j^{(n)}\big)^* y_j^{(n)} \Big\|^{1/2} \Big\}.\end{aligned}$$

Therefore, $\Big\{ \sum\limits_{i=1}^{N} b_i^* b_i \Big\}_{N=1}^{\infty}$ is a Cauchy sequence with respect to the uniform topology ensuring the vector (b_i) to belong to $V = l_2(A)$. Thus, V is self-dual and, by Theorem 4.3 of [7], A is finite-dimensional. ∎

COROLLARY 3.8. *Let A be a unital C^*-algebra. Let V be a countably generated Hilbert A-module, for which the topologies $\tau_{\widehat{V}}$ and $\tau_{V'}$ coincide. Then V is self-dual.*

The fact follows from Theorem 3.7 and Kasparov's stabilization theorem ([13], Theorem 2) directly.

Summing up, we have shown that for countably generated Hilbert C^*-modules over unital C^*-algebras the coincidence of both the toplogies $\tau_{\widehat{V}}$ and $\tau_{V'}$ on the Hilbert C^*-module forces self-duality of the Hilbert C^*-module, and vice versa.

4. THE MODULE SCHUR PROPERTY

In the following section we introduce a module analogue of the Schur property of Banach spaces. Recall that a Banach space X has the *Schur property* if every weak convergent sequence in X converges in norm. Let Y be a closed subspace of X. Then X has the Schur property if and only if both Y and the quotient space X/Y have the same property (cf. [2], [16]).

EXAMPLE 4.1. The Banach space $L^1[0,1]$ has the Schur property. Indeed, let $\{f_i\}$ be a weakly convergent sequence of $L^1[0,1]$. Then, for any functional α of $L^\infty[0,1] = L^1[0,1]'$ the sequence

$$\alpha(f_i) = \int_0^1 \alpha(x) f_i(x) \mathrm{d}x$$

converges. This, obviously, implies the sequence $\{f_i\}$ converges with respect to the L^1-norm

$$\|f\| = \int_0^1 |f(x)| \mathrm{d}x.$$

For classification results with respect to the classical Schur property see [16], [11].

DEFINITION 4.2. A Hilbert C^*-module V has the *module Schur property* if every $\tau_{\widehat{V}}$-convergent sequence in V converges in norm.

LEMMA 4.3. *Any $\tau_{\widehat{V}}$-convergent sequence $\{x_n\}$ of a Hilbert C^*-module V with a subsequence $\{x_{n(l)}\}$ which converges in norm to some element $x \in V$ admits x as its $\tau_{\widehat{V}}$-limit. Consequently, Hilbert C^*-modules with the module Schur property are sequentially $\tau_{\widehat{V}}$-complete.*

Proof. For any $z \in V$ and any $\varepsilon > 0$ there exists a number $N \in \mathbb{N}$ such that

$$\|\langle z, x_n - x_m \rangle\| < \varepsilon$$

for any $m, n > N$. So, there exists a number $L \in \mathbb{N}$ such that $n(l) > N$ for any $l > L$. The inequality

$$\|\langle z, x - x_m \rangle\| = \lim_{l \to \infty} \|\langle z, x_{n(l)} - x_m \rangle\| \leqslant \varepsilon$$

holds for any $m > N$, and $x \in V$ turns out to be the $\tau_{\widehat{V}}$-limit of the sequence $\{x_n\}$. ∎

It is known that a C^*-algebra has the Schur property if and only if it is finite dimensional (cf. Lemma 3.8 of [16]). Our next goal is to prove that a C^*-algebra with a strictly positive element possesses the module Schur property if and only if it is unital. The conclusion that a unital C^*-algebra has the module Schur property is obvious. But, before investigating the converse statement we will consider a couple of examples as a motivation.

EXAMPLE 4.4. Let $A = V = C_0(0, 1]$ and let us define a function $f_n \in A$ in the following way: f equals to zero at 0 and on the interval $[\frac{1}{n}, 1]$, f equals to 1 at the point $\frac{1}{2n}$, and f is linear on both the intervals $[0, \frac{1}{2n}]$ and $[\frac{1}{2n}, \frac{1}{n}]$. Obviously, the sequence $\{f_n\}$ converges to zero with respect to the topology $\tau_{\widehat{V}}$, but it does not converge in norm. So the C^*-algebra $C_0(0, 1]$ does not have the module Schur property.

EXAMPLE 4.5. Let $A = V = K(H)$ be the C^*-algebra of compact operators in a separable Hilbert space H and $B(H)$ be the set of all bounded linear operators on H. Consider a sequence $\{a_n\} \in A$ and assume that the sequence $\{ka_n\}$ converges in norm for any $k \in K(H)$. Then its limit with respect to the right strict topology belongs to the closure of $K(H)$ in $B(H)$, and this closure coincides with $B(H)$ (see, for instance, [18], [27]). However, the norm limit of $\{a_n\}$, whenever it exists, has to belong to $K(H)$. So any sequence from $K(H)$, whose right strict limit lies outside $K(H)$, $\tau_{\widehat{V}}$-converges, but does not converge in norm. Therefore, $K(H)$ does not have the module Schur property, too.

