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OPERATORS AND FRAMES

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ABSTRACT. Hilbert space frame theory has applications to various areas of pure mathematics, applied mathematics, and engineering. However, the question of how applying an invertible operator to a frame changes its properties has not yet been satisfactorily answered, and only partial results are known to date. In this paper, we will provide a comprehensive study of those questions, and, in particular, prove characterization results for (1) operators which generate frames with a prescribed frame operator; (2) operators which change the norms of the frame vectors by a constant multiple; (3) operators which generate equal norm nearly Parseval frames.

KEYWORDS: *Analysis operator, frame operator, Hilbert space frame, synthesis operator.*

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INTRODUCTION

To date, Hilbert space frame theory has broad applications in pure mathematics, see, for instance, [5], [8], [11], as well as in applied mathematics, computer science, and engineering. This includes time-frequency analysis [15], wireless communication [16], [20], image processing [19], coding theory [21], quantum measurements [14], sampling theory [13], and bioimaging [18], to name a few. A fundamental tool in frame theory are the analysis, synthesis and frame operators associated with a given frame. A deep understanding of these operators is indeed crucial to frame theory and its applications.

Let us start by recalling the basic definitions and notions of frame theory. Given a family of vectors $\{f_i\}_{i=1}^M$ in an N-dimensional Hilbert space $\mathcal{H}_N$, we say that $\{f_i\}_{i=1}^M$ is a *frame* if there exist constants $0 < A \leqslant B < \infty$ satisfying

$$A\|f\|^2 \leqslant \sum_{i=1}^{M} |\langle f, f_i\rangle|^2 \leqslant B\|f\|^2 \quad \text{for all } f \in \mathcal{H}_N.$$

The numbers A, B are called *lower and upper frame bounds* (respectively) for the frame. If $A = B$, $\{f_i\}_{i=1}^M$ is called an *A-tight frame*, and if $A = B = 1$, it is referred

to as a *Parseval frame*. The frame is an *equal norm frame*, if $\|f_i\| = \|f_k\|$ for all $1 \leqslant i,k \leqslant M$, and a *unit norm frame*, if $\|f_i\| = 1$ for all $1 \leqslant i \leqslant M$. The *analysis operator* $F : \mathcal{H}_N \to \ell_2(M)$ of the frame is defined by

$$F(f) = \{\langle f, f_i\rangle\}_{i=1}^M.$$

The adjoint of the analysis operator is referred to as the *synthesis operator* $F^* : \ell_2(M) \to \mathcal{H}_N$, which is

$$F^*(\{a_i\}_{i=1}^M) = \sum_{i=1}^M a_i f_i.$$

The *frame operator* is then the positive, self-adjoint invertible operator $S : \mathcal{H}_N \to \mathcal{H}_N$ given by $S = F^*F$, i.e.,

$$Sf = \sum_{i=1}^M \langle f, f_i\rangle f_i, \quad \text{for all } f \in \mathcal{H}.$$

Moreover, two frames $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$ in $\mathcal{H}$ are called *equivalent* (*unitarily equivalent*), if there exists an invertible (a unitary) operator $T : \mathcal{H} \to \mathcal{H}$ such that $g_i = Tf_i$ for all $i = 1, \ldots, M$. For further background on frame theory, we refer the reader to [10], [12].

Letting $\{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ with frame operator S, and letting T be an invertible operator on $\mathcal{H}$, a fundamental, but poorly understood, question in frame theory is the following: How do the frame properties of $\{Tf_i\}_{i=1}^M$ relate to the frame properties of $\{f_i\}_{i=1}^M$? Deserving particular attention are the following subproblems due to their relevance for theoretical as well as practical applicability of the generated frames.

(i) *Frame operator*. Often a particular frame operator is desired, which yields the question: Is it possible to classify the operators T under which the frame operator is invariant? More generally, can we classify those invertible operators such that the generated frame has a prescribed frame operator?

(ii) *Norm.* It is essential to have a means to control the norms of the generated frame elements due to, for instance, numerical stability issues. This raises the following question: Can we classify those invertible operators which map a frame to another frame which is equal norm? Another variant is the classification problem of all invertible operators T for which $\|f_i\| = c\|Tf_i\|$ for all $1 \leqslant i \leqslant M$.

(iii) *Parseval frame*. Parseval frames are crucial for applications, since from a numerical standpoint, they are optimally stable. Also for theoretical purposes those are the most useful frames to utilize, for instance, for the decomposition of mathematical objects. Often, however it is not possible to construct an "exact" Parseval frame, which leads to the problem of deriving a deep understanding of operators which map a frame to a nearly Parseval frame.

In this paper, we will provide comprehensive answers to these questions, which are in fact long-standing open problems or are close to such, giving proof to both their significance and their difficulty.

An answer to the first question complex will presumably, as we will indeed see, provide a way to index frames which possess the same frame operator. This in fact solves a problem which has been debated in frame theory for several years, see, for instance, [3].

The second group of problems has been discussed at meetings for many years but has not formally been stated in the literature. These are fundamental questions which often arise when one is trying to improve frame properties by applying an invertible operator to the frame.

Relating to the third problem complex, despite the significance of equal norm Parseval frames for applications, their class is one of the least understood classes of frames. The reason is that for any frame $\{f_i\}_{i=1}^M$ with frame operator S, the closest Parseval frame is the canonical Parseval frame $\{S^{-1/2}f_i\}_{i=1}^M$, which is rarely equal norm. Because of the difficulty of finding equal norm Parseval frames, the famous *Paulsen Problem* as stated below is still open to date. For stating it, recall that given $0 < \varepsilon < 1$, a frame $\{f_i\}_{i=1}^M$ is *ε-nearly Parseval*, if its frame operator S satisfies $(1-\varepsilon)\mathrm{Id} \leqslant S \leqslant (1+\varepsilon)\mathrm{Id}$. Also the frame is *$\varepsilon$-nearly equal norm*, provided that $|\|f_i\|^2 - N/M| < \varepsilon$ for all $i = 1, \dots, M$. Armed with these notions, we now state the Paulsen Problem.

PROBLEM 0.1 (Paulsen Problem). *Find a function $f(\varepsilon, N, M)$, such that for each frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$ which is both ε-nearly equal norm and ε-nearly Parseval, there exists an equal norm Parseval frame $\{g_i\}_{i=1}^M$ satisfying*

$$\sum_{i=1}^{M} \|f_i - g_i\|^2 \leqslant f(\varepsilon, N, M).$$

We refer the reader to [2], [5], [7] for some recent results on this problem. The question we deal with in this paper is closely related to the Paulsen Problem, and we anticipate our answer to provide a new direction of attack.

