J. OPERATOR THEORY
70:1(2013), 75–107
doi: 10.7900/jot.2011may05.1910

REGULARITY CRITERIA FOR BANACH FUNCTION ALGEBRAS

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Communicated by Şerban Strătilă

ABSTRACT. By means of a variant of a theorem of Yngve Domar we obtain decomposability criteria for bounded linear operators on Banach spaces depending on local growth conditions of the resolvent and local dimension properties of the spectrum. These criteria are then applied to obtain normality and regularity criteria for Banach function algebras of complex ultradifferentiable functions on perfect, compact subsets of the complex plane.

KEYWORDS: *Banach function algebras, decomposable operators.*

MSC (2010): Primary 46J10, 47B40; Secondary 46J15, 47A11.

1. INTRODUCTION AND PRELIMINARIES

For a perfect, compact subset K of the complex plane $\mathbb{C}$ and $k \in \mathbb{N} \cup \{\infty\}$ we denote by $D^k(K)$ the algebra of all k times continuously complex differentiable, complex valued functions on K. If k is finite then $D^k(K)$ is a normed algebra with respect to the sub-multiplicative norm given by

$$\|f\|_k := \sum_{j=0}^{k} \frac{1}{j!} \|f^{(j)}\|_K \quad (f \in D^k(K)),$$

where $\|\cdot\|_K$ is the uniform norm on K. In general these algebras are not complete and their completion need not to be semisimple [7]. A characterisation of the class of perfect, compact sets K for which the algebras $D^k(K)$ (or normed function algebras of complex ultradifferentiable functions on K) are complete and questions concerning the completion in the non-complete situation have been investigated in [14], [16], [9], [17] ....

Let $\mathfrak{M} = (M_p)_{p=0}^{\infty}$ be a sequence of bounded functions $M_p : K \to [1, \infty)$ with $M_0 \equiv 1$ satisfying the following condition for all non-negative integers p, q

with $0 \leqslant q \leqslant p$:

$$\forall z \in K : \quad \frac{M_p(z)}{p!} \geqslant \frac{M_q(z)M_{p-q}(z)}{q!(p-q)!}. \tag{1.1}$$

Let $D_1(K,\mathfrak{M})$ respectively $D_\infty(K,\mathfrak{M})$ denote the linear space of all $f \in D^\infty(K)$ for which

$$\|f\|_{1,\mathfrak{M}} := \sum_{p=0}^{\infty} \Big\| \frac{f^{(p)}}{M_p} \Big\|_K < \infty \quad \text{respectively} \quad \|f\|_{\infty,\mathfrak{M}} := \sup_{p\in\mathbb{N}_0} \Big\| \frac{f^{(p)}}{M_p} \Big\|_K < \infty. \tag{1.2}$$

Direct computation shows that $D_1(K,\mathfrak{M})$ is an algebra and that $\|\cdot\|_{1,\mathfrak{M}}$ is a submultiplicative norm on $D_1(K,\mathfrak{M})$. If the sequence $\mathfrak{M}$ satisfies the additional condition

$$C_{\mathfrak{M}} := \sup_{p\in\mathbb{N}} \Big\| \sum_{q=1}^{p-1} \frac{M_q M_{p-q} p!}{q!(p-q)!M_p} \Big\|_K < \infty \tag{1.3}$$

then $D_\infty(K,\mathfrak{M})$ is an algebra too, and $\|\cdot\|_{\infty,\mathfrak{M}}$ is a norm on $D_\infty(K,\mathfrak{M})$ for which the pointwise multiplication is continuous. Indeed, if $f, g \in D_\infty(K,\mathfrak{M})$, then we obtain by the Leibniz rule for all $p \in \mathbb{N}$,

$$\Big\| \frac{(fg)^{(p)}}{M_p} \Big\|_K \leqslant \Big\| \sum_{q=0}^{p} \frac{f^{(q)} g^{(p-q)}}{M_q M_{p-q}} \cdot \frac{M_q M_{p-q} p!}{q!(p-q)!M_p} \Big\|_K \leqslant \|f\|_{\infty,\mathfrak{M}} \|g\|_{\infty,\mathfrak{M}} (C_{\mathfrak{M}} + 2).$$

If the functions M_p are constant, then these algebras have been investigated by Dales and Davie in [16]. For the case of compact intervals see also Section 4.4 in [15] and the references given there.

We further introduce the spaces $\ell_s(C(K),\mathfrak{M})$, $s = 1, \infty$ as the spaces of all sequences $\mathbf{f} = (f_p)_{p=0}^\infty$ of continuous complex valued functions on K satisfying

$$|\mathbf{f}|_{1,\mathfrak{M}} := \sum_{p=0}^{\infty} \Big\| \frac{f_p}{M_p} \Big\|_K < \infty \quad \text{respectively} \quad |\mathbf{f}|_{\infty,\mathfrak{M}} := \sup_{p\in\mathbb{N}_0} \Big\| \frac{f_p}{M_p} \Big\|_K < \infty. \tag{1.4}$$

With respect to the multiplication suggested by the Leibniz rule

$$(f_p)_{p=0}^\infty (g_p)_{p=0}^\infty := \Big(\sum_{q=0}^{p} f_q g_{p-q} \binom{p}{q} \Big)_{p=0}^\infty$$

$(\ell_1(C(K),\mathfrak{M}), |\cdot|_{1,\mathfrak{M}})$ is a normed algebra and if (1.3) is fulfilled then $\ell_\infty(C(K),\mathfrak{M})$ is a normed algebra with respect to a submultiplicative norm that is equivalent to $|\cdot|_{\infty,\mathfrak{M}}$. Note that, for all $\mathbf{f} = (f_p)_{p=0}^\infty \in \ell_s(C(K),\mathfrak{M})$ and all $p \in \mathbb{N}_0$ we have $\|f_p\|_K \leqslant \|M_p\|_K \cdot |\mathbf{f}|_{s,\mathfrak{M}}$. Using this fact it follows easily that $(\ell_s(C(K),\mathfrak{M}), |\cdot|_{s,\mathfrak{M}})$ is complete and that the completion of $(D_s(K,\mathfrak{M}), \|\cdot\|_{s,\mathfrak{M}})$ may be identified with the closure of the image of the isometric algebra monomorphism

$$J_{s,\mathfrak{M}} : D_s(K,\mathfrak{M}) \to \ell_s(C(K),\mathfrak{M}), \quad f \mapsto (f^{(p)})_{p=0}^\infty,$$

of $D_s(K,\mathfrak{M})$ into $\ell_s(C(K),\mathfrak{M})$, $s = 1, \infty$.

In the same way the completion of $(D^k(K), \|\cdot\|_k)$ can be identified with the closure of the image of the isometric monomorphic embedding $J_k : f \mapsto (f, f', \ldots, f^{(k)})$ of $D^k(K)$ into $C(K)^{k+1}$, where $C(K)^{k+1}$ is endowed with the multiplication suggested by the Leibniz rule and with the submultiplicative norm $|\cdot|_k$ given by $|(f_0, \ldots, f_k)|_k := \sum_{p=0}^{k} \|f_p\|_K / p!$. Note that the radical of this commutative Banach algebra is given by

$$\operatorname{rad}(C(K)^{k+1}) = \{\mathbf{f} = (f_p)_{p=0}^k \in C(K)^{k+1} : f_0 \equiv 0\}$$

and consists of nilpotent elements. In particular, we have for the spectra $\sigma(\mathbf{f})$ and the spectral radius $r(\mathbf{f})$ of elements $\mathbf{f} \in C(K)^{k+1}$:

$$\forall \mathbf{f} = (f_p)_{p=0}^k \in C(K)^{k+1} : \quad \sigma(\mathbf{f}) = f_0(K), \quad \text{and} \quad r(\mathbf{f}) = \|f_0\|_K. \tag{1.5}$$

Following [16], a Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ of continuous functions on a compact Hausdorff space X will be called a *Banach function algebra on* X if it contains the constants and separates the points of X. A Banach function algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ on K is said to be *natural* if its character space $\Delta(\mathcal{A})$ coincides with the set $\{\delta_z : z \in K\}$ of all point evaluations in points of K. Recall that a Banach function algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ on K is *normal* on K if, for every finite open covering $U_1, \ldots, U_n$ of K there are functions $f_1, \ldots, f_n \in \mathcal{A}$ satisfying $f_1 + \cdots + f_n \equiv 1$ and $\operatorname{supp}(f_j) \subset U_j$, $j = 1, \ldots, n$. It is *regular* if and only if it is normal, when considered as a Banach function algebra on its character space. In particular every Banach function algebra that is normal and natural will be regular.

The aim of this article, which extends some results of the thesis [6] of the second author, is to obtain criteria for normality or regularity of the completions of the normed algebras of functions which have been defined above. As the functions in $D^k(K)$, $D_s(K, \mathfrak{M})$ are analytic in the interior of K, a necessary condition for normality will be that K has empty interior. We shall however see that there are examples of perfect, compact sets K of positive Lebesgue measure, for which the completions of these algebras are regular Banach function algebras on K.

In the following section we prove a variant of a theorem of Yngve Domar, which will be used in Section 4 to establish local and global decomposability criteria for bounded linear operators on Banach spaces by means of integrability criteria given in Section 3. Some simple growth properties of entire functions which are needed for resolvent estimates are given in Section 5. As normality and regularity of semisimple commutative Banach algebras are closely related to decomposability properties of multiplication operators (see Chapter VI of [13], [22], [2] and Chapter 4 of [27]), these decomposability results can then be applied in the last section to obtain normality and regularity criteria for Banach function algebras of the above mentioned type.

2. A VARIANT OF A THEOREM OF DOMAR

Recall that a function $u : \Omega \to [-\infty, \infty)$ on an open domain in $\mathbb{R}^N$ is said to be *subharmonic* if it is upper semicontinuous and if for every closed ball $B \subset \Omega$ and for every continuous function $h : B \to \mathbb{R}$ that is harmonic on the interior of B and satisfies $u \leqslant h$ on the boundary of B we have $u \leqslant h$ on all of B. For a Lebesgue measurable function $F : \Omega \to [0, \infty]$ let $\mathcal{M}(F)$ denote the set of all subharmonic functions u on Ω satisfying $u \leqslant F$ on Ω and define $M_F : \Omega \to [-\infty, \infty]$ by

$$M_F(x) := \sup\{u(x) : u \in \mathcal{M}(F)\}.$$

In [18], [19], Yngve Domar has obtained criteria for F to ensure the local boundedness of M_F on Ω which have been very useful in local spectral theory [28], [29], [25], [4].

The N-dimensional Lebesgue measure will be denoted by λ_N and we write V_N for the volume of the euclidean unit ball in $\mathbb{R}^N$. For subharmonic functions $u : \Omega \to [-\infty, \infty)$ and $n \in \mathbb{N}$ we consider the sets

$$E_n(u) := \{x \in \Omega : e^n \leqslant u(x) < e^{n+1}\}$$

and put $L_n(u) := \lambda_N(E_n(u))$. We shall need the following fact, which is a special case of Lemma 1.1 in [18] (with $\lambda = 1$ and D as defined below in that lemma).

LEMMA 2.1 (Domar). *Let $u : \Omega \to [-\infty, \infty)$ be a subharmonic function. If for some integer n and some $x_n \in \Omega$*

$$u(x_n) \geqslant e^n \quad \textit{and} \quad B_{r_n}(x_n) := \{x \in \mathbb{R}^N : |x - x_n| \leqslant r_n\} \subset \Omega,$$

where

$$r_n > D(L_{n-1}(u) + L_n(u))^{1/N} \quad \textit{and} \quad D := \Big(\frac{e^2}{(e-1)V_N}\Big)^{1/N},$$

then $B_{r_n}(x_n)$ contains a point x_{n+1} with $u(x_{n+1}) \geqslant e^{n+1}$.

The following variant of Domar's Theorem 2 in [18] is suited to our needs.

THEOREM 2.2. *Let $F : \Omega \to [0, \infty]$ be a Lebesgue measurable function on an open set $\Omega \subseteq \mathbb{R}^N$. Suppose that there exists a monotone increasing function $f : [0, \infty) \to [0, \infty)$ such that*

$$\int_a^\infty \frac{1}{f(t)} \mathrm{d}t < \infty \quad \textit{for some } a > 0 \tag{2.1}$$

and such that

$$\int_\Omega f(\log_+ F(x))^{N-1} \mathrm{d}\lambda_N(x) < \infty. \tag{2.2}$$

If $\Omega = \mathbb{R}^N$, then M_F is bounded. If $\Omega \neq \mathbb{R}^N$ then, for all $d > 0$ the function M_F is bounded on

$$A(d) := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) \geqslant d\}.$$

In particular M_F is locally bounded on Ω, i.e. bounded on every compact subset of Ω.

Proof. If u is subharmonic with $u \leqslant F$ on Ω then

$$f(n)^{N-1}L_n(u) = f(n)^{N-1}\lambda_N(E_n(u)) \leqslant \int\limits_{E_n(u)} f(\log_+ F(x))^{N-1}\mathrm{d}\lambda_N(x)$$

and hence

$$\sum_{n=1}^{\infty} f(n)^{N-1}L_n(u) \leqslant J := \int\limits_{\Omega} f(\log_+ F(x))^{N-1}\mathrm{d}\lambda_N(x) < \infty.$$

Using the Hölder inequality we obtain for all $u \in \mathcal{M}(F)$ and all $k \in \mathbb{N}$ with $f(k) > 0$:

$$\begin{aligned} S_k(u) &:= \sum_{n=k}^{\infty} L_n(u)^{1/N} = \sum_{n=k}^{\infty} \frac{1}{f(n)^{(N-1)/N}} \cdot f(n)^{(N-1)/N}L_n(u)^{1/N} \\ &\leqslant \Big(\sum_{n=k}^{\infty} \frac{1}{f(n)}\Big)^{(N-1)/N}\Big(\sum_{n=k}^{\infty} f(n)^{N-1}L_n(u)\Big)^{1/N} \leqslant \Big(\int\limits_k^{\infty} \frac{1}{f(t)}\mathrm{d}t\Big)^{(N-1)/N} J^{1/N}. \end{aligned}$$

Note that the right hand side of the exterior inequality is independent of u and converges to 0 for $k \to \infty$.

Let now x be an arbitrary point in Ω and put

$$d(x) := \sup\{r > 0 : B_r(x) \subset \Omega\}.$$

Let $k(x)$ be the smallest $k \in \mathbb{N}$ with the property

$$f(k) > 0 \quad \text{and} \quad 2DJ^{1/N}\Big(\int\limits_k^{\infty} \frac{1}{f(t)}\mathrm{d}t\Big)^{(N-1)/N} < d(x), \tag{2.3}$$

where D is the constant from Lemma 2.1. Assume that $M_F(x) > e^{k(x)+1}$. Then there exists some subharmonic function u on Ω satisfying $u(x) > e^{k(x)+1}$. Because of

$$\begin{aligned} \sum_{n=k(x)+1}^{\infty} D(L_{n-1}(u) + L_n(u))^{1/N} &\leqslant 2D \sum_{n=k(x)}^{\infty} L_n(u)^{1/N} \\ &\leqslant 2DJ^{1/N}\Big(\int\limits_{k(x)}^{\infty} \frac{1}{f(t)}\mathrm{d}t\Big)^{(N-1)/N} < d(x), \end{aligned}$$

we can find $r_n > D(L_{n-1}(u) + L_n(u))^{1/N}$, $n \geqslant k$, with

$$R := \sum_{n=k(x)+1}^{\infty} r_n < d(x).$$

By induction and using Lemma 2.1, one obtains a sequence $(x_n)_{n=k(x)+1}^{\infty}$ with $x_{k(x)+1} := x$, $|x_{n+1} - x_n| \leqslant r_n$ and $u(x_{n+1}) \geqslant e^{n+1}$ for all $n > k(x) + 1$. It follows that $x_n \in B_R(x) \subset \Omega$ for all $n > k(x)$. As u must be locally bounded, this is a contradiction. Hence we obtain $M_F(x) \leqslant e^{k(x)+1}$.