THEOREM 4.6. *A C^*-algebra with a strictly positive element, in particular, a separable C^*-algebra (cf.* 1.4.3 *of* [29]*), has the module Schur property if and only if it is unital. In particular, the Schur property for C^*-algebras is strictly stronger than the module Schur property.*

Proof. Suppose, a non-unital C^*-algebra A has a strictly positive element. That condition on A is equivalent to the existence of a countable approximative unit in A ([29], Proposition 3.10.5). For such A any element of its right multiplier algebra $RM(A)$ may be obtained just as a limit of a sequence of A converging with respect to the right strict topology, i.e. there is not any necessity to consider nets under these assumptions (cf. Section 2.3 of [31], and Section 5.5 of [21]). Now, to check that A does not have the module Schur property we will reason similarly as at Example 4.5. Let a sequence $\{a_n\}$ of A be $\tau_{\widehat{V}}$-convergent (here we identify $V = A$ again), i.e. the sequence $\{b^* a_n\}$ converges in norm for any $b \in A$. Therefore, the $\tau_{\widehat{V}}$-limit of $\{a_n\}$ belongs to the right strict closure of A inside of its bidual Banach space A^{**}, i.e. it belongs to the right multipliers $RM(A)$ of A. Because the algebra $RM(A)$ is strictly larger than A whenever A is not unital, we can find a sequence from A that converges with respect to the topology $\tau_{\widehat{V}}$ (for instance, to the identity of $RM(A)$), but does not converge in norm. Thus A does not have the module Schur property either.

Since unital C^*-algebras have the Schur property if and only if they are finite-dimensional, the module Schur property is weaker than the Schur property for C^*-algebras. ∎

The next example demonstrates the requirement of Theorem 4.6 to a C^*-algebra to have a strictly positive element to be essential.

EXAMPLE 4.7. Let H be a non-separable Hilbert space, $A = B(H)$ be the C^*-algebra of all linear bounded operators in H and A_0 denotes the set of operators $T \in B(H)$ such that both $\mathrm{Ker}(T)^\perp$ and $\overline{\mathrm{Range}(T)}$ are separable (closed) subspaces of H. Then A_0 is an involutive subalgebra of A, containing all compact operators on H. Moreover, A_0 is closed in norm. Indeed, let a sequence $\{T_n\}$ of A_0 converges in norm to a certain element T of $B(H)$. Consider (closed) separable subspaces

$$H_1 = \overline{\mathrm{span}}\Big\{ \bigcup \mathrm{Ker}(T_n)^\perp \Big\}, \quad H_2 = \overline{\mathrm{span}}\Big\{ \bigcup \overline{\mathrm{Range}(T_n)} \Big\} \tag{4.1}$$

of H, generated by the unions $\bigcup \mathrm{Ker}(T_n)^\perp$ and $\bigcup \overline{\mathrm{Range}(T_n)}$ respectively. Then, obviously, $H_1^\perp$ coincides with the intersection $\bigcap \mathrm{Ker}(T_n)$, so $H_1^\perp$ is included in $\mathrm{Ker}(T)$. This implies that $\mathrm{Ker}(T)^\perp$ is a subspace of H_1 and, consequently, is separable. On the other hand, $\overline{\mathrm{Range}(T)}$ belongs to the closure of the union

$\bigcup \overline{\mathrm{Range}(T_n)}$. Therefore, both $\mathrm{Ker}(T)^\perp$ and $\overline{\mathrm{Range}(T)}$ are separable subspaces of H and $T \in A_0$. Clearly, the C^*-algebra A_0 is not unital, but we claim that it is not even σ-unital. To verify this, let us suppose the opposite assertion to be true, and there is a countable approximate identity $\{T_n\}$ in A_0. Let H_1 be defined by (4.1). Then there is a non-zero vector $x \in H$ such that it is orthogonal to H_1. Let $p \in B(H)$ be the orthogonal projection onto the span of x. Then, actually, p belongs to A_0 and the sequence pT_n does not converge in norm to p. So we have a contradiction and, consequently, A_0 does not have any countable approximative identity. Nevertheless, we claim that $V = A_0$, considered as a Hilbert A_0-module, satisfies the module Schur property. Indeed, assume $\{T_n\}$ is a $\tau_{\widehat{V}}$-convergent sequence of V. Then both spaces H_1 and H_2, defined by (4.1), are separable. Let $u \in B(H)$ be a minimal partial isometry between H_1 and H_2. Then, in fact, u belongs to A_0, where $u|_{H_1^\perp} = 0$ and $u^*|_{H_2^\perp} = 0$. Thus the sequence $\{uu^*T_n\}$ converges in norm and coincides element-wise with the sequence $\{T_n\}$.