This paper is organized as follows. In Section 1, we first classify those operators which leave the frame operator invariant. A characterization of operators mapping frames to frames with comparable norms of the frame vectors is provided in Section 2. Section 3 is then devoted to the study of operators which generate nearly Parseval frames.

1. PRESCRIBED FRAME OPERATORS

1.1. INVARIANCE OF THE FRAME OPERATOR.

The first question in this section to tackle is the characterization of invertible operators which leave the frame operator invariant. More precisely, given a frame $\{f_i\}_{i=1}^M$ for a Hilbert space $\mathcal{H}_N$ with frame operator S, we aim to classify the invertible operators T on $\mathcal{H}_N$ for which the frame operator for $\{Tf_i\}_{i=1}^M$ equals S. We remark that not even unitary operators possess this property, the reason being that although a unitary operator

applied to a frame will maintain the eigenvalues of the frame operator, it will however in general not maintain the eigenvectors.

The exact mathematical formulation of this question is the following:

QUESTION 1.1. *Given a frame $\{f_i\}_{i=1}^M$ with frame operator S, can we classify the invertible operators T so that the frame operator of the generated frame $\{Tf_i\}_{i=1}^M$ equals S?*

We start with a well known result identifying the frame operator of the frame $\{Tf_i\}_{i=1}^M$. Since the proof is just one line, we include it for completeness.

THEOREM 1.2. *If $\{f_i\}_{i=1}^M$ is a frame for $\mathcal{H}_N$ with frame operator S, and T is an operator on $\mathcal{H}_N$, then the frame operator for $\{Tf_i\}_{i=1}^M$ equals TST^*. If T is invertible, then $\{Tf_i\}_{i=1}^M$ also constitutes a frame for $\mathcal{H}_N$.*

Proof. The claim follows from the fact that the frame operator for $\{Tf_i\}_{i=1}^M$ is given by

$$\sum_{i=1}^M \langle f, Tf_i\rangle Tf_i = T\Big(\sum_{i=1}^M \langle T^*f, f_i\rangle f_i\Big) = TST^*f. \quad \blacksquare$$

This leads to the following reformulation of Question 1.1:

QUESTION 1.3. *Given a positive, self-adjoint, invertible operator S, can we classify the invertible operators T for which $TST^* = S$?*

For the case of Parseval frames, the answer is well-known [9]. Since known proofs were highly non-trivial, we provide a trivial proof, which seems to have been overlooked in previous publications.

COROLLARY 1.4. *If $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$ are equivalent Parseval frames, then they are unitarily equivalent.*

In particular, in the case of Parseval frames, the desired set of operators in Question 1.1 *are the unitary operators.*

Proof. Since $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$ are equivalent, there exists an invertible operator T so that $f_i = Tg_i$ for every $i = 1, 2, \ldots, M$. By Theorem 1.2, the fact that both frames constitute Parseval frames implies that $T\,\mathrm{Id}\,T^* = \mathrm{Id}$. $\blacksquare$

1.2. GENERAL CHARACTERIZATION RESULT. Instead of directly providing an answer to Question 1.1 for any frame, we will now first state the generalization of this question whose answer will then include the solution to this problem. For this, we will start with the following consequence of Theorem 1.2.

COROLLARY 1.5. *Let $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$ be frames for $\mathcal{H}_N$ with frame operators S_1 and S_2, respectively. Then there exists an invertible operator T on $\mathcal{H}_N$ such that S_1 is the frame operator of $\{Tg_i\}_{i=1}^M$.*

Proof. Letting $T = S_1^{1/2} S_2^{-1/2}$, by Theorem 1.2, we obtain

$$TS_2T^* = (S_1^{1/2}S_2^{-1/2})S_2(S_1^{1/2}S_2^{-1/2})^* = S_1. \quad \blacksquare$$

Hence asking for the generation of frames with a prescribed frame operator can be formulated in terms of operator theory as the following generalization of Question 1.3 shows.

QUESTION 1.6. *Given two positive, invertible, self-adjoint operators S_1 and S_2 on a Hilbert space $\mathcal{H}$, can we classify the invertible operators T on $\mathcal{H}$ for which $S_1 = TS_2T^*$?*

A first classification result answering Question 1.6 is the following. In this context, we mention that condition (ii) shall also be compared with the choice of T in the proof of Corollary 1.5.

THEOREM 1.7. *Let S_1, S_2 be positive, self-adjoint, invertible operators on a Hilbert space $\mathcal{H}$, and let T be an invertible operator on $\mathcal{H}$. Then the following conditions are equivalent:*

(i) $S_2 = TS_1T^*$.

(ii) *There exists a unitary operator U on $\mathcal{H}$ such that $T = S_2^{1/2} U S_1^{-1/2}$.*

Proof. (i) $\Rightarrow$ (ii) We set $U = S_2^{-1/2} T S_1^{1/2}$, which is a unitary operator, since

$$(S_2^{-1/2}TS_1^{1/2})(S_2^{-1/2}TS_1^{1/2})^* = S_2^{-1/2}TS_1T^*S_2^{-1/2} = S_2^{-1/2}S_2S_2^{-1/2} = \mathrm{Id}.$$

Moreover, we have

$$S_2^{1/2}US_1^{-1/2} = S_2^{1/2}(S_2^{-1/2}TS_1^{1/2})S_1^{-1/2} = T.$$

(ii) $\Rightarrow$ (i) Since $T = S_2^{1/2} U S_1^{-1/2}$, we obtain

$$TS_1T^* = S_2^{1/2}US_1^{-1/2}S_1S_1^{-1/2}U^*S_2^{1/2} = S_2^{1/2}UU^*S_2^{1/2} = S_2^{1/2}\,\mathrm{Id}\,S_2^{1/2} = S_2,$$

which is (i). $\blacksquare$

This result is however not entirely satisfactory, since one might prefer to have an explicit construction of all invertible operators T satisfying $S_1 = TS_2T^*$.

1.3. CONSTRUCTIVE CLASSIFICATION. We start with some preparatory lemmata, the first being an easy criterion for identifying the eigenvectors of a positive, self-adjoint operator.

LEMMA 1.8. *Let $T : \mathcal{H}_N \to \mathcal{H}_N$ be an invertible operator on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be an orthonormal basis for $\mathcal{H}_N$. Then the following conditions are equivalent:*

(i) *$\{Te_j\}_{j=1}^N$ is an orthogonal set.*

(ii) *$\{e_j\}_{j=1}^N$ is an eigenbasis for T^*T with respective eigenvalues $\|Te_j\|^2$.*

In particular, T must map some orthonormal basis to an orthogonal set.

