J. OPERATOR THEORY
70:1(2013), 191–210
doi: 10.7900/jot.2011may31.1924

THE CROSSED-PRODUCT STRUCTURE OF C^*-ALGEBRAS ARISING FROM TOPOLOGICAL DYNAMICAL SYSTEMS

CYNTHIA FARTHING, NURA PATANI AND PAULETTE N. WILLIS

Communicated by Kenneth R. Davidson

ABSTRACT. We show that every topological k-graph constructed from a locally compact Hausdorff space Ω and a family of pairwise commuting local homeomorphisms on Ω satisfying a uniform boundedness condition on the cardinalities of inverse images may be realized as a semigroup crossed product in the sense of Larsen.

KEYWORDS: *Crossed product, topological higher-rank graph, product system of C^*-correspondences, Cuntz–Pimsner algebra.*

MSC (2010): Primary 46L05; Secondary 46L55, 37B10.

1. INTRODUCTION

In [4], Cuntz constructed the crossed product of a C^*-algebra A by an endomorphism α as a corner in an ordinary group crossed product. Since that time, there have been many efforts (see [14], [18], for example) to develop a theory of crossed products of C^*-algebras by single endomorphisms as well as by semigroups of endomorphisms. In [5], Exel proposed a new definition for the crossed product of A by α that depends not only on the pair (A, α) but also on the choice of a *transfer operator* (i.e., a positive continuous linear map $L : A \to A$ satisfying $L(\alpha(a)b) = aL(b)$). Exel shows that the Cuntz–Krieger algebra of a given $\{0,1\}$-matrix may be realized as the crossed product arising from the associated Markov sub-shift and a naturally defined transfer operator.

Extending Exel's construction to non-unital C^*-algebras, Brownlowe, Raeburn, and Vittadello in [2] model directed graph C^*-algebras as crossed products. In particular, they show that if E is a locally finite directed graph with no sources, then $C^*(E) \cong C_0(E^\infty) \rtimes_{\alpha,L} \mathbb{N}$ where E^∞ is the infinite-path space of E and α is the shift map on E^∞.

In another extension of Exel's construction, Larsen (in [13]) develops a theory of crossed products associated to dynamical systems (A, S, α, L) where A is a (not necessarily unital) C^*-algebra, S is an abelian semigroup with identity, α is an action of S by endomorphisms, L is an action of S by transfer operators, and for all $s \in S$, the maps α_s, L_s are extendible to $M(A)$ in an appropriate sense.

Given a locally compact Hausdorff space Ω and a family $\{T_i\}_{i=1}^k$ of local homeomorphisms of Ω that pairwise commute, Yeend ([21]) described the construction of an associated topological k-graph $(\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$. Motivated in part by the ideas in [2] described above, we show that if $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ is the topological k-graph constructed from the data $(\Omega, \{T_i\}_{i=1}^k)$, then $C^*(\Lambda)$ has a crossed product structure in the sense of Larsen in [13].

Given a general topological k-graph, it is not always the case than an associated graph C^*-algebra may be constructed. We show that Λ is compactly aligned, a condition that ensures $C^*(\Lambda)$ exists. This generalizes a result of Willis in [19] in which she essentially shows that the result holds when $k = 2$, Ω is compact, and the maps T_1, T_2 $*$-commute. In [3], Brownlowe et al. show that when Λ is a compactly aligned topological k-graph, the C^*-algebra $C^*(\Lambda)$ constructed from the boundary path groupoid is isomorphic to the Cuntz–Pimsner algebra $\mathcal{NO}_{X^\Lambda}$ where X^Λ is the topological k-graph correspondence associated to Λ. We show that the product system X^{Lar} associated to the dynamical system (in the sense of Larsen) arising from the data $(\Omega, \{T_i\}_{i=1}^k)$ is isomorphic to the topological k-graph correspondence X^Λ so that the associated Cuntz–Pimsner algebras are isomorphic.

To show that $\mathcal{NO}_X$ is isomorphic to the Larsen crossed product, we show that certain notions of covariance agree for representations of the product systems X^Λ and X^{Lar}. Our isomorphism result then follows from the universal properties of the associated C^*-algebras.

Brownlowe has shown in [1] that the C^*-algebra of a finitely-aligned discrete k-graph has a crossed product structure. He has suggested that the Cuntz–Nica–Pimsner algebra $\mathcal{NO}_X$ should be used to define a general crossed product by a quasi-lattice ordered semigroup of partial endomorphisms and partially-defined transfer operators. The fact that, in our setting, $\mathcal{NO}_X$ is isomorphic to the Larsen crossed product supports his proposal.

The paper is organized as follows: We begin with some preliminaries in Section 2. We state some necessary definitions about product systems of C^*-correspondences, various notions of Cuntz–Pimsner covariance appearing in the literature, and the C^*-algebras that are universal for such representations. We review several definitions about the topological k-graphs and the dynamical systems described by Larsen in [13], as well as the C^*-algebras associated to each of these constructions.

In Section 3, we define what we mean by a topological dynamical system and describe a uniform boundedness condition that is key to our results. We describe Yeend's construction of a topological k-graph from a topological dynamical system and show that this topological k-graph is always compactly aligned (in the sense of Definition 2.3 of [21]). We then show how an Exel–Larsen system may be associated to a topological dynamical system satisfying our uniform boundedness condition.

In Section 4, we define two product systems over $\mathbb{N}^k$ of C^*-correspondences: the topological k-graph correspondence X^Λ and the product system X^{Lar} associated to the Exel–Larsen system. We then show that these two product systems are isomorphic. Finally, in Section 5, we prove that the C^*-algebras associated to the topological k-graph and the Exel–Larsen system arising from a given topological dynamical system are isomorphic.

2. PRELIMINARIES

2.1. PRODUCT SYSTEMS OF C^*-CORRESPONDENCES. In this subsection, we give some key definitions for product systems of C^*-correspondences, many of which may be found in Section 2 of [17]. For more details on right-Hilbert C^*-modules and C^*-correspondences, we refer the reader to [12], [16].

Given a C^*-algebra A and a countable semigroup S with identity e, a *product system over S of C^*-correspondences* is a semigroup X equipped with a semigroup homomorphism $p : X \to S$ such that $X_s := p^{-1}(s)$ is an C^*-correspondence for each $s \in S$, $X_e = A$ (viewed as an C^*-correspondence), the multiplication in X implements isomorphisms $\beta_{s,t} : X_s \otimes_A X_t \to X_{st}$ for $s, t \in S \setminus \{e\}$, and multiplication in X by elements of $X_e = A$ induces maps $\beta_{s,e} : X_s \otimes_A X_e \to X_s$ and $\beta_{e,s} : X_e \otimes_A X_s \to X_s$ that give the right and left (respectively) actions of A on X_s. For each $s \in S$, $\beta_{s,e}$ is an isomorphism by Corollary 2.7 of [16].

For each $s \in S$ and $\xi, \eta \in X_s$, the operator $\Theta_{\xi,\eta} : X_s \to X_s$ defined by $\Theta_{\xi,\eta}(\zeta) := \xi \cdot \langle \eta, \zeta \rangle_A$ is adjointable with $\Theta^*_{\xi,\eta} = \Theta_{\eta,\xi}$. The space $\mathcal{K}(X_s) := \overline{\operatorname{span}}\{\Theta_{\xi,\eta} : \xi, \eta \in X_s\}$ is a closed two-sided ideal in $\mathcal{L}(X_s)$ which we call the *generalized compact operators on X_s*.

Given $s, t \in S$ with $s \neq e$, we have a homomorphism $\iota_s^{st} : \mathcal{L}(X_s) \to \mathcal{L}(X_{st})$ characterized by

$$\iota_s^{st}(T)(\xi\eta) = T(\xi)\eta \quad \text{for all } \xi \in X_s, \eta \in X_t, T \in \mathcal{L}(X_s).$$

Via the identification of $\mathcal{K}(X_e)$ with A, there also exists a homomorphism $\iota_e^s : \mathcal{K}(X_e) \to \mathcal{L}(X_s)$ given by $\iota_e^s = \phi_s$, where ϕ_s is the homomorphism of A to $\mathcal{L}(X_s)$ implementing the left action.

REMARK 2.1. For a C^*-algebra A, we will use the term A-correspondence to refer to a C^*-correspondence over A.