PROPOSITION 4.8. *Let a Hilbert C^*-module V be represented as a direct orthogonal sum $V = V_1 \oplus V_2$ of two Hilbert C^*-submodules. Then the following conditions are equivalent:*

(i) *V has the module Schur property.*
(ii) *Both V_1 and V_2 have the module Schur property.*

Proof. Obviously, (i) implies (ii) and we have just to verify the inverse assertion. So consider any $\tau_{\widehat{V}}$-convergent sequence $\{x_i\}$ of V. Then $x_i = x_i^{(1)} \oplus x_i^{(2)}$ with $x_i^{(1)} \in V_1$, $x_i^{(2)} \in V_2$. Obviously, the sequences $\{x_i^{(j)}\}$ are $\tau_{\widehat{V_j}}$-convergent sequences ($j = 1, 2$), and therefore, they converge with respect to the norm of V_j ($j = 1, 2$). Consequently,

$$\|x_n - x_m\| \leqslant \|x_n^{(1)} - x_m^{(1)}\| + \|x_n^{(2)} - x_m^{(2)}\|$$

for any n, m. Thus, $\{x_i\}$ is a Cauchy sequence with respect to the norm topology of V. ∎

COROLLARY 4.9. *Finitely generated Hilbert modules over unital C^*-algebras A have the module Schur property. In particular, all the Hilbert A-modules A^n ($n \in \mathbb{N}$) have the module Schur property.*

Proof. For the Hilbert A-modules A^n our statement follows from Proposition 4.8 and Theorem 4.6.

By Kasparov's stabilization theorem ([13], Theorem 2) any finitely generated module V is an direct orthogonal summand of the standard module $l_2(A)$. Therefore, by Theorem 1.3 of [22] it has to be projective whenever A is unital. So, V is a direct orthogonal summand of some Hilbert A-module A^n, $n < \infty$, and by Proposition 4.8 it has the module Schur property. ∎

COROLLARY 4.10. *Let A be C^*-algebra with a strictly positive element and with the module Schur property. Then finitely generated projective Hilbert modules over A have the module Schur property.*

The assertion follows from Theorem 4.6 and Proposition 4.8.

PROPOSITION 4.11. *The standard Hilbert module $V = l_2(A)$ does not have the module Schur property for any C^*-algebra A.*

Proof. For a unital C^*-algebra the statement is clear, because the standard basis $\{e_i\}$ (with all entries of e_i equal to zero except the i-th, which equals 1_A) $\tau_{\widehat{V}}$-converges to zero, but does not converge in norm.

For an arbitrary C^*-algebra A we can reason in the following way: let us fix an element $a \in A$ of norm one and consider the sequence $\{x_k\}$ from $l_2(A)$, where all entries of x_k are zero except the k-th entry that equals to a. Then for any $y = (b_i)$ from $l_2(A)$ one has

$$\|\langle y, x_k\rangle\| = \|b_k^* a\| \leqslant \|b_k\|$$

for each k, so $\{x_k\}$ converges to zero with respect to the topology $\tau_{\widehat{V}}$. On the other hand, this sequence does not converge in norm. ∎

Let us consider one example of a countably, but not finitely generated and non-standard Hilbert C^*-module without the module Schur property.

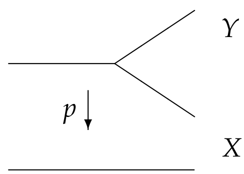

FIGURE 1. Example 4.12

EXAMPLE 4.12. Consider the projection map $p : Y \to X$ from Figure 1, where X is an interval, say $[-1, 1]$, and Y is the topological union of one interval with two copies of another half-interval with a branch point at 0. Then $C(Y)$ is a Banach $C(X)$-module with respect to the actions

$$(f\xi)(y) = f(y)\xi(p(y)), \quad f \in C(Y), \xi \in C(X).$$

Define the $C(X)$-valued inner product on $C(Y)$ by the formula

$$\langle f, g\rangle(x) = \frac{1}{\#p^{-1}(x)} \sum_{y \in p^{-1}(x)} \overline{f(y)} g(y), \tag{4.2}$$

where $\#p^{-1}(x)$ is the cardinality of $p^{-1}(x)$. It was shown in [28] that $C(Y)$ is a countably, but not finitely generated Hilbert $C(X)$-module with respect to the inner product (4.2). Now, for a point $x \in [0, 1]$ let us denote its pre-image $p^{-1}(x)$

intersected with the upper line of Y by $y_x^{(1)}$, and the intersection of $p^{-1}(x)$ with the lower line of Y by $y_x^{(2)}$. Consider functions $h_n \in C(X)$ which equal to zero to the left of the point $\frac{1}{2n}$ and to the right of the point $\frac{1}{n}$, equal to 1 at the point $\frac{3}{4n}$, and are linear on both the intervals $[\frac{1}{2n}, \frac{3}{4n}]$ and $[\frac{3}{4n}, \frac{1}{n}]$. We define a sequence $\{f_n\} \in C(Y)$ in the following way: $f_n = 0$ on $p^{-1}([-1,0])$, $f_n(y_x^{(1)}) = h_n(x)$ and $f_n(y_x^{(2)}) = -h_n(x)$ for $x \in [0,1]$. Then for any $g \in C(Y)$ one has

$$\begin{aligned}\|\langle g, f_n\rangle\| &= \max_{x\in[\frac{1}{2n},\frac{1}{n}]} \frac{1}{2}|\overline{g(y_x^{(1)})}f_n(y_x^{(1)}) + \overline{g(y_x^{(2)})}f_n(y_x^{(2)})| \\ &= \max_{x\in[\frac{1}{2n},\frac{1}{n}]} \frac{1}{2}|g(y_x^{(1)}) - g(y_x^{(2)})|h_n(x) \leqslant \max_{x\in[\frac{1}{2n},\frac{1}{n}]} \frac{1}{2}|g(y_x^{(1)}) - g(y_x^{(2)})|,\end{aligned}$$

and the latter sequence converges to zero if n goes to infinity. But, on the other hand, the sequence $\{f_n\}$ does not converge in norm. Thus $C(Y)$ does not have the module Schur property.

