J. OPERATOR THEORY
70:1(2013), 175–180
doi: 10.7900/jot.2011may16.1911

ON THE UNIQUENESS OF THE POLAR DECOMPOSITION OF BOUNDED OPERATORS IN HILBERT SPACES

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Communicated by Florian-Horia Vasilescu

ABSTRACT. It is well known as a fundamental result in the theory of the classical groups that the polar decomposition of a regular matrix exists and is uniquely determined. In this paper a generalization of the result above is given for a bounded operator in Hilbert spaces, in particular on the uniqueness of the polar decomposition.

KEYWORDS: *Polar decomposition, uniqueness, bounded linear operator, Hilbert space.*

MSC (2010): 47A05, 47L30.

1. INTRODUCTION

Let A be a regular matrix and A^* the adjoint matrix of A. We define $|A|$ by $\sqrt{A^*A}$. Then it is well known as a fundamental result in the theory of the classical groups (cf. [2], [4], [9]) that there exists a unitary matrix U satisfying

$$A = U|A| \tag{1.1}$$

and such a U is uniquely determined. The equation (1.1) is called the polar decomposition of A.

On the other hand, for a bounded operator in Hilbert spaces we have well known the existence of the polar decomposition, but as far as the authors know, there is no description at all on the uniqueness of the polar decomposition (cf. [3], [5], [6], [7], [8], [10], [11], [12]) except for [1]. We note that the statement in Section 2.6.3 of [1] is wrong, as a counterexample will be given in Section 2 of the present paper.

Our aim in the present paper is to generalize the result on the uniqueness of the polar decomposition for a regular matrix to the result for a bounded operator in Hilbert spaces.

Let H and K be Hilbert spaces over $\mathbb{C}$ with inner products $(\cdot,\cdot)_H$ and $(\cdot,\cdot)_K$, and norms $\|\cdot\|_H$ and $\|\cdot\|_K$, respectively. We sometimes omit subscripts H and K.

Let $\mathcal{L}(H, K)$ denote the space of all bounded linear operators from H into K. Let $A \in \mathcal{L}(H, K)$ and $A^* \in \mathcal{L}(K, H)$ its adjoint operator. We define $|A| \in \mathcal{L}(H, H)$ by $\sqrt{A^*A}$ as in Chapter VI of [10]. Let $U \in \mathcal{L}(H, K)$ be such that

$$\|U\Psi\|_K = \|\Psi\|_H \tag{1.2}$$

for all $\Psi \in (\ker U)^\perp$, where $\ker U$ denotes the kernel of U and $(\ker U)^\perp$ the orthogonal complement of $\ker U$. Such U is called a partially isometric operator.

Let $A \in \mathcal{L}(H, K)$. Then, $A = U|A|$ with a partially isometric operator U is called a polar decomposition of A. The following theorem is well known.

THEOREM 1.1. *Let $A \in \mathcal{L}(H, K)$. Then there is a partially isometric operator $U \in \mathcal{L}(H, K)$ such that $A = U|A|$ holds. In particular, U satisfying*

$$\ker U = \ker A \tag{1.3}$$

exists and is uniquely determined.

Let us write the partially isometric operator U satisfying $A = U|A|$ and (1.3) as U_0 throughout the present paper.

Our aim in the present paper is to prove the following theorem.

THEOREM 1.2. *Let H and K be Hilbert spaces over $\mathbb{C}$ and $A \in \mathcal{L}(H, K)$. Then the polar decomposition of A is unique if and only if either* $\ker A = \{0\}$ *or* $R(A)^\perp = \{0\}$, *where $R(A)$ denotes the range of A.*

Theorem above is a generalization of the result for a regular matrix.