2.2. REPRESENTATIONS OF PRODUCT SYSTEMS AND ASSOCIATED C^*-ALGEBRAS. Given a product system X over S of C^*-correspondences, a *(Toeplitz) representation* of X in a C^*-algebra B is a map $\psi : X \to B$ such that:

(i) For each $s \in S$, the pair $(\psi_s, \psi_e) := (\psi|_{X_s}, \psi|_{X_e})$ is a Toeplitz representation of X_s in the sense that $\psi_s : X_s \to B$ is linear and $\psi_e : A \to B$ is a homomorphism satisfying

$$\psi_s(\xi \cdot a) = \psi_s(\xi)\psi_e(a), \quad \psi_s(\xi)^*\psi(\eta) = \psi_e(\langle \xi, \eta \rangle_{X_s}), \quad \psi_s(a \cdot \xi) = \psi_e(a)\psi_s(\xi),$$

for $\xi, \eta \in X_s$, $a \in A$, and

(ii) $\psi(\xi\eta) = \psi(\xi)\psi(\eta)$, for $\xi, \eta \in X$.

For each $s \in S$, there is a homomorphism $\psi^{(s)} : \mathcal{K}(X_s) \to B$ satisfying

$$\psi^{(s)}(\Theta_{\xi,\eta}) = \psi_s(\xi)\psi_s(\eta)^* \quad \text{for } \xi, \eta \in X_s.$$

We say that a representation $\psi : X \to B$ is *Cuntz–Pimsner covariant* if for each $s \in S$ the (Toeplitz) representation (ψ_s, ψ_e) is Cuntz–Pimsner covariant, that is

$$\text{(CP-K)} \qquad \psi^{(s)}(\phi_s(a)) = \psi_e(a) \quad \text{for } a \in \phi_s^{-1}(\mathcal{K}(X_s)) \cap (\ker \phi_s)^{\perp}.$$

REMARK 2.2. Different definitions exist in the literature for Cuntz–Pimsner covariant representations. The one used above is sometimes referred to as the "Katsura convention" and differs from the definition originally introduced by Fowler in [8] where (CP-K) is instead required to hold for $a \in \phi^{-1}(\mathcal{K}(X))$. The two definitions coincide when the left action on each fibre is injective.

DEFINITION 2.3. For a product system X the *Cuntz–Pimsner algebra* $\mathcal{O}_X$ is the universal C^*-algebra generated by a representation $j^{\text{Fow}} : X \to \mathcal{O}_X$ that satisfies (CP-K).

A (Toeplitz) representation $\psi : X \to B$ is said to be *coisometric on* $K = \{K_s\}_{s \in S}$, where each K_s is an ideal in $\phi_s^{-1}(\mathcal{K}(X_s))$, if each (ψ_s, ψ_e) is coisometric on K_s; that is,

$$\text{(2.1)} \qquad \psi^{(s)}(\phi_s(a)) = \psi_e(a), \quad \text{for all } a \in K_s.$$

DEFINITION 2.4. The *relative Cuntz–Pimsner algebra* $\mathcal{O}(X, K)$ is the universal C^*-algebra generated by a representation $j^{\text{relCP}} : X \to \mathcal{O}(X, K)$ that is coisometric on $K = \{K_s\}_{s \in S}$.

A quasi-lattice ordered group (G, P) is a discrete group G and a subsemigroup P such that: $P \cap P^{-1} = \{e\}$, and any two elements $p, q \in G$ that have a common upper bound in P have a least upper bound $p \vee q \in P$ under the order $p \leqslant q \iff p^{-1}q \in P$. We write $p \vee q = \infty$ to indicate that $p, q \in G$ have no common upper bound in P, and we write $p \vee q < \infty$ otherwise. Since some of the results we discuss are true for arbitrary abelian semigroups with identity, we will use the notation S to highlight when this is the case. We will reserve the notation P to indicate that the semigroup is sitting inside a quasi-lattice ordered group (G, P).

DEFINITION 2.5. Given a quasi-lattice ordered group and a product system X over P of C^*-correspondences, we say that X is *compactly aligned* if whenever $p \vee q < \infty$, the map $\iota_p^{p\vee q}(S)\iota_q^{p\vee q}(T) \in \mathcal{K}(X_{p\vee q})$ for all $S \in \mathcal{K}(X_p), T \in \mathcal{K}(X_q)$.

A (Toeplitz) representation $\psi : X \to B$ is *Nica covariant* if, for each $p, q \in P$ and for all $S \in \mathcal{K}(X_p),\ T \in \mathcal{K}(X_q)$, we have

$$\text{(N)}\qquad \psi^{(p)}(S)\psi^{(q)}(T) = \begin{cases} \psi^{(p\vee q)}(\iota_p^{(p\vee q)}(S)\iota_q^{(p\vee q)}(T)) & \text{if } p \vee q < \infty, \\ 0 & \text{otherwise.} \end{cases}$$

In [17], Sims and Yeend introduced a new notion of Cuntz–Pimsner covariance for compactly aligned product systems. In order to define their notion of Cuntz–Pimsner covariance, we need to consider the space $\widetilde{X}$ which serves as a sort of "boundary" of X (see Remark 3.10 of [17]).

Given a quasi-lattice ordered group (G, P) and a product system X over P of C^*-correspondences, let $I_e = A$ and for $p \in P \setminus \{e\}$ let $I_p = \bigcap_{e<r\leqslant p} \ker(\phi_r)$. Note that I_p is an ideal of A. For $q \in P$, define

$$\widetilde{X}_q = \bigoplus_{p\leqslant q} X_p \cdot I_{p^{-1}q}.$$

Each $\widetilde{X}_q$ is an C^*-correspondence with left action implemented by $\widetilde{\phi}_q : A \to \mathcal{L}(\widetilde{X}_q)$ where $(\widetilde{\phi}_q(a)\xi)(p) = \phi_p(a)\xi(p)$, for $p \leqslant q$. There is a homomorphism $\widetilde{\iota}_p^q : \mathcal{L}(X_p) \to \mathcal{L}(\widetilde{X}_q)$ defined by

$$(\widetilde{\iota}_p^q(S)\xi)(r) = \iota_p^r(S)\xi(r).$$

Let (G, P) be a quasi-lattice ordered group and let X be a compactly aligned product system over P of C^*-correspondences such that $\widetilde{\phi}_q$ is injective for each $q \in P$. A (Toeplitz) representation $\psi : X \to B$ of X in a C^*-algebra B is said to be *Cuntz–Pimsner covariant* if

(CP-SY) for every finite $F \subset P$, and every choice $\{T_p \in \mathcal{K}(X_p) : p \in F\}$ such that $\sum_{p\in F} \widetilde{\iota}_p^s(T_p) = 0$ for large s, we have $\sum_{p\in F} \psi^{(p)}(T_p) = 0_B$.

See Definition 3.8 of [17] for the definition of *for large s*. If $\psi : X \to B$ satisfies both (CP-SY) and (N), then ψ is said to be *Cuntz–Nica–Pimsner covariant* or CNP-covariant.

DEFINITION 2.6. The *Cuntz–Nica–Pimsner algebra* $\mathcal{NO}_X$ is the universal C^*-algebra generated by a CNP-covariant representation $j^{\mathrm{CNP}} : X \to \mathcal{NO}_X$.

2.3. TOPOLOGICAL k-GRAPHS AND THEIR C^*-ALGEBRAS. For $k \in \mathbb{N}$, a *topological k-graph* is a pair (Λ, d) consisting of: (1) a small category Λ endowed with a second countable locally compact Hausdorff topology under which composition is continuous and open, the range map r is continuous, and the source map

s is a local homeomorphism; and (2) a continuous functor $d : \Lambda \to \mathbb{N}^k$, called the *degree map*, satisfying the *factorization property*: if $\lambda \in \Lambda$ with $d(\lambda) = m + n$, then there are unique $\mu, \nu \in \Lambda$ with $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. For $m \in \mathbb{N}^k$, let $\Lambda^m = d^{-1}(\{m\})$ denote the paths of degree m; we identify Λ^0 with the vertex space $\mathrm{Obj}(\Lambda)$. If U and V are subsets of Λ, then we define $UV = \{\lambda\mu \in \Lambda : \lambda \in U, \mu \in V, \text{ and } s(\lambda) = r(\mu)\}$. In particular, for $v \in \Lambda^0$ and $U \subseteq \Lambda$, we denote $\{v\}U$ by vU and $U\{v\}$ by Uv. For more details about topological k-graphs, see [20].