5. MODULE WEAK BANACH–SAKS PROPERTIES

In this section we search for module analogues for the different kinds of Banach–Saks properties for Banach spaces. As most promising we select a certain generalization of the weak Banach–Saks property.

Let us start with a short review of two Banach–Saks type properties for the classical situation. A Banach space X has the *Banach–Saks property* if from any bounded sequence $\{x_n\}$ of X there may be extracted a subsequence $\{x_{n(k)}\}$ such that

$$\lim_{k\to\infty} \left\| \frac{1}{k} \sum_{i=1}^{k} x_{n(i)} - x \right\| = 0 \tag{5.1}$$

for some element $x \in X$.

A Banach space is reflexive whenever it has the Banach–Saks property [4]. A C^*-algebra has the Banach–Saks property if and only if it is finite-dimensional [16], [3].

In functional analysis a weaker type of the Banach–Saks property gives very useful results for larger classes of Banach spaces. More precisely, a Banach space X has the *weak Banach–Saks property* if for any sequence $\{x_n\}$ of X which converges weakly to zero there may be selected a subsequence $\{x_{n(k)}\}$ such that the equality (5.1) is fulfilled with $x = 0$. We are interested in certain module analogues of the weak Banach–Saks property in this section. To give definitions, assume V to be a Hilbert C^*-module. Consider the following conditions on V:

(mBS1) Any $\tau_{\widehat{V}}$-null sequence $\{x_n\}$ of V admits a subsequence $\{x_{n(i)}\}$ which satisfies the equality (5.1) with $x = 0$.

(mBS2) Any $\tau_{\widehat{V}}$-convergent sequence $\{x_n\}$ of V admits a subsequence $\{x_{n(i)}\}$

such that the sequence $\frac{1}{k}\sum_{i=1}^{k} x_{n(i)}$ converges in norm.

The condition (5.1) with $x = 0$ may be rewritten for Hilbert C^*-modules in the following equivalent way

$$\lim_{k\to\infty}\left\|\frac{1}{k^2}\sum_{i,j=1}^{k}\langle x_{n(i)}, x_{n(j)}\rangle\right\| = 0\,. \tag{5.2}$$

Obviously, any $\tau_{\widehat{V}}$-null orthogonal sequence $\{x_k\}$ of norm one vectors in a Hilbert C^*-module V satisfies the condition (5.2).

LEMMA 5.1. *Condition* (mBS2) *implies condition* (mBS1)*, but not conversely, generally speaking.*

Proof. Clearly, condition (mBS2) implies condition (mBS1) as a particular case. To see the non-equivalence of these conditions consider the C^*-algebra $A = K(l_2)$ of all compact operators on a separable Hilbert space l_2 as a Hilbert C^*-module over itself. Then condition (mBS1) is fulfilled, however condition (mBS2) does not hold: indeed, for any sequence of minimal pairwise orthogonal projections $\{p_i\}$ with common least upper bound $1_{B(l_2)}$, where $1_{B(l_2)}$ denotes the unit of the C^*-algebra $B(l_2)$ of all bounded linear operators in l_2, the sequence of partial sums $\left\{q_k = \sum_{i=1}^{k} p_i\right\}$ is $\tau_{\widehat{V}}$-convergent to $1_{B(l_2)}$. For any subsequence $\{q_{k(j)}\}$ of it the sequence $\frac{1}{j}\sum_{i=1}^{j} q_{k(j)}$ converges to an infinite projection operator on l_2. So the latter sequence cannot converge in norm since infinite projection operators do not belong to A which is already norm-closed. ∎

LEMMA 5.2. *Any $\tau_{\widehat{V}}$-convergent sequence $\{x_n\}$ of a Hilbert C^*-module V with a subsequence $\{x_{n(i)}\}$ such that the sequence $\frac{1}{k}\sum_{i=1}^{k} x_{n(i)}$ converges in norm to some element $x \in V$ admits x as its $\tau_{\widehat{V}}$-limit. Consequently, Hilbert C^*-modules with condition* (mBS2) *are sequentially $\tau_{\widehat{V}}$-complete.*