If $\Omega = \mathbb{C}$ or if $J = 0$, then $k(x) = k_0 := \min\{n \in \mathbb{N} : f(n) > 0\}$ is independent of x and M_F must be bounded by e^{k_0+1}.

In the case $\Omega \neq \mathbb{C}$ and $J \neq 0$, then for all $d > 0$ and all $x \in A(d)$ we have $k(x) \leqslant k_d$ where k_d is the smallest integer $k \geqslant k_0$ satisfying

$$2DJ^{1/N}\Big(\int_k^\infty \frac{1}{f(t)}\mathrm{d}t\Big)^{(N-1)/N} < d$$

and we conclude that M_f is bounded on $A(d)$ by e^{k_d+1}. ∎

If the function F in Theorem 2.2 is upper semicontinuous, then, as noted in [18], the function M_F is again subharmonic by a result of Brelot [10] (see also [23]).

In [18] Domar has given the proof for the special case $f(t) = f_{0,\varepsilon}(t) := t^{1+\varepsilon}$, where $\varepsilon > 0$. Other interesting choices for f are the functions $f_{k,\varepsilon}$ given by

$$f_{k,\varepsilon}(t) := t\Big(\prod_{j=1}^{k} \mathrm{Lc}_j(t)\Big)\,\mathrm{Lc}_k(t)^\varepsilon, \tag{2.4}$$

where $k \in \mathbb{N}$, $\varepsilon > 0$ and where Lc_j denotes the j-fold compositional power of the $\log_+$. In this case, for $s > 0$ sufficiently large,

$$\int_s^\infty f_{k,\varepsilon}(t)^{-1}\mathrm{d}t = \frac{1}{\varepsilon}\,\mathrm{Lc}_{k,\varepsilon}(s)^{-\varepsilon}$$

and direct computation shows that in the case $\Omega \neq \mathbb{R}^N$, $J \neq 0$ we have

$$k_d \leqslant 1 + \mathrm{Ec}_k\,\Big(\frac{1}{\varepsilon^{1/\varepsilon}}\Big(\frac{2DJ^{1/N}}{d}\Big)^{N/((N-1)\varepsilon)}\Big)$$

for all $d > 0$, where now Ec_k denotes the k-fold compositional power of the exponential function. Hence, we obtain the growth behaviour

$$M_F(x) \leqslant e^2 \cdot \mathrm{Ec}_{k+1}\,\Big(\frac{1}{\varepsilon^{1/\varepsilon}}\Big(\frac{2DJ^{1/N}}{d(x)}\Big)^{N/((N-1)\varepsilon)}\Big), \quad (x \in \Omega).$$

We are mainly interested in the case of dimension 2, where we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$. Then condition (2.2) becomes

$$\int_\Omega f(\log_+ F(z))\mathrm{d}\lambda_2(z) < \infty. \tag{2.5}$$

If $\Omega = \mathbb{C}$, then, by Theorem 2.2 and the Liouville theorem for subharmonic functions (see for example Corollary 2.3.4 of [30]), M_F must be constant. Because of $M_F \leqslant F$ on $\mathbb{C}$, this implies $M_F \leqslant \inf_{z\in\mathbb{C}} F(z)$ and $\log_+ M_F \in f^{-1}(\{0\})$.

Let us also remark that condition (2.2) implies $\lambda_N(F^{-1}(\{\infty\})) = 0$.

The global integrability condition in Theorem 2.2 may be replaced by a localised form as follows:

COROLLARY 2.3. *Let $F : \Omega_0 \to [0,\infty]$ be a Lebesgue measurable function on an open set $\Omega_0 \subseteq \mathbb{R}^N$. Suppose that there exists a compact subset $K \subset \Omega_0$ such that for each $w \in \Omega_0 \setminus K$ there exists an open neighbourhood $\Omega \subseteq \Omega_0$ and a monotone increasing function $f : [0,\infty) \to [0,\infty)$ satisfying conditions* (2.1) *and* (2.2) *with respect to Ω. Then the function M_F is locally bounded on Ω_0.*

Proof. Let H be any compact subset of Ω_0 and fix an open set U and a compact set W such that $K \cup H \subset U \subset W \subset \Omega_0$. By our assumption and Theorem 2.2 each point in $W \setminus U$ has a compact neighbourhood on which the function M_F is bounded. As $W \setminus U$ is compact, we conclude that $C := \sup_{x \in W \setminus U} M_F(x) < \infty$. By the maximum principle for subharmonic functions we have $u \leqslant C$ on U for all $u \in \mathcal{M}(F)$ and hence, $M_F \leqslant C$ on W. ∎

COROLLARY 2.4. *Let $F : \Omega_0 \to [0,\infty]$ be a Lebesgue measurable function on an open set $\Omega_0 \subseteq \mathbb{R}^N$ and let S be a totally disconnected subset of Ω_0 which is closed in Ω_0 with respect to the relative topology. If for each $w \in \Omega_0 \setminus S$ there exists an open neighbourhood $\Omega \subseteq \Omega_0$ and a monotone increasing function $f : [0,\infty) \to [0,\infty)$ satisfying conditions* (2.1) *and* (2.2) *with respect to Ω, then the function M_F is locally bounded on Ω_0.*

Proof. Let H be an arbitrary compact subset of Ω_0. As S is totally disconnected there exists an open neighbourhood $\Omega_1 \subset \Omega_0$ of H such that $K_1 := \Omega_1 \cap S$ is compact. Hence we may apply the previous corollary to the open set Ω_1 and the compact set $K := K_1 \cup H$ and obtain the boundedness of M_F on H. ∎

Note that in these two corollaries the set $F^{-1}(\{\infty\})$ may have positive Lebesgue measure.

3. INTEGRABILITY CRITERIA

In this section we establish some criteria which help to verify the existence of the integral (2.5) in concrete situations. We first notice some elementary properties of the compositional powers Lc_k of the logarithm and of the functions $f_{k,\varepsilon}$.

LEMMA 3.1. *Let $k \geqslant 1$ be an integer and $\varepsilon > 0$ be real.*

(i) *For all $x \geqslant \mathrm{Ec}_k(1)$ we have*

$$\mathrm{Lc}_k'(x) = \Big(x \prod_{j=1}^{k-1} \mathrm{Lc}_j(x)\Big)^{-1}, \quad \text{and} \tag{3.1}$$

$$f_{k,\varepsilon}'(x) \leqslant (k+1) f_{k-1,\varepsilon}(\log(x)). \tag{3.2}$$

(ii) *For $a, c > 0$ and $b, d \in \mathbb{R}$ we have*

$$\lim_{x\to\infty} \frac{\mathrm{Lc}_k(ax+b)}{\mathrm{Lc}_k(cx+d)} = 1.$$

Proof. The proof of (3.1) follows with the chain rule, (3.2) is an obvious consequence of (3.1) and (ii) is easily obtained using induction with respect to k and L'Hospital's rule. ∎

LEMMA 3.2. *Let Q_a be a square in the complex plane of side length $a \in (0,1]$.*
(i) *For $0 < \alpha < 1$ we have*

$$\int_{Q_a} \mathrm{dist}(z, \partial Q_a)^{-\alpha} \mathrm{d}\lambda_2(z) = \frac{2^{1+\alpha} a^{2-\alpha}}{(2-\alpha)(1-\alpha)}.$$

(ii) *Let k be a non negative integer and let ε be a positive real number. There exist constants $C_1, C_2 > 0$, independent of a, such that, with the notation of Section 2 we have for all $\alpha > 0, c \geqslant 1$,*

$$\int_{Q_a} f_{k,\varepsilon}(\log(c\,\mathrm{dist}(z, \partial Q_a)^{-\alpha}))\mathrm{d}\lambda_2(z) \leqslant 2a^2(C_1 + f_{k,\varepsilon}(\log(c(2a^{-2})^{\alpha}))) \quad \textit{and} \tag{3.3}$$

$$\begin{aligned} &\int_{Q_a} f_{k,\varepsilon}(\log_+ \log(c\,\mathrm{dist}(z, \partial Q_a)^{-\alpha}))\mathrm{d}\lambda_2(z) \\ &\qquad \leqslant 2a^2(C_2 + f_{k,\varepsilon}(\log_+ \log(c(2a^{-2})^{\alpha}))). \end{aligned} \tag{3.4}$$

(iii) *Let $c, \alpha, \gamma, \varepsilon$ be positive numbers such that $c \geqslant 1$, $\gamma > 2$, and $\gamma\alpha - \varepsilon > 0$. For all integers $k \geqslant 1$ consider the function $h : [\mathrm{Ec}_k(1), \infty) \to [0, \infty)$ given by*

$$h(s) := f_{k,\varepsilon}(csf_{k-1,\alpha}(\log s)^{-\gamma}) \quad (s \geqslant \mathrm{Ec}_k(1)).$$

Then there exists some $a_0 > 0$ only depending on c, α, k, and ε such that for all $a \in (0, a_0)$, we have the estimate

$$\int_{Q_a} h(\mathrm{dist}(z, \partial Q_a)^{-1})\mathrm{d}\lambda_2(z) \leqslant \frac{8ca}{\gamma(1+\alpha) - 2 - \varepsilon} f_{k-1,\eta}\Big(\log\Big(\frac{2}{a}\Big)\Big)^{2-\gamma},$$

where $\eta = (\gamma\alpha - \varepsilon)/(\gamma - 2)$.

Proof. (i) follows by direct computation (see Lemma 3.7 of [4]).

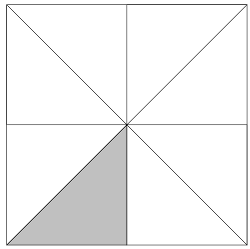

FIGURE 1

(ii) Let Δ denote the shaded triangle in Figure 1. By symmetry and using appropriate coordinates and partial integration, we have for all monotone increasing functions $h : (0,\infty) \to (0,\infty)$ for which $h(2/t^2)$ is integrable on $(0,1)$:

$$\begin{aligned}\int\limits_{Q_a} h(\operatorname{dist}(z,\partial Q_a)^{-1})\mathrm{d}\lambda_2(z) &= 8\int\limits_{\Delta} h(\operatorname{dist}(z,\partial Q_a)^{-1})\mathrm{d}\lambda_2(z) = 8\int\limits_0^{a/2}\Big(\frac{a}{2}-x\Big)h\Big(\frac{1}{x}\Big)\mathrm{d}x \\ &= 2a^2\int\limits_0^1 (1-t)h\Big(\frac{2}{at}\Big)\mathrm{d}t \leqslant 2a^2\int\limits_0^a h\Big(\frac{2}{t^2}\Big)\mathrm{d}t + 2a^2\int\limits_a^1 h\Big(\frac{2}{a^2}\Big)\mathrm{d}t \\ &\leqslant 2a^2\Big(h(2a^{-2}) + \int\limits_0^1 h(2t^{-2})\mathrm{d}t\Big).\end{aligned}$$

By applying this to the functions $s \mapsto f_{k,\varepsilon}(\log(cs^\alpha))$ and $s \mapsto f_{k,\varepsilon}(\log\log(cs^\alpha))$ we obtain the estimates (3.3) and (3.4).

(iii) From the proof of (ii) we get

$$I := \int\limits_{Q_a} h(\operatorname{dist}(z,\partial Q_a)^{-1})\mathrm{d}\lambda_2(z) \leqslant 2a^2\int\limits_0^1 h\Big(\frac{2}{at}\Big)\mathrm{d}t\,.$$

Note that, for $s \geqslant \mathrm{Ec}_k(1)$, we have $f_{k-1,\alpha}(\log s)^{-\gamma} \leqslant 1$ and hence

$$f_{k-1,\varepsilon}(\log(csf_{k-1,\alpha}(\log s)^{-\gamma})) \leqslant f_{k-1,\varepsilon}(\log(cs)).$$

Moreover, by Lemma 3.1(ii), there exists some $s_0 \geqslant \mathrm{Ec}_k(1)$, such that

$$\forall s \geqslant s_0 : \quad f_{k-1,\varepsilon}(\log(cs)) \leqslant 2f_{k-1,\varepsilon}(\log(s)).$$

Therefore, we obtain for $0 < a < a_0 := s_0^{-1}$,

$$
\begin{aligned}
I &\leqslant 2a^2 \int_0^1 h\Big(\frac{2}{at}\Big)\mathrm{d}t = 2a^2 \int_0^1 \frac{2cf_{k-1,\varepsilon}\Big(\log\Big(\frac{2c}{at}f_{k-1,\alpha}\Big(\log\Big(\frac{2}{at}\Big)\Big)^{-\gamma}\Big)\Big)}{atf_{k-1,\alpha}\Big(\log\Big(\frac{2}{at}\Big)\Big)^{\gamma}}\mathrm{d}t \\
&\leqslant 8ca \int_0^1 \frac{1}{t}\,\mathrm{Lc}_k\Big(\frac{2}{at}\Big)^{1+\varepsilon-\gamma(1+\alpha)}\Big[\prod_{j=1}^{k-1}\mathrm{Lc}_j\Big(\frac{2}{at}\Big)\Big]^{1-\gamma}\mathrm{d}t.
\end{aligned}
$$

With the substitution $u = \mathrm{Lc}_k(2/at)$, using (3.1) and the monotonicity of the functions Ec_j, we conclude finally

$$
\begin{aligned}
I &\leqslant 8ca \int_{\mathrm{Lc}_k(2a^{-1})}^{\infty} \Big[\prod_{j=1}^{k-1}\mathrm{Ec}_j(u)\Big]^{2-\gamma} u^{-\gamma(1+\alpha)+1+\varepsilon}\mathrm{d}u \\
&\leqslant 8ca\Big[\prod_{j=1}^{k-1}\mathrm{Lc}_j\Big(\frac{2}{a}\Big)\Big]^{2-\gamma} \int_{\mathrm{Lc}_k(2a^{-1})}^{\infty} u^{-\gamma(1+\alpha)+1+\varepsilon}\mathrm{d}u \\
&= \frac{8ca}{\gamma(1+\alpha)-2-\varepsilon} f_{k-1,\eta}\Big(\log\Big(\frac{2}{a}\Big)\Big)^{2-\gamma}
\end{aligned}
$$

for all $a \in (0, a_0)$. ∎

Let now K be a compact set in the complex plane $\mathbb{C}$. Given a closed square Q of side length a containing K and a sequence $\mathfrak{n} = (n_k)_{k=1}^{\infty}$ of integers $\geqslant 2$ we construct an *associated grid sequence* $\mathcal{G} = \mathcal{G}(K, Q, \mathfrak{n})$ inductively as follows:

(i) For $k = 0$ we define $a_0(\mathcal{G}) := a$, $\mathcal{Q}_0(\mathcal{G}) := \{Q\}$, $A_0(\mathcal{G}) := Q$, $N_0(\mathcal{G}) := \widetilde{N}_0(\mathcal{G}) := 1$ and $F_0(\mathcal{G}) := \widetilde{F}_0(\mathcal{G}) := a^2$.