Given a compactly aligned topological k-graph Λ, the *topological k-graph C^*-algebra* $C^*(\Lambda)$ is the full groupoid C^*-algebra $C^*(\mathcal{G}_\Lambda)$ of the boundary path groupoid $\mathcal{G}_\Lambda$ defined in Definition 4.1 of [21]. It is shown in Theorem 5.20 of [3] that $C^*(\Lambda)$ is isomorphic to the Cuntz–Nica–Pimsner algebra $\mathcal{NO}_{X^\Lambda}$ associated to the topological k-graph correspondence X^Λ (for details of the construction of X^Λ, see [17] for example), where $\mathcal{NO}_{X^\Lambda}$ is the universal C^*-algebra generated by a CNP-covariant representation $j^{\mathrm{CNP}} : X^\Lambda \to \mathcal{NO}_{X^\Lambda}$.

2.4. EXEL–LARSEN SYSTEMS AND THEIR RELATIVE CUNTZ–PIMSNER ALGEBRAS. Let A be a (not necessarily unital) C^*-algebra, S an abelian semigroup with identity e. Let $\alpha : S \to \mathrm{End}(A)$ be an action such that each α_s is *extendible,* meaning that it extends uniquely to an endomorphism $\overline{\alpha}_s$ of $M(A)$ such that

$$\overline{\alpha}_s(1_{M(A)}) = \lim \alpha_s(u_\lambda) \tag{2.2}$$

for some (and hence every) approximate unit (u_λ) in A and all $s \in S$. Finally, let L be an action of S by continuous, linear, positive maps $L_s : A \to A$ which have linear continuous extensions $\overline{L}_s : M(A) \to M(A)$ satisfying the *transfer operator identity*

$$L_s(\alpha_s(a)u) = a\overline{L}_s(u), \quad \text{for all } a \in A, u \in M(A), s \in S. \tag{2.3}$$

We call the quadruple (A, S, α, L) an *Exel–Larsen system.*

Given an Exel–Larsen system (A, S, α, L) there is an associated product system over S of C^*-correspondences which we will denote X^{Lar} (for details of the general construction, see [13]). The *Larsen crossed product* $A \rtimes_{\alpha,L} S$ is the relative Cuntz–Pimsner algebra of X^{Lar} and $K = \{K_s\}_{s\in S}$ where

$$K_s = \overline{A\alpha_s(A)A} \cap \phi_s^{-1}(\mathcal{K}(X_s^{\mathrm{Lar}})). \tag{2.4}$$

We denote by j^{Lar} the universal representation of X^{Lar} that generates $A \rtimes_{\alpha,L} S$.

3. CONSTRUCTIONS ASSOCIATED TO THE TOPOLOGICAL DYNAMICAL SYSTEM $(\Omega, \{T_i\}_{i=1}^k)$

DEFINITION 3.1. A *topological dynamical system (TDS)* is a pair $(\Omega, \{T_i\}_{i=1}^k)$ consisting of a locally compact Hausdorff space Ω and pairwise commuting local homeomorphisms $T_1, \dots, T_k : \Omega \to \Omega$. For each $m = (m_1, \dots, m_k) \in \mathbb{N}^k$, let $\Theta_m : \Omega \to \Omega$ be the local homeomorphism defined by

$$\Theta_m(x) = T_1^{m_1} \cdots T_k^{m_k}(x).$$

DEFINITION 3.2. Let X and Y be sets. A function $f : X \to Y$ has *uniformly bounded cardinality on inverse images* if there exists $N \in \mathbb{N}$ such that

$$\sup_{y \in Y} |\{x \in X : f(x) = y\}| \leqslant N.$$

The number N is called the uniform bound on the cardinality of the inverse image of f.

We say that a topological dynamical system $(\Omega, \{T_i\}_{i=1}^k)$ *satisfies condition (UBC)* if each T_i, $1 \leqslant i \leqslant k$, has uniformly bounded cardinalities on inverse images.

If $(\Omega, \{T_i\}_{i=1}^k)$ satisfies condition (UBC), then for each $m \in \mathbb{N}^k$, the local homeomorphism Θ_m also has uniformly bounded cardinality on inverse images.

EXAMPLE 3.3. (i) Let $\mathbb{T}$ denote the unit circle and fix $n_0 \in \mathbb{N}$. Define $T : \mathbb{T} \to \mathbb{T}$ by $z \mapsto z^{n_0}$. Then $(\mathbb{T}, T)$ is a TDS and for $m \in \mathbb{N}$, the local homeomorphism Θ_m is given by $z \mapsto z^{n_0^m}$. The system $(\mathbb{T}, T)$ satisfies condition (UBC) since

$$\sup_{y \in \mathbb{T}} |\{z \in \mathbb{T} : T(z) = z^{n_0} = y\}| \leqslant n_0.$$

(ii) Given any $n_1, \dots, n_k \in \mathbb{N}$ we may define $T_i : \mathbb{T} \to \mathbb{T}$ by $z \mapsto z^{n_i}$ to obtain k pairwise commuting local homeomorphisms. Then $(\mathbb{T}, \{T_i\}_{i=1}^k)$ is a topological dynamical system and for each $m \in \mathbb{N}^k$, the local homeomorphism Θ_m is given by $z \mapsto z^{n_1^{m_1} + \cdots + n_k^{m_k}}$. The system $(\mathbb{T}, \{T_i\}_{i=1}^k)$ satisfies condition (UBC) since, for each $i = 1, 2, \dots k$,

$$\sup_{y \in \mathbb{T}} |\{z \in \mathbb{T} : T_i(z) = z^{n_i} = y\}| \leqslant n_i.$$

(iii) Let A be a finite alphabet and for $n \in \mathbb{N}$, let A^n denote the space of words of length n. We let $A^{\mathbb{N}}$ denote the one-sided infinite sequence space, which is compact by Tychonoff's theorem. The *shift map* $\sigma : A^{\mathbb{N}} \to A^{\mathbb{N}}$ defined by

$$\sigma(x_1 x_2 x_3 \cdots) = x_2 x_3 \cdots$$

is a local homeomorphism of $A^{\mathbb{N}}$. Given a *block map* $d : A^n \to A$ for some $n \in \mathbb{N}$, we may define a *sliding block code* $\tau_d : A^{\mathbb{N}} \to A^{\mathbb{N}}$ via

$$\tau_d(x)_i = d(x_i \cdots x_{i+n-1}).$$

A function $\phi : A^{\mathbb{N}} \to A^{\mathbb{N}}$ is continuous and commutes with the shift map σ if and only if $\phi = \tau_d$ is the sliding block code associated to some block map d (see Lemma 3.3.3 and Lemma 3.3.7 of [19] for a proof, or Theorem 3.4 of [9] for an earlier proof in the two-sided setting). Exel and Renault prove in Theorem 14.3 of [6] that τ_d is a local homeomorphism whenever d is *progressive* (also called *right permutive*) in the sense that for each $x_1 \cdots x_{n-1} \in A^{n-1}$ the function $a \mapsto d(x_1 \cdots x_{n-1} a)$ is bijective. It follows that if τ_d is a sliding block code associated to a progressive block map, then $(A^{\mathbb{N}}, \{\sigma, \tau_d\})$ is a TDS and for $(a,b) \in \mathbb{N}^2$ the local homeomorphism $\Theta_{(a,b)}$ is given by

$$\Theta_{(a,b)}(x) = \sigma^a \tau_d^b(x).$$

A block map d is said to be regressive (also called *left permutive*) if for each $x_1 \cdots x_{n-1} \in A^{n-1}$ the function $a \mapsto d(ax_1 \cdots x_{n-1})$ is bijective. In Theorems 6.6 and 6.7 of [9], Hedlund shows that if τ_d is a sliding block code associated to a block map that is both progressive and regressive, then τ_d is $|A|^{n-1}$-to-1 and surjective. Therefore the system $(A^{\mathbb{N}}, \{\sigma, \tau_d\})$ satisfies condition (UBC) whenever the block map d is progressive and regressive.

(iv) Let Λ be a row-finite k-graph with no sources such that for each $i = 1, \ldots, k$,

$$|\Lambda^{e_i} v| < \infty, \quad \text{for all } v \in \Lambda^0.$$

Since Λ is row-finite with no sources, the boundary path space $\partial\Lambda$ coincides with the infinite path space Λ^∞ (see Examples 5.13, 1. of [7]). For each $i = 1, \ldots, k$, let $T_i : \partial\Lambda \to \partial\Lambda$ denote the shift by e_i, that is,

$$T_i(x)(n) = x(n + e_i) \quad \text{for } n \in \mathbb{N}^k.$$

Then $(\partial\Lambda, \{T_i\}_{i=1}^k)$ is a topological dynamical system and for $m \in \mathbb{N}^k$, the local homeomorphism Θ_m is given by $\Theta_m(x)(n) = x(n+m)$ for $n \in \mathbb{N}^k$. It is straightforward to see that the condition $|\Lambda^{e_i} v| < \infty$ for each $i = 1, \ldots, k$ and every $v \in \Lambda^0$ ensures the system $(\partial\Lambda, \{T_i\}_{i=1}^k)$ satisfies condition (UBC).