Proof. Consider any $\tau_{\widehat{V}}$-convergent sequence $\{x_n\}$ of V that admits a subsequence $\{x_{n(i)}\}$ such that the sequence $\frac{1}{k}\sum_{i=1}^{k} x_{n(i)}$ converges in norm to some $x \in V$. Then this sequence has to $\tau_{\widehat{V}}$-converge to the same element x, that means

$$\lim_{l\to\infty}\left\|\left\langle z, \frac{1}{l}\sum_{i=1}^{l}(x_{n(i)} - x)\right\rangle\right\| = 0$$

for any $z \in V$. Therefore, for any $z \in V$, any $\varepsilon > 0$ there exists an $L \in \mathbb{N}$ such that for any $k > l > L$ the inequality

$$\varepsilon > \left\| \left\langle z, \frac{1}{l} \sum_{i=1}^{l} (x_{n(i)} - x) \right\rangle - \left\langle z, \frac{1}{k} \sum_{i=1}^{k} (x_{n(i)} - x) \right\rangle \right\| = \frac{1}{k-l} \left\| \sum_{i=l+1}^{k} \langle z, (x_{n(i)} - x) \rangle \right\|$$

holds. Selecting $k = l + 1$ we arrive at

$$\lim_{l \to \infty} \| \langle z, (x_{n(l)} - x) \rangle \| = 0$$

and the subsequence $\{x_{n(l)}\}$ of the sequence $\{x_n\}$ $\tau_{\widehat{V}}$-converges to $x \in V$. Now, for any $z \in V$ we obtain

$$0 = \lim_{n \to \infty} \lim_{l \to \infty} \| \langle z, (x_n - x_{n(l)}) \rangle \| = \lim_{n \to \infty} \| \langle z, (x_n - x) \rangle \| .$$

Consequently, the sequence $\{x_n\}$ $\tau_{\widehat{V}}$-converges to $x \in V$ by definition, and this is equivalent to the assertion that the sequence $\{x_n - x\}$ $\tau_{\widehat{V}}$-converges to zero. ∎

The study of the property (mBS1) gives a rather surprisingly general result for Hilbert C^*-modules.

THEOREM 5.3. *Any Hilbert C^*-module has the property* (mBS1).

Proof. We will reason in the vein of the original work [1]. Let $\{x_k\}$ be a $\tau_{\widehat{V}}$-null sequence of a Hilbert C^*-module. By Proposition 3.3 we may suppose that the norms of all the elements x_k do not exceed one. Put $n(1) = 1$. By supposition for an arbitrary $\varepsilon_1 > 0$ there exists a number $n(2) > n(1)$ such that $\|\langle x_{n(1)}, x_i \rangle\| < \varepsilon_1$ whenever $i > n(2)$. Analogously for an arbitrary $\varepsilon_2 > 0$ there is a number $n(3) > n(2)$ such that $\|\langle x_{n(2)}, x_i \rangle\| < \varepsilon_2$ whenever $i > n(3)$ and so on. Set $\varepsilon_i = \frac{1}{i}$. Then we claim the subsequence $\{x_{n(i)}\}$ of $\{x_k\}$ satisfies the condition (5.2). Indeed

$$\begin{aligned}
\left\| \frac{1}{k^2} \sum_{i,j=1}^{k} \langle x_{n(i)}, x_{n(j)} \rangle \right\| &\leqslant \frac{1}{k^2} \Big(\sum_{i=1}^{k} \|x_{n(i)}\|^2 + 2 \sum_{1 \leqslant i < j \leqslant k} \|\langle x_{n(i)}, x_{n(j)} \rangle\| \Big) \\
&\leqslant \frac{1}{k} + \frac{2}{k^2} ((k-1)\varepsilon_1 + (k-2)\varepsilon_2 + \cdots + \varepsilon_{k-1}) \\
&= \frac{1}{k} + \frac{2}{k^2} \Big(k + \frac{k}{2} + \frac{k}{3} + \cdots + \frac{k}{k-1} - (k-1) \Big) \\
&= \frac{1}{k} + \frac{2}{k^2} + \frac{2}{k} \Big(\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1} \Big) = \\
&= \frac{1}{k} + \frac{2}{k^2} + \frac{2}{k} (C - 1 + \ln(k-1) + o(1)),
\end{aligned}$$

where $C \approx 0,5772$ is the Euler constant and, consequently, the right part of this estimate vanishes when k goes to infinity. ∎

It means that we have only one candidate for the module Banach–Saks property.

DEFINITION 5.4. We will call the property (mBS2) *the module Banach–Saks property*.

PROPOSITION 5.5. *If a Hilbert C^*-module has the module Schur property, then it has the module Banach–Saks property.*

Proof. Let $\{x_k\}$ be a $\tau_{\widehat{V}}$-convergent sequence of a Hilbert C^*-module V and let x be its limit in the completion of V with respect to the topology $\tau_{\widehat{V}}$. Then Lemma 4.3 ensures x has to belong to V actually. Thus the numerical sequence $\{\|x_k - x\|\}$ converges to zero. Without loss of generality (because one can pass to a subsequence) we can suppose that $\|x_k - x\| \leqslant \frac{1}{k^2}$. This implies

$$\Big\|\frac{1}{n}\Big(\sum_{k=1}^{n} x_k\Big) - x\Big\| = \Big\|\frac{1}{n}\sum_{k=1}^{n}(x_k - x)\Big\| \leqslant \frac{1}{n}\sum_{k=1}^{n}\|x_k - x\| \leqslant \frac{1}{n}\Big(\sum_{k=1}^{\infty}\frac{1}{k^2}\Big)$$

and the right part of this expression converges to zero. ∎

THEOREM 5.6. *A C^*-algebra with a strictly positive element, in particular, a separable C^*-algebra, has the module Banach–Saks property if and only if it is unital, if and only if it has the module Schur property.*

