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QUASI-REPRESENTATIONS OF FINSLER MODULES OVER C^*-ALGEBRAS

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Dedicated to the memory of Professor William B. Arveson

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ABSTRACT. We show that every Finsler module over a C^*-algebra has a quasi-representation into the Banach space $\mathbb{B}(\mathscr{H},\mathscr{K})$ of all bounded linear operators between some Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. We define the notion of completely positive φ-morphism and establish a Stinespring type theorem in the framework of Finsler modules over C^*-algebras. We also investigate the non-degeneracy and the irreducibility of quasi-representations.

KEYWORDS: *Finsler module, C^*-algebra, φ-morphism, quasi-representation, non-degenerate quasi-representation, irreducible quasi-representation.*

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1. INTRODUCTION

The notation of Finsler module is an interesting generalization of that of Hilbert C^*-module. It is a useful tool in the operator theory and the theory of operator algebras and may be served as a noncommutative version of the concept of Banach bundle, which is an essential concept in the Finsler geometry. In 1995 Phillips and Weaver [11] showed that if a C^*-algebra $\mathscr{A}$ has no nonzero commutative ideal, then any Finsler $\mathscr{A}$-module must be a Hilbert C^*-module. If $\mathscr{A}$ is the commutative C^*-algebra $C_0(X)$ of all continuous complex-valued functions vanishing at infinity on a locally compact Hausdorff space X, then any Finsler $\mathscr{A}$-module is isomorphic to the module of continuous sections of a bundle of Banach spaces over X. The concept of a φ-morphism between Finsler modules was introduced in [1].

The Gelfand–Naimark–Segal (GNS) representation theorem is one of the most useful theorems, which is applied in operator algebras and mathematical physics. That provides a procedure to construct representations of C^*-algebras. A generalization of GNS construction to a topological $*$-algebra established by

Borchers, Uhlmann and Powers leading to unbounded $*$-representations of $*$-algebras; see [12]. Another is a generalization of a positive linear functional to a completely positive map studied by Stinespring [14], see also [6].

Let $\mathscr{A}$ be a C^*-algebra and let $\mathscr{A}^+$ denote the positive cone of all positive elements of $\mathscr{A}$. We define a Finsler $\mathscr{A}$-module to be a right $\mathscr{A}$-module $\mathscr{E}$ equipped with a map $\rho : \mathscr{E} \to \mathscr{A}^+$ (denoted by $\rho_{\mathscr{A}}$ if there is an ambiguity) satisfying the following conditions:

(i) The map $\|\cdot\|_{\mathscr{E}} : x \longmapsto \|\rho(x)\|$ makes $\mathscr{E}$ into a Banach space.
(ii) $\rho(xa)^2 = a^*\rho(x)^2 a$, for all $a \in \mathscr{A}$ and $x \in \mathscr{E}$.

A Finsler module $\mathscr{E}$ over a C^*-algebra $\mathscr{A}$ is said to be full if the linear span of $\{\rho(x)^2 : x \in \mathscr{E}\}$ is dense in $\mathscr{A}$. For example, if $\mathscr{E}$ is a (full) Hilbert C^*-module over $\mathscr{A}$ (see [7]), then $\mathscr{E}$ together with $\rho(x) = \langle x, x\rangle^{1/2}$ is a (full) Finsler module over $\mathscr{A}$, since

$$\rho(xa)^2 = \langle xa, xa\rangle = a^*\langle x, x\rangle a = a^*\rho(x)^2 a.$$

In particular, every C^*-algebra $\mathscr{A}$ is a full Finsler module over $\mathscr{A}$ under the mapping $\rho(x) = (x^*x)^{1/2}$.

Our goal is to extend the notion of a representation of a Hilbert C^*-module to the framework of Finsler $\mathscr{A}$-modules. We show that every Finsler $\mathscr{A}$-module has a quasi-representation into the Banach space $\mathbb{B}(\mathscr{H}, \mathscr{K})$ of all bounded linear operators between some Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$. We define the notion of completely positive φ-morphism and establish a Stinespring type theorem in the framework of Finsler modules over C^*-algebras. We also introduce the notions of the nondegeneracy and the irreducibility of quasi-representations and study some interrelations between them.