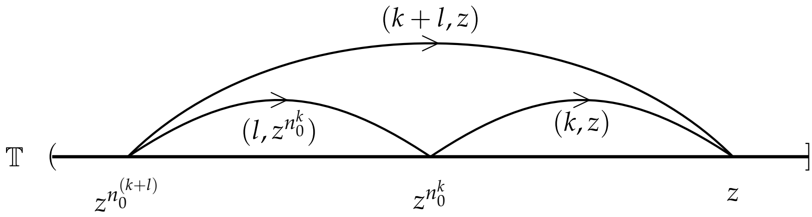
3.1. THE TOPOLOGICAL k-GRAPH ASSOCIATED TO $(\Omega, \{T_i\}_{i=1}^k)$. Given a topological dynamical system $(\Omega, \{T_i\}_{i=1}^k)$ with local homeomorphisms Θ_m as defined above, we construct a topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ as in Example 2.5(iv) of [21]. Specifically, we have:

(i) $\mathrm{Obj}(\Lambda) = \Omega$.
(ii) $\mathrm{Mor}(\Lambda) = \mathbb{N}^k \times \Omega$, with the product topology.
(iii) $r(n,x) = x$ and $s(n,x) = \Theta_n(x)$.
(iv) Composition is given by

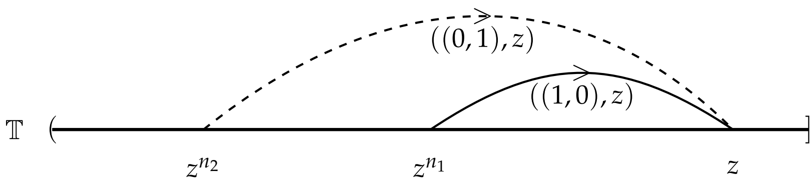
$$(n,x) \circ (m, \Theta_n(x)) = (n+m, x).$$

(v) The degree map is defined by $d(n,x) = n$.

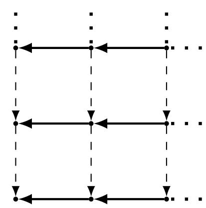
EXAMPLE 3.4. (i) Fix $n_0 \in \mathbb{N}$. Let $(\mathbb{T}, T)$ be the topological dynamical system described in Example 3.3(i). The associated topological 1-graph is visualized below.



(ii) For $n_1, n_2 \in \mathbb{N}$, we obtain a topological dynamical system $(\mathbb{T}, \{T_1, T_2\})$ in Example 3.3(ii) where $T_i : \mathbb{T} \to \mathbb{T}$ is given by $T_i(z) = z^{n_i}$. The 1-skeleton of the associated 2-graph is visualized below.



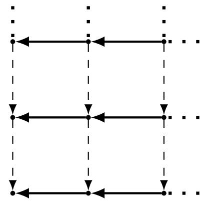
(iii) Let $\Lambda = \Omega_k$ be the discrete k-graph with $\mathrm{Obj}(\Omega_k) = \mathbb{N}^k$, $\mathrm{Mor}(\Omega_k) = \{(p,q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leqslant q\}$ (where $p \leqslant q$ if $p_i \leqslant q_i$ for all i), $r(p,q) = p$, $s(p,q) = q$, $d(p,q) = q - p$, and composition defined by $(p,q)(q,r) = (p,r)$. This is a locally-finite k-graph with no sources such that $|\Lambda^{e_i} v| < \infty$ for each $i = 1, \dots, k$ and every λ. For $k = 2$, the 1-skeleton of Λ is shown below:



Let $(\partial\Lambda, \{T_i\}^k)$ be the topological dynamical system presented in Example 3.3(iv). To visualize the associated topological k-graph Γ, note that, since $\Lambda = \Omega_k$, each $x \in \Lambda^\infty$ is uniquely determined by a point $p = r(x) = x(0) \in \mathrm{Obj}(\Lambda)$. So we may regard Λ^∞ as $\mathbb{N}^k$. Modifying notation to reflect this gives:

- $\mathrm{Obj}(\Gamma) = \mathbb{N}^k$;
- $\mathrm{Mor}(\Gamma) = \mathbb{N}^k \times \mathbb{N}^k$;
- $r(m,p) = p$ and $s(m,p) = p + m$;
- $d(m,p) = m$;
- $(m,p)(n,p+m) = (m+n,p)$.

The 1-skeleton of Γ is



Given a general topological dynamical system $(\Omega, \{T_i\}_{i=1}^k)$, it is important to verify that we may in fact construct the topological k-graph C^*-algebra $C^*(\Lambda)$ associated to the topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$. In order to establish this, we begin by showing that Λ is proper. We then prove that every proper topological k-graph is compactly aligned.

DEFINITION 3.5 ([21], Definition 6.4). A topological k-graph Λ is said to be *proper* if for all $m \in \mathbb{N}^k$, the map $r|_{\Lambda^m}$ is a proper map. That is, if for every $m \in \mathbb{N}^k$ and compact $U \subset \Lambda^0$, the set $U\Lambda^m$ is compact.

DEFINITION 3.6 ([21], Definition 2.3). A topological k-graph Λ is said to be *compactly aligned* if for all $p, q \in \mathbb{N}^k$ and for all compact $U \subset \Lambda^p$ and $V \subset \Lambda^q$, the set $U \vee V := U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q} \subset \Lambda^{p\vee q}$ is compact.

This compactly aligned condition ensures that the boundary path groupoid $\mathcal{G}_\Lambda$ is a locally compact r-discrete groupoid admitting a Haar system and hence that the associated C^*-algebra $C^*(\Lambda)$ may be defined.

LEMMA 3.7. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system. Then the associated topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ is proper.*

Proof. Fix $m \in \mathbb{N}^k$ and let $U \subset \Lambda^0$ be compact. We want to show that $U\Lambda^m$ is compact. Let $\mathcal{C} = \{C_i\}_{i\in I}$ be an open cover of $U\Lambda^m$. Note that $U\Lambda^m = \{m\} \times U$ so for each i, we have $C_i = \{m\} \times B_i$ for an open set B_i containing U. Then $\mathcal{B} = \{B_i\}_{i\in I}$ is an open cover of U.

Since U is compact, there is a finite subset J of I such that $U \subseteq \bigcup_{i\in J} B_i$. Then

$$U\Lambda^m = \{m\} \times U \subseteq \{m\} \times \bigcup_{i\in J} B_i = \bigcup_{i\in J}(\{m\} \times B_i) = \bigcup_{i\in J} C_i\,.$$

Then $\mathcal{C}' = \{C_i\}_{i\in J}$ is a finite subset of $\mathcal{C}$ that covers $U\Lambda^m$. It follows that $U\Lambda^m$ is compact and hence Λ is proper. ∎

To show that $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ is compactly aligned, we use the following lemma which is stated without proof in Remark 6.5 of [21].

LEMMA 3.8. *Every proper topological k-graph is compactly aligned.*

Proof. Let $p, q \in \mathbb{N}^k$ and let $U \subset \Lambda^p$ and $V \subseteq \Lambda^q$ be compact. Since s is continuous, $s(U)$ and $s(V)$ are both compact. By the assumption that Λ is proper, it follows that $s(U)\Lambda^{(p\vee q)-p}$ and $s(V)\Lambda^{(p\vee q)-q}$ are both compact. Moreover, since

$\Lambda * \Lambda$ has the relative topology inherited from the product topology on $\Lambda \times \Lambda$, the sets $U * s(U)\Lambda^{(p\vee q)-p}$ and $V * s(V)\Lambda^{(p\vee q)-q}$ are compact. Since the composition map is continuous, it follows that the images of these sets under the composition map, namely $U\Lambda^{(p\vee q)-p}$ and $V\Lambda^{(p\vee q)-q}$, are compact. Hence

$$U \vee V = U\Lambda^{(p\vee q)-p} \cap V\Lambda^{(p\vee q)-q}$$

is compact and therefore Λ is compactly aligned. ∎

PROPOSITION 3.9. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system. Then the associated topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ is compactly aligned.*

This follows directly from Lemma 3.7 and Lemma 3.8.

3.2. THE EXEL–LARSEN SYSTEM ASSOCIATED TO $(\Omega, \{T_i\}_{i=1}^k)$. Given a topological dynamical system $(\Omega, \{T_i\}_{i=1}^k)$ that satisfies condition (UBC), we may construct an *Exel–Larsen system*.