Proof. Theorem 4.6 and Proposition 5.5 ensure that a unital C^*-algebra has the module Banach–Saks property. To check the converse implication let us fix an arbitrary non-unital C^*-algebra A with a strictly positive element and consider $V = A$ as a Hilbert A-module. Then exactly for the same reasons as explained in the proof of Theorem 4.6 any element of the right multiplier algebra $RM(A)$ may be obtained just as a limit of a sequence from A with respect to the right strict topology, which coincides with the topology $\tau_{\widehat{V}}$. Therefore, one can find a sequence $\{x_k\}$ of A which converges with respect to the topology $\tau_{\widehat{V}}$ (for instance, to the unit of $RM(A)$), but which does not converge in norm. Then for any subsequence $\{x_{k(i)}\}$ of $\{x_k\}$ the sequence $\Big\{\frac{1}{n}\sum_{i=1}^{n} x_{k(i)}\Big\}$ of A has obviously the same $\tau_{\widehat{V}}$-limit in $RM(A)$ as the sequence $\{x_k\}$, so $\Big\{\frac{1}{n}\sum_{i=1}^{n} x_{k(i)}\Big\}$ cannot converge in norm. Thus A does not have the module Banach–Saks property. ∎

THEOREM 5.7. *Let a Hilbert C^*-module V admit a decomposition into a direct orthogonal sum $V = V_1 \oplus V_2$ of two Hilbert C^*-submodules. Then the following conditions are equivalent:*

(i) *V has the module Banach–Saks property.*

(ii) *Both V_1 and V_2 have the module Banach–Saks property.*

Proof. Obviously, (i) implies (ii) and we have just to ensure the inverse conclusion. Consider any $\tau_{\widehat{V}}$-convergent sequence $\{x_i\}$ of V. Then its elements admit decompositions as $x_i = x_i^{(1)} \oplus x_i^{(2)}$ with $x_i^{(1)} \in V_1$, $x_i^{(2)} \in V_2$. Both the sequences $\{x_i^{(j)}\}$ are $\tau_{\widehat{V}_j}$-convergent sequences ($j = 1, 2$). Therefore, there are subsequences

$\{x^{(j)}_{i(k)}\}$ of $\{x^{(j)}_i\}$ such that their means $\frac{1}{n}\sum\limits_{k=1}^{n} x^{(j)}_{i(k)}$ converge in norm to some elements $x^{(j)}$ of V_j, respectively for $j = 1,2$. Thus, for the sequence $\{x_{i(k)}\}$ and for $x = x^{(1)} \oplus x^{(2)}$ we have the following estimate:

$$\Big\| \frac{1}{n}\Big(\sum_{k=1}^{n} x_{i(k)} \Big) - (x^{(1)} \oplus x^{(2)}) \Big\| = \Big\| \Big(\frac{1}{n}\sum_{k=1}^{n} x^{(1)}_{i(k)} - x^{(1)} \Big) \oplus \Big(\frac{1}{n}\sum_{k=1}^{n} x^{(2)}_{i(k)} - x^{(2)} \Big) \Big\|$$
$$\leqslant \Big\| \frac{1}{n}\sum_{k=1}^{n} x^{(1)}_{i(k)} - x^{(1)} \Big\| + \Big\| \frac{1}{n}\sum_{k=1}^{n} x^{(2)}_{i(k)} - x^{(2)} \Big\|.$$

Since the upper bounds converge to zero as k tends to infinity, the sum in the first term converges to x in norm. So V has the module Banach–Saks property. ∎

COROLLARY 5.8. *Let A be a σ-unital C^*-algebra with the module Banach–Saks property. Then finitely generated Hilbert modules over A and finitely generated projective Hilbert modules over A have the module Banach–Saks property.*

Let A be a C^*-algebra and consider the infinite matrix algebra $M_\infty(A) = \bigcup\limits_{n=1}^{\infty} M_n(A)$. For vectors $x_1, \ldots, x_n$ of a Hilbert A-module V consider the Gram matrix based on them:

$$G(x_1, \ldots, x_n) = \begin{pmatrix} \langle x_1, x_1 \rangle & \langle x_1, x_2 \rangle \cdots & \langle x_1, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle \cdots & \langle x_2, x_n \rangle \\ \cdots & \cdots & \cdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle \cdots & \langle x_n, x_n \rangle \end{pmatrix},$$

with entries $g_{ij} = \langle x_i, x_j \rangle$, besides $G(x_1, \ldots, x_n) \in M_\infty(A)$. Then the condition (5.2) on a sequence $\{x_i\}$ of V with $x = 0$ can be equivalently reformulated as the assertion that $\Big\{ \frac{1}{k^2} \sum\limits_{i,j=1}^{k} g_{ij} \Big\}$ forms a sequence which converges to zero with respect to the norm topology. Instead of condition (5.2) one can require that the sequence $\{\frac{1}{n^2} G(x_1, \ldots, x_n)\}$ of $M_\infty(A)$ has to converge to zero in norm. For instance, this condition holds whenever $\{x_i\}$ is an orthogonal system of norm one vectors (or even an orthonormal base), because $\|G(x_1, \ldots, x_n)\| \leqslant \sum\limits_{i,j=1}^{n} \|g_{ij}\|$ (cf. [23]). Let us remark that neither the value $\Big\| \frac{1}{n^2} \sum\limits_{i,j=1}^{n} \langle x_i, x_j \rangle \Big\|$ nor the value $\Big\| \frac{1}{n^2} G(x_1, \ldots, x_n) \Big\|$ majorize each other, in general.