2. QUASI-REPRESENTATIONS OF FINSLER MODULES

We start our work by giving the definition of a φ-morphism of a Finsler module.

DEFINITION 2.1. Suppose that $(\mathscr{E}, \rho_{\mathscr{A}})$ and $(\mathscr{F}, \rho_{\mathscr{B}})$ are Finsler modules over C^*-algebras $\mathscr{A}$ and $\mathscr{B}$, respectively, and $\varphi : \mathscr{A} \to \mathscr{B}$ is a $*$-homomorphism of C^*-algebras. A (not necessarily linear) map $\Phi : \mathscr{E} \to \mathscr{F}$ is said to be a *φ-morphism of Finsler modules* if the following conditions are satisfied:

(i) $\rho_{\mathscr{B}}(\Phi(x)) = \varphi(\rho_{\mathscr{A}}(x))$;
(ii) $\Phi(xa) = \Phi(x)\varphi(a)$;

for all $x \in \mathscr{E}$ and $a \in \mathscr{A}$. In the case of Hilbert C^*-modules, Φ is assumed to be linear and then condition (ii) is deduced from (i).

Now we introduce the notion of a quasi-representation of a Finsler module. Due to $\mathbb{B}(\mathscr{H}, \mathscr{K})$ is a Hilbert C^*-module over $\mathbb{B}(\mathscr{H})$ via $\langle T, S\rangle = T^*S$, we can endow $\mathbb{B}(\mathscr{H},\mathscr{K})$ a Finsler structure by

$$\rho_0(T) = (T^*T)^{1/2}. \tag{2.1}$$

DEFINITION 2.2. Let $(\mathscr{E},\rho)$ be a Finsler module over a C^*-algebra $\mathscr{A}$. A map $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H},\mathscr{K})$, where $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ is a representation of $\mathscr{A}$ is called a *quasi-representation* of $\mathscr{E}$ if $\rho_0(\Phi(x)) = \varphi(\rho(x))$ for all $x \in \mathscr{E}$.

We are going to show that for every Finsler $\mathscr{A}$-module there is a quasi-representation into $\mathbb{B}(\mathscr{H},\mathscr{K})$ for some Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$, see also [9].

THEOREM 2.3. *Suppose $\mathscr{E}$ is a Finsler $\mathscr{A}$-module with the associated map ρ : $\mathscr{E} \to \mathscr{A}^+$. Then there is a quasi-representation $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H},\mathscr{K})$ for some Hilbert spaces $\mathscr{H}$ and $\mathscr{K}$.*

Proof. By the Gelfand–Naimark theorem for C^*-algebras, there is a representation $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ for some Hilbert space $\mathscr{H}$. We want to construct a Hilbert space $\mathscr{K}$. Put

$$\mathscr{K}_0 := \mathrm{span}\{\varphi(a)f : a \in \mathscr{A}, f : \mathscr{E} \to \mathscr{H} \text{ is a map with a finite support}\}$$

and define on $\mathscr{K}_0$ an inner product by

$$\langle \varphi(a)f, \varphi(b)g\rangle = \sum_{x\in\mathscr{E}} \langle \varphi(a)f(x), \varphi(b)g(x)\rangle.$$

Note that if $\langle \sum_{i=1}^n \varphi(a_i)f_i, \sum_{i=1}^n \varphi(a_i)f_i\rangle = 0$, then

$$\sum_{x\in\mathscr{E}} \left\langle \sum_{i=1}^n \varphi(a_i)f_i(x), \sum_{i=1}^n \varphi(a_i)f_i(x) \right\rangle = 0\,.$$

Thus $\sum\limits_{i=1}^n \varphi(a_i)f_i(x) = 0$ for each $x \in \mathscr{E}$, whence $\sum\limits_{i=1}^n \varphi(a_i)f_i = 0$.