Let P be the orthogonal projection from H onto $\ker A$. We will prove in Proposition 2.5 of the present paper that $A = U|A|$ is a polar decomposition of A, if and only if there is a partially isometric operator $V \in \mathcal{L}(\ker A, R(A)^\perp)$ such that

$$U = U_0 + VP. \tag{1.4}$$

Hence, if either $\ker A = \{0\}$ or $R(A)^\perp = \{0\}$, we see $V = 0$ and so $U = U_0$, which shows the uniqueness of the polar decomposition. Conversely, assume $\ker A \neq \{0\}$ and $R(A)^\perp \neq \{0\}$. Let $\Psi_0 \in \ker A$ and $\Phi_0 \in R(A)^\perp$ such that $\|\Psi_0\|_H = \|\Phi_0\|_K = 1$. We can easily define the partially isometric operator $V \in \mathcal{L}(\ker A, R(A)^\perp)$ such that

$$V(a\Psi_0) = a\Phi_0, \quad a \in \mathbb{C}$$

and $\ker V = \{\Psi_0\}^\perp$. Since $U_0 + VP \neq U_0$, we can see that the polar decomposition of A is not unique. Thus we have only to prove Proposition 2.5 to complete the proof of theorem. We will prove Proposition 2.5 in the next section.

2. PROOF OF THEOREM

We first give a counterexample of the statement in Section 2.6.3 of [1] that if $\ker A \neq \{0\}$, the polar decomposition of A is not unique.

EXAMPLE 2.1. Let $A \in \mathcal{L}(\mathbb{C}^2, \mathbb{C})$ defined by $A = (a, 0)$ for a complex constant $a \neq 0$. Then, $\ker A$ is the linear span of ${}^t(0,1) \in \mathbb{C}^2$, where ${}^t(0,1)$ is the transposed vector of $(0,1)$. So $\ker A \neq \{0\}$.

We will show that the polar decomposition of A is unique. We can easily see

$$|A| = \begin{pmatrix} |a| & 0 \\ 0 & 0 \end{pmatrix}.$$

Let $U = (u_1, u_2) \in \mathcal{L}(\mathbb{C}^2, \mathbb{C})$ be a partially isometric operator such that $A = U|A|$. From $A = U|A|$ we have

$$U = (a/|a|, u_2).$$

So we can easily see that $\ker U$ is the linear span of ${}^t(-|a|u_2/a, 1)$ and that $(\ker U)^\perp$ is the linear span of ${}^t(1, |a|\overline{u}_2/\overline{a})$, where $\overline{u}_2$ is the complex conjugate of $u_2 \in \mathbb{C}$.

Let $w \in (\ker U)^\perp$. Then ${}^t w = \beta\, {}^t(1, |a|\overline{u}_2/\overline{a})$ for a $\beta \in \mathbb{C}$. We can easily see

$$\|w\|^2 = |\beta|^2(1 + |u_2|^2), \quad \|Uw\|^2 = |\beta|^2(1 + |u_2|^2)^2.$$

Since U is partially isometric, we have $\|Uw\|^2 = \|w\|^2$, which shows $u_2 = 0$. So U is determined uniquely as $U = (a/|a|, 0)$.

Let H and K be Hilbert spaces over $\mathbb{C}$. Let $A \in \mathcal{L}(H,K)$ and U_0 the partially isometric operator defined in Introduction. We can easily have

$$\ker |A| = \ker A, \quad \text{and} \tag{2.1}$$

$$H = \ker |A| \oplus \overline{R(|A|^*)} = \ker |A| \oplus \overline{R(|A|)} = \ker A \oplus \overline{R(|A|)}, \tag{2.2}$$

where $\overline{R(|A|)}$ denotes the closure of $R(|A|)$.

LEMMA 2.2. *Let $A \in \mathcal{L}(H,K)$ and suppose that $A = U|A|$, with U partially isometric. We write*

$$D(U) := \{\Psi \in H : \|U\Psi\|_K = \|\Psi\|_H\}. \tag{2.3}$$

Then we obtain

$$D(U) = (\ker U)^\perp \tag{2.4}$$

and $R(|A|) \subset D(U)$.