(ii) For $k \geqslant 1$ put $a_k(\mathcal{G}) := a_{k-1}(\mathcal{G})/n_k$ and denote by $\mathcal{Q}_k(\mathcal{G})$ the collection of all squares of side length $a_k(\mathcal{G})$ contained in Q obtained by subdividing each square in $\mathcal{Q}_{k-1}(\mathcal{G})$ into $n_k(\mathcal{G})^2$ closed sub-squares. $N_k(\mathcal{G})$ denotes the number of all squares in $\mathcal{Q}_k(\mathcal{G})$ having non-empty intersection with K while $\widetilde{N}_k(\mathcal{G})$ will be the number of $R \in \mathcal{Q}_k(\mathcal{G})$ such that the interior of R has non-empty intersection with K. Finally $F_k(\mathcal{G}) := N_k(\mathcal{G})a_k(\mathcal{G})^2$ respectively $\widetilde{F}_k(\mathcal{G}) := \widetilde{N}_k(\mathcal{G})a_k(\mathcal{G})^2$ denotes the area of the union $A_k(\mathcal{G})$ respectively $\widetilde{A}_k(\mathcal{G})$ of these squares.

We call

$$\overline{\dim}_{\mathcal{G}}(K) := \limsup_{k\to\infty} \frac{\log N_k(\mathcal{G})}{\log(a_k(\mathcal{G})^{-1})}$$

the *upper $\mathcal{G}$-dimension of* K. Obviously we have

$$\underline{\dim}_{\mathrm{B}}(K) \leqslant \overline{\dim}_{\mathcal{G}}(K) \leqslant \overline{\dim}_{\mathrm{B}}(K),$$

where $\underline{\dim}_{\mathrm{B}}(K)$and $\overline{\dim}_{\mathrm{B}}(K)$ denote the lower and upper box dimension of K in the sense of [20]. In particular, if the box dimension of K exists, i.e. if $\underline{\dim}_{\mathrm{B}}(K) =$

$\overline{\dim}_{\mathrm{B}}(K)$, then $\overline{\dim}_{\mathcal{G}}(K)$ coincides with the box dimension of K. If the sequence $(\widetilde{N}_k(\mathcal{G}))_{k=0}^{\infty}$ is not eventually 0 we may also consider the *upper $\mathcal{G}$-constant*

$$\overline{\mathrm{D}}_{\mathcal{G}}(K) := \limsup_{k\to\infty} \frac{\log \widetilde{N}_k(\mathcal{G})}{\log(a_k(\mathcal{G})^{-1})}.$$

Obviously, $\overline{\mathrm{D}}_{\mathcal{G}}(K) \leqslant \overline{\dim}_{\mathcal{G}}(K)$.

If $h : (0,\infty) \to \mathbb{R}$ is a function satisfying

$$\lim_{t\to\infty} \frac{h(t)}{\log t} = 0$$

then we say that K has *upper $\mathcal{G}$-dimension* (respectively upper $\mathcal{G}$-constant) $\leqslant d$ with asymptotic bound h if there exists some $k_0 \in \mathbb{N}$ such that for all $k \geqslant k_0$ we have

$$\frac{\log N_k(\mathcal{G})}{\log(a_k(\mathcal{G})^{-1})} \leqslant d + \frac{h(a_k(\mathcal{G})^{-1})}{\log(a_k(\mathcal{G})^{-1})}$$

respectively

$$\frac{\log \widetilde{N}_k(\mathcal{G})}{\log(a_k(\mathcal{G})^{-1})} \leqslant d + \frac{h(a_k(\mathcal{G})^{-1})}{\log(a_k(\mathcal{G})^{-1})}.$$

PROPOSITION 3.3. *Let K be a compact set contained in a compact square Q of side length $a > 0$ and let $g : (0,\infty) \to (0,\infty)$ be a continuous function. In each of the following situations the integral*

$$I := \int_Q f_{m,\varepsilon}(g(\mathrm{dist}(z,K)^{-1}))\mathrm{d}\lambda_2(z)$$

(with the convention $f_{m,\varepsilon}(g(\infty)) = \infty$) is finite:

(i) *K admits an associated grid sequence $\mathcal{G} = \mathcal{G}(K, Q, (n_j)_{j=1}^{\infty})$ with upper $\mathcal{G}$-dimension (respectively upper $\mathcal{G}$-constant) $d < 2$ satisfying*

$$\lim_{j\to\infty} n_j^{1/j} = 1, \quad \textit{and} \qquad (3.5)$$
$$g(t) = Ct^{\alpha} \quad (t > 0),$$

with constants $C > 0$, $\alpha < \mu_0 := \min\{1, 2-d\}$, $m = 0$ and $\varepsilon > 0$ with $\alpha(1+\varepsilon) < \mu_0$.

(ii) *K admits an associated grid sequence $\mathcal{G} = \mathcal{G}(K, Q, (q^k)_{k=1}^{\infty})$ with $2 \leqslant q \in \mathbb{N}$, having upper $\mathcal{G}$-dimension (respectively upper $\mathcal{G}$-constant) 2 with asymptotic bound h, where*

$$h(t) := -\log_+(C_b f_{m,\delta}(\log t) f_{m-1,\delta}(\log\log t)) \quad (t > \mathrm{Ec}_{m+1}(1)), \qquad (3.6)$$

$m \geqslant 1$, $0 < \varepsilon < \delta$, $C_b > 0$, and

$$g(t) = \log(ct^{\alpha}) \quad (t > \mathrm{Ec}_{m+1}(1))$$

with constants $c > 1, \alpha > 0$.

(iii) *K admits an associated grid sequence $\mathcal{G} = \mathcal{G}(K, Q, (p^{q^{k-1}})_{k=1}^{\infty})$ with $2 \leqslant p, q \in \mathbb{N}$, having upper $\mathcal{G}$-dimension (respectively upper $\mathcal{G}$-constant) 2 with asymptotic bound h, where*

$$h(t) := -\log_+(C_c f_{m,\delta}(\log\log t) f_{m-1,\delta}(\log\log\log t)) \quad (t > \mathrm{Ec}_{m+2}(1)),$$

$m \in \mathbb{N}$, $\varepsilon \in (0, \delta)$, $C_c > 0$, and

$$g(t) = \log\log(ct^{\alpha}) \quad (t > \mathrm{Ec}_{m+2}(1))$$

with constants $c > 1, \alpha > 0$.

(iv) *$m \geqslant 1$, $\alpha, \delta, \varepsilon > 0$ are positive constants satisfying $3\alpha - \delta - \varepsilon > 0$, K admits an associated grid sequence $\mathcal{G} = \mathcal{G}(K, Q, (q)_{k=1}^{\infty})$, $2 \leqslant q \in \mathbb{N}$, having upper $\mathcal{G}$-dimension (respectively upper $\mathcal{G}$-constant) 1 with asymptotic bound h, where*

$$h(t) := \log_+(C_d \,\mathrm{Lc}_m(t)^{\delta})$$

with $C_d > 0$ and g is a non-negative continuous function satisfying

$$g(t) = \mathrm{O}\Big(\frac{t}{f_{m-1,\alpha}(\log t)^3}\Big) \quad \textit{for } t \to \infty.$$

Proof. The proof will be given for the sequences $(N_j(\mathcal{G}))_{j=0}^{\infty}$. In the other situations the proof is obtained by replacing this sequence by $(\widetilde{N}_j(\mathcal{G}))_{j=0}^{\infty}$, $A_j(\mathcal{G})$ by $\widetilde{A}_j(\mathcal{G})$, and $F_j(\mathcal{G})$ by $\widetilde{F}_j(\mathcal{G})$.

Note that in all of the four cases we have $F_k(\mathcal{G}) \to 0$ for $k \to \infty$ and hence $\lambda_2(K) = 0$. It follows that, for $j_0 > 1$, we have

$$I = I(j_0) + \sum_{j=j_0+1}^{\infty} \int\limits_{A_{j-1}(\mathcal{G}) \setminus A_j(\mathcal{G})} f_{m,\varepsilon}(g(\mathrm{dist}(z, K)^{-1}))\mathrm{d}\lambda_2(z),$$

where

$$I(j_0) := \int\limits_{Q \setminus A_{j_0}(\mathcal{G})} f_{m,\varepsilon}(g(\mathrm{dist}(z, K)^{-1}))\mathrm{d}\lambda_2(z)$$

is finite. As $A_{j-1}(\mathcal{G}) \setminus \mathrm{int} A_j(\mathcal{G})$ is the union of $n_j^2 N_{j-1} - N_j$ squares of side length $a_j(\mathcal{G})$ we obtain the estimate

$$I \leqslant I(j_0) + \sum_{j=j_0+1}^{\infty} (n_j^2 N_{j-1}(\mathcal{G}) - N_j(\mathcal{G}))M_j, \tag{3.7}$$

where

$$M_j = \max\Big\{ \int\limits_R f_{m,\varepsilon}\Big(g\Big(\frac{1}{\mathrm{dist}(z, K)}\Big)\Big)\mathrm{d}\lambda_2(z); R \in \mathcal{Q}_j(\mathcal{G}), R \subset A_{j-1}(\mathcal{G}) \setminus \mathrm{int} A_j(\mathcal{G})\Big\}.$$

As g is monotone we have for $j > j_0$:

$$M_j \leqslant \mathcal{I}_j := \int\limits_{R_j} f_{m,\varepsilon}(g(\mathrm{dist}(z, \partial R_j)^{-1}))\mathrm{d}\lambda_2(z), \tag{3.8}$$

where R_j is a square of side length $a_j(\mathcal{G})$. For the proof it suffices to show that the series

$$S(j_0) := \sum_{j=j_0+1}^{\infty} (n_j^2 N_{j-1}(\mathcal{G}) - N_j(\mathcal{G}))\mathcal{I}_j$$

converges for some j_0 which is large enough.

In the situation of (i), where $m = 0$ and $0 < \alpha(1+\varepsilon) < \mu_0 \leqslant 1$, we fix $j_0 \in \mathbb{N}$ such that $a_{j_0}(\mathcal{G}) < 1$ and obtain from Lemma 3.2(i)

$$\mathcal{I}_j \leqslant C^{1+\varepsilon} \frac{2^{1+\alpha(1+\varepsilon)} a_j(\mathcal{G})^{2-\alpha(1+\varepsilon)}}{(2-\alpha(1+\varepsilon))(1-\alpha(1+\varepsilon))}$$

for all $j \geqslant j_0$. Hence, with a constant $C_1 > 0$, we have

$$\begin{aligned} S(j_0) &\leqslant C_1 \sum_{j=j_0+1}^{\infty} (n_j^2 N_{j-1}(\mathcal{G}) - N_j(\mathcal{G})) a_j(\mathcal{G})^{2-\alpha(1+\varepsilon)} \\ &= C_1 \sum_{j=j_0+1}^{\infty} (F_{j-1}(\mathcal{G}) - F_j(\mathcal{G})) a_j(\mathcal{G})^{-\alpha(1+\varepsilon)}. \end{aligned} \tag{3.9}$$

Fix now $\delta > 0$ with $d + \delta + \alpha(1+\varepsilon) < 2$. By the definition of d there exists some $j_1 \geqslant j_0$ such that for all $j \geqslant j_1$ we have

$$\frac{\log(N_j(\mathcal{G}))}{\log(a_j(\mathcal{G})^{-1})} < d + \delta.$$

Thus, for $j > j_1$,

$$F_j(\mathcal{G}) a_j(\mathcal{G})^{-\alpha(1+\varepsilon)} = N_j(\mathcal{G}) a_j(\mathcal{G})^2 a_j(\mathcal{G})^{-\alpha(1+\varepsilon)} \leqslant a_j(\mathcal{G})^{2-d-\delta-\alpha(1+\varepsilon)} \to 0$$

as $j \to \infty$. Therefore the sum in (3.9) coincides with $F_{j_0}(\mathcal{G}) a_{j_0+1}(\mathcal{G})^{-\alpha(1+\varepsilon)} + S$, where

$$\begin{aligned} S &= \sum_{j=j_0+1}^{\infty} F_j(\mathcal{G})(a_{j+1}(\mathcal{G})^{-\alpha(1+\varepsilon)} - a_j(\mathcal{G})^{-\alpha(1+\varepsilon)}) \\ &= \sum_{j=j_0+1}^{\infty} F_j(\mathcal{G}) a_j(\mathcal{G})^{-\alpha(1+\varepsilon)} (n_{j+1}^{\alpha(1+\varepsilon)} - 1). \end{aligned}$$

Because of $a_j(\mathcal{G}) \leqslant 2^{-j} a$ for all j and by (3.5) we obtain

$$\begin{aligned} \limsup_{j\to\infty} (F_j(\mathcal{G}) a_j(\mathcal{G})^{-\alpha(1+\varepsilon)} (n_{j+1}^{\alpha(1+\varepsilon)} - 1))^{1/j} &\leqslant \limsup_{j\to\infty} (a_j(\mathcal{G})^{2-d-\delta-\alpha(1+\varepsilon)})^{1/j} \\ &\leqslant \Big(\frac{1}{2}\Big)^{2-d-\delta-\alpha(1+\varepsilon)} < 1. \end{aligned}$$

This shows that S and hence also I must be finite.

