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COVARIANT REPRESENTATIONS OF C^*-DYNAMICAL SYSTEMS WITH COMPACT GROUPS

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ABSTRACT. Let (A, G, σ) be a C^*-dynamical system, where G is compact. We show that every irreducible covariant representation (π, U) of (A, G, σ) is induced from an irreducible covariant representation (π_0, U_0) of a subsystem (A, G_0, σ) such that π_0 is a factor representation. We show that if (π, U) is an irreducible covariant representation of (A, G_P, σ) with $\ker \pi = P$, then π is a homogenous representation. Hence, (A, G, σ) satisfies the strong-EHI property.

KEYWORDS: *Crossed product, compact group, irreducible representation, induced representation, strong-EHI.*

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1. INTRODUCTION

Let G be a locally compact group, A a C^*-algebra, and σ a point-wise norm continuous homomorphism of G into the automorphism group of A. We call the triple (A, G, σ) a C^*-dynamical system. Given a C^*-dynamical system we can construct the crossed product C^*-algebra $A \times_\sigma G$ that encodes the action of G on A. It is well known that there exists a one to one correspondence between the nondegenerate covariant representations of the system (A, G, σ) and the nondegenerate $*$-representations of $A \times_\sigma G$. Therefore, the study of representations of $A \times_\sigma G$ is equivalent to that of covariant representations of (A, G, σ).

Our goal is to study induced covariant representations of systems involving compact groups. The study of induced representations was initiated by Mackey in [8], [9] in the context of unitary representations of locally compact groups. Using Mackey's approach, Takesaki extended the theory to covariant representations of C^*-dynamical systems in [12]. Subsequently, Rieffel recast that theory in terms of Hilbert modules and Morita equivalence with [11]. It follows from Proposition 5.4 in [13] that the construction of induced representations for crossed products by Rieffel is equivalent to that of Takesaki.

The importance of induced representations arises from the fact that the fundamental structure of a crossed product $A \times_\sigma G$ is reflected in the structure of the orbit space for the G-action on $\operatorname{Prim} A$ together with the subsystems (A, G_P, σ), where G_P is the stability group at $P \in \operatorname{Prim} A$. In particular, one gets a complete description of the primitive ideal space and its topology for transformation group C^*-algebra $C_0(X) \times_\sigma G$ when G is abelian. In many important cases we also get a characterization of when $A \times_\sigma G$ is GCR or CCR. Williams presents all these results and more in his book [13].

Although induced representations have been studied extensively there remain many natural questions in the theory. We outline below two questions for which answers are not known. Using structure theorems obtained in this paper we give a positive answer to both questions in the case of separable C^*-dynamical systems with compact groups.

One of the key ingredients in building the connection between $\operatorname{Prim} A \times_\sigma G$ and the G-action on $\operatorname{Prim} A$ is establishing that every primitive ideal of $A \times_\sigma G$ is induced from a stability group ([13], p. 235). The latter result was conjectured by Effros and Hahn, and systems for which the conjecture holds are called EH-regular. The proof that the Effros–Hahn conjecture holds for amenable groups is due to Gootman, Rosenberg and Sauvageot and it is one of the major results in the theory [7]. There exists a stronger notion of EH-regularity namely the requirement that every irreducible representation of $A \times_\sigma G$ is induced from a stability group. The latter requirement is known to hold in a number of instances ([13], Theorem 8.16), but the general case remains open.

Another natural question that arises is the connection between irreducibility of a representation of a subsystem and irreducibility of the induced representation. Following the nomenclature proposed by Echterhoff and Williams in [4], we say that (A, G, σ) satisfies strong Effros–Hahn induction property (strong-EHI), if, for each primitive ideal P of A and a covariant irreducible representation (π, U) of (A, G_P, σ) with $\ker \pi = P$ the corresponding induced representation of (A, G, σ) is irreducible. It was shown in [4] that the strong-EHI property holds in many instances including when A is a type I C^*-algebra or G is an abelian group. Nevertheless, we do not know if the strong-EHI holds in general even with an additional assumption that G is amenable.

In this paper, we use Takesaki's approach to the theory of induced representations for crossed products. As in [12] we will often assume basic countability conditions. These assumptions are necessary since the direct integral decomposition theory works best in the separable case. If G is a second countable, locally compact group acting on a separable C^*-algebra A, then we call (A, G, σ) a separable system.

In Section 2, we give a brief background about topological and Borel dynamical systems necessary for Section 3. In Section 3, we study Borel dynamical systems. In particular, we prove that if G is a compact group and (Γ, μ) is an ergodic standard Borel G-measure space, then G acts transitively on (Γ, μ). The last statement is false if G is not compact. For instance, the action of $\mathbb{Z}$ on $\mathbb{T}$ by an irrational rotation is ergodic, but it is not transitive.

In Section 4, we study the structure of covariant representations of (A, G, σ). Given a system of imprimitivity $\mathbb{A}$ for a covariant representation (π, U) there exists an essentially unique standard Borel G-measure space (Γ, μ) such that $L^\infty(\Gamma, \mu)$ is isomorphic to $\mathbb{A}$. If G acts ergodically on $\mathbb{A}$, then the corresponding action on (Γ, μ) is also ergodic. In particular, by the main result in Section 3, if G acts ergodically on $\mathbb{A}$, then G acts transitively on (Γ, μ) and we can identify the space Γ with the right coset space G_0/G for an appropriate closed subgroup G_0 of G. Then following Mackey's construction it can be shown that (π, U) is induced from a representation of (A, G_0, σ) ([12], Theorem 4.2). Our key result in this section is Theorem 4.2 regarding covariant factor representations of C^*-dynamical systems with compact groups. This theorem extends a similar result for finite groups obtained by Arias and Latremoliere ([2], Theorem 3.4). As a corollary of Theorem 4.2 we show that every irreducible representation of (A, G, σ) is induced from a stability group.