Proof. For any $1 \leqslant j,k \leqslant N$, we have

$$\langle (T^*T)e_j, e_k\rangle = \langle Te_j, Te_k\rangle.$$

Hence, $\{Te_j\}_{j=1}^N$ is an orthogonal set if and only if $\langle (T^*T)e_j, e_k\rangle = 0$ for all $1 \leqslant j \neq k \leqslant N$. This in turn is equivalent to the condition that $T^*Te_j = \lambda_j e_j$ for all $j = 1, 2, \ldots, N$ with

$$\lambda_j = \langle (T^*T)e_j, e_j\rangle = \langle Te_j, Te_j\rangle = \|Te_j\|^2.$$

This is (ii), and the lemma is proved. ∎

In the language of frames, the next lemma will describe the impact of invertible operators on the eigenvalues and eigenvectors of a frame.

LEMMA 1.9. *Let S_1, S_2 be positive, self-adjoint, invertible operators on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be the eigenbasis for S_1 with corresponding eigenvalues $\{\lambda_j\}_{j=1}^N$. Further, let T be an invertible operator on $\mathcal{H}_N$ satisfying $S_1 = T^*S_2T$. Then the following conditions hold:*

(i) *$\{S_2^{1/2}T^*e_j\}_{j=1}^N$ is an orthogonal set.*

(ii) *$\|S_2^{1/2}T^*e_j\|^2 = \lambda_j$ for all $j = 1, \ldots, N$.*

Proof. We have

$$\langle S_2^{1/2}T^*e_j, S_2^{1/2}T^*e_k\rangle = \langle TS_2T^*e_j, e_k\rangle = \langle S_1e_j, e_k\rangle = \begin{cases} 0 & \text{if } j \neq k, \\ \lambda_j & \text{if } j = k. \end{cases}$$

The claims follow immediately. ∎

The property which is a crucial ingredient of Lemma 1.9 will be fundamental for the following characterization results. Hence we manifest a formal notation of it.

DEFINITION 1.10. Let $\mathcal{E} = \{e_j\}_{j=1}^N$ and $\mathcal{G} = \{g_j\}_{j=1}^N$ be orthonormal bases for $\mathcal{H}_N$, and let $\Lambda = \{\lambda_j\}_{j=1}^N$ and $\Gamma = \{\gamma_j\}_{j=1}^N$ be sequences of positive constants. An operator $T : \mathcal{H}_N \to \mathcal{H}_N$ is called *admissible for* $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$, if there exists some orthonormal basis $\{h_j\}_{j=1}^N$ for $\mathcal{H}_N$ satisfying

$$T^*e_j = \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\gamma_k}}\langle h_j, g_k\rangle g_k, \quad \text{for all } j = 1, \ldots, N.$$

This notion now allows us to formulate the main classification theorem answering Question 1.6 in a constructive manner.

THEOREM 1.11. *Let S_1 and S_2 be positive, self-adjoint, invertible operators on a Hilbert space $\mathcal{H}_N$, and let $\mathcal{E} = \{e_j\}_{j=1}^N$ and $\mathcal{G} = \{g_j\}_{j=1}^N$ be eigenvectors with eigenvalues $\Lambda = \{\lambda_j\}_{j=1}^N$ and $\Gamma = \{\gamma_j\}_{j=1}^N$ for S_1 and S_2, respectively. Further, let T be an invertible operator on $\mathcal{H}_N$. Then the following conditions are equivalent:*

(i) $S_1 = TS_2T^*$.

(ii) *T is an admissible operator for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$.*

Proof. (i) $\Rightarrow$ (ii) By Lemma 1.9, $\{S_2^{1/2}T^*e_j\}_{j=1}^N$ is an orthogonal set satisfying $\|S_2^{1/2}T^*e_j\|^2 = \lambda_j$ for all $j = 1, 2, \ldots, N$. Hence,

$$\{h_j\}_{j=1}^N := \Big\{\frac{1}{\sqrt{\lambda_j}} S_2^{1/2}T^*e_j\Big\}_{j=1}^N$$

is an orthonormal set. Thus, for each $j = 1, 2, \ldots, N$,

$$\frac{1}{\sqrt{\lambda_j}} S_2^{1/2}T^*e_j = \sum_{k=1}^N \langle h_j, g_k\rangle g_k.$$

This implies

$$T^*e_j = \sqrt{\lambda_j} S_2^{-1/2} \sum_{j=1}^N \langle h_j, g_k\rangle g_k = \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_k.$$

Hence, we proved that T is admissible for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$.

(ii) $\Rightarrow$ (i) Now assume that T is admissible for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$, which implies that

$$S_2^{1/2}T^*e_j = S_2^{1/2} \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_j = \sum_{k=1}^N \sqrt{\gamma_k}\sqrt{\frac{\lambda_j}{\gamma_k}} \langle h_j, g_k\rangle g_k = \sqrt{\lambda_j} h_j.$$

Hence, $\{S_2^{1/2}T^*e_j\}_{j=1}^N$ is an orthogonal set, and, for all $j = 1, \ldots, N$, we have

$$\|S_2^{1/2}T^*e_j\|^2 = \lambda_j.$$

By Lemma 1.8, we obtain

$$(S_2^{1/2}T^*)^*(S_2^{1/2}T^*)e_j = \lambda_j e_j \quad \text{for all } j = 1, 2 \ldots, N,$$

which yields

$$S_1 = (S_2^{1/2}T^*)^*(S_2^{1/2}T^*) = (TS_2^{1/2})S_2^{1/2}T^* = TS_2T^*,$$

i.e., condition (i). ∎

1.4. BACK TO THE FRAMES SETTING. We now go back to the frame setting, and state Theorem 1.11 in this language, thereby answering our main question concerning the characterization of invertible operators generating frames with a prescribed frame operator.

THEOREM 1.12. *Let $\{f_i\}_{i=1}^M$ and $\{g_i\}_{i=1}^M$ be frames for a Hilbert space $\mathcal{H}_N$ with frame operators S_1 and S_2 having eigenvectors $\mathcal{E} = \{e_j\}_{j=1}^N$ and $\mathcal{G} = \{g_j\}_{j=1}^N$ with eigenvalues $\Lambda = \{\lambda_j\}_{j=1}^N$ and $\Gamma = \{\gamma_j\}_{j=1}^N$ for S_1 and S_2, respectively. Further, let T be an invertible operator on $\mathcal{H}_N$. Then the following conditions are equivalent:*

(i) $S_1 = TS_2T^*$.

(ii) *T is an admissible operator for $(\mathcal{E}, \mathcal{G}, \Lambda, \Gamma)$.*

We finally return to Question 1.1, and provide a constructive characterization of those invertible operators which leave the frame operator invariant.