We conclude this section with the observation that for Hilbert C^*-modules over some C^*-algebras which admit the classical Banach–Saks property of Banach spaces the module Banach–Saks property is admitted, too. This follows from the definition of the module Banach–Saks property and from Proposition 3.3. For a complete classification of Hilbert C^*-modules with the classical Banach–Saks property we refer to Propositions 2.3, 2.5 of [11]. There are also Hilbert C^*-modules over non-commutative C^*-algebras satisfying the module Banach–Saks property.

EXAMPLE 5.9. Take an infinite-dimensional Hilbert space H and consider the C^*-algebra $A = K(H)$ of all compact linear operators on H. Select some projection $p \in A$ with finite-dimensional range and set $V = pA$ as a (right) Hilbert A-module. Then V admits the classical Banach–Saks property and, hence, the module Banach–Saks property. Note, that $\langle pA, pA \rangle = A$.

6. DUAL MODULES AND MODULE SCHUR AND BANACH–SAKS PROPERTIES

In this section we establish an interrelation between self-duality of countably generated Hilbert C^*-modules and their properties to have the module Schur or the module Banach–Saks property, respectively. The main goal is an alternative characterization of C^*-dual Hilbert C^*-modules of representatives of this class.

Suppose X denotes the completion of V with respect to the topology $\tau_{\widehat{V}}$, and X_s denotes the subset of X consisting of all equivalence classes of $\tau_{\widehat{V}}$-Cauchy sequences from V. (Two Cauchy sequences are supposed to be equivalent if and only if their difference is a $\tau_{\widehat{V}}$-null sequence.) Let us show how the set X_s can be canonically identified with a subset of the dual module V', similarly like V is canonically embeddable into its C^*-dual A-module V'. Let $\{x_i\}$ be a $\tau_{\widehat{V}}$-Cauchy sequence from V, i.e. the sequence $\{\langle x_i, z\rangle\}$ converges in norm to some element $x(z) \in A$ for any $z \in V$ since $\|\langle u, v\rangle\| = \|\langle v, u\rangle\|$ for any $u, v \in V$. As we see, this construction defines an A-linear functional $\langle x, \cdot\rangle : V \to A$. Now, for this functional $\langle x, \cdot\rangle$, for $z \in V$ and for any $\varepsilon > 0$ there exists a number i_0 such that

$$\|x(z)\| \leqslant \|\langle x_i, z\rangle\| + \varepsilon \leqslant \|x_i\|\|z\| + \varepsilon$$

whenever $i > i_0$, where $\|x_i\| < C$ for some constant C and for all i by Proposition 3.3. Thus, the functional $\langle x, \cdot\rangle$ is bounded and belongs to the dual module V'.

PROPOSITION 6.1. *The topology $\tau_{\widehat{V}}$ has a countable base of neighborhoods of zero provided V is a countably generated Hilbert A-module.*

Proof. Let $V = \overline{\mathrm{span}}_A\{x_i : i \in \mathbb{N}\}$, i.e. the sequence $\{x_i\}_{i=1}^{\infty}$ generates V over A. Then we claim the countable family of sets

$$U_{x_{i_1},\ldots,x_{i_n},\frac{1}{N}} = \Big\{y \in V : \|\langle x_{i_1}, y\rangle\| < \frac{1}{N}, \ldots, \|\langle x_{i_n}, y\rangle\| < \frac{1}{N}\Big\}, \quad 1 < i_1, \ldots, i_n < \infty, n, N \in \mathbb{N} \tag{6.1}$$

forms the base of neighborhoods of zero for the topology $\tau_{\widehat{V}}$. Indeed, let us consider any $\tau_{\widehat{V}}$-neighborhood of zero of the form

$$U_{x,\varepsilon} = \{y \in V : \|\langle x, y\rangle\| < \varepsilon\},$$

where $x \in V$, $\varepsilon > 0$. It is just enough to check that any of these open sets contains one of the sets (6.1). By supposition, for any $\delta > 0$ there are $a_1, \dots, a_n \in A$ such that

$$\Big\| x - \sum_{i=1}^{n} x_i a_i \Big\| < \delta .$$

Consider a neighborhood $U_{x_1,\dots,x_n,\frac{1}{N}}$. By Proposition 3.5 we can assume that the norms of all the elements of the generating set do not exceed one, whenever N is large enough. Therefore, for any $z \in U_{x_1,\dots,x_n,\frac{1}{N}}$ one can deduce

$$\begin{aligned} \|\langle x, z\rangle\| &\leqslant \Big\| \Big\langle x - \sum_{i=1}^{n} x_i a_i, z \Big\rangle \Big\| + \Big\| \Big\langle \sum_{i=1}^{n} x_i a_i, z \Big\rangle \Big\| \leqslant \delta \|z\| + \sum_{i=1}^{n} \|a_i\| \|\langle x_i, z\rangle\| \\ &\leqslant \Big(\delta + \frac{1}{N} \sum_{i=1}^{n} \|a_i\| \Big) \|z\| . \end{aligned}$$