Proof. We can easily see $(\ker U)^\perp \subset D(U)$, because U is partially isometric. Let $\Psi \in D(U)$ and write

$$\Psi = \Psi_1 + \Psi_2, \quad \Psi_1 \in \ker U,\ \Psi_2 \in (\ker U)^\perp.$$

Then we have

$$\|\Psi\|_H^2 = \|U\Psi\|_K^2 = \|U\Psi_2\|_K^2 = \|\Psi_2\|_H^2,$$

which shows $\Psi_1 = 0$. Consequently we have $\Psi \in (\ker U)^\perp$ and so $D(U) \subset (\ker U)^\perp$. Hence we obtain (2.4).

Let $\Psi \in H$. It follows from $A = U|A|$ that

$$(U|A|\Psi, U|A|\Psi)_K = (A\Psi, A\Psi)_K = (|A|\Psi, |A|\Psi)_H,$$

which shows $|A|\Psi \in D(U)$. ∎

LEMMA 2.3. *Let $A \in \mathcal{L}(H, K)$ and suppose that $A = U|A|$, with U partially isometric. Then, there is a partially isometric operator $V \in \mathcal{L}(\ker A, R(A)^{\perp})$ satisfying* (1.4).

Proof. We see from (2.4) that $D(U)$ is a closed vector subspace in H and so

$$(U\Psi_1, U\Psi_2) = (\Psi_1, \Psi_2), \quad \Psi_1,\ \Psi_2 \in D(U) \tag{2.5}$$

holds from a polarization argument.

Using (2.2) and (2.4), we set

$$D_0 := \ker A \cap D(U) = R(|A|)^{\perp} \cap D(U), \tag{2.6}$$

$$G_0 := \ker A \cap D(U)^{\perp} = \ker A \cap \ker U. \tag{2.7}$$

Then

$$\ker A = D_0 \oplus G_0. \tag{2.8}$$

Let V be the restriction $U \restriction \ker A$ of U with the domain $\ker A$. Let $\Psi \in \ker A$. Then from (2.8) we can write

$$\Psi = \Psi_1 + \Psi_2, \quad \Psi_1 \in D_0,\ \Psi_2 \in G_0.$$

Then we have $V\Psi = V\Psi_1 + V\Psi_2 = U\Psi_1$ from (2.7). Let $\Phi \in H$. The relation $|A|\Phi \in D(U)$ follows from Lemma 2.2. So from (2.5) and (2.6) we have

$$(V\Psi, A\Phi)_K = (U\Psi_1, A\Phi)_K = (U\Psi_1, U|A|\Phi)_K = (\Psi_1, |A|\Phi)_H = 0,$$

which shows

$$R(V) \subset R(A)^{\perp}.$$

Thus we could see $V \in \mathcal{L}(\ker A, R(A)^{\perp})$.

Because of $V = U \restriction \ker A$ we have

$$\ker V = \ker A \cap \ker U = G_0$$

from (2.7), which shows

$$(\ker V)^{\perp} = D_0 = \ker A \cap (\ker U)^{\perp} \quad \text{in } \ker A \tag{2.9}$$

from (2.4), (2.6) and (2.8). Hence, since U is partially isometric, so is V.

Let $\Psi \in H$ and write

$$\Psi = \Psi_1 + \Psi_2, \quad \Psi_1 \in \ker A,\ \Psi_2 \in \overline{R(|A|)}$$

by means of (2.2). Then we obtain

$$(U - U_0)\Psi = U\Psi - U_0\Psi = U\Psi_1 + U\Psi_2 - U_0\Psi_2 = U\Psi_1 = VP\Psi$$

because of $\ker U_0 = \ker A$ and $A = U|A| = U_0|A|$. Thus we could complete the proof. ∎

LEMMA 2.4. *Let $V \in \mathcal{L}(\ker A, R(A)^{\perp})$ be a partially isometric operator and set $U := U_0 + VP \in \mathcal{L}(H, K)$. Then U is partially isometric and satisfies $A = U|A|$.*

Proof. From (2.2) we have

$$\overline{R(|A|)} = (\ker A)^\perp.$$

So we obtain $P|A| = 0 \in \mathcal{L}(H,H)$. Consequently $U|A| = U_0|A| = A$ holds.