THEOREM 1.13. *Let $\{f_i\}_{i=1}^M$ be a frame for a Hilbert space $\mathcal{H}_N$ with frame operator S having eigenvectors $\mathcal{E} = \{e_j\}_{j=1}^N$ with respective eigenvalues $\Lambda = \{\lambda_j\}_{j=1}^N$. Further, let T be an invertible operator on $\mathcal{H}_N$. Then the following conditions are equivalent:*

(i) $S = TST^*$.

(ii) *There exists some orthonormal basis $\{h_j\}_{j=1}^N$ for $\mathcal{H}_N$ such that*

$$T^* e_j = \sum_{k=1}^N \sqrt{\frac{\lambda_j}{\lambda_k}} \langle h_j, e_k \rangle e_k.$$

We remark that this theorem provides a way to index frames which possess the same frame operator, solving a problem which has been debated in frame theory since several years, see, for instance, [3].

2. PRESCRIBED NORMS

We now focus on the second question, namely to derive a classification of all invertible operators which map frames to frames such that the norms of its frame elements are a fixed multiple of the norms of the original frame vectors. Formalizing, we face the following problem:

QUESTION 2.1. *Given a constant $c > 0$ and a frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$, can we classify the invertible operators $T : \mathcal{H}_N \to \mathcal{H}_N$ which satisfy $\|Tf_i\| = c\|f_i\|$ for all $i = 1, 2, \ldots, M$?*

2.1. MAIN CLASSIFICATION RESULT.

We first observe that without loss of generality we can assume that each frame vector is non-zero, since for a zero vector f_i, say, the condition $\|Tf_i\| = c\|f_i\|$ is trivially fulfilled. Furthermore, note that a solution to Question 2.1 for a particular $c > 0$ immediately implies a solution for any $c > 0$ just by multiplying the operators by an appropriate constant.

We start with a very simple lemma.

LEMMA 2.2. *Let T be an invertible operator on $\mathcal{H}$, and let $f \in \mathcal{H}$. Then the following conditions are equivalent:*

(i) $\|Tf\|^2 = c^2\|f\|^2$.

(ii) $\langle (T^*T - c^2\mathrm{Id})f, f \rangle = 0$.

Proof. We have

$$\langle (T^*T - c^2\mathrm{Id})f, f \rangle = \langle T^*Tf, f \rangle - \langle c^2 f, f \rangle = \|Tf\|^2 - \|cf\|^2 = \|Tf\|^2 - c^2\|f\|^2.$$

The result is immediate from here. ∎

Another angle of thought is provided by the following remark.

REMARK 2.3. Let T_1 and T_2 be operators satisfying $T_1^*T_1 = c^2T_2^*T_2$. Then, for any frame $\{f_i\}_{i=1}^M$,

$$\|T_1f_i\|^2 = \langle T_1^*T_1f_i, f_i\rangle = \langle c^2T_2^*T_2f_i, f_i\rangle = c^2\|T_2f_i\|^2. \tag{2.1}$$

The remark shows that considering the situation of operators which map equal norm frames to equal norm frames, equation (2.1) needs to be satisfied by the two analysis operators F_1, F_2, say, of the frames.

The following definition is required for our main result. It will give rise to a special class of vectors associated to a frame and an orthonormal basis.

DEFINITION 2.4. Let $\mathcal{F} = \{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $\mathcal{E} = \{e_j\}_{j=1}^N$ be an orthonormal basis for $\mathcal{H}_N$. Then we define

$$\mathcal{H}(\mathcal{F}, \mathcal{E}) = \operatorname{span}\{(|\langle f_i, e_j\rangle|^2)_{j=1}^N\}_{i=1}^M \subset \mathcal{H}_N.$$

We can now state the main result of this section, which will subsequently solve Question 2.1 completely.

THEOREM 2.5. *Let $\{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $\{c_i\}_{i=1}^M$ be positive scalars. Further, let T be an invertible operator on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be the eigenvectors for T^*T with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Then the following conditions are equivalent:*

(i) *We have*

$$c_i^2\|f_i\|^2 = \|Tf_i\|^2 \quad \textit{for all } i = 1, 2, \ldots, M.$$

(ii) *We have*

$$\Big\langle \sum_{j=1}^N (\lambda_j - c_i^2)e_j, \sum_{j=1}^N |\langle f_i, e_j\rangle|^2 e_j \Big\rangle = 0 \quad \textit{for all } i = 1, 2, \ldots, N.$$

Proof. By Lemma 2.2, (i) is equivalent to

$$\langle (T^*T - c_i^2\mathrm{Id})f_i, f_i\rangle = 0 \quad \text{for all } i = 1, 2, \ldots, N.$$

But, for all $i = 1, 2, \ldots, N$,

$$\begin{aligned} 0 = \langle (T^*T - c_i^2\mathrm{Id})f_i, f_i\rangle &= \Big\langle \sum_{j=1}^N (\lambda_j - c_i^2)\langle f_i, e_j\rangle e_j, \sum_{j=1}^N \langle f_i, e_j\rangle e_j \Big\rangle \\ &= \sum_{j=1}^N (\lambda_j - c_i^2)|\langle f_ie_j\rangle|^2 = \Big\langle \sum_{j=1}^N (\lambda_j - c_i^2)e_j, \sum_{j=1}^N |\langle f_i, e_j\rangle|^2 e_j \Big\rangle. \end{aligned}$$

The result is immediate from here. ∎

Now we can answer Question 2.1. The result follows directly from Theorem 2.5. We want to caution the reader that T^*T is *not* the frame operator for the frame $\{Tf_i\}_{i=1}^M$. In fact, as discussed before, the frame operator for this frame is TST^*, where S is the frame operator for $\{f_i\}_{i=1}^M$.

THEOREM 2.6. *Let $\mathcal{F} = \{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $c > 0$. Further, let T be an invertible operator on $\mathcal{H}_N$, and let T^*T have the orthonormal basis $\mathcal{E} = \{e_j\}_{j=1}^N$ as eigenvectors with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Then the following conditions are equivalent:*

(i) $\|Tf_i\| = c\|f_i\|$, *for all* $i = 1, 2, \ldots, M$.

(ii) $(\lambda_1 - c^2, \lambda_2 - c^2, \ldots, \lambda_N - c^2) \perp \mathcal{H}(\mathcal{F}, \mathcal{E})$.

In particular, if $\mathcal{H}(\mathcal{F}, \mathcal{E}) = \mathcal{H}_N$, then $\lambda_i = c^2$ for all $i = 1, 2, \ldots, N$, and hence T is a multiple of a unitary operator.

Given a frame $\{f_i\}_{i=1}^M$, Theorem 2.6 now provides us with a unique method for constructing all operators T so that $\|Tf_i\| = c\|f_i\|$ for all $i = 1, 2, \ldots, M$, detailed in the following remark.