In (ii) we choose $j_1 > 1$ large enough, such that for all $j \geqslant j_0$ the conditions $a_j(\mathcal{G})^{-1} > \mathrm{Ec}_{m+1}(1)$ and

$$\frac{\log N_j(\mathcal{G})}{\log(a_j(\mathcal{G})^{-1})} \leqslant 2 + \frac{h(a_j(\mathcal{G})^{-1})}{\log(a_j(\mathcal{G})^{-1})} \tag{3.10}$$

are satisfied. Thus, by (3.3), we get in (3.8)

$$M_j \leqslant 2a_j(\mathcal{G})^2(C_1 + f_{m,\varepsilon}(\log(c2^\alpha a_j(\mathcal{G})^{-2\alpha}))) \quad (j \geqslant j_0).$$

Direct computation (using Lemma 3.1(ii) and $a_j(\mathcal{G}) = q^{-(j(j+1))/2}a < 1$) shows that there exist a constant $C_0 \geqslant 1$ and some $j_0 \geqslant j_1$ such that for all $j \geqslant j_0$ we have

$$\begin{aligned} &f_{m,\varepsilon}(\log(c2^\alpha a_j(\mathcal{G})^{-2\alpha})) \leqslant C_0 j f_{m,\varepsilon}(j) \quad \text{and} \\ &\exp(-h(a_j(\mathcal{G})^{-1})) \geqslant \frac{1}{C_0}(j+1) f_{m,\delta}(j) f_{m-1,\delta}(\log(j+1)). \end{aligned} \tag{3.11}$$

Hence, we obtain in the situation of (ii),

$$\begin{aligned} S(j_0) &\leqslant 2 \sum_{j=j_0+1}^{\infty} (F_{j-1}(\mathcal{G}) - F_j(\mathcal{G}))(C_1 + f_{m,\varepsilon}(\log(c2^\alpha a_j(\mathcal{G})^{-2\alpha}))) \\ &\leqslant 2C_1 F_{j_0}(\mathcal{G}) + 2C_0 \sum_{j=j_0+1}^{\infty} (F_{j-1}(\mathcal{G}) - F_j(\mathcal{G})) j f_{m,\varepsilon}(j). \end{aligned} \tag{3.12}$$

By (3.10) and (3.11) we have for $j \geqslant j_0$,

$$F_j(\mathcal{G}) = N_j(\mathcal{G}) a_j(\mathcal{G})^2 \leqslant C_0((j+1) f_{m,\delta}(j) f_{m-1,\delta}(\log(j+1)))^{-1}$$

and therefore,

$$F_j(\mathcal{G}) j f_{m,\varepsilon}(j) \leqslant C_0 f_{m-1,\delta}(\log(j+1))^{-1} \to 0 \tag{3.13}$$

for $j \to \infty$. Moreover by the mean value theorem and Lemma 3.1(i) there exists some $x_j \in [j, j+1]$ with

$$\begin{aligned} F_j(\mathcal{G})((j+1) f_{m,\varepsilon}(j+1) - j f_{m,\varepsilon}(j)) &\leqslant F_j(\mathcal{G})(f_{m,\varepsilon}(x_j) + x_j(m+1) f_{m-1,\varepsilon}(\log(x_j))) \\ &= (m+2) F_j(\mathcal{G}) x_j f_{m-1,\varepsilon}(\log(x_j)) \\ &\leqslant \frac{C_0(m+2)(j+1) f_{m-1,\varepsilon}(\log(j+1))}{(j+m+1) f_{m,\delta}(j) f_{m-1,\delta}(\log(j+1))} \\ &\leqslant \frac{C_0(m+2)}{f_{m,\delta}(j)}. \end{aligned}$$

Now, by (3.12) and (3.13), we obtain with

$$C(j_0) := 2F_{j_0}(\mathcal{G})(C_1 + C_0(j_0+1) f_{m,\varepsilon}(j_0+1))$$

the needed estimate

$$\begin{aligned} S(j_0) &\leqslant C(j_0) + 2C_0 \sum_{j=j_0+1}^{\infty} F_j(\mathcal{G})((j+1) f_{m,\varepsilon}(j+1) - j f_{m,\varepsilon}(j)) \\ &\leqslant C(j_0) + 2C_0^2(m+2) \sum_{j=j_0+1}^{\infty} f_{m,\delta}(j)^{-1} < \infty. \end{aligned}$$

(iii) In this case we have $a_j(\mathcal{G}) = a \cdot p^{-(q^j-1)/(q-1)}$. Similar as in the proof for (ii), we find some constant $C_0 \geqslant 1$, $j_0 \in \mathbb{N}$ large enough to ensure the following inequalities for all $j \geqslant j_0$:

$$\begin{aligned} a_j(\mathcal{G})^{-1} &> \mathrm{Ec}_{m+2}(1), \\ \frac{\log N_j(\mathcal{G})}{\log(a_j(\mathcal{G})^{-1})} &\leqslant 2 + \frac{h(a_j(\mathcal{G})^{-1})}{\log(a_j(\mathcal{G})^{-1})}, \\ f_{m,\varepsilon}(\log\log(c2^\alpha a_j(\mathcal{G})^{-2\alpha})) &\leqslant C_0 j f_{m,\varepsilon}(j), \\ \exp(-h(a_j(\mathcal{G})^{-1})) &\geqslant \frac{1}{C_0}(j+1) f_{m,\delta}(j) f_{m-1,\delta}(\log(j+1)). \end{aligned} \tag{3.14}$$

By (3.4) we now get in (3.8),

$$\mathcal{I}_j \leqslant 2a_j(\mathcal{G})^2(C_2 + f_{m,\varepsilon}(\log\log(c2^\alpha a_j(\mathcal{G})^{-2\alpha}))) \quad (j \geqslant j_0).$$

From the inequalities in (3.14) we obtain (as in the proof of (ii)) the estimate

$$S(j_0) \leqslant C'(j_0) + 2C_0^2(m+1) \sum_{j=j_0+1}^{\infty} f_{m,\delta}(j)^{-1} < \infty$$

with the constant $C'(j_0) = 2F_{j_0}(\mathcal{G})(C_2 + C_0(j_0+1)f_{m,\varepsilon}(j_0+1))$.

(iv) There is some $j_0 \in \mathbb{N}$ such that for all $j \geqslant j_1$ Lemma 3.2(iii) is applicable (with $\gamma = 3$) and we obtain for $j \geqslant j_0$,

$$\mathcal{I}_j \leqslant C a_j(\mathcal{G}) f_{m-1,3\alpha-\varepsilon}\Big(\log\Big(\frac{2}{a_j(\mathcal{G})}\Big)\Big)^{-1},$$

where $C > 0$ does not depend on j. Having chosen j_0 large enough, and using $a_j(\mathcal{G}) = aq^{-j}$, we may, by Lemma 3.1(ii), also assume that, with some constant $C_0 > 1$, we have for $j \geqslant j_0$,

$$\begin{aligned} &\frac{1}{C_0} f_{m-1,3\alpha-\varepsilon}\Big(\log\Big(\frac{2}{a_j(\mathcal{G})}\Big)\Big) \leqslant f_{m-1,3\alpha-\varepsilon}(j) \leqslant C_0 f_{m-1,3\alpha-\varepsilon}\Big(\log\Big(\frac{2}{a_j(\mathcal{G})}\Big)\Big), \\ &f_{m-1,3\alpha-\varepsilon}(j+1) \leqslant C_0 f_{m-1,3\alpha-\varepsilon}(j), \quad \text{and} \\ &\mathrm{Lc}_m\Big(\frac{1}{a_{j+1}(\mathcal{G})}\Big)^\delta \leqslant C_0 \,\mathrm{Lc}_{m-1}(j)^\delta. \end{aligned}$$

Moreover, if j_0 is chosen large enough, we obtain from the asymptotic grid condition

$$N_j(\mathcal{G})a_j(\mathcal{G}) \leqslant C_d \,\mathrm{Lc}_m(a^{-1}q^j)^\delta \leqslant C_0 C_d \,\mathrm{Lc}_{m-1}(j)^\delta$$

for all $j \geqslant j_0$. Hence,

$$N_j(\mathcal{G})a_j(\mathcal{G}) f_{m-1,3\alpha-\varepsilon}(j)^{-1} \leqslant C_d C_0 f_{m-1,3\alpha-\varepsilon-\delta}(j)^{-1} \to 0 \quad \text{for } j \to \infty. \tag{3.15}$$

By means of (3.15), we have in this situation,

$$\begin{aligned} S(j_0) &\leqslant C \sum_{j=j_0+1}^{\infty} (n_j^2 N_{j-1}(\mathcal{G}) - N_j(\mathcal{G})) a_j(\mathcal{G}) f_{m-1,3\alpha-\varepsilon}\Big(\log\Big(\frac{2}{a_j(\mathcal{G})}\Big)\Big)^{-1} \\ &\leqslant C_0 C \sum_{j=j_0+1}^{\infty} (q N_{j-1}(\mathcal{G}) a_{j-1}(\mathcal{G}) - N_j(\mathcal{G}) a_j(\mathcal{G})) f_{m-1,3\alpha-\varepsilon}(j)^{-1} \\ &= C_0 C (N_{j_0}(\mathcal{G}) a_{j_0}(\mathcal{G}) f_{m-1,3\alpha-\varepsilon}(j_0+1)^{-1} + S), \end{aligned}$$

where, by (3.15),

$$\begin{aligned} S &= \sum_{j=j_0+1}^{\infty} N_j(\mathcal{G}) a_j(\mathcal{G}) f_{m-1,3\alpha-\varepsilon}(j)^{-1} \Big(q \frac{f_{m-1,3\alpha-\varepsilon}(j+1)}{f_{m-1,3\alpha-\varepsilon}(j)} - 1 \Big) \\ &\leqslant C_0^2 C_d q \sum_{j=j_0+1}^{\infty} f_{m-1,3\alpha-\varepsilon-\delta}(j)^{-1}. \end{aligned}$$

As this sum is finite we also obtain $I < \infty$ in (iv). ∎

REMARK 3.4. (i) Many examples of compact sets in $\mathbb{C}$ with $\overline{\dim}_{\mathrm{B}} K < 2$ and hence satisfying the condition in Proposition 3.3(i) can be found in the standard references on fractal geometry (see for example [20]).

(ii) An example of a compact set K of upper box dimension 2 satisfying the condition in Proposition 3.3(ii) for all $m \geqslant 1$ can be constructed as follows: Let Q be a square of side length 1 and let p, q be integers with $1 < p+1 < q$. Set $\mathcal{Q}_0(\mathcal{G}) := \{Q\}$. For $1 \leqslant k \in \mathbb{N}$ let $\mathcal{Q}_k(\mathcal{G})$ be the the collection of all squares of side length $q^{-(k(k+1))/2}$ obtained by subdividing each square in $\mathcal{Q}_{k-1}(\mathcal{G})$ into q^k closed sub-squares. Put $K_0 := Q$ and, inductively for $k \geqslant 1$, let K_k be the compact set which is obtained from K_{k-1} by deleting from each square $R \in \mathcal{Q}_{k-1}$ with $R \subset K_{k-1}$ the interior of the union of pq of the q^{2k} squares from $\mathcal{Q}_k$ contained in R. For the compact set $K := \bigcap_{k=1}^{\infty} K_k$ and the associated grid sequence $\mathcal{G} = \mathcal{G}(K, Q, (q^k)_{k=1}^{\infty})$ we then have $\widetilde{N}_k(\mathcal{G}) = q^{k^2}(q-p)^k$ and hence

$$\frac{\log \widetilde{N}_k(\mathcal{G})}{-\log a_k(\mathcal{G})} = 2 - k \frac{\log\Big(\frac{q}{q-p}\Big)}{-\log a_k(\mathcal{G})},$$

where $-\log a_k(\mathcal{G}) = (k(k+1)/2)\log q$. From this it follows that for all $m \geqslant 1$ the set K has $\mathcal{G}$-constant 2 with asymptotic bound h (h as in (3.6)).

(iii) It is also possible to construct examples of compact sets K satisfying a condition as in (iii) of Proposition 3.3 but not the one in 3.3(ii).

(iv) The asymptotic condition for K in Proposition 3.3(iv) is satisfied for all rectifiable arcs in $\mathbb{C}$. This follows easily from [25] (proof of Proposition 5.4 and formula (5.1)).

4. APPLICATIONS TO LOCAL SPECTRAL THEORY

Throughout this section, $(E, \|\cdot\|_E)$ will be a (complex) Banach space and $\mathcal{L}(E)$ the Banach algebra of all bounded linear operators on E endowed with the operator norm. Recall that an operator $T \in \mathcal{L}(E)$ is said to have the *single valued extension property* (respectively *Bishop's property* (β), [8]) on an open set $\Omega \subseteq \mathbb{C}$ if, for each open subset U of Ω the map α_T^U from the Fréchet space $\mathcal{O}(U, E)$ of all analytic E-valued functions on U into itself defined by

$$\alpha_T^U(f)(z) := (z - T)f(z) \quad \text{for all } z \in U,\ f \in \mathcal{O}(U, E)$$

is injective (respectively is injective with closed range; equivalently, if for every sequence $(f_n)_{n=1}^{\infty} \subset \mathcal{O}(U, E)$ for which $(z - T)f_n(z) \to 0$ uniformly for each compact subset of U the sequence itself tends to 0 uniformly on each compact subset of U as $n \to \infty$).

Let $T \in \mathcal{L}(E)$ and let S be a closed subset of the spectrum $\sigma(T)$ of T. The operator T is *residually decomposable* (briefly *S-decomposable*) [31], [32] with *residuum* S if for each open cover $\mathbb{C} = U_1 \cup U_2$ with $S \subset U_1$ and $\overline{U}_2 \cap S = \emptyset$ there exist closed T-invariant subspaces X_1, X_2 with $E = X_1 + X_2$ and $\sigma(T|_{X_j}) \subset U_j$ for $j = 1, 2$. By the duality results in Section 3 of [3] this is the case if and only if T and its transpose T^* both have property (β) on $\mathbb{C} \setminus S$. Recall that T is said to be *decomposable* in the sense of C. Foiaş [21] if T is residually decomposable with residuum $S = \emptyset$. Moreover, by Theorem IV.4.26 of [32], $T \in \mathcal{L}(E)$ is S-decomposable if and only if there exists a map $\mathcal{E} : \mathcal{C}\ell_S(\mathbb{C}) \to \mathrm{Lat}(T)$, from the set $\mathcal{C}\ell_S(\mathbb{C})$ of all closed subsets of $\mathbb{C}$ that either contain S or are disjoint to S into the set $\mathrm{Lat}(T)$ of all closed T-invariant subspaces of E, with the following properties:

(i) $\mathcal{E}(\emptyset) = \{0\}$, $\mathcal{E}(\mathbb{C}) = E$.

(ii) For every family $(F_n)_{n=1}^{\infty}$ of sets in $\mathcal{C}\ell_S(\mathbb{C})$ we have

$$\bigcap_{n=1}^{\infty} \mathcal{E}(F_n) = \mathcal{E}\Big(\bigcap_{n=1}^{\infty} F_n\Big).$$

(iii) For every finite open cover $\bigcup_{k=0}^{n} U_k = \mathbb{C}$ of $\mathbb{C}$ such that $S \subset U_0$ and $\overline{U}_k \cap S = \emptyset$ for $k = 1, \ldots, n$, we have

$$E = \mathcal{E}(\overline{U}_0) + \cdots + \mathcal{E}(\overline{U}_n).$$

(iv) For all $F \in \mathcal{C}\ell_S(\mathbb{C})$ we have $\sigma(T|_{\mathcal{E}(F)}) \subset F$.

Such a map $\mathcal{E}$ is called a *S-spectral capacity* for T. It is uniquely determined by the properties (i)–(iv) ([32], Theorem IV.1.9). We refer to the monographs [13], [32], [27] for further results on these classes of operators.

PROPOSITION 4.1. *Let $\Omega \subset \mathbb{C}$ be an open set, let T be an operator in $\mathcal{L}(E)$ and let $K_e \subseteq \sigma(T)$ be a totally disconnected compact set. If for every $z \in \Omega \setminus K_e$ there is some* *neighbourhood U_z of z and some monotone increasing function $f_z : [0,\infty) \to [0,\infty)$ satisfying condition* (2.1) *and*

$$\int_{U_z} f_z(\log_+ \log_+(\|(\zeta - T)^{-1}\|))\mathrm{d}\lambda_2(\zeta) < \infty, \tag{4.1}$$

then T has Bishop's property (β) on Ω.

Note, that condition (4.1) can only be satisfied if $\sigma(T) \cap U_z$ has planar Lebesgue measure 0.

The proof of this proposition is similar to that of Theorem 3.3 in [4]. One has only to replace in that proof the original theorem of Domar by its variant Theorem 2.2 and to take f_z instead of $t \mapsto t^{1+\varepsilon(z)}$.