We will show that $U_0 + VP \in \mathcal{L}(H,K)$ is partially isometric. From Theorem VI.10 in [10] we know

$$R(U_0) = \overline{R(A)}. \tag{2.10}$$

We note that $U_0U_0^* \in \mathcal{L}(K,K)$ is the projection onto $R(U_0)$ (cf. Section VI.4 in [10]). So we can see $U_0U_0^*VP = 0 \in \mathcal{L}(H,K)$ from $R(V) \subset R(A)^\perp$ and (2.10). Consequently,

$$(U_0^*VP\Psi, U_0^*VP\Psi)_H = 0$$

for all $\Psi \in H$, which shows

$$U_0^*VP = 0 \in \mathcal{L}(H,H). \tag{2.11}$$

Hence we get

$$\|(U_0 + VP)\Psi\|_K^2 = \|U_0\Psi\|_K^2 + \|VP\Psi\|_K^2. \tag{2.12}$$

Let $\Psi \in \ker(U_0 + VP) \subset H$. This is equivalent to $\Psi \in \ker U_0 = \ker A$ and $VP\Psi = 0$ from (2.12), which is also equivalent to $\Psi \in \ker A$ and $V\Psi = 0$ from $P\Psi = \Psi$. Hence we have

$$\ker(U_0 + VP) \subset \ker V (\subset \ker A).$$

Conversely, let $\Psi \in \ker V (\subset \ker A)$. Then we have $U_0\Psi = 0$ and $P\Psi = \Psi$. So we have $\Psi \in \ker(U_0 + VP)$ from (2.12). Thus we could see

$$\ker(U_0 + VP) = \ker V (\subset \ker A). \tag{2.13}$$

Let $\Psi \in \ker(U_0 + VP)^\perp \subset H$. Then we can write

$$\Psi = \Psi_1 + \Psi_2, \quad \Psi_1 \in \ker A,\ \Psi_2 \in (\ker A)^\perp. \tag{2.14}$$

It follows from (2.13) that $\ker(U_0 + VP)^\perp \supset (\ker A)^\perp$. So we have

$$\Psi_1 \in \ker A \cap \ker(U_0 + VP)^\perp, \quad \Psi_2 \in (\ker A)^\perp \cap \ker(U_0 + VP)^\perp.$$

Consequently from (2.12)–(2.14), and the properties of U_0, P and V we get

$$\begin{aligned}\|(U_0 + VP)\Psi\|_K^2 &= \|U_0\Psi_2\|_K^2 + \|VP\Psi_1\|_K^2 = \|U_0\Psi_2\|_K^2 + \|V\Psi_1\|_K^2 \\ &= \|\Psi_2\|_H^2 + \|\Psi_1\|_H^2 = \|\Psi\|_H^2,\end{aligned}$$

where we used the assumption that V is partially isometric. Thus we could prove that $U_0 + VP$ is partially isometric. ∎

We obtain the following from Lemmas 2.3 and 2.4.

PROPOSITION 2.5. *Let $A \in \mathcal{L}(H,K)$. Then $A = U|A|$, with U partially isometric, if and only if there is a partially isometric operator $V \in \mathcal{L}(\ker A, R(A)^\perp)$ satisfying* (1.4).

Acknowledgements. The authors thank the referee for helping to get the proof of theorem elegant and the result generalized. The first author is grateful to Shinshu University, Faculty of Science of Shinshu University for Grant-in-Aid and JSPS (Japan Society for the Promotion of Science) for Grant-in-Aid for Scientific Research No. 23540195.

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Received May 16, 2011; revised June 17, 2011.