REMARK 2.7. Let $\mathcal{F} = \{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$, and let $c > 0$. First, we choose any orthonormal basis $\mathcal{E} = \{e_j\}_{j=1}^N$ for $\mathcal{H}_N$, and consider $\mathcal{H}(\mathcal{F}, \mathcal{E})$. We distinguish two cases:

Case $\mathcal{H}(\mathcal{F}, \mathcal{E}) = \mathcal{H}_N$. In this case only unitary operators T can map $\{f_i\}_{i=1}^M$ to an equal norm frame and satisfy that the operator T^*T has $\mathcal{E}$ as its eigenvectors.

Case $\mathcal{H}(\mathcal{F}, \mathcal{E}) \neq \mathcal{H}_N$. In this case, choose a vector

$$\sum_{j=1}^N a_j e_j = (a_1, a_2, \ldots, a_N) \in \mathcal{H}(\mathcal{F}, \mathcal{E})^\perp,$$

which satisfies $c^2 + a_j > 0$ for all $j = 1, 2, \ldots, N$. Set $\lambda_j := c^2 + a_j$ for all $j = 1, 2, \ldots, N$. Then choose any operator T on $\mathcal{H}_N$ such that $\{Te_j\}_{j=1}^N$ forms an orthogonal set and satisfies

$$\|Te_j\|^2 = \lambda_j \quad \text{for all } j = 1, 2, \ldots, N.$$

By Lemma 1.8,

$$T^*Te_j = (c^2 + a_j)e_j = \lambda_j e_j \quad \text{for all } j = 1, 2, \ldots, N.$$

Moreover, by our choice of $\{\lambda_j\}_{j=1}^N$,

$$(\lambda_1 - c^2, \lambda_2 - c^2, \ldots, \lambda_N - c^2) \perp \mathcal{H}(\mathcal{F}, \mathcal{E}).$$

Since

$$\|Tf_i\|^2 - c^2\|f_i\|^2 = \langle (T^*T - c^2\mathrm{Id})f_i, f_i\rangle = \sum_{j=1}^N (\lambda_j - c^2)|\langle f_i, e_j\rangle|^2$$

$$= \Big\langle \sum_{j=1}^N (\lambda_j - c^2)e_j, \sum_{j=1}^N |\langle f_i, e_j\rangle|^2 e_j \Big\rangle = \Big\langle \sum_{j=1}^N a_j e_j, \sum_{j=1}^N |\langle f_i, e_j\rangle|^2 e_j \Big\rangle = 0,$$

it follows that $\|Tf_i\|^2 = c^2\|f_i\|^2$ for all $i = 1, 2, \ldots, M$.

2.2. GENERATING EQUAL NORM FRAMES. Now we regard Theorem 2.6 from a different standpoint. In fact, for a given invertible operator T, Theorem 2.6 identifies all unit norm frames which T maps to equal norm frames. Namely, this is the family of frames for which there exists some $c > 0$ such that

$$(\lambda_1 - c^2, \lambda_2 - c^2, \dots, \lambda_N - c^2) \perp \mathcal{H}(\mathcal{F}, \mathcal{E}).$$

Based on this observation, we derive several rather surprising corollaries from Theorem 2.6.

COROLLARY 2.8. *Let $c > 0$, let $\{f_i\}_{i=1}^M$ be a unit norm frame for $\mathcal{H}_N$, and let T be an invertible operator on $\mathcal{H}_N$ such that T^*T has eigenvectors $\{e_j\}_{j=1}^N$ with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Then the following are true:*

(i) *The operator T maps $\{f_i\}_{i=1}^M$ to an equal norm frame with $\|Tf_i\| = c$ if and only if*

$$(\lambda_1 - c^2, \lambda_2 - c^2, \dots, \lambda_N - c^2) \perp \mathcal{H}(\mathcal{F}, \mathcal{E}).$$

(ii) *The operator T maps $\{f_i\}_{i=1}^M$ to an equal norm Parseval frame if and only if*

$$\Big(\frac{1}{\sqrt{\lambda_1}} - \frac{N}{M}, \frac{1}{\sqrt{\lambda_2}} - \frac{N}{M}, \dots, \frac{1}{\sqrt{\lambda_N}} - \frac{N}{M}\Big) \perp \mathcal{H}(\mathcal{F}, \mathcal{E}).$$

Proof. (i) This follows directly from Theorem 2.6.

(ii) The operator T maps $\{f_i\}_{i=1}^M$ to an equal norm Parseval frame if and only if $T = S^{-1/2}$ (S the frame operator of the frame) and $\{S^{-1/2}f_i\}_{i=1}^M$ is equal norm with $\|S^{-1/2}f_i\|^2 = N/M$ for all $i = 1, 2, \dots, N$. The claim then follows from Theorem 2.6. ∎

COROLLARY 2.9. *Let $\mathcal{F} = \{f_i\}_{i=1}^N$ be a unit norm frame for $\mathcal{H}_N$ (i.e. a unit norm linearly independent set). Then for every non-unitary operator T on $\mathcal{H}_N$ which maps $\{f_i\}_{i=1}^N$ to an equal norm spanning set, we have*

$$\mathcal{H}(\mathcal{F}, \mathcal{E}) \neq \mathcal{H}_N.$$

Proof. Towards a contradiction, assume that $\mathcal{H}(\mathcal{F}, \mathcal{E}) = \mathcal{H}_N$. Then, by Corollary 2.8(ii), $\{f_i\}_{i=1}^M$ is not equivalent to an equal norm Parseval frame, a contradiction. ∎

COROLLARY 2.10. *Every invertible operator T on a Hilbert space $\mathcal{H}_N$ maps some equal norm Parseval frame to an equal norm frame.*

Proof. Let T be an invertible operator on $\mathcal{H}_N$, and let $\{e_j\}_{j=1}^N$ be an eigenbasis for T^*T with respective eigenvalues $\{\lambda_j\}_{j=1}^N$. Set

$$c^2 = \frac{1}{N}\sum_{j=1}^N \lambda_j \quad \text{and} \quad f = \sum_{i=1}^N e_j.$$

Then

$$\langle T^*T - c^2\mathrm{Id}\, f, f\rangle = \sum_{j=1}^{N}(\lambda_j - c^2)\cdot 1 = 0,$$

which means

$$(1,1,\dots,1) \perp (\lambda_1 - c^2, \lambda_2 - c^2, \dots, \lambda_N - c^2).$$

Next, consider the frame

$$\{f_i\}_{i=1}^{2^N} = \Big\{ \sum_{j=1}^{N} \varepsilon_j e_j \Big\}_{\{\varepsilon_j\}\in\{1,-1\}^N}.$$

For every $g = \sum\limits_{j=1}^{N} a_j e_j$, we obtain

$$\sum_{i=1}^{2^N} |\langle g, f_i\rangle|^2 = \sum_{i=1}^{2^N} \Big| \sum_{j=1}^{N} \varepsilon_j a_j \Big|^2 = 2^N \sum_{j=1}^{N} |a_j|^2 = 2^N \|g\|^2.$$

Thus $\Big\{ \frac{1}{\sqrt{2^N}} f_i \Big\}_{i=1}^{2^N}$ forms an equal norm Parseval frame, and we have $\mathcal{H}(\mathcal{F},\mathcal{E}) = \mathrm{span}\{(1,1,\dots,1)\}$. By Theorem 2.6, this implies that $\{Tf_i\}_{i=1}^{2^N}$ is an equal norm frame with $\|Tf_i\|^2 = c^2$ for all $i = 1,2,\dots,2^N$. ∎

We now provide an example of an equal norm Parseval frame and a non-unitary operator T, which maps it to a unit norm frame.