If T satisfies the conditions of Proposition 4.1, then so do its dual operator T^*, the operators $L_{\mathfrak{R}}(T)$ and $R_{\mathfrak{R}}(T)$ of left- and right-multiplication by T on every closed subalgebra $\mathfrak{R}$ of $\mathcal{L}(E)$ containing all rational functions of T with poles only outside of $\sigma(T)$ and also the duals of these multiplication operators. Hence, we obtain the following decomposability statement.

COROLLARY 4.2. *Let Ω, K_e and $T \in \mathcal{L}(E)$ be as in Proposition* 4.1 *and let $\mathfrak{R}$ be any closed subalgebra of $\mathcal{L}(E)$ containing all rational functions of T with poles only outside of $\sigma(T)$, then the operators T, T^*, $L_{\mathfrak{R}}(T)$ and $R_{\mathfrak{R}}(T)$ are S-decomposable with $S := \sigma(T) \setminus \Omega$.*

With this corollary and Proposition 1.4.4 in [27] we obtain:

COROLLARY 4.3. *If in the situation of Corollary* 4.2 *we have $\sigma(T) \subset \Omega$, then the operators T, T^*, $L_{\mathfrak{R}}(T)$ and $R_{\mathfrak{R}}(T)$ are even super-decomposable in the sense of Definition* 1.4.1 *of* [27].

The following proposition shows how decomposability properties of an operator $T \in \mathcal{L}(E)$ may be obtained from local dimension properties of $\sigma(T)$ and local growth properties of the resolvent of T.

PROPOSITION 4.4. *Let T be an operator in $\mathcal{L}(E)$ and let S be a closed subset of $\sigma(T)$. Assume that for each $z \in \sigma(T) \setminus S$ there exists a compact square $Q_z \subset \mathbb{C} \setminus S$ with centre z and a continuous function $g_z : (0,\infty) \to (0,\infty)$ satisfying*

$$\log_+ \log_+ \|(\zeta - T)^{-1}\| \leqslant g_z(\operatorname{dist}(\zeta, \sigma(T))^{-1}) \quad (\zeta \in Q_z \setminus \sigma(T))$$

such that one of the four conditions in Proposition 3.3 *is satisfied for Q_z, $K_z := (Q_z \cap \sigma(T)) \cup \partial Q_z$ and g_z. Then T is S-decomposable. If S is totally disconnected, then T is super-decomposable.*

This is now an immediate consequence of Propositions 3.3, 4.1 and Corollaries 4.2, 4.3.

COROLLARY 4.5. *Let $T \in \mathcal{L}(E)$ be a bounded linear operator and let S be a closed subset of $\sigma(T)$. Suppose, that for each $z \in \sigma(T) \setminus S$ there exists a compact square Q_z*

with centre z such that there are some $\alpha_z, m_z \in \mathbb{N}$ and constants $C_z, \delta_z, c_z > 0$ with

$$\forall \zeta \in \mathrm{int} Q_z \setminus \sigma(T): \quad \|(\zeta - T)^{-1}\| \leqslant c_z \mathrm{dist}(\zeta, K_z)^{-\alpha_z},$$

(where $K_z := (Q_z \cap \sigma(T)) \cup \partial Q_z$) and such that the dimension condition in Proposition 3.3(iii) *is satisfied for K_z with respect to $Q = Q_z$, $m = m_z$, $C_c = C_z$ and $\delta = \delta_z$. Then T is S-decomposable. In particular, if S is totally disconnected, then T is super-decomposable.*

For a compact set $K \subset \mathbb{C}$, we denote by $R(K)$ the uniform closure of $\mathrm{Rat}(K)$.

REMARK 4.6. It seems to be still an open problem (see [12]) if every hyponormal operator T on a separable, infinite dimensional Hilbert space satisfying $R(\sigma(T)) = C(\sigma(T))$ has a non-trivial hyperinvariant subspace. Notice, that in this case we have

$$\forall \zeta \in \mathbb{C} \setminus \sigma(T): \quad \|(\zeta - T)^{-1}\| \leqslant \mathrm{dist}(\zeta, \sigma(T))^{-1}.$$

Hence, if there is some closed subset $S \subset \sigma(T)$ such that $\sigma(T)$ satisfies locally outside of S a dimension condition as in the previous corollary, and if $\sigma(T) \setminus S$ contains more than one point, then T is S-decomposable and has a non-trivial hyperinvariant subspace.

In this connection a result from [11] is of interest:

REMARK 4.7. Let $K \subset \mathbb{C}$ be a compact set and S be a closed subset of K such that $R(S) = C(S)$. If $K \setminus S$ has planar Lebesgue measure 0, then $R(K) = C(K)$.

By Mergelyan's theorem this applies to situations, where the complement of S in $\mathbb{C}$ is connected and $\mathrm{int} S = \emptyset$ and thus in particular to situations, where S is totally disconnected.

The following easy fact will be useful later:

LEMMA 4.8. *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach function algebra on a compact set $K \subset \mathbb{C}$ containing* $\mathrm{Rat}(K)$. *If the operator L_{id} of multiplication by the variable is decomposable on $\mathcal{A}$, then $\mathcal{A}$ is normal. If in addition* $\mathrm{Rat}(K)$ *is dense in $\mathcal{A}$, then $\mathcal{A}$ is regular.*

Proof. As the inclusion mapping $\mathcal{A} \hookrightarrow C(K)$ is continuous and intertwines the operators of multiplication by id_K on $\mathcal{A}$ and $C(K)$ we have for the spectral capacity $\mathcal{E}$ of $L_{\mathrm{id}}|_{\mathcal{A}}$ and all closed subsets F of $\mathbb{C}$:

$$\mathcal{E}(F) \subseteq \{f \in \mathcal{A} : \mathrm{supp}(f) \subseteq F\},$$

see for example Proposition 1.2.17 in [27]. Hence, by the decomposability, $\mathcal{A}$ is normal. Moreover, for all $f \in \mathrm{Rat}(K)$, the operator L_f of multiplication by f is decomposable by Corollary II.1.11 in [13]. The set $\mathrm{Dec}(\mathcal{A})$ of all $f \in \mathcal{A}$ for which L_f is decomposable on $\mathcal{A}$ is a closed subalgebra of $\mathcal{A}$ (see [5] or Proposition 4.4.9 in [27]). Hence, if $\mathrm{Rat}(K)$ is dense in $\mathcal{A}$, we have $\mathrm{Dec}(\mathcal{A}) = \mathcal{A}$ and $\mathcal{A}$ is regular by [22]. ∎

5. SOME GROWTH ESTIMATES FOR ENTIRE FUNCTIONS

For the resolvent estimates that are needed in order to be able to apply the results of the previous section we need some elementary growth estimates for certain entire functions.

LEMMA 5.1. *Let $z \mapsto f(z) = \sum_{p=0}^{\infty} a_p z^p$ be an entire function and let $(b_p)_{p=0}^{\infty}$ be a monotone decreasing null sequence in $(0,1]$ satisfying $b_0 = 1$ and suppose that $|a_p| \leqslant b_p^p$ for all $p \in \mathbb{N}_0$. For $r > 0$ denote by $b_*(r)$ the smallest $p \in \mathbb{N}$ with $b_p \leqslant 1/2r$. Then we have the following estimate*

$$M_f(r) := \sup_{|z| \leqslant r} |f(z)| \leqslant b_*(r) r^{b_*(r)-1} + 2^{1-b_*(r)} \quad (r > 1). \tag{5.1}$$

Proof. By the definition of $b_*(r)$ and $M_f(r)$ we have for all $r > 1$ the estimate

$$\begin{aligned} M_f(r) &\leqslant \sum_{p=0}^{b_*(r)-1} |a_p| r^p + \sum_{p=b_*(r)}^{\infty} b_p^p r^p \leqslant \frac{r^{b_*(r)} - 1}{r-1} + \sum_{p=b_*(r)}^{\infty} 2^{-p} \\ &\leqslant b_*(r) r^{b_*(r)-1} + 2^{1-b_*(r)}. \quad \blacksquare \end{aligned}$$

COROLLARY 5.2. *Suppose that $0 < s \in \mathbb{R}$. In the situation and with the notation of the previous lemma we have in particular:*

(i) *If $b_0 = 1$ and $b_p = p^{-s}$ for $p > 0$, then*

$$\forall r > 1: \quad M_f(r) \leqslant (2 + 2^{1/s}) \exp\Big(\Big(2^{1/s} + \frac{1}{s}\Big) r^{1/s} \log r\Big).$$

(ii) *If $b_0 = 1$ and $b_p = \log(e+p)^{-s}$ for $p > 0$, then*

$$\forall r \geqslant e: \quad M_f(r) < \exp(2 \log(r) \exp((2r)^{1/s})).$$

(iii) *If $a_p = \Big(\prod_{j=0}^{p} \log(j+e)\Big)^{-s}$, then there are constants $C_1, C_2, C_3 \geqslant 1$ such that we have*

$$\forall r \geqslant 1: \quad M_f(r) \leqslant C_1 \exp(C_2 \exp(C_3 r^{(1/s)+\varepsilon})).$$

Proof. (i) In this case we have $b_*(r) - 1 < (2r)^{1/s} \leqslant b_*(r)$ and the statement follows easily from Lemma 5.1.

(ii) Now we have

$$b_*(r) - 1 < \exp((2r)^{1/s}) - e \leqslant b_*(r)$$

and the statement follows with the help of Lemma 5.1.

(iii) Fix $\delta \in (0,1)$ with $((1-\delta)s)^{-1} < s^{-1} + \varepsilon$. For all $p \geqslant 2$ in $\mathbb{N}$ let p_δ be the unique integer with $p_\delta < \delta p \leqslant p_\delta + 1$. Then we get for $p \geqslant e/\delta$:

$$a_p^{-1} \geqslant \prod_{j=p_\delta}^{p} (\log(j+e))^s \geqslant (\log(p_\delta + e))^{s(1-\delta)p} \geqslant (\log(\delta p))^{s(1-\delta)p}.$$

With $b_p := 1$ for $0 \leqslant p < e/\delta$ and $b_p := (\log(\delta p))^{-s(1-\delta)}$ for $p \geqslant e/\delta$ we have $a_p \leqslant b_p^p$ for all $p \in \mathbb{N}_0$. By means of Lemma 5.1 and some obvious estimates one now obtains easily the statement in (iii). ∎

6. BASIC PROPERTIES OF NORMED ALGEBRAS OF COMPLEX DIFFERENTIABLE AND ULTRADIFFERENTIABLE FUNCTIONS

LEMMA 6.1. *Let $K \subset \mathbb{C}$ be a perfect, compact set and $k \in \mathbb{N}$.*

(i) *$D^k(K)$ is a Banach function algebra on K if and only if the linear operator $T : f \mapsto (f', \dots, f^{(k)})$ with domain $\mathrm{D}(T) = D^k(K) \subset C(K)$ to $C(K)^k$ is closed.*

(ii) *The following are equivalent:*

(a) *The completion $\widehat{D}^k(K)$ of $D^k(K)$ is a Banach function algebra on K.*

(b) *The completion of $D^k(K)$ is semisimple.*

(c) *The linear operator $T : f \mapsto (f', \dots, f^{(k)})$ with domain $D^k(K) \subset C(K)$ to $C(K)^k$ is closable.*

Proof. As $D^k(K)$ contains the constants and separates the points of K, (i) follows immediately from the fact that $J_k(D^k(K))$ coincides with the graph $G(T)$ of T.

(ii) "(a) $\Rightarrow$ (b)" is obvious.

If T is not closable, then there exists some $\mathbf{f} = (f_0, f_1, \dots, f_k)$ in $\overline{G(T)} = \overline{J_k(D^k(K))}$ with $f_0 \equiv 0$ and $f_j \not\equiv 0$ for some $j \in \{1, \dots, k\}$. As $\mathbf{f}$ is nilpotent, this can not happen if $\overline{J_k(D^k(K))}$ is semisimple.

If T is closable with closure $\overline{T}$, then the domain $\mathrm{D}(\overline{T}) \subset C(K)$ of $\overline{T}$, endowed with the graph norm, is isometrically isomorphic to $\overline{G(T)} = \overline{J_k(D^k(K))}$. As $\mathrm{D}(\overline{T})$ contains the constants and separates the points of K, we see that $\widehat{D}^k(K) = \mathrm{D}(\overline{T})$ is a Banach function algebra. ∎

REMARK 6.2. If the operator $\mathrm{d}/\mathrm{d}z$ is closed (respectively closable), then the operator T in the statement of Lemma 6.1 is easily seen to be closed (respectively closable). In this case $D^k(K)$ (respectively $\widehat{D}^k(K)$) is then a Banach function algebra for all $k \geqslant 1$. Moreover, by (1.5), we have for all $f \in \widehat{D}^k(K)$,

$$r(f) = r(J_k(f)) = \|f\|_K.$$

REMARK 6.3. Suppose that the operator T in the statement of Lemma 6.1 is closable and that the uniform closure of $D^k(K)$ is a natural Banach function algebra, then the Banach function algebra $\widehat{D}^k(K)$ is natural. This is a consequence of the theorem in [24].

For the algebras $D^k(K)$ we can show:

PROPOSITION 6.4. *Let $K \subset \mathbb{C}$ be a perfect compact set, $k \in \mathbb{N}$ and write again J_k for the canonical isometric embedding of $D^k(K)$ into the Banach algebra $C(K)^{k+1}$ endowed with the multiplication suggested by the Leibniz rule.*

(i) *The operator $L_{J_k(\mathrm{id}_K)}$ of multiplication by $J_k(\mathrm{id}_K)$ is $C(K)$-spectral in the sense of* [13] *and hence decomposable.*

(ii) *The operator L_{id_K} of multiplication by id_K on the completion $\widehat{D}^k(K)$ has property (β).*

(iii) *If $\mathrm{d}/\mathrm{d}\zeta$ is closable and if S is a closed, totally disconnected subset of K such that in $K \setminus S$ locally (as in Corollary* 4.5*) the dimension condition of Proposition* 3.3(iii) *holds, then the completion $\widehat{D}^k(K)$ is a regular, natural Banach function algebra.*

Proof. (i) We have $L_{J_k(\mathrm{id}_K)} = S + N$, where S is the operator of multiplication by $(\mathrm{id}_K, 0, \ldots, 0)$ and N is the nilpotent operator of multiplication by $(0, 1, 0, \ldots, 0)$ commuting with S. A continuous $C(K)$ functional calculus for S is given by $f \mapsto \Phi(f)$, where $\Phi(f)$ is the operator of multiplication by $(f, 0, \ldots, 0)$. Hence $J_k(\mathrm{id}_K)$ is $C(K)$-spectral.

(ii) As L_{id_K} is similar to a restriction of $L_{J_k(\mathrm{id}_K)}$, it inherits property (β) from that operator.

(iii) Lemma 6.1(ii) and Remark 6.2 imply that $\widehat{D}^k(K)$ is a Banach function algebra containing $\mathrm{Rat}(K)$. Hence, by Remark 4.7, the uniform closure of $\widehat{D}^k(K)$ coincides with $C(K)$. By Remark 6.3, the Banach algebra $\widehat{D}^k(K)$ is natural. Corollary 4.5 shows that L_{id_K} is decomposable on $\widehat{D}^k(K)$. Therefore, $\widehat{D}^k(K)$ is normal (by Lemma 4.8) and thus regular. ∎

Useful criteria for the semi-simplicity of $\widehat{D}^k(K)$ or for the completeness of $D^k(K)$ can be found in [16], [9], [17]. We mention also the following fact:

REMARK 6.5. If K is a perfect, compact set such that $\mathrm{d}/\mathrm{d}z$ is closable in $C(K)$, then K does not contain any totally disconnected subset $A \neq \emptyset$ that is *clopen* in K (i.e. closed and open in the relative topology of K). In particular, if $z \in K$ and $\varepsilon > 0$, then $\dim_H(U_\varepsilon(z) \cap K) \geqslant 1$.