In Section 5, we study irreducible representations (π, U) of (A, G_P, σ) with $\ker \pi = P$, where $P \in \operatorname{Prim} A$. We show that in this case π must be a homogeneous representation. Using a theorem of [4], we get that (A, G, σ) satisfies the strong-EHI property.

2. PRELIMINARIES

Suppose that G is a topological (respectively Borel) group; that is, G is a topological (respectively Borel) space and a group such that the map $(s, t) \in G \times G \mapsto s^{-1}t \in G$ is continuous (respectively Borel). When G is a topological group, G is often considered as a Borel group equipped with the Borel structure determined by its topology. Let Γ be a topological (respectively Borel) space. Suppose that an anti-homomorphism of G into the group of all homeomorphisms (respectively Borel-automorphisms) of Γ is given. Denote the homeomorphism (respectively Borel-automorphism) of Γ corresponding to $s \in G$ by $\gamma \in \Gamma \mapsto \gamma \cdot s \in \Gamma$. If the map: $(\gamma, s) \in \Gamma \times G \mapsto \gamma \cdot s \in \Gamma$ is continuous (respectively Borel), then Γ is said to be a topological (respectively Borel) G-space. By a measure μ on a Borel space Γ, we shall mean a complete measure determined by a σ-finite measure on the Borel sets of Γ. For each $s \in G$ define a measure $s(\mu)$ on Γ by $s(\mu)(E) = \mu(E \cdot s)$. We say that μ is quasi-invariant if $s(\mu)$ is equivalent to μ for each $s \in G$ and we call the measure space (Γ, μ) a G-measure space.

If a quasi-invariant measure μ on a Borel G-space Γ satisfies the condition that $\mu(E) = 0$ or $\mu(\Gamma - E) = 0$ for every Borel set E of Γ with $\mu(E \triangle (E \cdot s)) = 0$ for every $s \in G$, then μ is said to be ergodic. Given a unital C^*-dynamical system (A, G, σ) we say the group action is ergodic if the only fixed elements of A under the group action are the scalars. Similarly, given a W^*-dynamical system $(\mathbb{A}, G, \tau)$

(defined at the beginning of Section 3) we say that the action of G on $\mathbb{A}$ is ergodic if the only fixed elements are the scalars. Suppose that (Γ, μ) is a standard Borel G-measure space such that the corresponding action of G on $L^\infty(\Gamma, \mu)$ given by $(\tau_s f)(\gamma) = f(\gamma \cdot s)$ is continuous in the strong operator topology (SOT), then we can form a W^*-dynamical system $(L^\infty(\Gamma, \mu), G, \tau)$. In this case, (Γ, μ) is an ergodic G-measure space if and only if the action of G on $L^\infty(\Gamma, \mu)$ is ergodic.

REMARK 2.1. We can express the continuity of group action on $L^\infty(\Gamma, \mu)$ on the space level by requiring that $\mu(E \triangle (E \cdot s)) \to 0$ as $s \to e$ for each measurable set E with $\mu(E) < \infty$ ([3], p. 285).

Let (Γ, μ) be a Borel G-measure space. For each $\gamma \in \Gamma$, define

$$O_\gamma = \{\gamma \cdot s : s \in G\}$$

to be the orbit of γ under the group action. If there is $\gamma \in \Gamma$ such that $\mu(\Gamma - O_\gamma) = 0$, then (Γ, μ) is said to be transitive. Clearly, transitivity implies ergodicity. As mentioned in the introduction, the converse is not true in general.

3. ERGODIC ACTIONS OF COMPACT GROUPS

In Lemma 3.4, we will connect ergodicity and transitivity for a compact group acting on an abelian C^*-algebra. Our first goal is to identify a suitable C^*-algebra in the von Neumann algebra context.

Let G be a locally compact group, $\mathbb{A}$ a von Neumann algebra, and τ a pointwise SOT-continuous homomorphism of G into the automorphism group of $\mathbb{A}$. We call the triple $(\mathbb{A}, G, \tau)$ a W^*-dynamical system. Given a W^*-dynamical system $(\mathbb{A}, G, \tau)$ the set $\mathbb{A}^c$ of $x \in \mathbb{A}$ such that the function $s \mapsto \tau_s(x)$ is norm continuous is a G-invariant C^*-algebra and it is σ-weakly dense in $\mathbb{A}$ ([3], Proposition III.3.2.4). Since $\mathbb{A}^c$ is unital it follows from the double commutant theorem that $\mathbb{A}^c$ is SOT-dense in $\mathbb{A}$. We will use $\mathbb{A}^c$ to study W^*-dynamical systems of the form $(L^\infty(\Gamma, \mu), G, \tau)$. First, we need the following version of the spectral theorem.

LEMMA 3.1. *Let N be a masa on a separable Hilbert space $\mathcal{K}$ and $\zeta \in \mathcal{K}$ be a cyclic, separating vector for N. Suppose M is a unital C^*-subalgebra of N such that $\overline{M\zeta} = \mathcal{K}$. Then there exists a compact Hausdorff space Y, finite Radon measure ν and unitary $V : \mathcal{K} \to L^2(Y, \nu)$ such that $VNV^* = L^\infty(Y, \nu)$ and $VMV^* = C(Y)$.*

Proof. Let $\rho : M \to C(Y)$ be the Gelfand isomorphism. Define a positive linear functional ϕ on M by $\phi(x) = \langle x\zeta, \zeta \rangle$. Then there is a finite Radon measure ν on Y such that

$$\phi(x) = \int_Y \rho(x) \mathrm{d}\nu$$

for all $x \in M$.