Let us consider the closure $\overline{\mathscr{K}}_0$ of $\mathscr{K}_0$ to get a Hilbert space, which is denoted by $\mathscr{K}$. For any $y \in \mathscr{E}$ and $h \in \mathscr{H}$, the map $h_y : \mathscr{E} \to \mathscr{H}$ defined by

$$h_y(x) = \begin{cases} h & x = y, \\ 0 & x \neq y, \end{cases}$$

has a finite support. For $x \in \mathscr{E}$, define $\Phi(x) : \mathscr{H} \to \mathscr{K}$ by $\Phi(x)h = \varphi(\rho(x))h_x$. We show that $\Phi(x) \in \mathbb{B}(\mathscr{H},\mathscr{K})$. Clearly $\Phi(x)$ is linear. Also $\Phi(x)$ is bounded, since

$$\begin{aligned}\|\Phi(x)h\|^2 &= \langle \Phi(x)h, \Phi(x)h\rangle = \langle \varphi(\rho(x))h_x, \varphi(\rho(x))h_x\rangle \\ &= \sum_{y\in\mathscr{E}} \langle \varphi(\rho(x))h_x(y), \varphi(\rho(x))h_x(y)\rangle = \langle \varphi(\rho(x))h, \varphi(\rho(x))h\rangle \\ &\leqslant \|\varphi(\rho(x))\|^2\|h\|^2\,,\end{aligned}$$

whence $\|\Phi(x)\| \leqslant \|\varphi(\rho(x))\|$.

Further,

$$\begin{aligned}\langle \Phi(x)^*\Phi(x)h,h'\rangle &= \langle \Phi(x)h,\Phi(x)h'\rangle = \langle \varphi(\rho(x))h_x,\varphi(\rho(x))h'_x\rangle \\ &= \sum_{y\in\mathscr{E}} \langle \varphi(\rho(x))h_x(y),\varphi(\rho(x))h'_x(y)\rangle \\ &= \langle \varphi(\rho(x))h,\varphi(\rho(x))h'\rangle = \langle \varphi(\rho(x)^2)h,h'\rangle,\end{aligned}$$

for all $h,h' \in \mathscr{H}$ and $x \in \mathscr{E}$. Hence $\Phi(x)^*\Phi(x) = \varphi(\rho(x)^2)$. Hence

$$(\Phi(x)^*\Phi(x))^{1/2} = \varphi(\rho(x)). \tag{2.2}$$

It follows from (2.1) and equality (2.2) that $\rho_0(\Phi(x)) = \varphi(\rho(x))$. ∎

REMARK 2.4. If Φ is surjective and $\mathbb{B}(\mathscr{H},\mathscr{K})$ is a full Finsler $\mathbb{B}(\mathscr{H})$-module, then by Theorem 3.4(iv) of [1], φ is surjective.

In the next section the notion of completely positive φ-morphism is introduced and a construction of Stinespring's theorem for Finsler modules is given.

3. A STINESPRING TYPE THEOREM FOR FINSLER MODULES

The Stinespring theorem was first introduced in the work of Stinespring in 1995 that described the structure of completely positive maps of a C^*-algebra into the C^*-algebra of all bounded linear operators on a Hilbert space; see [14]. Recently Asadi [3] proved this theorem for Hilbert C^*-modules. Further, Bhat et al. [4] improved the result of [3] with omitting a technical condition. In this section we intend to establish a Stinespring type theorem in the framework of Finsler modules over C^*-algebras.

Let $(\mathscr{E},\rho_{\mathscr{A}})$ be a Finsler module over a C^*-algebra $\mathscr{A}$. A map $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H},\mathscr{K})$ is called completely positive if there is a completely positive map $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ such that (i) and (ii) of Definition 2.1 hold with $\rho_{\mathscr{B}} = \rho_0$.