Indeed, if $z \in K$ and $\dim_H(U_\varepsilon(z) \cap K) < 1$ for some $\varepsilon > 0$, then $U_\varepsilon(z) \cap K$ is totally disconnected (see Section 2.2 of [20]) and K contains a clopen relative neighbourhood of z in K. If there is a clopen subset $A \neq \emptyset$ in K, then it is easy to show the existence of a sequence $(f_n)_{n=1}^{\infty}$ in $D^1(K)$ such that $f_n \to 0$ uniformly on K and $f_n' \to \chi_K$ uniformly on K, where χ_A is the characteristic function of A.

In the following, K will usually denote a perfect, compact set in $\mathbb{C}$ and $\mathfrak{M} = (M_p)_{p=0}^{\infty}$ will be a sequence of bounded, positive functions with $M_0 \equiv 1$ satisfying (1.1) and, in the case $s = \infty$, when considering the spaces $\ell_\infty(C(K), \mathfrak{M})$, $D_\infty(K, \mathfrak{M})$ or $\widehat{D}_\infty(K, \mathfrak{M})$ we shall also assume that (1.3) holds. For convenience, we also use the notations m_p and μ_p for the functions

$$(6.1) \qquad z \mapsto m_p(z) := \Big(\frac{M_p(z)}{p!}\Big)^{1/p}, \quad z \mapsto \mu_p(z) := \log m_p(z), \quad (p \in \mathbb{N}_0).$$

If the operator $\mathrm{d}/\mathrm{d}z$ is closable in $C(K)$ with closure d_z, we may consider the spaces

$$\widetilde{D}_s(K,\mathfrak{M}) := \Big\{ f \in \bigcap_{p=0}^{\infty} \mathrm{D}(d_z^p) : \|f\widetilde{\|}_{s,\mathfrak{M}} < \infty \Big\},$$

$s = 1, \infty$, where

$$\|f\widetilde{\|}_{1,\mathfrak{M}} := \sum_{p=0}^{\infty} \Big\| \frac{d_z^p f}{M_p} \Big\|_K, \quad \text{respectively} \quad \|f\widetilde{\|}_{\infty,\mathfrak{M}} := \sup_{p\in\mathbb{N}_0} \Big\| \frac{d_z^p f}{M_p} \Big\|_K.$$

Then, $(\widetilde{D}_s(K,\mathfrak{M}), \|\cdot\widetilde{\|}_{s,\mathfrak{M}})$ is a normed algebra (with respect to some equivalent submultiplicative norm in the case $s = \infty$).

LEMMA 6.6. *Let $K, \mathfrak{M}$ be as above.*
(i) *If $D^1(K)$ is complete, then $D_s(K,\mathfrak{M})$ is a Banach function algebra on K.*
(ii) *If the differentiation operator* $\mathrm{d}/\mathrm{d}z$ *is closable in $C(K)$, then $\widetilde{D}_s(K,\mathfrak{M})$ and the completion $\widehat{D}_s(K,\mathfrak{M})$ of $D_s(K,\mathfrak{M})$ are Banach function algebras on K.*

Proof. In the situation of (i), $\mathrm{d}/\mathrm{d}z$ is closed and hence $D_s(K,\mathfrak{M}) = \widetilde{D}_s(K,\mathfrak{M})$. Therefore (i) is a consequence of (ii).

Suppose now that $\mathrm{d}/\mathrm{d}z$ is closable in $C(K)$ with closure d_z. Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $\widetilde{D}_s(K,\mathfrak{M})$ and let $\mathbf{g} = (g_p)_{p=0}^{\infty}$ be the limit of $(J_{s,\mathfrak{M}} f_n)_{n=1}^{\infty}$ in $\ell_s(C(K),\mathfrak{M})$, where $J_{s,\mathfrak{M}} : \widetilde{D}_s(K,\mathfrak{M}) \to \ell_s(C(K),\mathfrak{M})$ again denotes the isometric monomorphism defined by $J_{s,\mathfrak{M}} f := (d_z^p f)_{p=0}^{\infty}$, $f \in \widetilde{D}_s(K,\mathfrak{M})$. In particular $\|d_z^p f_n - g_p\|_K \to 0$ for all $p \in \mathbb{N}_0$. By induction we see $g_0 \in \bigcap_{p=0}^{\infty} \mathrm{D}(d_z^p)$. Thus $f_n \to g_0 \in \widetilde{D}_s(K,\mathfrak{M})$. Therefore, $\widetilde{D}_s(K,\mathfrak{M})$ is complete and $\widehat{D}_s(K,\mathfrak{M})$ is a closed subalgebra of $\widetilde{D}_s(K,\mathfrak{M})$. As both of them contain the polynomials, they are Banach function algebras on K. ∎

REMARK 6.7. Let K and $\mathfrak{M}$ be as above. As $D_s(K,\mathfrak{M})$ is an algebra, the set $\mathrm{Rat}(K)$ of all rational functions with poles off K is contained in $D_s(K,\mathfrak{M})$, for $s = 1$ respectively $s = \infty$, if and only if for all $z \in \mathbb{C} \setminus K$ the function $\zeta \mapsto (z - \zeta)^{-1}$ is in $D_s(K,\mathfrak{M})$ which in turn is equivalent to

$$\sum_{p=0}^{\infty} \Big\| \frac{1}{(z-\cdot)^{p+1} m_p^p} \Big\|_K < \infty \text{ respectively } \sup_{p\in\mathbb{N}_0} \Big\| \frac{1}{(z-\cdot)^{p+1} m_p^p} \Big\|_K < \infty \quad (z \in \mathbb{C} \setminus K).$$

Sufficient conditions for this are

$$\lim_{p\to\infty} \|m_p^{-1}\|_K = 0, \quad \text{and} \tag{6.2}$$

$$\lim_{p\to\infty} \Big\| \frac{m_p^p}{m_{p+1}^{p+1}} \Big\|_K = 0. \tag{6.3}$$

In the case considered in [16], where all the functions M_p and m_p are constants, each of these two conditions is also necessary.

LEMMA 6.8. *Let $K \subset \mathbb{C}$ be a perfect, compact set and let $\mathfrak{M}$ be as above with the additional property that the functions M_p, $p \in \mathbb{N}_0$, are continuous on K. If* $\mathrm{Rat}(K) \subset D_s(K, \mathfrak{M})$ *then*

$$\lim_{p\to\infty} \|m_p^{-1}\|_{\partial K} = 0 \quad \text{and} \quad \lim_{p\to\infty} \Big\| \frac{m_p^p}{m_{p+1}^{p+1}} \Big\|_{\partial K} = 0. \tag{6.4}$$

Hence, if $\mathrm{int} K = \emptyset$, *then* $\mathrm{Rat}(K) \subset D_s(K, \mathfrak{M})$ *is equivalent to* (6.2) *and to* (6.3).

Proof. If $\lim\limits_{p\to\infty} \|1/m_p\|_{\partial K} \neq 0$ then there exists some $C > 0$ and some subsequence $(p_k)_{k=1}^{\infty}$ of $(p)_{p=0}^{\infty}$ and a sequence $(z_k)_{k=1}^{\infty}$ of points in ∂K such that $1/m_{p_k}(z_k) > C$ for all k. Passing to some subsequence, we may assume (by the compactness of ∂K) that $z_k \to z_0$ for some point $z_0 \in \partial K$. Hence, there is some $k_0 > 0$ such that $|z_k - z_0| < C/3$ for all $k \geqslant k_0$. Fix $w_0 \in \mathbb{C} \setminus K$ with $|z_0 - w_0| < C/3$ and consider the rational function $h \in \mathrm{Rat}(K)$ with $h(z) := (z - w_0)^{-1}$ for all $z \in K$. It follows that for $k \geqslant k_0$ we have

$$\Big\| \frac{h^{(p_k)}}{M_{p_k}} \Big\|_K \geqslant \frac{|h^{(p_k)}(z_k)|}{M_{p_k}(z_k)} = \frac{1}{|z_k - w_0|^{p_k+1} m_{p_k}(z_k)^{p_k}} \geqslant \frac{3}{2C} \cdot \Big(\frac{3}{2}\Big)^{p_k} \to \infty$$

for $k \to \infty$. Hence, $h \notin D_\infty(K, \mathfrak{M})$.

If the second condition in (6.4) is violated, then it follows with similar arguments that for some $w_0 \in \mathbb{C} \setminus K$ the rational function $z \mapsto (z - w_0)^{-1}$ is not in $D_\infty(K, \mathfrak{M})$. ∎

If $\mathrm{Rat}(K) \subset D_s(K, \mathfrak{M})$, then K coincides with the spectra of the identity function id_K and of the operator by multiplication with id_K in $\widehat{D}_s(K, \mathfrak{M})$ and, in the case that $\mathrm{d}/\mathrm{d}\zeta$ is closable in $C(K)$, also in $\widetilde{D}_s(K, \mathfrak{M})$. The same is true for $J_{s,\mathfrak{M}}(\mathrm{id}_K)$ and the operator of multiplication by $J_{s,\mathfrak{M}}(\mathrm{id}_K)$ in $\ell_s(C(K), \mathfrak{M})$.

In the proof of the following fact we use some arguments of the proof of Theorem 3.3 in [1].

PROPOSITION 6.9. *Let K be a (not necessarily perfect) compact set and let $\mathfrak{M}$ be a sequence of bounded functions $M_p : K \to [1, \infty)$ satisfying* (1.1) *and*

$$\lim_{p\to\infty} \max_{0<q<p} \Big\| \frac{m_{p-q}^{p-q} m_q^q}{m_p^p} \Big\|_K = 0. \tag{6.5}$$

Then

$$\mathrm{rad}(\ell_1(C(K), \mathfrak{M})) = \{\mathbf{f} = (f_p)_{p=0}^{\infty} \in \ell_1(C(K), \mathfrak{M}) : f_0 \equiv 0\}. \tag{6.6}$$

In particular, the spectrum and the spectral radius in $\ell_1(C(K), \mathfrak{M})$ are given by

$$\sigma(\mathbf{f}) = f_0(K) \quad \text{and} \quad r(\mathbf{f}) = \|f_0\|_K \quad \text{for all} \quad \mathbf{f} = (f_p)_{p=0}^{\infty} \in \ell_1(C(K), \mathfrak{M}). \tag{6.7}$$

Proof. An induction argument shows that for all $\mathbf{f} = (f_p)_{p=0}^\infty \in \ell_1(C(K), \mathfrak{M})$ with $f_0 \equiv 0$ and $n \in \mathbb{N}$ we have $\mathbf{f}^n = (g_p)_{p=0}^\infty$ where $g_p \equiv 0$ for $0 \leqslant p < n$ and

$$g_p = \sum_{\substack{p_1,\ldots,p_n \geqslant 1 \\ p_1+\cdots+p_n=n}} p! \prod_{j=1}^n \frac{f_{p_j}}{p_j!}.$$

Hence,

$$|\mathbf{f}^n|_{1,\mathfrak{M}} = \sum_{p=n}^\infty \Big\| \frac{g_p}{M_p} \Big\|_K \leqslant \sum_{p=n}^\infty \sum_{\substack{p_1,\ldots,p_n \geqslant 1 \\ p_1+\cdots+p_n=n}} \Big\| \frac{p!}{M_p} \prod_{j=1}^n \frac{M_{p_j}}{p_j!} \Big\|_K \cdot \prod_{j=1}^n \Big\| \frac{f_{p_j}}{M_{p_j}} \Big\|_K$$
$$\leqslant A_n |\mathbf{f}|_{1,\mathfrak{M}}^n,$$

where A_n is defined by

$$A_n := \sup \Big\{ \Big\| \frac{1}{m_p^p} \prod_{j=1}^n m_{p_j}^{p_j} \Big\|_K : p \geqslant n,\, p_1,\ldots,p_n \in \mathbb{N},\, \sum_{j=1}^n p_j = p \Big\}.$$

Therefore $\mathbf{f}$ will be quasi-nilpotent, if we can show

$$\lim_{n\to\infty} A_n^{1/n} = 0. \tag{6.8}$$

Let $\varepsilon > 0$ be arbitrary. By condition (6.5) there is some $n_0 \in \mathbb{N}$ such that for all $p \geqslant n_0$ and $0 \leqslant q \leqslant p$,

$$\Big\| \frac{m_{p-q}^{p-q} m_q^q}{m_p^p} \Big\|_K < \varepsilon.$$

Let $n \geqslant n_0$ and $p_1, \ldots, p_{n+1} \in \mathbb{N}$ with $p := p_1 + \cdots + p_{n+1} \geqslant n+1$. For all $z \in K$ we then have

$$\frac{1}{m_p(z)^p} \prod_{j=1}^{n+1} m_{p_j}(z)^{p_j} = \frac{m_{p-p_{n+1}}(z)^{p-p_{n+1}} m_{p_{n+1}}(z)^{p_{n+1}}}{m_p(z)^p \cdot m_{p-p_{n+1}}(z)^{p-p_{n+1}}} \prod_{j=1}^n m_{p_j}(z)^{p_j} \leqslant \varepsilon A_n.$$

Hence, $A_{n+1} \leqslant \varepsilon A_n$. This shows

$$\lim_{n\to\infty} \frac{A_n}{A_{n+1}} = 0$$

which implies (6.8) and we conclude that $\mathbf{f}$ is quasi-nilpotent.

The mapping $f \mapsto \Psi(f) := (\delta_{p,0} f)_{p=0}^\infty$ defines an isometric monomorphism $\Psi : C(K) \to \ell_1(C(K), \mathfrak{M})$. Hence, $r(\Psi(f)) = \|f\|_K$ for all $f \in C(K)$. As every element $\mathbf{f} = (f_p)_{p=0}^\infty$ is of the form $\mathbf{f} = \Psi(f_0) + ((1-\delta_{0,p}) f_p)_{p=0}^\infty$ and $((1-\delta_{0,p}) f_p)_{p=0}^\infty$ is quasi-nilpotent by the first part of the proof, we obtain (6.6) and (6.7). ∎

The following is now an immediate consequence of Lemma 6.9 and the main result of Honary in [24].

COROLLARY 6.10. *Let $K \subset \mathbb{C}$ be a perfect, compact set such that the operator* $\mathrm{d}/\mathrm{d}\zeta$ *is closable in* $C(K)$ *and let* $\mathfrak{M}$ *be a sequence as in Proposition* 6.9. *Then the character spaces of* $\widehat{D}_1(K,\mathfrak{M})$ *and* $\widetilde{D}_1(K,\mathfrak{M})$ *coincide with that of the uniform closure* $A_1(K,\mathfrak{M})$ *of* $D_1(K,\mathfrak{M})$ *in* $C(K)$*. In particular,* $\widehat{D}_1(K,\mathfrak{M})$ *and* $\widetilde{D}_1(K,\mathfrak{M})$ *are natural if* $A_1(K,\mathfrak{M})$ *is natural.*

Notice, that Theorem 3.3 in [1] is a direct consequence of Theorem 1.6 in [16] and Corollary 6.10.