Let $\pi_\phi : M \to \mathcal{B}(L^2(Y,\nu))$ be the corresponding GNS representation with 1_Y as the cyclic vector. Since ζ is a separating vector then the map $V : M\zeta \to \pi_\phi(M)1_Y$ given by $V(x\zeta) = \pi_\phi(x)1_Y$ is well defined. Clearly, V is an isometry. Hence, we can extend V to a unitary from $\mathcal{K}$ onto $L^2(Y,\nu)$. Moreover, $\pi_\phi(x) = VxV^*$ for all $x \in M$ so that $VMV^* = \pi_\phi(M) = C(Y)$. To see that $VNV^* = L^\infty(Y,\nu)$ let $x_1 \in M$ and $x_2 \in N$, then

$$(Vx_1V^*)(Vx_2V^*) = (Vx_2V^*)(Vx_1V^*).$$

So $(Vx_2V^*) \subseteq (VMV^*)' = C(Y)' = L^\infty(Y,\nu)$. Conversely, if $T \in L^\infty(Y,\nu) \subseteq (VNV^*)'$, then $T(VxV^*) = (VxV^*)T$, for all $x \in N$. So $(V^*TV)x = x(V^*TV)$, for all $x \in N$. Thus $V^*TV \in N' = N$ and $T = V(V^*TV)V^* \in VNV^*$. ∎

Suppose that (Γ,μ) is a standard Borel G-measure space with the group action continuous in the sense of Remark 2.1. Consider the corresponding W^*-dynamical system $(L^\infty(\Gamma,\mu), G, \tau)$. Then $L^\infty(\Gamma,\mu)^{\mathrm{c}}$ is SOT-dense in $L^\infty(\Gamma,\mu)$. Let $\zeta \in L^2(\Gamma,\mu)$ be a cyclic, separating vector for $L^\infty(\Gamma,\mu)$. Then we can apply Lemma 3.1 to $N = L^\infty(\Gamma,\mu)$, $M = L^\infty(\Gamma,\mu)^{\mathrm{c}}$, and ζ.

COROLLARY 3.2. *Let G be a locally compact group and let (Γ,μ) be a standard Borel G-measure space. Then there is a compact Hausdorff space Y together with a finite Radon measure ν and a unitary $V : L^2(\Gamma,\mu) \to L^2(Y,\nu)$ such that $VL^\infty(\Gamma,\mu)V^* = L^\infty(Y,\nu)$ and $VL^\infty(\Gamma,\mu)^{\mathrm{c}}V^* = C(Y)$.*

Consider the W^*-dynamical system $(L^\infty(Y,\nu), G, \tau')$, where $\tau'_s(VfV^*) = V(\tau_s f)V^*$ for all $s \in G$ and $f \in L^\infty(\Gamma,\mu)$. Then by construction we get that $L^\infty(Y,\nu)^{\mathrm{c}} = C(Y)$. In particular, $(C(Y), G, \tau')$ is a C^*-dynamical system. Hence, there is an action of G on Y so that Y is a topological G-space and

$$(\tau'_s f)(y) = f(y \cdot s)$$

for all $y \in Y, s \in G$ and $f \in C(Y)$ ([13], Proposition 2.7). We would like to show that the above equality extends to projections in $L^\infty(Y,\nu)$.

LEMMA 3.3. *In the above situation, let E be a Borel subset of Y and $s \in G$. Then $(\tau'_s\chi_E)(y) = \chi_E(y \cdot s)$ for almost all $y \in Y$. In particular, ν is a quasi-invariant measure.*

Proof. By the Urysohn lemma there is a sequence $\{f_i\}$ in $C(Y)$ with $0 \leqslant f_i \leqslant 1$ such that $f_i(y) \to \chi_E(y)$ for almost every y. We can assume, without the loss generality, that $\chi_E(y) = 1$ (respectively 0) whenever $f_i(y) \to 1$ (respectively 0). It follows from the dominated convergence theorem that $f_i \to \chi_E$ in the strong operator topology as multiplication operators. Since an automorphism of a von Neumann algebra is SOT-continuous on bounded sets ([3], Proposition III.2.2.2), then $\tau'_s f_i \to \tau'_s\chi_E$ in the strong operator topology. In particular, $\tau'_s f_i \to \tau'_s\chi_E$ in $L^1(Y,\mu)$. Therefore, there exists a subsequence such that $\tau'_s f_{i_j}(y) \to \tau'_s\chi_E(y)$ for almost every y. By replacing the original sequence with the subsequence we can assume without loss of generality that $\tau'_s f_i \to \tau'_s\chi_E$ almost everywhere. Since $f_i \in C(Y)$ then $(\tau'_s f_i)(y) = f_i(y \cdot s)$ for all $y \in Y$ and i. It follows $(\tau'_s\chi_E)(y) = \chi_E(y \cdot s)$

for almost all $y \in Y$. In particular, $\nu(E) = 0 \iff \chi_E = 0 \iff \tau'_s(\chi_E) = 0 \iff \chi_{(E\cdot s^{-1})} = 0 \iff \nu(E \cdot s^{-1}) = 0$. ∎

Let $L^\infty(\Gamma, \mu)$ and $C(Y)$ be as in Corollary 3.2. Suppose the action of G on $L^\infty(\Gamma, \mu)$ is ergodic, then the action of G on $C(Y)$ must also be ergodic. In general, ergodic actions are far from being transitive. However, we will show that if G is a compact group, then the two notions coincide. The first part of Lemma 3.4 is similar, with a different proof, to a result by Albeverio and Høegh-Krohn ([1], Lemma 2.1).