LEMMA 3.10. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system. For $m \in \mathbb{N}^k$, define $\alpha_m \in \mathrm{End}(C_0(\Omega))$ by*

$$\alpha_m(f) = f \circ \Theta_m, \quad \text{for } f \in C_0(\Omega)$$

so that α is an action of $\mathbb{N}^k$ on $C_0(\Omega)$. Then α extends uniquely to an endomorphism $\overline{\alpha}_s$ of $C_b(\Omega)$ satisfying (2.2).

Proof. First note that each α_m is nondegenerate, i.e., $\alpha_m(C_0(\Omega))C_0(\Omega) = C_0(\Omega)$. To see this, it is enough to show that for $g \in C_c(\Omega)$ there is $f \in C_c(\Omega)$ such that $\alpha(f)g = g$. Since Θ_m is continuous and g is compactly supported, the set $\Theta_m(\mathrm{supp}(g))$ is compact. By Urysohn's lemma for locally compact Hausdorff spaces, we may choose $f \in C_c(\Omega)$ such that $f|_{\Theta_m(\mathrm{supp}(g))} = 1$. It follows that $\alpha_m(f)g = g$.

Since $\alpha_m(C_0(\Omega))C_0(\Omega) = C_0(\Omega)$, the unique strictly continuous extension $\overline{\alpha}_m$ defined by $\overline{\alpha}_m(f) = f \circ \Theta_m$, for $f \in C_b(\Omega)$ is unital (see Proposition 1.1.13 of [10]) so that $\overline{\alpha}_m$ satisfies (2.2) as desired. ∎

LEMMA 3.11. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system that satisfies condition* (UBC). *For $m \in \mathbb{N}^k$, $f \in C_0(\Omega)$, and $x \in \Omega$ define L_m by*

$$L_m(f)(x) = \begin{cases} \sum_{\Theta_m(y)=x} f(y) & \text{if } x \in \Theta_m(\Omega), \\ 0 & \text{else,} \end{cases}$$

and similarly define $\overline{L}_m$ for $f \in C_b(\Omega)$. Then each L_m is a continuous, linear, positive map on $C_0(\Omega)$ with continuous linear extension $\overline{L}_m$ satisfying

$$L_m(\alpha_m(f)g) = f\overline{L}_m(g).$$

Proof. Fix $m \in \mathbb{N}^k$. To see that L_m maps $C_0(\Omega)$ into $C_0(\Omega)$, let $x \in \Omega$. Then there is an open neighborhood V of x and an open neighborhood U_y for each $y \in \Theta_m^{-1}(\{x\})$ such that $\Theta_m|_{U_y} : U_y \to V$ is a homeomorphism. Condition (UBC) ensures that there are finitely many such sets U_y. It follows then that $L_m(f)$ is the finite sum of the functions $f|_{U_y}$ and is hence in $C_0(\Omega)$.

It is straightforward to see that, for each $m \in \mathbb{N}^k$, L_m is continuous, linear, and positive. We must show that $\overline{L}_m$ is a continuous linear extension of L_m satisfying (2.3). By an argument similar to the one above, we know that $\overline{L}_m$ maps $C_b(\Omega)$ to $C_b(\Omega)$. Suppose $f, g \in C_b(\Omega)$, $a, b \in \mathbb{C}$, $x \in \Theta_m(\Omega)$. Then

$$\begin{aligned}(a\overline{L}_m(f) - b\overline{L}_m(g))(x) = a\overline{L}_m(f)(x) - b\overline{L}_m(g)(x) &= \sum_{\Theta_m(y)=x} af(y) - \sum_{\Theta_m(y)=x} bg(y)\\ &= \sum_{\Theta_m(y)=x} (af - bg)(y) = \overline{L}_m(af - by)(x).\end{aligned}$$

If $x \notin \Theta_m(\Omega)$, then both $(a\overline{L}_m(f) - b\overline{L}_m(g))(x)$ and $\overline{L}_m(af - by)(x)$ are zero. Thus, $\overline{L}_m$ is linear. Continuity of $\overline{L}_m$ follows from the fact that $(\Omega, \{T_i\}_{i=1}^k)$ satisfies condition (UBC) because for any $x \in \Theta_m(\Omega)$, we have

$$|\overline{L}_m(f)(x)| = \Big| \sum_{\Theta_m(y)=x} f(y) \Big| \leqslant \sum_{\Theta_m(y)=x} |f(y)| \leqslant N_m \cdot \|f\|_\infty,$$

where $N_m \in \mathbb{N}$ is the uniform bound on the cardinality of the inverse image of Θ_m. If $x \notin \Theta_m(\Omega)$, then $|\overline{L}_m(f)(x)| = 0$, and the inequality

$$|\overline{L}_m(f)(x)| \leqslant N_m \|f\|_\infty$$

holds for all $x \in \Omega$. Taking the supremum over all $x \in \Omega$ gives that

$$\|\overline{L}_m(f)\| \leqslant N_m \|f\|_\infty$$

so that $\overline{L}_m$ is bounded. Since $\overline{L}_m$ is a linear map on a normed space, it follows that it is continuous.

Now if $f \in C_0(\Omega)$, $g \in C_b(\Omega)$, and $x \in \Theta_m(\Omega)$, it follows that

$$\begin{aligned}L_m(\alpha(f)g)(x) &= \sum_{\Theta_m(y)=x} (\alpha_m(f)g)(y) = \sum_{\Theta_m(y)=x} f(\Theta_m(y))g(y)\\ &= f(x) \sum_{\Theta_m(y)=x} g(y) = (f\overline{L}_m(g))(x).\end{aligned}$$

When $x \notin \Theta_m(\Omega)$, both $L_m(\alpha(f)g)(x)$ and $(f\overline{L}_m(g))(x)$ are zero. It follows that $L_m(\alpha_m(f)g) = f\overline{L}_m(g)$. ∎

The above result shows that L is an action by transfer operators. It follows from Lemma 3.10 and Lemma 3.11 that $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ is an Exel–Larsen system.

EXAMPLE 3.12. (i) Let $A = \{0, 1, 2, 3\}$ and define $d : A^2 \to A$ via $(a, b) \mapsto a + b \mod 4$. It is straightforward to see that d is both progressive and regressive and hence the associated sliding block code τ_d is a local homeomorphism that has uniformly bounded cardinalities on inverse images. Then $(A^{\mathbb{N}}, \{\sigma, \tau_d\})$ from Example 3.3(iii) is a topological dynamical system that satisfies condition (UBC). Define $\alpha : \mathbb{N}^2 \to C(A^{\mathbb{N}})$ by $\alpha_{(m,n)}(f) = f \circ \Theta_{(m,n)}$. For $(m,n) \in \mathbb{N}^2$, since $\Theta_{(m,n)}$ is surjective, we may define $L_{(m,n)} : C(A^{\mathbb{N}}) \to C(A^{\mathbb{N}})$ by

$$L_{(m,n)}(f)(x) = \sum_{\Theta_{(m,n)}(y)=x} f(y).$$

The quadruple $(A^{\mathbb{N}}, \mathbb{N}^2, \alpha, L)$ is an Exel–Larsen system.

(ii) Let $\Lambda = \Omega_k$ and let $(\partial\Lambda, \{T_i\}_{i=1}^k)$ be the topological dynamical system in Example 3.3(iv). We define the Exel–Larsen system $(C_0(\Lambda^\infty), \mathbb{N}^k, \alpha, L)$ by setting

$$\alpha_m(f)(x) = f \circ \sigma_m(x),$$
$$L_m(f)(x) = \begin{cases} \sum\limits_{\sigma_m(y)=x} f(y) & \text{if } x \in \sigma_m(\Lambda^\infty), \\ 0 & \text{else .} \end{cases}$$

Again regarding Λ^∞ as $\mathbb{N}^k$, it is straightforward to show that

$$\alpha_m(f)(n) = f(n+m),$$
$$L_m(f)(n) = \begin{cases} f(n-m) & \text{if } n-m \in \mathbb{N}^k, \\ 0 & \text{else,} \end{cases}$$

so that we obtain the Exel–Larsen system $(C_0(\mathbb{N}^k), \mathbb{N}^k, \alpha, L)$.

4. THE ASSOCIATED PRODUCT SYSTEMS

Associated to the topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ and the Exel–Larsen system $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ are two product systems over $\mathbb{N}^k$ of $C_0(\Omega)$-correspondences, denoted X^Λ and X^{Lar} respectively. We show in Theorem 4.4 that the two product systems are, in fact, isomorphic.