For the analogue to Proposition 6.9 for $\ell_\infty(C(K),\mathfrak{M})$ we need a stronger assumption on the sequence $\mathfrak{M}$. In the proof we follow the idea of the proof of Lemma 3.4 in [16].

PROPOSITION 6.11. *Let K be a compact set and let $\mathfrak{M}$ be a sequence of bounded functions $M_p : K \to [1,\infty)$ satisfying* (1.1) *and*

$$\lim_{p\to\infty} \Big\| \sum_{q=1}^{p-1} B_{p,q} \Big\|_K = 0, \tag{6.9}$$

where

$$B_{p,q}(z) := \binom{p}{q} \frac{M_{p-q}(z)M_q(z)}{M_p(z)} \quad (z \in K).$$

Then

$$\operatorname{rad}(\ell_\infty(C(K),\mathfrak{M})) = \{\mathbf{f} = (f_p)_{p=0}^\infty \in \ell_\infty(C(K),\mathfrak{M}) : f_0 \equiv 0\}.$$

In particular, the spectrum and the spectral radius in $\ell_\infty(C(K),\mathfrak{M})$ are given by

$$\sigma(\mathbf{f}) = f_0(K) \quad \textit{and} \quad r(\mathbf{f}) = \|f_0\|_K \quad \textit{for all} \quad \mathbf{f} = (f_p)_{p=0}^\infty \in \ell_\infty(C(K),\mathfrak{M}). \tag{6.10}$$

Proof. For the proof it suffices to show that every $\mathbf{f} = (f_p)_{p=0}^\infty \in \ell_\infty(C(K),\mathfrak{M})$ for which f_0 has no zero on K has an inverse in $\ell_\infty(C(K),\mathfrak{M})$. For such an element $\mathbf{f}$ we have $\delta := \inf_{z\in K} |f_0(z)| > 0$. We may assume $\delta = 1$. Note that $\mathbf{1} := (\delta_{0,p})_{p=0}^\infty$ is the unit element in $\ell_\infty(C(K),\mathfrak{M})$. Hence $\mathbf{f}$ will be invertible if we can show that the sequence $\mathbf{g} = (g_p)_{p=0}^\infty$ of continuous functions, inductively defined by

$$g_0 := \frac{1}{f_0} \quad \text{and} \quad g_{p+1} := -\frac{1}{f_0} \sum_{q=1}^{p+1} \binom{p}{q} f_q g_{p-q} \quad (p \in \mathbb{N}_0),$$

is in $\ell_\infty(C(K),\mathfrak{M})$. With the notation $\alpha_p := |f_p|/M_p$, $\beta_p := |g_p|/M_p$, $p \in \mathbb{N}_0$, we have $\beta_0 \leqslant 1$ and hence

$$\begin{aligned} \beta_{p+1}(z) &\leqslant \alpha_{p+1}(z) + \sum_{q=1}^{p} B_{p,q}(z)\alpha_q(z)\beta_{p+1-q}(z) \\ &\leqslant |\mathbf{f}|_{\infty,\mathfrak{M}} \Big(1 + \Big\| \sum_{q=0}^{p} B_{p+1,q} \Big\|_K \cdot \max_{1\leqslant q\leqslant p} \|\beta_q\|_K\Big). \end{aligned}$$

Choose n such that $\Big\| \sum\limits_{q=0}^{p} B_{p+1,q} \Big\|_K \leqslant (2|\mathbf{f}|_{\infty,\mathfrak{M}})^{-1}$ for all $p \geqslant n$ and let m be the maximum of $2|\mathbf{f}|_{\infty,\mathfrak{M}}$ and $\max\limits_{1\leqslant q\leqslant n} \|\beta_q\|_K$. Then we have for $p \geqslant n$:

$$\|\beta_{p+1}\|_K \leqslant \frac{1}{2} \max_{1\leqslant q\leqslant p} \|\beta_q\|_K + |\mathbf{f}|_{\infty,\mathfrak{M}}.$$

By induction on p we obtain $\|\beta_p\|_K \leqslant m$ for all $p \in \mathbb{N}_0$, and so $\mathbf{g} \in \ell_\infty(C(K), \mathfrak{M})$. ∎

With the theorem of Honary [24] we obtain:

COROLLARY 6.12. *Let $K \subset \mathbb{C}$ be a perfect, compact set such that the operator $\mathrm{d}/\mathrm{d}\zeta$ is closable in $C(K)$ and let $\mathfrak{M}$ be a sequence as in Proposition 6.11. Then the character spaces of $\widehat{D}_\infty(K, \mathfrak{M})$ and $\widetilde{D}_\infty(K, \mathfrak{M})$ coincide with that of the uniform closure $A_\infty(K, \mathfrak{M})$ of $D_\infty(K, \mathfrak{M})$ in $C(K)$. In particular, $\widehat{D}_\infty(K, \mathfrak{M})$ and $\widetilde{D}_\infty(K, \mathfrak{M})$ are natural if $A_\infty(K, \mathfrak{M})$ is natural.*

As an example for situations in which condition (6.9) is satisfied, we show:

LEMMA 6.13. *Let $K \subset \mathbb{C}$ be a perfect, compact set and let $\alpha, \beta : K \to [0, \infty)$ be two continuous functions satisfying $\alpha(z) + \beta(z) > 0$ for all $z \in K$. The sequence $\mathfrak{M} = (M_p)_{p=0}^\infty$ given by $M_0 \equiv 1$ and*

$$M_p(z) := p! \cdot p^{p\alpha(z)} \cdot (\log(p + e))^{p\beta(z)} \quad (z \in K, p \in \mathbb{N}).$$

satisfies conditions (1.1) *and* (6.9), *hence also the algebra condition* (1.3) *and* (6.5).

Proof. With the notation of Proposition 6.11, we have in this situation

$$B_{p,q}(z) = B_{\alpha,p,q}^{\alpha(z)} B_{\beta,p,q}^{\beta(z)} \quad (z \in K),$$

where

$$B_{\alpha,p,q} = \frac{q^q (p-q)^{p-q}}{p^p} < 1 \quad \text{and} \quad B_{\beta,p,q} = \frac{(\log(e+q))^q (\log(e+p-q))^{p-q}}{(\log(e+p))^p}$$

for $1 \leqslant q < p$. Using elementary calculus, one easily shows that the functions $q \mapsto B_{\alpha,p,q}$ and $q \mapsto B_{\beta,p,q}$ are strictly monotone decreasing on $(0, p/2]$ and obtains $B_{\alpha,p,q} < 1$ and $B_{\beta,p,q} < 1$ for $1 \leqslant q < p$. In particular, $\mathfrak{M}$ satisfies (1.1).

By the assumptions on the functions α and β there are compact subsets K_α, K_β of K with $K = K_\alpha \cup K_\beta$ and

$$c_\alpha := \min_{z\in K_\alpha} \alpha(z) > 0, \quad c_\beta := \min_{z\in K_\beta} \beta(z) > 0.$$

With

$$S_{\alpha,p} := \sum_{q=1}^{p-1} B_{\alpha,p,q}^{c_\alpha} \quad \text{and} \quad S_{\beta,p} := \sum_{q=1}^{p-1} B_{\beta,p,q}^{c_\beta}$$

we obtain

$$\Big\| \sum_{q=1}^{p-1} B_{p,q} \Big\|_K \leqslant \max\{S_{\alpha,p}, S_{\beta,p}\}.$$

For $0 < \varepsilon < 1$ we fix q_0 such that $2^{1-c_\alpha}/(1-2^{-c_\alpha}) < \varepsilon/2$ and $p_0 > q_0$ with $2q_0^{1+c_\alpha}p_0^{-c_\alpha} < \varepsilon/2$. For all $p \geqslant 2p_0$ we then obtain

$$S_{\alpha,p} \leqslant 2 \sum_{1\leqslant q\leqslant p/2} \Big(\frac{q}{p}\Big)^{qc_\alpha} \leqslant 2 \sum_{q=1}^{q_0-1} \Big(\frac{q_0}{p}\Big)^{qc_\alpha} + 2 \sum_{q_0<q\leqslant p/2} \Big(\frac{q}{p}\Big)^{qc_\alpha} 2^{-qc_\alpha} < \varepsilon.$$

For the estimate of $S_{\beta,p}$ we proceed similar as in the proof of Lemma 3.3 in [16]. Fix $\delta \in (0,1)$ with $4\delta^{c_\beta}/(1-\delta^{c_\beta}) < \varepsilon$ and then $p_1 > 2/\varepsilon$ such that

$$\forall p \geqslant p_1 : \quad \frac{\log(e+1+\log p)}{\log(e+p)} < \min\{\delta, e^{-2/c_\beta}\}.$$

for all $p \geqslant p_1$ let q_p denote the smallest integer greater than $\log p$. We obtain the following and hence, (6.9) is satisfied:

$$\begin{aligned} S_{\beta,p} &\leqslant 2 \sum_{1\leqslant q\leqslant p/2} B_{\beta,p,q}^{c_\beta} \leqslant 2 \sum_{q=1}^{q_p} \Big(\frac{\log(e+q)}{\log(e+p)}\Big)^{qc_\beta} + 2 \sum_{q_p<q\leqslant p/2} B_{\beta,p,q_p}^{c_\beta} \\ &< 2 \sum_{q=1}^{q_p} \delta^{qc_\beta} + p\Big(\frac{\log(e+1+\log p)}{\log(e+p)}\Big)^{q_pc_\beta} < \frac{\varepsilon}{2} + p\exp(-2\log p) < \varepsilon. \quad \blacksquare \end{aligned}$$

For the sequence $\mathfrak{M} = (M_p)_{p=0}^\infty$ with

$$M_p(z) := p!^{1+\alpha(z)} \prod_{j=1}^{p} (\log(e+j))^{\beta(z)} \quad (p \in \mathbb{N}_0, z \in K)$$

with α and β as in the previous lemma the same statement can be proved.

REMARK 6.14. If $\mathfrak{M}$ satisfies $\mathrm{Rat}(K) \subset \widehat{D}_s(K,\mathfrak{M})$ and $R(K) = C(K)$, then the algebra $A_s(K,\mathfrak{M})$ coincides with $C(K)$ and is hence natural ($s = 1,\infty$).

We also note the following fact which follows from Lemma 6.6 with some standard methods (cf. the proof of Theorem 1.8 in [16]).

REMARK 6.15. If $K \subset \mathbb{C}$ is a perfect, compact set such that $\mathrm{d}/\mathrm{d}z$ is closable in $C(K)$ and if $\mathfrak{M}$ is a sequence satisfying (1.1) and $\mathrm{Rat}(K) \subset D_s(K,\mathfrak{M})$ (and in the case $s = \infty$ also (1.3)), then the closure $R_s(K,\mathfrak{M})$ of $\mathrm{Rat}(K)$ in $\widehat{D}_s(K,\mathfrak{M})$ is a natural Banach function algebra.

7. NORMALITY AND REGULARITY CRITERIA

For a compact set $K \subset \mathbb{C}$ and a point $z \in K$ we call

$$d_K(z) := \limsup_{\varepsilon\to 0} \overline{\dim}_{\mathrm{B}}(K \cap \overline{U_\varepsilon(z)})$$

the *upper box dimension of* K *at* z. If K is in addition perfect and $\mathfrak{M} = (M_p)_{p=0}^\infty$ a sequence of bounded functions $M_p : K \to [1,\infty)$ satisfying $M_0 \equiv 1$ and (1.1) then, with the notation introduced in (6.1), we define for all $k \in \mathbb{N}$,

$$\Lambda^k_{\mathfrak{M}}(z) := \limsup_{\varepsilon\to 0}\limsup_{p\to\infty}\Big\|\frac{\mathrm{Lc}_k(p)}{\mu_p}\Big\|_{K\cap\overline{U_\varepsilon(z)}}. \tag{7.1}$$

THEOREM 7.1. *Let $K \subset \mathbb{C}$ be a perfect, compact set, for which the operator* $\mathrm{d}/\mathrm{d}z$ *is closable in $C(K)$, and let $\mathfrak{M}$ be a sequence as above, which in addition satisfies* (6.2)*. Denote by Ω_0 the set of all points $z \in K$ such that for each $\varepsilon > 0$ there exists a closed square $Q_z \subset U_\varepsilon(z)$ with $z \in \mathrm{int}Q_z$ such that for Q_z and $K_z := \partial Q_z \cup (Q_z \cap K)$, the asymptotic dimension condition in part* (ii) *of Proposition* 3.3 *is satisfied with some $m \in \mathbb{N}$ and some $\delta > 0$. The sets*

$$\Omega_1 := \{z \in \Omega_0 : \Lambda^1_{\mathfrak{M}}(z) < \infty\} \quad \textit{and} \quad \Omega_2 := \{z \in K : d_K(z) + \Lambda^2_{\mathfrak{M}}(z) < 2\}$$

are open for the relative topology on K. If $S := K \setminus (\Omega_1 \cup \Omega_2)$ is totally disconnected, then $\widehat{D}_1(K,\mathfrak{M})$ and $\widetilde{D}_1(K,\mathfrak{M})$ are normal Banach function algebras. The maximal regular subalgebra of $\widehat{D}_1(K,\mathfrak{M})$ contains $R_1(K,\mathfrak{M})$. If in addition $\mathfrak{M}$ satisfies condition (6.5)*, then $\widehat{D}_1(K,\mathfrak{M})$ and $\widetilde{D}_1(K,\mathfrak{M})$ are regular.*

Proof. As $\mathrm{d}/\mathrm{d}z$ is closable in $C(K)$ we must have $d_K(z) \geqslant 1$ for all $z \in K$ (cf. Remark 6.5). It follows immediately from the definitions of Ω_0, $\Lambda^k_{\mathfrak{M}}$, d_K, that the sets Ω_1, Ω_2 are relatively open in K. Because of (6.2) the algebra $D_1(K,\mathfrak{M})$ contains $\mathrm{Rat}(K)$ and ω, defined by the following, is an entire function:

$$\omega(\zeta) := \sum_{p=0}^{\infty}\Big\|\frac{1}{m_p}\Big\|_K^p \zeta^{p+1} \quad (\zeta \in \mathbb{C}).$$

Let z be an arbitrary point in Ω_1. Then there exist constants $\varepsilon_0 \in (0,1)$, $C > 0$ and $p_0 \in \mathbb{N}$ such that $\log p \leqslant C\mu_p(\zeta)$ for all $p \geqslant p_0$ and all $\zeta \in K \cap \overline{U_{\varepsilon_0}(z)}$. In particular we have

$$\Big\|\frac{1}{m_p}\Big\|_K \leqslant p^{-1/C} \quad (p \geqslant p_0). \tag{7.2}$$