EXAMPLE 2.11. Let $f_1,\dots,f_4$ be the vectors in $\mathbb{R}^3$ defined by

$$f_1 = \frac{1}{2}(1,1,1),\ f_2 = \frac{1}{2}(-1,1,1), \boldsymbol{f}_3 = \frac{1}{2}(1,-1,1),\ f_4 = \frac{1}{2}(1,1,-1).$$

$\{f_i\}_{i=1}^4$ is an equal norm Parseval frame for $\mathbb{R}^3$. If we let $\{e_j\}_{j=1}^3$ denote the standard unit vector basis, then

$$\mathcal{H}(\mathcal{F},\mathcal{E}) = \mathrm{span}\{(1,1,1)\}.$$

We next choose a vector g such that $g \perp \mathcal{H}(\mathcal{F},\mathcal{E})$ by $g = (1,-1,0)$. Let now, for instance, $c = 2$, and set

$$\lambda_1 = 1 + c^2 = 5,\ \lambda_2 = -1 + c^2 = 3,\ \lambda_3 = 0 + c^2 = 4.$$

Then define an operator T such that $\{Te_j\}_{j=1}^3$ is an orthogonal set and $\|Te_j\| = \lambda_j$. One example of such an operator is defined by

$$T(1,0,0) = (5,0,0),\ T(0,1,0) = \Big(0, \sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}\Big),\ T(0,0,1) = (0,\sqrt{2},-\sqrt{2}).$$

Thus

$$Tf_1 = \frac{1}{2}\Big(\sqrt{5}, \sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}\Big), \qquad Tf_2 = \frac{1}{2}\Big(-\sqrt{5}, \sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}\Big),$$
$$Tf_3 = \frac{1}{2}\Big(\sqrt{5}, -\sqrt{\frac{3}{2}} + \sqrt{2}, \sqrt{\frac{3}{2}} - \sqrt{2}\Big), \quad Tf_4 = \frac{1}{2}\Big(\sqrt{5}, \sqrt{\frac{3}{2}} - \sqrt{2}, \sqrt{\frac{3}{2}} + \sqrt{2}\Big),$$

and we indeed obtain

$$\|Tf_i\|^2 = \frac{1}{4}(5+3+4) = 3$$

as desired.

3. GENERATING NEARLY PARSEVAL FRAMES

We finally tackle the question of deriving a deep understanding of operators which map a frame to a nearly Parseval frame.

3.1. PARSEVAL FRAMES AND DETERMINANTS.

As a first step, we will draw an interesting connection between Parseval frames and determinants. We remark that this result is closely related to results by Cahill [4], see also [6], who pioneered the utilization of the Plücker embedding from algebraic geometry for characterization results in frame theory.

We start by pointing out some important, but simple consequences of the well known arithmetic-geometric mean inequality, which we first state for reference.

THEOREM 3.1 (Arithmetic-geometric mean inequality). *Let $\{x_j\}_{j=1}^N$ be a sequence of positive real numbers, then*

$$\Big(\prod_{j=1}^N x_j\Big)^{1/n} \leqslant \frac{1}{N}\sum_{j=1}^N x_j$$

with equality if and only if $x_j = x_k$ for every $j, k = 1, \dots, N$.

In terms of positive self-adjoint matrices, this results reads as follows.

COROLLARY 3.2. *Let S be a positive, self-adjoint $N \times N$ matrix such that $\mathrm{Tr}(S) = N$ and $\det(S) = 1$, then S is the identity matrix.*

We now use this result to draw a connection between Parseval frames and determinants, namely of the matrix representation of the associated frame operator.

THEOREM 3.3. *Let $\{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ with frame operator S. If $\det(S) \geqslant 1$ and $\sum_{i=1}^M \|f_i\|^2 = N$, then $\{f_i\}_{i=1}^M$ constitutes a Parseval frame.*

Proof. Set

$$\{g_i\}_{i=1}^M = \Big\{\frac{f_i}{\det(S)}\Big\}_{i=1}^M.$$

Now, let $\{\lambda_j\}_{j=1}^N$ denote the eigenvalues of S, and let the eigenvalues of the frame operator for $\{g_i\}_{i=1}^M$ be denoted by $\{\lambda_j'\}_{j=1}^N$. Then we obtain

$$\sum_{j=1}^N \lambda_j' = \frac{\sum_{j=1}^N \lambda_j}{\det(S)^2} = \frac{N}{\det(S)^2},$$

which implies

$$\frac{\sum_{j=1}^N \lambda_j'}{N} = \frac{1}{\det(S)^2} \leqslant 1 = \prod_{j=1}^N \lambda_j'.$$

However, this contradicts the arithmetic-geometric mean inequality unless $\lambda_j' = 1$ for all $j = 1, 2, \dots N$, i.e., unless $\{f_i\}_{i=1}^M$ constitutes a Parseval frame. ∎

3.2. CHARACTERIZATION OF UNITARY OPERATORS. We will next provide a classification of unitary operators as those operators of determinant one which map some Parseval frame to a set of vectors with the same norms of the frame vectors.

THEOREM 3.4. *Let T be an operator on $\mathcal{H}_N$. Then the following conditions are equivalent:*

(i) *T is unitary.*

(ii) $|\det(T)| = 1$, *and there exists some Parseval frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$ such that $\|f_i\| = \|Tf_i\|$ for all $i = 1, 2, \dots, M$.*

(iii) $|\det(T)| = 1$, *and there exists some Parseval frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$ such that $\sum_{i=1}^M \|Tf_i\|^2 = N$.*

(iv) $|\det(T)| = 1$, *and there exists some frame $\{f_i\}_{i=1}^M$ for $\mathcal{H}_N$ such that $\sum_{i=1}^M \|f_i\|^2 = N$ and $\{Tf_i\}_{i=1}^M$ is a Parseval frame for $\mathcal{H}_N$.*

Proof. (i) ⇒ (ii) ⇒ (iii) are clear.