THEOREM 3.1. *Let $(\mathscr{E},\rho)$ be a Finsler module over a unital C^*-algebra $\mathscr{A}$, let $\mathscr{H},\mathscr{K}$ be Hilbert spaces and let $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H},\mathscr{K})$ be a completely positive map associated to a completely positive map $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$. Then there exist Hilbert spaces $\mathscr{H}',\mathscr{K}'$ and isometries $V : \mathscr{H} \to \mathscr{H}', W : \mathscr{K} \to \mathscr{K}'$, a $*$-homomorphism $\theta : \mathscr{A} \to \mathbb{B}(\mathscr{H}')$ and a θ-morphism $\Psi : \mathscr{E} \to \mathbb{B}(\mathscr{H}',\mathscr{K}')$ such that $\varphi(a) = V^*\theta(a)V, \Phi(x) = W^*\Psi(x)V$ for all $x \in \mathscr{E}$ and $a \in \mathscr{A}$.*

Proof. By Theorem 4.1 of [10] there exist a Hilbert space $\mathscr{H}' = \mathscr{A} \otimes \mathscr{H}$, a representation $\theta : \mathscr{A} \to \mathbb{B}(\mathscr{H}')$ and an isometry $V : \mathscr{H} \to \mathscr{H}'$ defined by $V(h) = 1 \otimes h$ such that $\varphi(a) = V^*\theta(a)V$. We may consider a minimal Stinespring representation for θ, where $\mathscr{H}'$ is the closed linear span of $\{\theta(a)Vh : a\in\mathscr{A}, h\in\mathscr{H}\}$.

Now, we put $\mathscr{K}'$ to be the closed linear span of $\{\Phi(x)h : x \in \mathscr{E}, h \in \mathscr{H}\}$ and define the mapping $\Psi : \mathscr{E} \to \mathbb{B}(\mathscr{H}',\mathscr{K}'), x \mapsto \Psi(x)$, where $\Psi(x)$:

$\mathrm{span}\{\theta(a)Vh, a \in \mathscr{A}, h \in \mathscr{H}\} \to \mathscr{K}'$ is defined by $\Psi(x)\Big(\sum\limits_{i=1}^{n} \theta(a_i)Vh_i\Big) = \sum\limits_{i=1}^{n} \Phi(xa_i)h_i$ for $x \in \mathscr{E}, a_i \in \mathscr{A}, h_i \in \mathscr{H}$.

The map $\Psi(x)$ is well-defined and bounded, since

$$\begin{aligned}
&\Big\|\Psi(x)\Big(\sum_{i=1}^{n}\theta(a_i)Vh_i\Big)\Big\|^2 \\
&\quad = \Big\|\sum_{i=1}^{n}\Phi(xa_i)h_i\Big\|^2 = \sum_{i,j=1}^{n}\langle\Phi(xa_j)^*\Phi(xa_i)h_i, h_j\rangle \\
&\quad = \sum_{i,j=1}^{n}\langle\varphi(a_j^*)\Phi(x)^*\Phi(x)\varphi(a_i)h_i, h_j\rangle = \sum_{i,j=1}^{n}\langle\varphi(a_j^*)\varphi(\rho(x)^2)\varphi(a_i)h_i, h_j\rangle \\
&\quad = \sum_{i,j=1}^{n}\langle\varphi(a_j^*\rho(x)^2a_i)h_i, h_j\rangle = \sum_{i,j=1}^{n}\langle V^*\theta(a_j^*\rho(x)^2a_i)Vh_i, h_j\rangle \\
&\quad = \sum_{i,j=1}^{n}\langle\theta(\rho(x)^2)\theta(a_i)Vh_i, \theta(a_j)Vh_j\rangle \leqslant \|\theta(\rho(x)^2)\|\Big\|\sum_{i=1}^{n}\theta(a_i)Vh_i\Big\|^2 \\
&\quad \leqslant \|\rho(x)\|^2\Big\|\sum_{i=1}^{n}\theta(a_i)Vh_i\Big\|^2 = \|x\|^2\Big\|\sum_{i=1}^{n}\theta(a_i)Vh_i\Big\|^2.
\end{aligned}$$