DEFINITION 4.1. The *topological k-graph correspondence* X^Λ associated to a topological k-graph Λ is the product system over $\mathbb{N}^k$ of $C_0(\Omega)$-correspondences such that:

(i) For each $m \in \mathbb{N}^k$, X_m^Λ is the topological graph correspondence associated to the topological graph

$$E_m = (\Lambda^0, \Lambda^m, r|_{\Lambda^m}, s|_{\Lambda^m}) = (\Omega, \{m\} \times \Omega, r_m, s_m).$$

In particular, X_m^Λ is a completion of $C_c(\{m\} \times \Omega)$ (see [11] for details), and the $C_0(\Omega)$-bimodule operations and $C_0(\Omega)$-valued inner product are given by

$$(f \cdot \xi \cdot g)(m,x) = f(r(m,x))\xi(m,x)g(s(m,x)) = f(x)\xi(m,x)g(\Theta_m(x)), \text{ and}$$
$$\langle \xi, \eta \rangle_m(x) = \sum_{(m,y)\in s_m^{-1}(x)} \overline{\xi(m,y)}\eta(m,y).$$

(ii) For $m,n \in \mathbb{N}^k$, $\beta^\Lambda_{m,n} : X_m^\Lambda \otimes_{C_0(\Omega)} X_n^\Lambda \to X^\Lambda_{m+n}$ is defined by

$$\beta^\Lambda_{m,n}(\xi \otimes \eta)(m+n,x) = \xi(m,x)\eta(n,\Theta_m(x)).$$

REMARK 4.2. It is important to note that a topological k-graph correspondence is not a C^*-correspondence, but is instead a product system over $\mathbb{N}^k$ of C^*-correspondences. We use the terminology *topological k-graph correspondence* to agree with the existing notions of *graph correspondence* (see [15] for example) and *topological graph correspondence* (see [11]). The Cuntz–Pimsner algebras of a graph correspondence and a topological graph correspondence are isomorphic to the graph C^*-algebra and topological graph C^*-algebra, respectively. Similarly, a generalization of the Cuntz–Pimsner algebra of the topological k-graph correspondence is isomorphic to the topological k-graph C^*-algebra (see Theorem 5.20 of [3]).

DEFINITION 4.3. Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system that satisfies condition (UBC), and let $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ be the Exel–Larsen system constructed from $(\Omega, \{T_i\}_{i=1}^k)$ as in Section 3.2. The *product system* X^{Lar} *associated to* $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ is the product system over $\mathbb{N}^k$ of $C_0(\Omega)$-correspondences such that :

(i) For each $m \in \mathbb{N}^k$, $X_m^{\mathrm{Lar}} = \{m\} \times C_0(\Omega)$ with $C_0(\Omega)$-bimodule operations

$$f \cdot (m,g) \cdot h = (m, fg\alpha_m(h)),$$

where $(fg\alpha_m(h))$ is defined by $x \mapsto f(x)g(x)h(\Theta_m(x))$. The $C_0(\Omega)$-valued inner product on X_m^{Lar} is given by

$$\langle (m,f),(n,g)\rangle_m(x) = L_m(f^*g)(x) = \begin{cases} \sum\limits_{\Theta_m(y)=x} \overline{f(y)}g(y) & \text{if } x \in \Theta_m(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

(ii) For $m,n \in \mathbb{N}^k$, $\beta^{\mathrm{Lar}}_{m,n} : X_m^{\mathrm{Lar}} \otimes_{C_0(\Omega)} X_n^{\mathrm{Lar}} \to X^{\mathrm{Lar}}_{m+n}$ is defined by

$$\beta^{\mathrm{Lar}}_{m,n}((m,f)\otimes(n,g)) = (m+n, f\alpha_m(g)).$$

4.1. THE ISOMORPHISM OF PRODUCT SYSTEMS.

THEOREM 4.4. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system satisfying condition* (UBC)*, let $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ be the associated topological k-graph, and let $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ be the associated Exel–Larsen system. Then the topological*

k-graph correspondence X^Λ is isomorphic to the product system X^{Lar} associated to the Exel–Larsen system.

To show that the product systems are isomorphic we must show there is a map $\psi : X^{\mathrm{Lar}} \to X^\Lambda$ satisfying:

(i) for each $m \in \mathbb{N}^k$, the map $\psi_m = \psi|_{X^{\mathrm{Lar}}_m} : X^{\mathrm{Lar}}_m \to X^\Lambda_m$ is a $C_0(\Omega)$-correspondence isomorphism that preserves inner product, and

(ii) ψ respects the multiplication in the semigroups X^{Lar} and X^Λ.

The following lemmas are helpful in proving our result.

LEMMA 4.5. *Let $E = (E^0, E^1, r, s)$ be a topological graph such that the source map $s : E^1 \to E^0$ has uniformly bounded cardinalities on inverse images. Then the associated graph correspondence $X_E = C_0(E^1)$ as an algebraic $C_0(E^0)$-bimodule.*

Proof. By Lemma 1.6 of [11], $C_c(E^1)$ is dense in X_E, hence $X_E \subseteq C_0(E^1)$. For the reverse containment, let $\xi \in C_0(E^1)$. Since $s : E^1 \to E^0$ has uniformly bounded cardinalities on inverse images, there is $M \in \mathbb{N}$ such that $|\{e \in E^1 : s(e) = v\}| \leqslant M$ for every $v \in E^0$.

To see that the map $v \mapsto \sum_{e \in s^{-1}(v)} |\xi(e)|^2$ is in $C_0(E^0)$, note that for $v \in E^0$ there is a neighborhood V of v and finitely many open sets $U_{e_1}, \dots, U_{e_n}$, $n \leqslant M$, such that s restricts to a homeomorphism from U_{e_i} onto V. It follows that $v \mapsto \sum_{e \in s^{-1}(v)} |\xi(e)|^2$ is a finite sum of continuous functions vanishing at infinity and is therefore in $C_0(E^0)$; hence $\xi \in X_E$. ∎

LEMMA 4.6. *Fix $m \in \mathbb{N}^k$. For each $f \in C_0(\Omega)$, the function $\widetilde{f} : \{m\} \times \Omega \to \mathbb{C}$ defined by*

$$\widetilde{f}(m, x) = f(x)$$

is an element of X^Λ_m.

Proof. Since $X^\Lambda_m = X_{E^\Lambda_m}$, by Lemma 4.5 it is sufficient to show that $\widetilde{f} \in C_0(E^1_m)$ where $E^1_m = \{m\} \times \Omega$. Note that $\widetilde{f}$ is the composition of f with the homeomorphism $(m, x) \mapsto x$ of $\{m\} \times \Omega$ onto Ω. Then $\widetilde{f}$ is continuous since f is. If $\varepsilon > 0$ and K is a compact set such that $|f(x)| \leqslant \varepsilon$ for $x \in \Omega \setminus K$, then $|\widetilde{f}(m, x)| < \varepsilon$ for $(m, x) \in E^1_m \setminus (\{m\} \times K)$. Hence $\widetilde{f} \in X^\Lambda_m$ as desired. ∎

By a similar argument, we see that for each $\xi \in X^\Lambda_m$, the function $\widehat{\xi} = (m, \eta)$ where

$$\eta(x) = \xi(m, x)$$

is an element of X^{Lar}_m.

We define $\psi : X^{\mathrm{Lar}} \to X^\Lambda$ by letting $\psi_m : X^{\mathrm{Lar}}_m \to X^\Lambda_m$ be given by

$$\psi_m(m, f) = \widetilde{f}$$

for each $m \in \mathbb{N}^k$.

Proof of Theorem 4.4. Fix $m \in \mathbb{N}^k$. Straightforward arguments show that $\psi_m : X_m^{\mathrm{Lar}} \to X_m^{\Lambda}$ is an injective $C_0(\Omega)$-correspondence morphism preserving the inner product. For surjectivity, let $\xi \in X_m^{\Lambda}$. Then $(m, \widehat{\xi}) \in \{m\} \times C_0(\Omega)$ satisfies $\psi_m(m, \widehat{\xi}) = \xi$ since

$$\psi_m(m, \widehat{\xi})(m, x) = \widetilde{\widehat{\xi}}(m, x) = \widehat{\xi}(x) = \xi(m, x).$$

To see that ψ respects the semigroup multiplication, let $(m, f) \in X_m^{\mathrm{Lar}}, (n, g) \in X_n^{\mathrm{Lar}}, (m+n, x) \in \{m+n\} \times \Omega$. Then

$$\begin{aligned}(\psi_m(m,f)\psi_n(n,g))(m+n,x) &= \psi_m(m,f)(m,x)\psi_n(n,g)(n,\Theta_m(x)) \\ &= \widetilde{f}(m,x)\widetilde{g}(n,\Theta_m(x)) = f(x)g(\Theta_m(x)) = f\alpha_m(g)(x) \\ &= \psi_{m+n}(m+n, f\alpha_m(g))(m+n,x).\end{aligned}$$

Hence $\psi_m(m,f)\psi_n(n,g) = \psi_{n+m}(n+m, f\alpha_m(g))$ as desired. ∎

REMARK 4.7. We showed in Proposition 3.9 that the topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$ is compactly aligned. By Proposition 5.15 of [3], this happens if and only if X^{Λ} is compactly aligned in the sense of Definition 2.5. In [19], Willis shows that if $k = 2$, Ω is compact, and T_1 and T_2 $*$-commute (meaning that whenever $T_1(x) = T_2(y)$, there is a unique $z \in \Omega$ with $T_1(z) = y$ and $T_2(z) = x$), then the product system X^{Lar} constructed from the Exel–Larsen system $(C(\Omega), \mathbb{N}^2, \alpha, L)$ is compactly aligned. Theorem 4.4 together with Proposition 3.9 then imply that the $*$-commuting restriction may be lifted and Willis' result holds for arbitrary $k \in \mathbb{N}$ and locally compact Ω.