(iii) ⇒ (i) By Theorem 1.2, the frame operator of $\{Tf_i\}_{i=1}^M$ equals TT^*, since $\{f_i\}$ is a Parseval frame. Now, let $\lambda_1, \dots, \lambda_N$ denote the eigenvalues of TT^*. Then, it follows that

$$\sum_{j=1}^N \lambda_j = \sum_{i=1}^M \|Tf_i\|^2 = \sum_{i=1}^M \|f_i\|^2 = N,$$

where the last equality follows from the fact that $\{f_i\}_{i=1}^M$ is a Parseval frame. On the other hand, we have

$$\prod_{j=1}^N \lambda_j = \det(TT^*) = |\det(T)|^2 = 1.$$

Hence, we have equality in the arithmetic-geometric mean inequality. Thus, all λ_j's coincide, which is only possible provided that $\lambda_j = 1$ for every $j = 1, \dots, N$. This implies $TT^* = \mathrm{Id}$.

(iv) $\Leftrightarrow$ (i) This follows by applying the previous argument to T^{-1}. ∎

One might wonder whether the assumption in Theorem 3.4 that either $\{f_i\}_{i=1}^M$ or $\{Tf_i\}_{i=1}^M$ is a Parseval frame is indeed necessary. This is in fact the case, even for a linearly independent set. In the following we will construct such an illuminating example in $\mathbb{R}^4$ by using the results of the previous section.

EXAMPLE 3.5. In $\mathbb{R}^4$, let $\{e_j\}_{j=1}^4$ denote the standard unit orthonormal basis, and define

$$f_1 = (1,1,2,2),\ f_2 = (1,-1,2,2),\ f_3 = (1,1,-2,2),\ f_4 = (1,1,2,-2).$$

An easy computation shows that

$$\mathcal{H}(\mathcal{F},\mathcal{E}) = \mathrm{span}\{(1,1,2,2)\}.$$

Now, let

$$a_1 = 1,\ a_2 = 1,\ a_3 = -\frac{1}{2},\ a_4 = -\frac{1}{2},$$

and

$$\lambda_1 = 1 + a_1 = 2,\ \lambda_2 = 1 + a_2 = 2,\ \lambda_3 = 1 + a_3 = -\frac{1}{2},\ \lambda_4 = 1 + a_4 = -\frac{1}{2}.$$

This choice ensures that, for every $i = 1, 2, \dots, 4$,

$$\Big\langle \sum_{j=1}^4 (\lambda_j - 1) e_j, \sum_{j=1}^4 |\langle f_i, e_j\rangle|^2 e_j \Big\rangle = 0.$$

By Remark 2.7, if we choose any — in particular, a non-unitary — operator T on $\mathbb{R}^4$ such that $\{Te_j\}_{j=1}^4$ an orthogonal set with $\|Te_j\|^2 = \lambda_i$ for $j = 1, 2, 3, 4$, then $\|Tf_i\| = \|f_i\|$ for all $i = 1, 2, \dots, 4$ and $\det T = \prod_{j=1}^4 \lambda_j = 1$.

3.3. EXTENSION OF THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY. We now proceed to analyze when an invertible operator can map an equal norm frame to a nearly Parseval frame. For the next proposition, we first need the following special case of a result from [17].

THEOREM 3.6 ([17]). *Let $N \geqslant 2$, and let $x_j \geqslant 0$ for all $j = 1, 2, \dots, N$. Then*

$$\frac{1}{N(N-1)} \sum_{1 \leqslant j < k \leqslant N} (x_j^{1/2} - x_k^{1/2})^2 \leqslant \frac{\sum_{j=1}^N x_j}{N} - \Big(\prod_{j=1}^N x_j\Big)^{1/N}.$$

The following quantitative version of the arithmetic-geometric mean inequality will be crucial for our main result in this section.

THEOREM 3.7. *Let $N \geqslant 2$, and let $0 \leqslant x_j \leqslant N$ for all $j = 1, 2, \ldots, N$. If*

$$\frac{\sum_{j=1}^N x_j}{N} - \Big(\prod_{j=1}^N x_j\Big)^{1/N} < \varepsilon,$$

then there exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with

$$|x_j - x_k| \leqslant f(\varepsilon) \quad \text{for all } j, k = 1, 2, \ldots, N$$

and

$$1 - f(\varepsilon) \leqslant x_j \leqslant 1 + f(\varepsilon) \quad \text{for all } j = 1, 2, \ldots, N.$$

Moreover, f is bounded by

$$f(\varepsilon) \leqslant 2\varepsilon^{1/2} N^{3/2}.$$

Proof. By Theorem 3.6,

$$\frac{1}{N(N-1)} \sum_{1 \leqslant j < k \leqslant N} (x_j^{1/2} - x_k^{1/2})^2 \leqslant \frac{\sum_{j=1}^N x_j}{N} - \Big(\prod_{j=1}^N x_j\Big)^{1/N} < \varepsilon.$$

Therefore, for all $1 \leqslant j < k \leqslant N$,

$$|x_j^{1/2} - x_k^{1/2}|^2 \leqslant \sum_{1 \leqslant \tilde{j} < \tilde{k} \leqslant N} (x_{\tilde{j}}^{1/2} - x_{\tilde{k}}^{1/2})^2 \leqslant N(N-1)\varepsilon.$$

Since $x_j \leqslant N$, it follows that $x_j^{1/2} \leqslant N^{1/2}$ for all $j = 1, 2, \ldots, N$. Thus,

$$|x_j - x_k|^2 = |x_j^{1/2} - x_k^{1/2}|^2 \, |x_j^{1/2} + x_k^{1/2}|^2 \leqslant N(N-1)\varepsilon^2 4N \leqslant 4N^3\varepsilon,$$

which implies

$$|x_j - x_k| \leqslant 2N^{3/2}\varepsilon^{1/2}.$$

Further, for any $1 \leqslant j \leqslant N$, we obtain

$$x_j = \frac{\sum_{k=1}^N x_k + x_j - x_k}{N} \leqslant \frac{\sum_{k=1}^N x_k}{N} + \frac{\sum_{k=1}^N |x_j - x_k|}{N} \leqslant 1 + \frac{N 2N^{3/2}\varepsilon^{1/2}}{N} = 1 + 2N^{3/2}\varepsilon^{1/2}.$$

The inequality $x_j \geqslant 1 - 2N^{3/2}\varepsilon^{1/2}$ can be similarly proved. ∎

The bound on f in Theorem 3.7 is certainly not optimal. We believe that the *optimal bound* is of the order of εN; our intuition is supported by the following example.