LEMMA 3.4. *Let G be a compact group. Let X be a compact, Hausdorff topological G-space. Suppose the action of G on $C(X)$ given by $(\sigma_s f)(x) = f(x \cdot s)$ is ergodic. Then the action of G on X is transitive.*

Moreover, there exists a closed subgroup G_0 of G such that the right coset space G_0/G with the quotient topology is homeomorphic to X.

Proof. Recall that O_x denotes the orbit of x, for each $x \in X$. Since the map $s \mapsto x \cdot s$ is continuous from G to X and G is compact, then O_x is compact for each $x \in X$. In particular, O_x is closed for each $x \in X$.

Fix $x_0 \in X$. Suppose there is $x_1 \in X - O_{x_0}$, then O_{x_0} and O_{x_1} are disjoint closed subsets of X. By the Urysohn lemma there exists a continuous function $f : X \to [0,1]$ such that $f(x_0 \cdot s) = 0$ and $f(x_1 \cdot s) = 1$ for all $s \in G$. Define a function $g : X \to [0,1]$ by $g(x) = \int_G f(x \cdot s) \mathrm{d}\mu(s)$ integrating with respect to the Haar measure. We want to show that g is continuous. To this end, let $\varepsilon > 0$ be given; extend f to $\overline{f} : X \times G \to [0,1]$ by defining $\overline{f}(x,s) = f(x \cdot s)$. Then $\overline{f}$ is continuous function with compact support so we can find a finite open cover $\{F_i \times G_i\}_{i=1}^n$ of $X \times G$ such that $|f(x \cdot s) - f(y \cdot t)| < \varepsilon$ whenever (x,s) and (y,t) are both in $F_i \times G_i$ for some $i = 1, \ldots, n$. Given any $x \in X$ define $F_x = \bigcap\{F_i : x \in F_i\}$. It is not hard to check that $|f(x \cdot s) - f(y \cdot s)| < \varepsilon$ for all $y \in F_x$ and $s \in G$. Then $|g(x) - g(y)| \leqslant \int_G |f(x \cdot s) - f(y \cdot s)| \mathrm{d}\mu(s) \leqslant \varepsilon$ for all $y \in F_x$. It follows that g is continuous.

Moreover, g is G-invariant and hence must be constant on X. But $g(x_0) = 0$ and $g(x_1) = 1$, contradiction. It follows that $O_{x_0} = X$.

To prove the second part, let $G_{x_0} = \{s \in G : x_0 \cdot s = x_0\}$. Then G_{x_0} is a closed subgroup of G and the right coset space G_{x_0}/G is compact in the quotient topology. Moreover, it is easy to see that the map $G_{x_0} \cdot s \mapsto x_0 \cdot s$ is a continuous bijection from G_{x_0}/G onto X. Since G_{x_0}/G is compact and X is Hausdorff it follows that G_{x_0}/G is in fact homeomorphic to X. ∎

COROLLARY 3.5. *Let G be a second countable compact group. Let X be a compact, Hausdorff topological G-space. Suppose the action of G on $C(X)$ given by $(\sigma_s f)(x) = f(x \cdot s)$ is ergodic. Then X is a second countable topological space.*

Applying Lemma 3.4 to $(C(Y), G, \tau')$ we see that G acts transitively on Y. We are now ready to prove the main result of this section.

THEOREM 3.6. *Let G be a second countable, compact group. Let (Γ, μ) be a standard Borel G-measure space. Suppose the action of G on (Γ, μ) is ergodic and the corresponding action of G on $L^\infty(\Gamma, \mu)$ is* SOT*-continuous. Then G acts transitively on (Γ, μ).*

Proof. We know by Corollary 3.2 that there is a compact, Hausdorff space Y together with a Radon measure ν and a unitary $V : L^2(\Gamma, \mu) \to L^2(Y, \nu)$ such that $VL^\infty(\Gamma, \mu)V^* = L^\infty(Y, \nu)$ and $VL^\infty(\Gamma, \mu)^c V^* = C(Y)$. We define the action of G on $L^\infty(Y, \nu)$ as in Lemma 3.3, then (Y, ν) becomes a Borel G-measure space. Since G is a second countable, compact group then Y is a second countable topological space by Corollary 3.5. In particular, (Y, ν) is a standard Borel G-measure space.

It follows from Mackey's Theorem 2 in [10] that there are invariant Borel subsets $Y' \subseteq Y$ and $\Gamma' \subseteq \Gamma$ and a Borel isomorphism $\theta : Y' \to \Gamma'$ such that:

(i) $\mu(\Gamma - \Gamma') = \nu(Y - Y') = 0$.
(ii) $\theta(y \cdot s) = \theta(y) \cdot s$ for all $y \in Y', s \in G$.

By Lemma 3.4, we know that G acts transitively on Y. In particular, $Y' = Y$. Let y be any point in Y. Then Γ' is equal to the orbit of $\theta(y)$. ∎

4. COVARIANT REPRESENTATIONS OF (A, G, σ) WITH G COMPACT

Our goal in this section is to show that an irreducible representation (π, U) of a separable system is induced from an irreducible representation (π_0, U_0) of a subsystem with the key additional property that π_0 is a factor representation of A. As a corollary, we get a strengthening of the GRS theorem for compact groups. First, we need to describe the construction of induced representations and systems of imprimitivity.