The mapping Ψ is a θ-morphism, since for all $a, b \in \mathscr{A}$ and $h, g \in \mathscr{H}$

$$\begin{aligned}
&\langle\Psi(x)^*\Psi(x)(\theta(a)Vh), \theta(b)Vg\rangle \\
&\qquad = \langle\Psi(x)(\theta(a)Vh), \Psi(x)(\theta(b)Vg)\rangle = \langle\Phi(xa)h, \Phi(xb)g\rangle \\
&\qquad = \langle\Phi(x)\varphi(a)h, \Phi(x)\varphi(b)g\rangle = \langle\Phi(x)^*\Phi(x)\varphi(a)h, \varphi(b)g\rangle \\
&\qquad = \langle\varphi(\rho(x)^2)\varphi(a)h, \varphi(b)g\rangle = \langle\varphi(b^*\rho(x)^2a)h, g\rangle \\
&\qquad = \langle V^*\theta(b^*\rho(x)^2a)Vh, g\rangle = \langle\theta(\rho(x)^2)\theta(a)Vh, \theta(b)Vg\rangle,
\end{aligned}$$

whence $\Psi(x)^*\Psi(x) = \theta(\rho(x)^2)$. Moreover

$$\Psi(x)\theta(a)(\theta(b)Vh) = \Psi(x)(\theta(ab)Vh) = \Phi(x(ab))h = \Phi((xa)b)h = \Psi(xa)(\theta(b)Vh),$$

so that $\Psi(x)\theta(a) = \Psi(xa)$.

Since $\mathscr{K}' \subseteq \mathscr{K}$ we can consider a map W as the orthogonal projection of $\mathscr{K}$ onto $\mathscr{K}'$. Hence $W^* : \mathscr{K}' \to \mathscr{K}$ is the inclusion map, whence for any $k' \in \mathscr{K}'$ we have $WW^*(k') = W(k') = k'$, that is $WW^* = I_{\mathscr{K}'}$.

Finally we observe that $W^*\Psi(x)Vh = \Psi(x)Vh = \Psi(x)(\theta(1)Vh) = \Phi(x)h$, that is $W^*\Psi(x)V = \Phi(x)$. ∎

4. NONDEGENERATE AND IRREDUCIBLE QUASI-REPRESENTATIONS

In this section we define the notions of nondegenerate and irreducible quasi-representations of Finsler modules and describe relations between the nondegeneracy and the irreduciblity, see [2]. Throughout this section we assume that the quasi-representations satisfy condition (ii) of Definition 2.1.

DEFINITION 4.1. Let $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H}, \mathscr{K})$ be a quasi-representation of a Finsler module $\mathscr{E}$ over a C^*-algebra $\mathscr{A}$. The map Φ is said to be *nondegenerate* if $\overline{\Phi(\mathscr{E})\mathscr{H}} = \mathscr{K}$ and $\overline{\Phi(\mathscr{E})^*\mathscr{K}} = \mathscr{H}$ (or equivalently, if there exist $\xi \in \mathscr{H}, \eta \in \mathscr{K}$ such that $\Phi(\mathscr{E})\xi = 0$ and $\Phi(\mathscr{E})^*\eta = 0$, then $\xi = \eta = 0$). Recall that a representation $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ of a C^*-algebra $\mathscr{A}$ is nondegenerate if $\overline{\varphi(\mathscr{A})\mathscr{H}} = \mathscr{H}$ (or equivalently, if there exists $\xi \in \mathscr{H}$ such that $\varphi(\mathscr{A})\xi = 0$, then $\xi = 0$), see Definition A.1. of [13].

THEOREM 4.2. *If $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H}, \mathscr{K})$ is a nondegenerate quasi-representation, then $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ is a nondegenerate representation. If $\mathscr{E}$ is full and φ is nondegenerate, then Φ is also nondegenerate.*

Proof. Suppose that Φ is nondegenerate and $\varphi(\mathscr{A})\xi = 0$. It follows from the Hewitt–Cohen factorization theorem that $\Phi(\mathscr{E})\xi = \Phi(\mathscr{E}\mathscr{A})\xi = \Phi(\mathscr{E})\varphi(\mathscr{A})\xi = 0$. We conclude that $\xi = 0$. Thus φ is nondegenerate.