EXAMPLE 3.8. Fix K, N and let

$$x_1 = K, \; x_2 = \frac{1}{K}, \text{ and } x_j = 1 \quad \text{for all } j = 3, 4, \ldots, N.$$

Then $\prod_{j=1}^N x_j = 1$. Also, since $KN > N$, we can conclude that

$$\frac{\sum_{j=1}^N x_j}{N} = \frac{K}{N} + \frac{1}{KN} + 1 - \frac{2}{N} \leqslant 1 + \frac{K}{N} - \frac{1}{N} = 1 + \varepsilon,$$

where $\varepsilon = (K-1)/N$. For N large, ε is arbitrarily small. Moreover, $\lambda_1 = K \approx \varepsilon N$, which implies

$$f(\varepsilon) \geqslant \varepsilon N.$$

3.4. MAIN RESULTS. We now head towards our main results. The next proposition gives a first estimate of how close an equal norm frame is to being Parseval in terms of the determinant of the frame operator.

PROPOSITION 3.9. *Let $\{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ with frame operator S satisfying $\|f_i\|^2 = N/M$ for all $i = 1, \dots, M$ and $(1-\varepsilon)^N \leqslant |\det(S)|$. Then*

$$(1 - f(\varepsilon))\mathrm{Id} \leqslant S \leqslant (1 + f(\varepsilon))\mathrm{Id},$$

where

$$f(\varepsilon) \leqslant 2\varepsilon^{1/2} N^{3/2}.$$

Proof. Let $\{\lambda_j\}_{j=1}^N$ denote the eigenvalues of S. By hypothesis,

$$(1-\varepsilon)^N \leqslant |\det(S)| = \prod_{j=1}^N \lambda_j,$$

which implies

$$1 - \varepsilon \leqslant \Big(\prod_{j=1}^N \lambda_j\Big)^{1/N}. \tag{3.1}$$

On the other hand,

$$\frac{\sum_{j=1}^N \lambda_j}{N} = \frac{\sum_{i=1}^M \|f_i\|^2}{N} = \frac{\sum_{i=1}^M N/M}{N} = 1. \tag{3.2}$$

This implies $\lambda_j \leqslant N$ for all $j = 1, 2, \dots, N$, and combining (3.1) and (3.2), we obtain

$$\frac{\sum_{j=1}^N \lambda_j}{N} - \Big(\prod_{j=1}^N \lambda_j\Big)^{1/N} < \varepsilon.$$

The result now follows from Theorem 3.7. ∎

Our first main theorem, which we will state in the sequel, now provides sufficient conditions for when there exists a mapping of an arbitrary frame to an equal norm nearly Parseval frame. It also in a certain sense weakens the assumption of Proposition 3.9.

THEOREM 3.10. *Let $\{f_i\}_{i=1}^M$ be a frame for $\mathcal{H}_N$ with $(1-\varepsilon)^N \leqslant |\det(S)|$. Further, let T be an operator with $|\det(T)| \geqslant 1$ and $\|Tf_i\|^2 = N/M$ for all $i = 1, 2, \dots, M$. Also, let S_1 be the frame operator of $\{Tf_i\}_{i=1}^M$ and denote its eigenvalues by $\{\mu_j\}_{j=1}^N$. Then*

$$(1-\varepsilon)\frac{\sum_{j=1}^N \mu_j}{N} \leqslant \Big(\prod_{j=1}^N \mu_j\Big)^{1/N} \leqslant \frac{\sum_{j=1}^N \mu_j}{N}.$$

Moreover,

$$(1 - f(\varepsilon))\mathrm{Id} \leqslant S_1 \leqslant (1 + f(\varepsilon))\mathrm{Id}.$$

Proof. By hypothesis,

$$|\det(TST^*)| = |\det(T)|^2 |\det(S)| \geqslant |\det(S)| \geqslant (1 - \varepsilon)^N.$$

Thus

$$\prod_{j=1}^{N} \mu_j \geqslant (1 - \varepsilon)^N, \quad \text{hence } 1 - \varepsilon \leqslant \Big(\prod_{j=1}^{N} \mu_j\Big)^{1/N}. \tag{3.3}$$

Since $\|Tf_i\|^2 = M/N$ for all $i = 1, 2, \ldots, M$,

$$\sum_{j=1}^{N} \mu_j = N, \quad \text{hence } \frac{\sum_{j=1}^{N} \mu_j}{N} = 1. \tag{3.4}$$

By (3.3) and (3.4), it follows that

$$(1 - \varepsilon)\frac{\sum_{j=1}^{N} \mu_j}{N} = 1 - \varepsilon \leqslant \Big(\prod_{j=1}^{N} \mu_j\Big)^{1/N} \leqslant \frac{\sum_{j=1}^{N} \mu_j}{N} = 1.$$

The *moreover* part is immediate by Theorem 3.7. ∎

Our final main result provides a generalization of Theorem 3.10, in the sense that it gives a bound on TT^* in the situation that T maps an equal norm frame to an equal norm frame.

THEOREM 3.11. *Let $\{f_i\}_{i=1}^{M}$ be a frame for $\mathcal{H}_N$, and let T be an invertible operator on $\mathcal{H}_N$ with $|\det(T)| \geqslant 1$ and satisfying:*

(i) $\|f_i\|^2 = \|Tf_i\|^2 = N/M$, *and*

(ii) $(1 - \varepsilon)^N \leqslant |\det(S)|$.

Then

$$\frac{1 - f(\varepsilon)}{1 + f(\varepsilon)}\mathrm{Id} \leqslant TT^* \leqslant \frac{1 + f(\varepsilon)}{1 - f(\varepsilon)}\mathrm{Id}.$$

Proof. Since $(1 - \varepsilon)^N \leqslant |\det(S)| \leqslant |\det(TST^*)|$, applying Proposition 3.9 to both $\{f_i\}_{i=1}^{M}$ and $\{Tf_i\}_{i=1}^{M}$ yields

$$(1 - f(\varepsilon))\mathrm{Id} \leqslant S \leqslant (1 + f(\varepsilon))\mathrm{Id} \tag{3.5}$$

and

$$(1 - f(\varepsilon))\mathrm{Id} \leqslant TST^* \leqslant (1 + f(\varepsilon))\mathrm{Id}. \tag{3.6}$$

By applying T from the left and T^* from the right to (3.5), we obtain

$$(1 - f(\varepsilon))TT^* \leqslant T(1 - f(\varepsilon))T^* \leqslant TST^*.$$

Combining with (3.6), we can conclude that

$$(1 - f(\varepsilon))TT^* \leqslant (1 + f(\varepsilon))\mathrm{Id},$$

which implies

$$TT^* \leqslant \frac{1+f(\varepsilon)}{1-f(\varepsilon)}\mathrm{Id}.$$

Similar arguments lead to the other inequality. ∎

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