In this section we will assume that (A, G, σ) is a separable system and all Hilbert spaces are separable. A covariant representation of (A, G, σ) on a Hilbert space $\mathcal{H}$ is a pair (π, U), where π is a non-degenerate representation of A on $\mathcal{H}$ and U is a SOT-continuous homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ such that

$$U(s)\pi(a)U(s)^* = \pi(\sigma_s a)$$

for all $a \in A$ and $s \in G$.

Let G_0 be a closed subgroup of G and G_0/G be the corresponding right coset space endowed with the quotient topology. Let (π_0, U_0) be a covariant representation of (A, G_0, σ) on a Hilbert space $\mathcal{H}_0$. Following Mackey and Takesaki, we can construct a new covariant representation (π, U) of (A, G, σ), which is called the induced covariant representation.

The construction of the induced covariant representation is simplified when G is compact. If G is a locally compact group, G_0/G does not always admit a G-invariant measure so the construction of induced representations for such groups involves the use of a quasi-invariant measure on G_0/G. However, if G is compact, there exists a unique (up to a scalar multiple) G-invariant Radon measure on G_0/G ([6], Corollary 2.51). Since the induced representation is independent, up to a unitary equivalence, of the choice of the quasi-invariant measure ([8], Theorem 2.1) we can assume that we have a G-invariant measure on G_0/G.

We now describe induced covariant representations following the construction given in [12]. Let G_0 be a closed subgroup of a compact group G and let (π_0, U_0) be a covariant representation of (A, G_0, σ) on a Hilbert space $\mathcal{H}_0$. Let μ be a fixed G-invariant Radon measure on G_0/G. Let $\mathcal{H}$ denote the induced representation space that is, $\mathcal{H}$ is the space of all $\mathcal{H}_0$-valued functions ξ on G satisfying the following conditions:

(i) $\langle \xi(s), h_0 \rangle$ is Borel function of s for all $h_0 \in \mathcal{H}_0$.
(ii) $\xi(ts) = U_0(t)\xi(s)$ for all $t \in G_0$ and all $s \in G$.
(iii) $\int\limits_{G_0/G} \langle \xi(s), \xi(s) \rangle \mathrm{d}\mu(\bar{s}) < \infty$.

Define U to be the homomorphism of G into the unitary group of $\mathcal{B}(\mathcal{H})$ given by

$$(U(t)\xi)(s) = \xi(st)$$

for all $\xi \in \mathcal{H}$ and $s, t \in G$. For each $a \in A$, define an operator $\pi(a)$ on $\mathcal{H}$ by

$$(\pi(a)\xi)(s) = \pi_0(\sigma_s a)\xi(s)$$

for all $\xi \in \mathcal{H}$ and $s \in G$. Then (π, U) is easily checked to be a covariant representation of (A, G, σ):

$$\begin{aligned} U(t)\pi(a)U(t^{-1})\xi(s) &= (\pi(a)U(t^{-1})\xi)(st) = \pi_0(\sigma_{st}a)(U(t^{-1})\xi)(st) \\ &= \pi_0(\sigma_{st}a)\xi(s) = (\pi(\sigma_t a)\xi)(s) \end{aligned}$$

for all $s, t \in G$ and $a \in A$. Since the G-invariant measure μ is unique up to a scalar multiple, the induced representation is independent of the choice of the measure.

Let (π, U) be a covariant representation of (A, G, σ) on $\mathcal{H}$. We say that (π, U) is irreducible if the only operators that commute with $\pi(a)$ and $U(s)$ for all $a \in A, s \in G$ are the scalars. Note that (π, U) is an irreducible representation of (A, G, σ) if and only if $\pi \times_\sigma U$ is an irreducible representation of $A \times_\sigma G$.

Following [12], we define a system of imprimitivity for (π, U) to be a commutative von Neumann algebra $\mathbb{A}$ acting on $\mathcal{H}$ such that:

(i) $\mathbb{A} \subseteq \pi(A)'$.
(ii) $U(s)\mathbb{A}U(s)^* = \mathbb{A}$ for all $s \in G$.

Note that (ii) implies that G acts by automorphisms on $\mathbb{A}$. Since U is assumed to be strongly continuous, for each $x \in \mathbb{A}$ the map $s \mapsto U(s)xU(s)^*$ is continuous in the strong operator topology. We obtain a W^*-dynamical system

$(\mathbb{A}, G, \mathrm{Ad}\, U)$. We call $\mathbb{A}$ an ergodic system of imprimitivity if the action of G on $\mathbb{A}$ is ergodic. In particular, if (π, U) is an irreducible covariant representation, then $\mathbb{A}$ is always an ergodic system of imprimitivity. Given a system of imprimitivity $\mathbb{A}$ for (π, U), not necessarily ergodic, there exists a standard Borel G-measure space (Γ, μ) and an isomorphism i of the algebra $L^\infty(\Gamma, \mu)$ onto $\mathbb{A}$ such that

$$U(s)i(f)U(s)^* = i(\tau_s f)$$

for each $f \in L^\infty(\Gamma, \mu)$ and $s \in G$, where $(\tau_s f)(\gamma) = f(\gamma \cdot s)$ ([10], Theorem 4). In the above situation we say that the system of imprimitivity $\mathbb{A}$ for (π, U) is based on the G-measure space (Γ, μ) with respect to i. As in [12] we say that a system of imprimitivity $\mathbb{A}$ is transitive if the corresponding Borel G-measure space is transitive. It follows from Theorem 2 in [10] that the definition of transitivity is independent of the choice of G-space (Γ, μ).