Suppose that $\Phi(\mathscr{E})\xi = 0$ for some $\xi \in \mathscr{H}$. Then for any $x \in \mathscr{E}$ we have $\|\Phi(x)\xi\|^2 = \langle \Phi(x)^*\Phi(x)\xi, \xi\rangle = \langle \varphi(\rho(x)^2)\xi, \xi\rangle = \|\varphi(\rho(x))\|^2 = 0$. Since $\mathscr{E}$ is a full Finsler $\mathscr{A}$-module, $a = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\rho(x_{i,n})^2$ for some $k_n \in \mathbb{N}$, $x_{i,n} \in \mathscr{E}$ and $\lambda_{i,n} \in \mathbb{C}$. Hence

$$\varphi(a)\xi = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\varphi(\rho(x_{i,n}))^2\xi = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\varphi(\rho(x_{i,n}))\varphi(\rho(x_{i,n}))\xi = 0,$$

whence $\xi = 0$. ∎

REMARK 4.3. The second result of Theorem 4.2 may fail, if the condition of being full is dropped. To see this take $\mathscr{A}$ to be a nondegenerate von Neumann algebra acting on a Hilbert space, which has a nontrivial central projection P. Hence the identity map $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ is assumed to be nondegenerate.

Put $\mathscr{E} = \mathscr{A}P = \{aP : a \in \mathscr{A}\}$ as a Finsler $\mathscr{A}$-module equipped with $\rho(aP) = |aP|$. Clearly $\mathscr{A}P$ is not full. The identity map $\Phi : \mathscr{A}P \to \mathbb{B}(\mathscr{H})$ satisfies the following:

(i) $\rho_0\Phi(aP) = \rho_0(aP) = |aP| = \varphi(|aP|) = \varphi\rho(aP)$, where ρ_0 is defined as in (2.1).

(ii) $\Phi(aPb) = \Phi(aP)\varphi(b)$ for all $b \in \mathscr{A}$.

Hence Φ is a quasi-representation of $\mathscr{E}$, which is not nondegenerate, since

$$\overline{\Phi(\mathscr{E})\mathscr{H}} = \overline{\mathscr{A}P(\mathscr{H})} = \overline{P(\mathscr{A}\mathscr{H})} \subseteq \overline{P(\mathscr{H})} = P(\mathscr{H}) \neq \mathscr{H}.$$

In the following corollary we investigate a condition under which the representation φ and the quasi-representation Φ are nondegenerate.

COROLLARY 4.4. *If $\varphi(\rho(x)) = I_{\mathscr{H}}$, then both Φ and φ are nondegenerate.*

Proof. Suppose $\Phi(\mathscr{E})\xi = 0$ for some $\xi \in \mathscr{H}$. Then for all $x \in \mathscr{E}$ we have $\|\Phi(x)\xi\|^2 = \langle \Phi(x)^*\Phi(x)\xi, \xi\rangle = \langle \varphi(\rho(x)^2)\xi, \xi\rangle = \|\xi\|^2 = 0$, so that $\xi = 0$. The nondegeneracy of φ follows from Theorem 4.2. ∎

DEFINITION 4.5. Let $\Phi : \mathscr{E} \to B(\mathscr{H}, \mathscr{H}')$ be a quasi-representation of a Finsler module $\mathscr{E}$ over a C^*-algebra $\mathscr{A}$ and let $\mathscr{K}, \mathscr{K}'$ be closed subspaces of $\mathscr{H}$ and $\mathscr{H}'$, respectively. A pair of subspaces $(\mathscr{K}, \mathscr{K}')$ is said to be *Φ-invariant* if $\Phi(\mathscr{E})\mathscr{K} \subseteq \mathscr{K}'$ and $\Phi(\mathscr{E})^*\mathscr{K}' \subseteq \mathscr{K}$. The quasi-representation Φ is said to be *irreducible* if $(0,0)$ and $(\mathscr{H}, \mathscr{H}')$ are the only Φ-invariant pairs. Recall that a representation $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ of a C^*-algebra $\mathscr{A}$ is irreducible if 0 and $\mathscr{H}$ are only closed subspaces of $\mathscr{H}$ which are φ-invariant, i.e. are invariant for $\varphi(\mathscr{A})$.