5. THE LARSEN CROSSED PRODUCT AND $C^*(\Lambda)$

In this section, we show that $C^*(\Lambda)$, the C^*-algebra associated to the topological k-graph $\Lambda = (\Lambda(\Omega, \{T_i\}_{i=1}^k), d)$, is isomorphic to $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$, the C^*-algebra associated to the Exel–Larsen system $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$. To accomplish this, we show that $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ is isomorphic to $\cong \mathcal{NO}_{X^{\Lambda}}$, which was shown to be isomorphic to $C^*(\Lambda)$ in [3].

REMARK 5.1. In Example 7.1(iii) of [21], Yeend describes the associated topological k-graph C^*-algebra in the case where the maps $\{T_i\}_{i=1}^k$ are homeomorphisms. Surjectivity of the maps ensures that the associated topological k-graph Λ has no sources. As a result, the boundary path groupoid is amenable. Since the maps are homeomorphisms, there is an induced action α of $\mathbb{Z}^k$ on $C_0(\Omega)$ defined by

$$\alpha_m(f)(x) = f(\Theta_m(x)),$$

with universal crossed product $(C_0(\Omega) \rtimes_\alpha \mathbb{Z}^k, j_{C_0(\Omega)}, j_{\mathbb{Z}^k})$. Yeend asserts that the topological k-graph C^*-algebra is isomorphic to this crossed product. The main result in this section, Theorem 5.7, generalizes this to the setting where the maps are local homeomorphisms that are not necessarily surjective.

PROPOSITION 5.2. *Let* $(\Omega, \{T_i\}_{i=1}^k)$ *be a topological dynamical system that satisfies condition* (UBC) *and let* $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ *be the Exel–Larsen system described in Subsection* 3.2. *Let* X^{Lar} *be the product system over* $\mathbb{N}^k$ *of* $C_0(\Omega)$*-correspondences from Definition* 4.3. *Let* $K = \{K_m\}_{m\in\mathbb{N}^k}$ *be the family of ideals defined by* (2.4). *Let* $\psi : X^{\mathrm{Lar}} \to B$ *be a (Toeplitz) representation of* X^{Lar} *in a* C^**-algebra* B. *Then* ψ *is Cuntz–Pimsner covariant in the sense of* (CP-K) *if and only if it is coisometric on* K.

Proof. In the proof of Lemma 3.10, $\alpha_m(C_0(\Omega))C_0(\Omega) = C_0(\Omega)$ hence

$$\begin{aligned}K_m &= \overline{C_0(\Omega)\alpha_m(C_0(\Omega))C_0(\Omega)} \cap \phi_m^{-1}(\mathcal{K}(X_m^{\mathrm{Lar}}))\\ &= C_0(\Omega) \cap \phi_m^{-1}(\mathcal{K}(X_m^{\mathrm{Lar}})) = \phi_m^{-1}(\mathcal{K}(X_m^{\mathrm{Lar}})).\end{aligned}$$

Recall from the construction of X^{Lar} in Definition 4.3 that the left action on each X_m^{Lar} is given by multiplication. Thus ϕ_m is injective so that $(\ker\phi_m)^\perp = C_0(\Omega)$ and hence coisometric on K is equivalent to (CP-K). ∎

COROLLARY 5.3. *Let* $(\Omega, \{T_i\}_{i=1}^k)$ *be a topological dynamical system satisfying condition* (UBC) *and let* $(C_0(\Omega), \mathbb{N}^k, \alpha, L)$ *be the Exel–Larsen system described in Subsection* 3.2. *Let* X^{Lar} *be the product system over* $\mathbb{N}^k$ *of* $C_0(\Omega)$*-correspondences from Definition* 4.3. *Then*

$$C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k \cong \mathcal{O}_{X^{\mathrm{Lar}}}.$$

Proof. Since a representation $\psi : X \to B$ is coisometric on K, where $K = \{K_m\}_{m\in\mathbb{N}^k}$ is the family of ideals defined by (2.4), if and only if it is Cuntz–Pimsner covariant in the sense of (CP-K), it follows that $j^{\mathrm{Fow}} : X^{\mathrm{Lar}} \to \mathcal{O}_{X^{\mathrm{Lar}}}$ is coisometric on K and j^{Lar} satisfies (CP-K). It follows from the universal properties of $\mathcal{O}_{X^{\mathrm{Lar}}}$ and the Larsen crossed product $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ that there are unique surjective homomorphisms

$$\Pi_{j^{\mathrm{Lar}}} : \mathcal{O}_{X^{\mathrm{Lar}}} \to C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k, \quad \Pi_{j^{\mathrm{Fow}}} : C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k \to \mathcal{O}_{X^{\mathrm{Lar}}},$$

such that $j^{\mathrm{Lar}} = \Pi_{j^{\mathrm{Lar}}} \circ j^{\mathrm{Fow}}$ and $j^{\mathrm{Fow}} = \Pi_{j^{\mathrm{Fow}}} \circ j^{\mathrm{Lar}}$. Since $\mathcal{O}_{X^{\mathrm{Lar}}}$ and $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ are generated by j^{Fow} and j^{Lar} respectively, it follows that $\Pi_{j^{\mathrm{Lar}}}$ and $\Pi_{j^{\mathrm{Fow}}}$ take generators to generators and hence

$$C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k \cong \mathcal{O}_{X^{\mathrm{Lar}}}. \quad ∎$$

We now show that for any representation $\psi : X^\Lambda \to B$, Cuntz–Pimsner covariance in the sense of (CP-K) is equivalent to CNP-covariance. In order for CNP-covariance to make sense for a representation of X^Λ, we must have that X^Λ is compactly aligned. Recall that X^Λ is the topological k-graph correspondence associated to the topological k-graph $\Lambda = (\Lambda(\Omega,\Theta), d)$ which we showed in Proposition 3.9 is compactly aligned. By Proposition 5.15 of [3], since Λ is compactly aligned, so is X^Λ.

PROPOSITION 5.4. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system and let Λ be the associated topological k-graph. Let X^Λ be the topological k-graph correspondence. Let $\psi : X^\Lambda \to B$ be a (Toeplitz) representation of X^Λ in a C^*-algebra B. Then ψ is Cuntz–Pimsner covariant in the sense of* (CP-K) *if and only if it is* CNP*-covariant.*

We would like to apply Corollary 5.2 of [17] to obtain the desired result. In order to do so, we need to establish that the left action on each fibre is by compact operators.