Suppose that $\mathbb{A}$ is an ergodic system of imprimitivity for (π, U) on a Hilbert space $\mathcal{H}$. Then we can assume $\mathcal{H} = L^2(\Gamma, \mu) \otimes \mathcal{H}_0$ for some Hilbert space $\mathcal{H}_0$ and $\mathbb{A} = L^\infty(\Gamma, \mu) \otimes I_{\mathcal{H}_0}$ ([12], p. 285). The action of $\mathbb{A}$ on $\mathcal{H}$ is given by

$$i(f)\xi)(\gamma) = f(\gamma)\xi(\gamma)$$

for all $f \in L^\infty(\Gamma, \mu)$ and $\xi \in L^2(\mathcal{H}_0, \Gamma, \mu)$. Since $\pi(A) \subseteq \mathbb{A}'$, there exists a $\mathrm{Rep}(A : \mathcal{H}_0)$-valued measurable function $\gamma \in \Gamma \mapsto \pi_\gamma \in \mathrm{Rep}(A : \mathcal{H}_0)$ such that

$$(\pi(a)\xi)(\gamma) = \pi_\gamma(a)\xi(\gamma)$$

for each $a \in A$, $\xi \in \mathcal{H}$ and almost all $\gamma \in \Gamma$. Since the action of G on $\mathbb{A}$ is SOT-continuous in $\mathcal{B}(\mathcal{H})$, then the corresponding action of G on $L^\infty(\Gamma, \mu)$ is SOT-continuous in $\mathcal{B}(L^2(\Gamma, \mu))$. The action of G on $L^\infty(\Gamma, \mu)$ is ergodic. Hence, by Theorem 3.6, G acts transitively on (Γ, μ). We obtain the following result.

PROPOSITION 4.1. *Let (π, U) be an irreducible covariant representation of a separable system (A, G, σ), where G is compact. If $\mathbb{A}$ is a system of imprimitivity for (π, U), then $\mathbb{A}$ is transitive.*

Let (π, U) be a covariant representation of (A, G, σ) on $\mathcal{H}$. Suppose that $\mathbb{A}$ is a transitive system of imprimitivity for (π, U), then by Theorem 6.1 of [9], the associated G-measure space (Γ, μ) can be identified with the right coset space G_0/G of a closed subgroup of G. As it was shown in [9], there exists a representation U_0 of G_0 such that U is equivalent to the induced representation by U_0. Takesaki showed that this result can be extended to covariant representations, i.e. there exists a covariant representation (π_0, U_0) of (A, G_0, σ) such that (π, U) is equivalent to the representation induced by (π_0, U_0) ([12], Theorem 4.2).

A natural choice for a system of imprimitivity for (π, U) is the center of the commutant of $\pi(A)$, which we denote $\mathbf{Z}(\pi(A)')$. If (π, U) is a factor representation, then $\mathbf{Z}(\pi(A)')$ is automatically an ergodic system of imprimitivity for (π, U). In this case, (π, U) is particularly easy to describe, using Theorem 3.6 in Section 3 and Theorem 5.2 in [12].

THEOREM 4.2. *Let (π, U) be a factor (respectively irreducible) representation of a separable system (A, G, σ), where G is compact. Then there exists a closed subgroup G_0 of G and a unique covariant representation (π_0, U_0) of the subsystem (A, G_0, σ) such that (π, U) is equivalent to the representation induced by (π_0, U_0), where the uniqueness is up to equivalence. Moreover,*

(i) *(π_0, U_0) is a factor (respectively irreducible) representation.*

(ii) *π_0 is a factor representation.*

(iii) *There is an isomorphism $i : L^\infty(G_0/G, \mu) \to \mathbf{Z}(\pi(A)')$ given by $(i(f)\xi)(s) = f(\bar{s})\xi(s)$.*

We can view Theorem 4.2 as a generalization of a similar result for finite groups obtained by Arias and Latremoliere ([2], Theorem 3.4). Let G be a finite group and (π, U) be an irreducible representation of (A, G, σ). Then we know by Theorem 4.2 that (π, U) is induced from an irreducible representation (π_0, U_0) of (A, G_0, σ), where π_0 is a factor representation. Define an action of G_0 on $\pi_0(A)'$ by $\tau_s(T) = U_0(s)TU_0(s)^*$ for all $s \in G_0$ and $T \in \pi_0(A)'$. Since G_0 is finite and acts ergodically on $\pi_0(A)'$, then $\pi_0(A)'$ must be finite dimensional. It follows that π_0 is a direct sum of finitely many equivalent irreducible representations.

Next, we give a corollary that strengthens the GRS theorem in the case of compact groups. Let P be a primitive ideal of A and define $G_P := \{s \in G : \sigma_s P = P\}$. Note that G_P is a closed subgroup of G.

COROLLARY 4.3. *Let (π, U) be an irreducible representation of (A, G, σ). Then there exists a primitive ideal P of A and a covariant representation (π_P, U_P) of the subsystem (A, G_P, σ) such that (π, U) is induced by (π_P, U_P). Moreover,* $\ker \pi_P = P$.

Proof. By Theorem 4.2, there exists a closed subgroup G_0 of G and a covariant representation (π_0, U_0) of the subsystem (A, G_0, σ) such that (π, U) is induced by (π_0, U_0). Since A is separable and π_0 is a factor representation, $\ker \pi_0 \in \operatorname{Prim} A$. Let $P := \ker \pi_0$. Then $G_0 \subseteq G_P$. We take (π_P, U_P) to be the representation of (A, G_P, σ) induced by the representation (π_0, U_0) of the subsystem (A, G_0, σ).