THEOREM 4.6. *Suppose that the quasi-representation $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H}, \mathscr{K})$ constructed in Theorem* 2.3 *is irreducible. Then so is $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$. If $\mathscr{E}$ is full and φ is irreducible, then Φ is irreducible.*

Proof. Suppose that Φ is irreducible and a closed subspace $\mathscr{K}$ of $\mathscr{H}$ is φ-invariant. Consider $\mathscr{K}' = \overline{\Phi(\mathscr{E})\mathscr{K}}$. Clearly $\Phi(\mathscr{E})\mathscr{K} \subseteq \mathscr{K}'$. Due to $\overline{\varphi(\mathscr{A})\mathscr{K}} \subseteq \mathscr{K}$ we observe that $\overline{\varphi(\rho(x)^2)\mathscr{K}} \subseteq \mathscr{K}$, whence $\overline{\Phi(x)^*\Phi(x)\mathscr{K}} \subseteq \mathscr{K}$ for all $x \in \mathscr{E}$. Now let $x \neq y$. In the notation of Theorem 2.3 we have

$$\begin{aligned}\langle \Phi(x)^*\Phi(y)h, h'\rangle &= \langle \Phi(y)h, \Phi(x)h'\rangle = \langle \varphi(\rho(y))h_y, \varphi(\rho(x))h'_x\rangle \\ &= \sum_{z\in\mathscr{E}} \langle \varphi(\rho(y))h_y(z), \varphi(\rho(x))h'_x(z)\rangle = 0,\end{aligned}$$

for all $h, h' \in \mathscr{H}$. Put $h' = \Phi(x)^*\Phi(y)h$ to get $\langle \Phi(x)^*\Phi(y)h, \Phi(x)^*\Phi(y)h\rangle = 0$. So that $\Phi(x)^*\Phi(y)h = 0$. Therefore $\Phi(x)^*\Phi(y)\mathscr{K} = 0\mathscr{K} \subseteq \mathscr{K}$. It follows that $\Phi(E)^*\overline{\Phi(E)\mathscr{K}} \subseteq \overline{\Phi(E)^*\Phi(E)\mathscr{K}} \subseteq \mathscr{K}$. Since Φ is irreducible, we conclude that $(\mathscr{K}, \mathscr{K}') = (0,0)$ or $(\mathscr{K}, \mathscr{K}') = (\mathscr{H}, \mathscr{H}')$, hence either $\mathscr{K} = 0$ or $\mathscr{K} = \mathscr{H}$. This implies that φ is irreducible.

Now assume that φ is irreducible. It follows from Remark 4.1.4 of [8] that φ is nondegenerate. By Theorem 4.2, Φ is nondegenerate.

Consider $(\mathscr{K}, \mathscr{K}')$ as a Φ-invariant pair of subspaces. Any $a \in \mathscr{A}$ can be represented as $a = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{k_n} \lambda_{i,n}\rho(x_{i,n})^2$ for some $k_n \in \mathbb{N}$, $x_{i,n} \in \mathscr{E}$ and $\lambda_{i,n} \in \mathbb{C}$. Hence

$$\varphi(a)\mathscr{K} = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\varphi(\rho(x_{i,n}))^2\mathscr{K} = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\Phi(x_{i,n})^*\Phi(x_{i,n})\mathscr{K} \subseteq \mathscr{K},$$

Hence either $\mathscr{K} = 0$ or $\mathscr{K} = \mathscr{H}$.

If $\mathscr{K} = 0$ then $\Phi(\mathscr{E})^*\mathscr{K}' \subseteq \mathscr{K} = 0$, and for every $\xi' \in \mathscr{K}'$ we have $0 = \langle \Phi(x)^*\xi', \xi \rangle = \langle \xi', \Phi(x)\xi \rangle$ for $x \in \mathscr{E}$ and $\xi \in \mathscr{H}$, so that $\mathscr{K}' \perp \overline{\Phi(\mathscr{E})\mathscr{H}} = \mathscr{H}'$. Since $\mathscr{K}' \subseteq \mathscr{H}'$, we have $\mathscr{K}' = 0$.