The numerical expression turns out to be less than the selected ε, whenever both δ and $\frac{1}{N}$ are small enough and $n \in \mathbb{N}$ is fixed. So we are done. ∎

COROLLARY 6.2. *Let V be a countably generated Hilbert C^*-module. Then under the notations above X_s coincides with the completion X of V with respect to the topology $\tau_{\widehat{V}}$ and, consequently, X can be isometrically embedded into the the C^*-dual module V' extending the canonical embedding of V into V'.*

THEOREM 6.3. *Let V be a countably generated Hilbert module over a unital C^*-algebra A. Then the completion X of V with respect to the topology $\tau_{\widehat{V}}$ coincides with the dual module V' extending the canonical embedding of V into V'.*

Proof. We already know that X is included into V' via the canonical embedding of V into V', and we should just check the coincidence in the particular case. First, let us suppose, V is the standard module $l_2(A)$. Then we can use the characterization of the dual module $l_2(A)'$ described in Proposition 2.5.5 of [21]. For any $\beta = (b_i) \in l_2(A)'$, where $\Big\| \sum_{i=1}^{n} b_i^* b_i \Big\| \leqslant C$ for all n and some finite constant C, let us consider the finite vectors $\alpha_n = (b_1, \dots, b_n, 0, 0, \dots) \in l_2(A)$. Then for any $y = (y_i) \in l_2(A)$ we have

$$\|\langle \alpha_n - \alpha_m, y\rangle\| = \Big\| \sum_{i=n+1}^{m} b_i^* y_i \Big\| \leqslant \Big\| \sum_{i=n+1}^{m} b_i^* b_i \Big\|^{1/2} \Big\| \sum_{i=n+1}^{m} y_i^* y_i \Big\|^{1/2} \leqslant C^{1/2} \Big\| \sum_{i=n+1}^{m} y_i^* y_i \Big\|^{1/2}$$

and this expression goes to zero provided m, n go to infinity. Thus, $\{\alpha_n\}$ is a $\tau_{\widehat{V}}$-Cauchy sequence and, moreover, its $\tau_{\widehat{V}}$-limit, obviously, coincides with β. This shows the desirable identification of X and V' which extends the canonical embedding of V into V'.

Now, assume V is an arbitrary countably generated Hilbert A-module. Then by Kasparov's stabilization theorem there exists a Hilbert A-module W such that $l_2(A) = V \oplus W$. Denote by $P_V : l_2(A) \to l_2(A)$ the corresponding orthogonal projection onto the first summand, so $\mathrm{Range}(P_V) = V$. For any functional $f \in V'$ one can define a functional $\widetilde{f} \in l_2(A)'$ as the composition $\widetilde{f} = fP_V$. Because $l_2(A)'$ coincides with the $\tau_{\widehat{l_2(A)}}$-completion of $l_2(A)$ there is a sequence $\{z_i\}$ of $l_2(A)$, where $z_i = x_i \oplus y_i$, such that its $\tau_{\widehat{l_2(A)}}$-limit is $\widetilde{f}$. Thus, the sequence $\{x_i\}$ of V converges to f with respect to the topology $\tau_{\widehat{V}}$. ∎

REMARK 6.4. Theorem 6.3 is true also for σ-unital C^*-algebras A, but the proof involves more complicated techniques and is left to the reader. Actually, it is the corollary of the crucial general result Theorem 6.4 of [9] and Proposition 3.3. Let us also emphasize for $V = A$ that in the non-unital, σ-unital case the theorem above is equivalent to the well-known fact that the completion of A with respect to the left strict topology coincides with the left multiplier algebra $LM(A)$.

Applying Theorem 6.3 we get:

COROLLARY 6.5. *Let V be a countably generated Hilbert module over a unital C^*-algebra. Then V is self-dual whenever it has the module Schur property.*

COROLLARY 6.6. *Let V be a countably generated self-dual Hilbert module over a unital C^*-algebra. Then V has the property* (mBS1) *if and only if it has the module Banach–Saks property.*

Proof. Suppose the property (mBS1) holds for V. Consider any $\tau_{\widehat{V}}$-convergent sequence $\{x_i\}$ of V. Then by supposition and by Theorem 6.3 its $\tau_{\widehat{V}}$-limit x belongs to V. Then the sequence $\{y_i = x_i - x\}$ is a $\tau_{\widehat{V}}$-null sequence. Hence, it admits a subsequence $\{y_{i(k)}\}$ satisfying the equality (5.1), so the sequence $\frac{1}{n} \sum_{k=1}^{n} x_{i(k)}$ converges to x in norm. ∎

COROLLARY 6.7. *Any countably generated self-dual Hilbert module over a unital C^*-algebra possesses the module Banach–Saks property.*

The proof follows from Theorem 5.3 and Corollary 6.6.

As a final result we can state, that the module Schur and the module Banach–Saks properties are different in general, because any standard Hilbert module over a finite dimensional C^*-algebra is self-dual and, consequently, has the module Banach–Saks property, however it does not have the module Schur property by Proposition 4.11.

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