In addition, it follows from Lemma 5.1 in the next section that $\ker \pi_P = \bigcap_{r \in G_P} \sigma_r P = P$. ∎

5. STRONG-EHI

In this section we assume that (A, G, σ) is a separable system and G is compact. Our goal is to show that such systems satisfy strong-EHI. Let π be a representation of A on a separable Hilbert space $\mathcal{H}$. If E is a projection in $\pi(A)'$, then we define π^E to be the subrepresentation of π acting on $E\mathcal{H}$.

Let G_0 be a closed subgroup of G and (π_0, U_0) be a covariant representation of (A, G_0, σ) on $\mathcal{H}_0$. Let (π, U) be the covariant representation of (A, G, σ) on $\mathcal{H}$

induced by (π_0, U_0), then there is a natural family of projections in $\pi(A)'$ associated with Borel subsets of G_0/G. Consider the map $i : L^\infty(G_0/G, \mu) \to \pi(A)'$ given by $(i(f)\xi)(s) = f(\bar{s})\xi(s)$. For each nonzero Borel subset E of G_0/G we define π^E to be the subrepresentation of π acting on $i(\chi_E)\mathcal{H}$.

LEMMA 5.1. *In the context of the last paragraph, let $Q := \ker \pi_0$ and F be an open subset of G_0/G. Then* $\ker \pi^F = \bigcap\limits_{s\in q^{-1}(F)} \sigma_{s^{-1}} Q$.

Proof. Clearly, $\bigcap\limits_{s\in q^{-1}(F)} \sigma_{s^{-1}} Q \subseteq \ker \pi^F$. Suppose there is $a \in A$ such that $a \notin \bigcap\limits_{s\in q^{-1}(F)} \sigma_{s^{-1}} Q$. We will show that $\pi^F(a) \neq 0$. Let $s \in q^{-1}(F)$ such that $\pi_0(\sigma_s a) \neq 0$. Choose a unit vector $h \in \mathcal{H}_0$ and $\varepsilon > 0$ so that

$$\|\pi_0(\sigma_s a)h\| \geqslant 2\varepsilon.$$

As in Lemma 6.19 in [13], we will construct $\xi \in C(G, \mathcal{H}_0) \cap \mathcal{H}$ such that

$$\|\xi(s) - h\| \leqslant \frac{\varepsilon}{\|a\|}.$$

To this end, using the strong continuity of U_0, we can find an open neighborhood $N \subseteq G_0$ of e such that $\|U_0(t)h - h\| < \varepsilon/\|a\|$ for all $t \in N$. We can assume without loss of generality that $N = N^{-1}$ (replaced with $N \cap N^{-1}$). Let M be an open set in G such that $N = G_0 \cap M$. By the Urysohn lemma we can find a function $g \in C(G)$ such that $g(e) = 1$ and $g(t) = 0$ for all t in the complement of M in G. Note that $g^{-1}((\frac{1}{2}, \infty))$ is an open neighborhood of e in G therefore its intersection with G_0 is open in the relative topology of G_0. Since G_0 is compact every nonempty open set has a positive measure with respect to the Haar measure. In particular, $\mu_{G_0}(g^{-1}((\frac{1}{2}, \infty)) \cap G_0) > 0$ so we can assume that $\int\limits_{G_0} g(t)\mathrm{d}\mu_{G_0}(t) = 1$. Let $f(r) = g(rs^{-1})$. Define $\xi : G \to \mathcal{H}_0$ by

$$\xi(r) = \int\limits_{G_0} f(tr)U_0(t^{-1})(h)\mathrm{d}\mu_{G_0}(t).$$

It is routine to verify, using for instance the dominated convergence theorem, that $\xi \in C(G, \mathcal{H}_0)$ and ξ satisfies all the conditions of an element of $\mathcal{H}$. Then

$$\begin{aligned}\|\xi(s) - h\| &= \Big\| \int\limits_{G_0} f(ts)(U_0(t^{-1})h - h)\mathrm{d}\mu_{G_0}(t)\Big\| = \Big\| \int\limits_{G_0} g(t)(U_0(t^{-1})h - h)\mathrm{d}\mu_{G_0}(t)\Big\| \\ &= \Big\| \int\limits_{N} g(t)(U_0(t^{-1})h - h)\mathrm{d}\mu_{G_0}(t)\Big\| \leqslant \frac{\varepsilon}{\|a\|}.\end{aligned}$$

It follows that $\|\pi_0(\sigma_s a)\xi(s) - \pi_0(\sigma_s a)h\| \leqslant \|\pi_0(\sigma_s a)\| \cdot \|\xi(s) - h\| \leqslant \|a\| \cdot (\varepsilon/\|a\|) = \varepsilon$. By the reverse triangle inequality,

$$\|\pi_0(\sigma_s a)\xi(s)\| \geqslant \varepsilon.$$

Since $\pi_0(\sigma_{s_j}a) \to \pi_0(\sigma_s a)$ whenever $s_j \to s$ and $\xi \in C(G, \mathcal{H}_0)$, there exists an open neighborhood $F_s \subseteq G_0/G$ of $G_0 s$ such that

$$\|\pi_0(\sigma_t a)\xi(t)\| > \frac{\varepsilon}{2}$$

for all $t \in q^{-1}(F_s)$. Then $\pi^F(a)(\chi_{q^{-1}(F_s \cap F)}\xi) \neq 0$. ∎

We call π a homogeneous representation if $\ker \pi^E = \ker \pi$ for every nonzero projection $E \in \pi(A)'$. It follows from Lemma G.3 in [13] that π is a homogeneous representation if $\ker \pi^E = \ker \pi$ for every nonzero projection $E \in \pi(A)' \cap \pi(A)''$. A very useful structure theory developed by Effros in [5] allows us to decompose an arbitrary representation into a direct integral of homogeneous representations. The value of homogeneous representations is highlighted in the following result:

THEOREM 5.2 (Echterhoff and Williams, [4]). *Let (A, G, σ) be a separable system. Suppose that ρ is a homogeneous representation of A with* $\ker \rho = P$, *and that $\rho \times_\sigma V$ is an irreducible representation of $A \times_\sigma G_P$. Then the representation of $A \times_\sigma G$ induced by $\rho \times_\sigma V$ is irreducible.*

We say that (A, G, σ) satisfies the strong-EHI if given $P \in \operatorname{Prim} A$ and an irreducible covariant representation (π_P, U_P) of (A, G_P, σ) with $\ker \pi_P = P$, then the corresponding induced representation of (A, G, σ) is irreducible. We would like to use Theorem 5.2 to prove the strong-EHI property for separable systems involving compact groups. To this end we prove the following theorem.

THEOREM 5.3. *Let (A, G, σ) be a separable system, where G is a compact group. Suppose P is a primitive ideal of A and (π, U) is an irreducible covariant representation of (A, G_P, σ) on $\mathcal{H}$ with* $\ker \pi = P$. *Then π is a homogeneous representation of A.*

Proof. Note that G_P is a closed subgroup of G so G_P is compact. By Theorem 4.2, there exists a closed subgroup G_0 of G_P and an irreducible covariant representation (π_0, U_0) of the subsystem (A, G_0, σ) such that (π, U) is equivalent to the representation induced by (π_0, U_0). Moreover, there is an isomorphism $i : L^\infty(G_0/G_P, \mu) \to \mathbf{Z}(\pi(A)')$ given by $(i(f)\xi)(s) = f(\bar{s})\xi(s)$. Let E be a Borel subset of G_0/G_P of nonzero measure. By Lemma G.3 in [13], it is enough to show that $\ker \pi^E = \ker \pi$.

Let $Q = \ker \pi_0$. Suppose F is an open subset of G_0/G_P. Let $F' := \{s^{-1} : s \in q^{-1}(F)\}$. By Lemma 5.1, $\ker \pi^F = \bigcap_{s \in F'} \sigma_s Q$. Since G_P is compact and F' is open, there is $\{t_j\}_{1 \leqslant j \leqslant n} \subseteq G_P$ such that $G_P = \bigcup t_j F'$. Then by Lemma 5.1,

$$P = \bigcap_{r \in G_P} \sigma_r Q = \bigcap \sigma_{t_j}\Big(\bigcap_{s \in F'} \sigma_s Q\Big) = \bigcap \sigma_{t_j}(\ker \pi^F).$$

Since P is a prime ideal and P is G_P-invariant, it follows that $P = \ker \pi^F$. In particular, $\|\pi^F(a)\| = \|\pi(a)\|$ for all $a \in A$.

Now let K be a compact subset of G_0/G_P of nonzero measure. By a simple compactness argument we can find $G_0 s \in K$ such that every open neighborhood of $G_0 s$ intersects with K in a set of positive measure. We claim that $\ker \pi^K \subseteq \ker \pi_0 \circ \sigma_s$. To this end, suppose that $\pi_0(\sigma_s a) \neq 0$ for some $a \in A$. Then, as in Lemma 5.1, we can construct a function $\zeta \in C(G_P, \mathcal{H}_0) \cap \mathcal{H}$ such that

$$\|\pi_0(\sigma_s a)\xi(s)\| \geqslant \varepsilon.$$

Similarly, there exists an open neighborhood $F_s \subseteq G_0/G_P$ of $G_0 s$ such that

$$\|\pi_0(ta)\xi(t)\| > \frac{\varepsilon}{2}$$

for all $t \in q^{-1}(F_s)$. It follows that $\pi^K(a)(\chi_{q^{-1}(F_s \cap K)}\xi) \neq 0$.

Next we want to show that $\ker \pi_0 \circ \sigma_s \subseteq P$. Suppose $\pi_0(\sigma_s a) = 0$ for some $a \in A$. Let $\varepsilon > 0$ be given. Since $\pi_0(\sigma_{s_j} a) \to 0$ whenever $s_j \to s$ we can find an open neighborhood F' of s in G_P such that $\|\pi_0(\sigma_t a)\| < \varepsilon$ for all $t \in F'$. Then $\|\pi(a)\| = \|\pi^{q(F')}(a)\| < \varepsilon$. Thus $\pi(a) = 0$ as claimed. It follows that $\ker \pi^K = P$.

Finally, if E a nonzero Borel subset of G_0/G_P, then we can choose a compact subset $K \subseteq E$ such that $\mu(K) > 0$. Suppose $\pi^E(a) = 0$. Then $\pi^K(a) = 0$. It follows $\|\pi(a)\| = \|\pi^K(a)\| = 0$. So $\ker \pi^E = P$. ∎

Combining Theorem 5.2 and Theorem 5.3, we obtain the following corollary.

COROLLARY 5.4. *Let (A, G, σ) be a separable C^*-dynamical system, where G is compact. Then (A, G, σ) satisfies the strong-*EHI *property.*

As mentioned in the introduction it remains unknown whether the strong-EHI property holds for an arbitrary C^*-dynamical system. We can inquire about a weaker property of C^*-dynamical systems, the EHI property [4]. We say that (A, G, σ) satisfies the EHI if, given $P \in \operatorname{Prim} A$ and a primitive ideal J in $A \times_\sigma G_P$ with $\operatorname{Res} J = P$, then J induces to a primitive ideal in $A \times_\sigma G$. However, even with an additional assumption that G is amenable it is not known whether all separable C^*-dynamical systems satisfy the EHI property.

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