If $\mathscr{K} = \mathscr{H}$, then $\mathscr{H}' = \overline{\Phi(\mathscr{E})\mathscr{H}} = \overline{\Phi(\mathscr{E})\mathscr{K}} \subseteq \mathscr{K}'$. Hence $\mathscr{K}' = \mathscr{H}'$. Therefore Φ is irreducible. ∎

REMARK 4.7. The result may fail, if the condition of being full is dropped. The closed subspace $P(\mathscr{H})$ in Remark 4.3 when $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ is irreducible provides a counterexample.

Next we present some conditions under which the quasi-representation Φ is nondegenerate and irreducible.

COROLLARY 4.8. *Let $\mathscr{E}$ be a full Finsler $\mathscr{A}$-module and let $\varphi : \mathscr{A} \to \mathbb{B}(\mathscr{H})$ be irreducible. Then the quasi-representation $\Phi : \mathscr{E} \to \mathbb{B}(\mathscr{H}, \mathscr{K})$ is nondegenerate and irreducible.*

Proof. Since φ is irreducible, it is nondegenerate. Since $\mathscr{E}$ is full, by Theorem 4.2, Φ is nondegenerate and by Theorem 4.6, Φ is irreducible. ∎

THEOREM 4.9. *Let $\mathscr{E}$ be a full Finsler $\mathscr{A}$-module. Then $\Phi(\mathscr{E})$ is a subset of the space $\mathbb{K}(\mathscr{H}, \mathscr{H}')$ of all compact operators from $\mathscr{H}$ into $\mathscr{H}'$ if and only if $\varphi(\mathscr{A}) \subseteq \mathbb{K}(\mathscr{H})$.*

Proof. Suppose $\varphi(\mathscr{A}) \subseteq \mathbb{K}(\mathscr{H})$. Applying the Hewitt–Cohen factorization theorem we have $\Phi(\mathscr{E}) = \Phi(\mathscr{E}\mathscr{A}) = \Phi(\mathscr{E})\varphi(\mathscr{A}) \subseteq \mathbb{K}(\mathscr{H}, \mathscr{H}')$.

Conversely, suppose that $\Phi(\mathscr{E}) \subseteq \mathbb{K}(\mathscr{H}, \mathscr{H}')$. Since $\mathscr{E}$ is full we have

$$\varphi(a) = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\varphi(\rho(x_{i,n}))^2 = \lim_{n\to\infty} \sum_{i=1}^{k_n} \lambda_{i,n}\Phi(x_{i,n})^*\Phi(x_{i,n}) \in \mathbb{K}(\mathscr{H}),$$

where $a = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{k_n} \lambda_{i,n}\rho(x_{i,n})^2$ for some $k_n \in \mathbb{N}$, $x_{i,n} \in \mathscr{E}$ and $\lambda_{i,n} \in \mathbb{C}$. ∎

In the next two examples we illustrate the considered situations in the notation of Theorem 2.3.

EXAMPLE 4.10. By Theorem 1.10.2 of [5] the identity map $\varphi : \mathbb{K}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$ is irreducible. It is known that the C^*-algebra $\mathbb{K}(\mathscr{H})$ is a full Finsler module over $\mathbb{K}(\mathscr{H})$ with $\rho(T) = |T|$. Hence the quasi-representation $\Phi : \mathbb{K}(\mathscr{H}) \to \mathbb{B}(\mathscr{H}, \mathscr{K})$ is nondegenerate and irreducible.

EXAMPLE 4.11. Consider $\varphi = I : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H})$. Then $\varphi(\mathbb{B}(\mathscr{H}))^c = \{T \in \mathbb{B}(\mathscr{H}) : \varphi(S)T = T\varphi(S), \text{ for all } S \in \mathbb{B}(\mathscr{H})\} = \{T \in \mathbb{B}(\mathscr{H}) : ST = TS, \text{ for all } S \in \mathbb{B}(\mathscr{H})\} = \mathbb{C}I$. Hence φ is irreducible. Also $\mathbb{B}(\mathscr{H})$ is a full Finsler $\mathbb{B}(\mathscr{H})$-module, so that the quasi-representation $\Phi : \mathbb{B}(\mathscr{H}) \to \mathbb{B}(\mathscr{H}, \mathscr{K})$ is nondegenerate and irreducible.

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