LEMMA 5.5. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system and Λ be the associated topological k-graph. Then the left action of $C_0(\Omega)$ on each fibre X_m^Λ of the topological k-graph correspondence is by compact operators.*

Proof. It follows from Proposition 1.24 of [11], that $\phi_m^{-1}(\mathcal{K}(X_m)) = C_0(\Omega_{\text{fin}})$ where

$$\Omega_{\text{fin}} = \{v \in \Omega : v \text{ has a neighborhood } V \text{ such that } r_m^{-1}(V) \text{ is compact}\}.$$

Since Ω is a locally compact space, every point $v \in \Omega$ has a compact neighborhood V. The range map is given by projection onto Ω so that $r_m^{-1}(V) = \{m\} \times V$, which is compact. It follows that $\Omega_{\text{fin}} = \Omega$. ∎

Proof of Proposition 5.4. Recall that $(\mathbb{Z}^k, \mathbb{N}^k)$ is a quasi-lattice ordered group such that each pair $s, t \in \mathbb{N}^k$ has a least upper bound and that X^Λ is compactly aligned. It follows from the construction of X^Λ in Definition 4.1 that the left action on each fibre is given by multiplication and is therefore injective. By Lemma 5.5, the left action on each fibre is by compact operators. Then by Corollary 5.2 of [17], ψ is CNP-covariant if and only if

$$\psi^{(m)} \circ \phi_m = \psi_0 \quad \text{for all } m \in \mathbb{N}^k.$$

It follows that ψ is CNP-covariant if and only if ψ satisfies (CP-K). ∎

COROLLARY 5.6. *Let $(\Omega, \{T_i\}_{i=1}^k)$ be a topological dynamical system and let Λ be the topological k-graph described in Subsection* 3.1. *Let X^Λ be the associated topological k-graph correspondence as in Definition* 4.1. *Then*

$$\mathcal{N}\mathcal{O}_{X^\Lambda} \cong \mathcal{O}_{X^\Lambda}.$$

Proof. Since a representation $\psi : X^\Lambda \to B$ is Cuntz–Pimsner covariant in the sense of (CP-K) if and only if it is CNP-covariant, it follows that $j^{\text{Fow}} : X^\Lambda \to \mathcal{O}_{X^\Lambda}$ is CNP-covariant and j^{CNP} satisfies (CP-K). It follows from the universal properties of $\mathcal{O}_{X^\Lambda}$ and $\mathcal{N}\mathcal{O}_{X^\Lambda}$ that there are unique surjective homomorphisms

$$\Pi_{j^{\text{CNP}}} : \mathcal{O}_{X^\Lambda} \to \mathcal{N}\mathcal{O}_{X^\Lambda}, \quad \Pi_{j^{\text{Fow}}} : \mathcal{N}\mathcal{O}_{X^\Lambda} \to \mathcal{O}_{X^\Lambda},$$

such that $j^{\text{CNP}} = \Pi_{j^{\text{CNP}}} \circ j^{\text{Fow}}$ and $j^{\text{Fow}} = \Pi_{j^{\text{Fow}}} \circ j^{\text{CNP}}$. Since $\mathcal{O}_{X^\Lambda}$ and $\mathcal{N}\mathcal{O}_{X^\Lambda}$ are generated by j^{Fow} and j^{CNP} respectively, it follows that $\Pi_{j^{\text{CNP}}}$ and $\Pi_{j^{\text{Fow}}}$ take generators to generators and hence

$$\mathcal{N}\mathcal{O}_{X^\Lambda} \cong \mathcal{O}_{X^\Lambda}. \quad \blacksquare$$

THEOREM 5.7. *Let* $(\Omega, \{T_i\}_{i=1}^k)$ *be a topological dynamical system satisfying condition* (UBC). *Let* $C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k$ *be the Larsen crossed product and* $C^*(\Lambda)$ *be the topological k-graph* C^**-algebra associated to* $(\Omega, \{T_i\}_{i=1}^k)$. *Then*

$$C^*(\Lambda) \cong C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k.$$

Proof. By Theorem 5.20 of [3], we have that $C^*(\Lambda) \cong \mathcal{N}\mathcal{O}_{X^\Lambda}$. By Theorem 4.4, $X^{\mathrm{Lar}} \cong X^\Lambda$ and hence the Cuntz–Pimsner algebras $\mathcal{O}_{X^{\mathrm{Lar}}}$ and $\mathcal{O}_{X^\Lambda}$ are isomorphic. Thus, by Corollary 5.3

$$C^*(\Lambda) \cong \mathcal{N}\mathcal{O}_{X^\Lambda} \cong \mathcal{O}_{X^\Lambda} \cong \mathcal{O}_{X^{\mathrm{Lar}}} \cong C_0(\Omega) \rtimes_{\alpha,L} \mathbb{N}^k. \quad \blacksquare$$

EXAMPLE 5.8. It is known that $C^*(\Omega_k) \cong \mathcal{K}(\ell^2(\mathbb{N}^k))$. The result above, together with our description of the Exel–Larsen system $(C_0(\mathbb{N}^k), \mathbb{N}^k, \alpha, L)$ in Example 3.12(ii) gives that $\mathcal{K}(\ell^2(\mathbb{N}^k)) \cong C_0(\mathbb{N}^k) \rtimes_{\alpha,L} \mathbb{N}^k$.

Acknowledgements. The third author was supported by the NSF Mathematical Sciences Postdoctoral Fellowship DMS-1004675, the University of Iowa Graduate College Fellowship as part of the Sloan Foundation Graduate Scholarship Program, and the University of Iowa Department of Mathematics NSF VIGRE grant DMS-0602242.

REFERENCES

[1] N. BROWNLOWE, Realizing the C^*-algebra of a higher-rank graph as an Exel crossed product, *J. Operator Theory*, to appear.

[2] N. BROWNLOWE, I. RAEBURN, S.T. VITTADELLO, Exel's crossed product for non-unital C^*-algebras, *Math. Proc. Cambridge Philos. Soc.* **149**(2010), 423–444.

[3] T. CARLSEN, N. LARSEN, A. SIMS, S.T. VITTADELLO, Co-universal algebras associated to product systems, and gauge-invariant uniqueness theorems, *Proc. London Math. Soc.*, to appear.

[4] J. CUNTZ, The internal structure of simple C^*-algebras, in *Operator Algebras and Applications, Part I (Kingston, Ont., 1980)*, Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, RI 1982, pp. 85–115.

[5] R. EXEL, A new look at the crossed-product of a C^*-algebra by an endomorphism, *Ergodic Theory Dynam. Systems* **23**(2003), 1733–1750.

[6] R. EXEL, J. RENAULT, Semigroups of local homeomorphisms and interaction groups, *Ergodic Theory Dynam. Systems* **27**(2007), 1737–1771.

[7] C. FARTHING, P.S. MUHLY, T. YEEND, Higher-rank graph C^*-algebras: an inverse semigroup and groupoid approach, *Semigroup Forum* **71**(2005), 159–187.

[8] N.J. FOWLER, Discrete product systems of Hilbert bimodules, *Pacific J. Math.* **204**(2002), 335–375.

[9] G.A. HEDLUND, Endomorphisms and automorphisms of the shift dynamical systems, *Math. Systems Theory* **3**(1969), 320–375.

[10] K.K. JENSEN, K. THOMSEN, *Elements of KK-theory*, Birkhäuser, Boston 1991.

[11] T. KATSURA, A class of C^*-algebras generalizing both graph algebras and homeomorphism C^*-algebras. I. Fundamental results, *Trans. Amer. Math. Soc.* **356**(2004), 4287–4322 (electronic).

[12] E.C. LANCE, *Hilbert C^*-Modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.

[13] N.S. LARSEN, Crossed products by abelian semigroups via transfer operators, *Ergodic Theory Dynam. Systems* **30**(2010), 1147–1164.

[14] W.L. PASCHKE, The crossed product of a C^*-algebra by an endomorphism, *Proc. Amer. Math. Soc.* **80**(1980), 113–118.

[15] I. RAEBURN, *Graph Algebras*, CBMS Reg. Conf. Ser. Math., vol. 103, Amer. Math. Soc., Providence RI 2005.

[16] I. RAEBURN, D.P. WILLIAMS, *Morita Equivalence and Continuous-trace C^*-Algebras*, Math. Surveys Monogr., vol. 60, Amer. Math. Soc., Providence, RI 1998.

[17] A. SIMS, T. YEEND, C^*-algebras associated to product systems of Hilbert bimodules, *J. Operator Theory* **64**(2010), 349–376.

[18] P.J. STACEY, Crossed products of C^*-algebras by $*$-endomorphisms, *J. Austral. Math. Soc. Ser. A* **54**(1993), 204–212.

[19] P.N. WILLIS, C^*-algebras of labeled graphs and $*$-commuting endomorphisms, Ph.D. Dissertation, The University of Iowa, Iowa 2010.

[20] T. YEEND, Topological higher-rank graphs and the C^*-algebras of topological 1-graphs, *Contemp. Math.* **414**(2006), 231–244.

[21] T. YEEND, Groupoid models for the C^*-algebras of topological higher-rank graphs, *J. Operator Theory* **57**(2007), 95–120.

CYNTHIA FARTHING, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF IOWA, IOWA CITY, 52242, U.S.A.
E-mail address: cynthia-farthing@uiowa.edu

NURA PATANI, SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, ARIZONA STATE UNIVERSITY, TEMPE, 85287, U.S.A.
E-mail address: nura.patani@asu.edu

PAULETTE N. WILLIS, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, 77204, U.S.A.
E-mail address: pnwillis@math.uh.edu

Received May 31, 2011.