Because of $z \in \Omega_0$ there exists some closed square $Q_z \subset U_{\varepsilon_0}(z)$ with centre z such that with $Q = Q_z$, $K = K_z := \partial Q_z \cup (Q_z \cap K)$, the asymptotic dimension condition in part (ii) of Proposition 3.3 is satisfied for some $m \in \mathbb{N}$ and some $\delta_z > 0$. Put $\eta := \mathrm{dist}(Q_z, \mathbb{C} \setminus U_{\varepsilon_0}(z))^{-1}$. For all $w \in Q_z \setminus K_z$ we have

$$\begin{aligned}\|(w-\cdot)^{-1}\|_{1,\mathfrak{M}} &\leqslant \sum_{p=0}^{\infty}\Big(\Big\|\frac{1}{(w-\cdot)^{p+1}m_p^p}\Big\|_{K\setminus U_{\varepsilon_0}(z)} + \Big\|\frac{1}{(w-\cdot)^{p+1}m_p^p}\Big\|_{\overline{U_{\varepsilon_0}(z)}\cap K}\Big)\\ &\leqslant \omega(\eta) + \sum_{p=0}^{\infty}\Big\|\frac{1}{m_p}\Big\|^p_{\overline{U_{\varepsilon_0}(z)}\cap K}\mathrm{dist}(w,K_z)^{-p-1}.\end{aligned}$$

With the help of (7.2) and Corollary 5.2(i) we see that for each $\alpha > 1/C$ there exists a constant $c > 0$ such that

$$\|(w-\cdot)^{-1}\|_{1,\mathfrak{M}} \leqslant \exp\Big(\frac{c}{\mathrm{dist}(w,K_z)^\alpha}\Big)$$

for all $w \in Q_z \backslash K_z$. By Proposition 3.3(ii), we see that the following integral is finite:

$$\int_{Q_z} f_{m,\delta_z/2}(\log_+ \log_+(\|(w-\cdot)^{-1}\|_{1,\mathfrak{M}}))\mathrm{d}\lambda_2(w).$$

Consider now an arbitrary point z in Ω_2 and fix $\delta > 0$ with $d_K(z) + \Lambda^2_{\mathfrak{M}}(z) < 2 - 2\delta$. Hence, there exists constants $\varepsilon_0 > 0$, $p_0 \in \mathbb{N}$ and $C \in (0,1)$ such that $\log\log p \leqslant C\mu_p(\zeta)$ for all $p \geqslant p_0$ and all $\zeta \in K \cap \overline{U_{\varepsilon_0}(z)}$. In particular we have

$$\left\| \frac{1}{m_p} \right\|_{K \cap \overline{U_{\varepsilon_0}(z)}} \leqslant \log(e+p)^{-1/C} \quad (p \geqslant p_0). \tag{7.3}$$

Fix now $\alpha \in (C, 2 - \Lambda^2_{\mathfrak{M}}(z) - 2\delta)$. By the definition of $\Lambda^2_{\mathfrak{M}}(z)$ there exists some closed square $Q_z \subset U_{\varepsilon_0}(z)$ with centre z such that for Q_z and $K_z := \partial Q_z \cup (Q_z \cap K)$, we have $\dim_B K_z < \Lambda^2_{\mathfrak{M}}(z) + \delta$. With $\eta := \operatorname{dist}(Q_z, \mathbb{C} \setminus U_{\varepsilon_0}(z))^{-1}$ we obtain (as in the case $z \in \Omega_1$) the estimate

$$\|(w-\cdot)^{-1}\|_{1,\mathfrak{M}} \leqslant \omega(\eta) + \sum_{p=0}^{\infty} \left\| \frac{1}{m_p} \right\|^p_{\overline{U_{\varepsilon_0}(z)} \cap K} \operatorname{dist}(w, K_z)^{-p-1} \quad (w \in Q_z \setminus K_z).$$

With the help of (7.3) and Corollary 5.2(ii) one easily obtains the existence of a constant $c > 0$ such that

$$\log\log \|(w-\cdot)^{-1}\|_{1,\mathfrak{M}} \leqslant c\operatorname{dist}(w, K_z)^{-\alpha}$$

for all $w \in Q_z \setminus K_z$. Now we may apply part (i) of Proposition 3.3 and see that the following integral is finite:

$$\int_{Q_z} (\log_+ \log_+(\|(w-\cdot)^{-1}\|_{1,\mathfrak{M}}))^{1+\delta}\mathrm{d}\lambda_2(w).$$

As $\|(w-\cdot)^{-1}\|_{1,\mathfrak{M}}$ coincides with the norm of the resolvent $(w - L_{\mathrm{id}_K})^{-1}$ of the operator L_{id_K} of multiplication by id_K on each of the three Banach function algebras $R_1(K,\mathfrak{M})$, $\widehat{D}_1(K,\mathfrak{M})$, $\widetilde{D}_1(K,\mathfrak{M})$, we conclude from Proposition 4.1 that these multiplication operators are decomposable. By Lemma 4.8 this implies that $\widehat{D}_1(K,\mathfrak{M})$, $\widetilde{D}_1(K,\mathfrak{M})$ and $R_1(K,\mathfrak{M})$ are normal. As $R_1(K,\mathfrak{M})$ is a natural Banach function algebra (by Remark 6.15), it is even regular. ∎

The following variant of Theorem 7.1 for $D_\infty(K,\mathfrak{M})$ can be obtained with only minor changes in the proof.

THEOREM 7.2. *Let $K \subset \mathbb{C}$ be a perfect, compact set, for which the operator* $\mathrm{d}/\mathrm{d}z$ *is closable in $C(K)$, and let $\mathfrak{M}$ be a sequence as above, which in addition satisfies* (6.2) *and* (1.3). *Define Ω_0, Ω_1 and Ω_2 as in the statement of Theorem* 7.1. *Then the sets Ω_1 and Ω_2 are open for the relative topology on K. If $S := K \setminus (\Omega_1 \cup \Omega_2)$ is totally disconnected, then $\widehat{D}_\infty(K,\mathfrak{M})$ and $\widetilde{D}_\infty(K,\mathfrak{M})$ are normal Banach function algebras. The maximal regular subalgebra of $\widehat{D}_\infty(K,\mathfrak{M})$ contains $R_\infty(K,\mathfrak{M})$. If in addition $\mathfrak{M}$ satisfies condition* (6.9), *then $\widehat{D}_\infty(K,\mathfrak{M})$ and $\widetilde{D}_\infty(K,\mathfrak{M})$ are regular.*

EXAMPLE 7.3. Let $K \subset \mathbb{C}$ be a perfect, compact set for which the operator $\mathrm{d}/\mathrm{d}z$ is closable in $C(K)$. Let $\alpha, \beta : K \to [0, \infty)$ be two continuous functions and define the sequence $\mathfrak{M} = (M_p)_{p=0}^{\infty}$ by $M_0 \equiv 1$ and

$$M_p(z) := p! \cdot p^{p\alpha(z)} \cdot \log(e+p)^{p\beta(z)} \quad (p \in \mathbb{N},\, z \in K).$$

Suppose that the following three conditions are satisfied:

(i) $\beta \geqslant 1$ on $\{z \in K : \alpha(z) = 0\}$ and the set $S := \{z \in K; \beta(z) = 1 \text{ and } \alpha(z) = 0\}$ is totally disconnected.

(ii) For all $z \in \{\zeta \in K \setminus S : \alpha(\zeta) = 0\}$ we have $d_K(z) + \beta(z)^{-1} < 2$.

(iii) For all $z \in \{\zeta \in K \setminus S : \alpha(\zeta) \neq 0\}$ the asymptotic dimension condition in part (ii) of Proposition 3.3 is satisfied for some $m \in \mathbb{N}$ and some $\delta > 0$.

Then the Banach function algebras $R_s(K, \mathfrak{M})$, $\widehat{D}_s(K, \mathfrak{M})$, and $\widetilde{D}_s(K, \mathfrak{M})$ are regular for $s = 1$ and $s = \infty$.

Proof. By Lemma 6.13, $\mathfrak{M}$ satisfies conditions (1.1), (6.9) and therefore also (1.3) and (6.5).

By the continuity of α and β we have $\Lambda^1_{\mathfrak{M}}(z) = \alpha(z)^{-1}$ if $\alpha(z) > 0$ and $\Lambda^2_{\mathfrak{M}}(z) \leqslant \beta(z)^{-1}$ if $\alpha(z) = 0$. Hence, by Theorem 7.1 respectively Theorem 7.2, the algebras $\widehat{D}_s(K, \mathfrak{M})$ and $\widetilde{D}_s(K, \mathfrak{M})$ are natural, regular Banach function algebras containing $R_s(K, \mathfrak{M})$ as a regular closed subalgebra ($s = 1, \infty$). ∎

For the sequence $\mathfrak{M} = (M_p)_{p=0}^{\infty}$ with

$$M_p(z) := p!^{1+a(z)} \prod_{j=1}^{p} (\log(e+j))^{\beta(z)} \quad (p \in \mathbb{N}_0, z \in K)$$

with α and β as in the previous example a similar statement can be proved.

E. Bishop has given in [7] an example of a non-rectifiable, compact Jordan arc K, for which $\mathrm{d}/\mathrm{d}z$ is not closable in $C(K)$. Moreover, by Theorem 10.5 of Dales and Feinstein in [17], the operator $\mathrm{d}/\mathrm{d}z$ cannot be closed if K is any non-rectifiable Jordan arc. On the other hand, constructions as given in [26] show, that there exist compact Jordan arcs K of positive planar Lebesgue measure (which are of course not rectifiable) such that the union of all rectifiable subarcs of K is dense in K. In this situation, $\mathrm{d}/\mathrm{d}z$ must be closable in $C(K)$ by Theorem 6.3 in [17]. Moreover, for the examples constructed in [26] there is a totally disconnected closed subset S of K such that all points in $K \setminus S$ are contained in some rectifiable subarcs of K. It follows from the previous example, that in this situation the algebras $\widehat{D}_s(K, \mathfrak{M})$ and $\widetilde{D}_s(K, \mathfrak{M})$ are natural, regular Banach function algebras containing $R_s(K, \mathfrak{M})$ as a regular closed subalgebra ($s = 1, \infty$) for the sequence $\mathfrak{M} = (M_p)_{p=0}^{\infty}$ with

$$M_p(z) := p!(\log(e+p))^{p(1+\mathrm{dist}(z,S))} \quad (z \in K, p \in \mathbb{N}_0).$$

Acknowledgements. The second author was supported by HEC–DAAD. The first author thanks Professor H.G. Dales for a valuable hint to the literature.

REFERENCES

[1] M. ABTAHI, T.G. HONARY, On the maximal ideal space of Dales–Davie algebras of infinitely differentiable functions, *Bull. London Math. Soc. (6)* **39**(2007), 940–948.

[2] E. ALBRECHT, Decomposable systems of operators in harmonic analysis, in *Toeplitz Centennial (Tel Aviv, 1981)*, Oper. Theory Adv. Appl., vol. 4, Birkhäuser Verlag, Basel 1982, pp. 19–35.

[3] E. ALBRECHT, J. ESCHMEIER, Analytic functional models and local spectral theory, *Proc. London Math. Soc.* (3) **75**(1997), 323–348.

[4] E. ALBRECHT, W. RICKER, Local spectral theory for operators with thin spectrum, in *Spectral Analysis and its Applications.* I. *Colojoară Anniversary Volume*, Theta Ser. Adv. Math., vol. 2, Theta Foundation, Bucharest 2003, pp. 1–25.

[5] C. APOSTOL, Decomposable multiplication operators, *Rev. Roumaine Math. Pures Appl.* **17**(1972), 323–333.

[6] T. ATHAR, Local spectral methods in the theory of Banach function algebras, Ph.D. Dissertation, Universität des Saarlandes, Saarbrücken 2010.

[7] E. BISHOP, Approximation by a polynomial and its derivatives on certain closed sets, *Proc. Amer. Math. Soc.* **9**(1958), 946–953.

[8] E. BISHOP, A duality theorem for an arbitrary operator, *Pacific J. Math.* **9**(1959), 379–397.

[9] W.J. BLAND, J.F. FEINSTEIN, Completions of normed algebras of differentiable functions, *Studia Math.* **170**(2005), 89–111.

[10] M. BRELOT, Minorantes sous-harmoniques, extrémales et capacités, *J. Math. Pures Appl.* **24**(1945), 1–32.

[11] L. BROWN, L. RUBEL, Rational approximation and swiss cheeses of positive area, *Kodai Math. J.* **5**(1982), 132–133.

[12] S.W. BROWN, Hyponormal operators with thick spectra have invariant subspaces, *Ann. of Math. (2)* **125**(1987), 93–103.

[13] I. COLOJOARĂ, C. FOIAŞ, *Theory of Generalized Spectral Operators*, Gordon and Breach, New York-London-Paris 1968.

[14] H.G. DALES, Boundaries and peak points for algebras of functions, Ph.D. Dissertation, Univ. of Newcastle-Upon Tyne, Newcastle-Upon Tyne 1970.

[15] H.G. DALES, *Banach Algebras and Automatic Continuity*, Londom Math. Soc. Monographas, vol. 24, Oxford Sci. Publ., The Clarendon Press, Oxford Univ. Press, New York, 2000.

[16] H.G. DALES, A.M. DAVIE, Quasianalytic Banach function algebras, *J. Funct. Anal.* **13**(1973), 28–50.

[17] H.G. DALES, J.F. FEINSTEIN, Normed algebras of differentiable functions on compact plane sets, *Indian J. Pure Appl. Math.* **41**(2010), 153–187.

[18] Y. DOMAR, On the existence of a largest subharmonic minorant of a given function, *Ark. Mat.* **3**(1957), 429–440.

[19] Y. DOMAR, Uniform boundedness in families related to subharmonic functions, *J. London Math. Soc. (2)* **38**(1988), 485–491.

[20] K.J. FALCONER, *Fractal Geometry. Mathematical Foundations and Applications*, John Wiley and Sons, Chichester 1990.

[21] C. FOIAŞ, Spectral maximal spaces and decomposable operators in Banach spaces, *Arch. Math. (Basel)* **14**(1963), 341–349.

[22] ŞT. FRUNZĂ, A characterization of regular Banach algebras, *Rev. Roumaine Math. Pures Appl.* **18**(1973), 1057–1059.

[23] J.W. GREEN, Approximately subharmonic functions, *Proc. Amer. Math. Soc.* **3**(1952), 829–833.

[24] T.G. HONARY, Relations between Banach function algebras and their uniform closure, *Proc. Amer. Math. Soc.* **109**(1990), 337–342.

[25] K. KELLAY, M. ZARRABI, Normality, non-quasianalyticity and invariant subspaces, *J. Operator Theory* **46**(2001), 221–250.

[26] T. LANCE, E. THOMAS, Arcs with positive measure and a space filling curve, *Amer. Math. Monthly (2)* **98**(1991), 124–127.

[27] K.B. LAURSEN, M.M. NEUMANN, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford 2000.

[28] V. PELLER, Invariant subspaces and Toeplitz operators, *J. Soviet Math.* **5**(1984), 2533–2539.

[29] V. PELLER, Spectrum, similarity, invariant subspaces of Toeplitz operators, *Izv. Akad. Nauk. SSSR* **50**(1986), 776–787.

[30] T. RANSFORD, *Potential Theory in the Complex Plane*, London Math. Soc. Student Texts, vol. 28, Cambridge Univ. Press, Cambridge 1995.

[31] F.-H. VASILESCU, Residually decomposable operators in Banach spaces, *Tôhoku Math. J.* **21**(1969), 509–522.

[32] F.-H. VASILESCU, *Analytic Functional Calculus and Spectral Decompositions*, D. Reidel Publ. Comp., Dordrecht 1982.

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Received May 5, 2011.