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ISOMORPHISMS OF NONCOMMUTATIVE DOMAIN ALGEBRAS. II

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ABSTRACT. We classify aspherical Popescu's noncommutative domain algebras in terms of their defining symbols. An aspherical noncommutative domain algebra is defined by its one-dimensional spectrum not being the unit ball of a hermitian space. We first use the geometry of the spectra of noncommutative domain algebras, together with Sunada's classification of Reinhardt domains in $\mathbb{C}^n$, to show that isomorphisms between aspherical domains must be linear in the generators. We then employ a new combinatorial argument to show that the existence of such an isomorphism implies that the defining symbols must be equivalent, in the sense that they can be obtained from each others by permutation and rescaling of their indeterminates. This paper uses and greatly expand on previous work of the authors.

KEYWORDS: *Non-self-adjoint operator algebras, disk algebra, weighted shifts, biholomorphisms, Reinhardt domains.*

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1. INTRODUCTION

Noncommutative domain algebras, introduced by Popescu [7], provide a generalization of noncommutative disk algebras and serve as universal operator algebras for a large class of noncommutative domains, i.e. noncommutative analogues of domains in $\mathbb{C}^n$. This paper addresses the question of classification of these operator algebras up to completely isometric isomorphisms. In our previous work [3] on this problem, we suggested an approach in terms of spectral theory: each noncommutative domain algebra naturally gives rise to countably many spectra, and an isomorphism between such algebras induces (contravariantly) biholomorphic maps between the spectra. Exploiting this construction, we were able to show that many examples of noncommutative domains algebras were not isomorphic and we could characterize the noncommutative disk algebras in terms of their symbols. However, the classification of the noncommutative domain algebras was not done in terms of their defining symbols in general.

Our present work uses the classification of Reinhardt domains by Sunada in [10] and new combinatorial techniques to provide a complete classification of a large class of noncommutative domain algebras in terms of their defining symbol. This class consists of the aspherical noncommutative domain algebras whose symbol is polynomial, as we shall define in the first section of this paper. In addition, we show that our techniques can be used to generalize results from several complex variables analysis to the setup of noncommutative domain algebras, by proving a version of the Cartan's lemma [8], [9]. We expect that one could use the work in this paper to establish other such generalizations. The results of this paper fit in the broader context that nonselfadjoint operators tend to preserve the objects used to construct them. We refer to [1], [5] for examples.

Let us introduce the objects of interest in this paper, as well as the notations we will use throughout our exposition. Popescu noncommutative domains are defined by means of a symbol, which is a special type of formal power series. Let $\mathbb{F}_n^+$ be the free semigroup on n generators $g_1,\ldots,g_n$ and identity g_0. If $Y_1,\ldots,Y_n$ belong to any ring, and if $\alpha \in \mathbb{F}_n^+$ is written as $\alpha = g_{i_1}\cdots g_{i_n}$, then we write Y_α for the product $Y_{i_1}\cdots Y_{i_n}$. We shall denote the free formal power series f with coefficients $(a_\alpha)_{\alpha\in\mathbb{F}_n^+}$ by $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$ where it is understood that $X_1,\ldots,X_n$ are the indeterminates of the ring of free formal power series — in other words, the ring of the semigroup $\mathbb{F}_n^+$ where the generators $g_1,\ldots,g_n$ correspond to $X_1,\ldots,X_n$.

A formal power series $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$ with real coefficients a_α ($\alpha \in \mathbb{F}_n^+$) is *regular positive* if:

$$(1.1)\quad a_{g_0}=0;\ a_{g_i}>0 \quad \text{if } i=1,\ldots,n;\ a_\alpha \geqslant 0 \text{ for all } \alpha\in\mathbb{F}_n^+;\ \sup_{n\in\mathbb{N}^*}\left(\left|\sum_{|\alpha|=n} a_\alpha^2\right|^{1/n}\right)<\infty.$$

These conditions are sufficient for the existence of the noncommutative domain algebra of symbol f, which we now define.

Let us be given a regular positive free formal power series $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$ in n indeterminates, and a Hilbert space $\mathcal{H}$. We define the noncommutative domain:

$$\mathcal{D}_f(\mathcal{H}) = \left\{(T_1,\ldots,T_n) \in \mathcal{B}(\mathcal{H}) : \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha T_\alpha T_\alpha^* \leqslant 1\right\}$$

where $\mathcal{B}(\mathcal{H})$ is the von Neumann algebra of all bounded linear operators on $\mathcal{H}$ and 1 is its identity. The interior of $\mathcal{D}_f(\mathbb{C}^k)$ will be denoted by $\mathbb{D}_f^k$. Popescu proved in [7] that certain weighted shifts on the full Fock space $\ell^2(\mathbb{F}_n^+)$ of $\mathbb{C}^n$ provide a model for all n-tuples in all domains $\mathcal{D}_f(\mathcal{H})$. Specifically, as shown in [7], one can find positive real numbers $(b_\alpha)_{\alpha\in\mathbb{F}_n^+}$ and define linear operators W_i^f on $\ell^2(\mathbb{F}_n^+)$ such that:

$$W_i^f \delta_\alpha = \sqrt[2]{\frac{b_\alpha}{b_{g_i\alpha}}}\delta_{g_i\alpha},$$

where $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$ is the canonical basis of $\ell^2(\mathbb{F}_n^+)$. Then $(W_1^f, \dots, W_n^f) \in \mathcal{D}_f(\ell^2(\mathbb{F}_n^+))$ and the coefficients $(a_\alpha)_{\alpha\in\mathbb{F}_n^+}$ can be recovered from the coefficients $(b_\alpha)_{\alpha\in\mathbb{F}_n^+}$. We then define:

$$\mathcal{A}(\mathcal{D}_f) = \overline{\operatorname{span}\{W_\alpha : \alpha \in \mathbb{F}_n^+\}}$$

i.e. $\mathcal{A}(\mathcal{D}_f)$ is the norm closure in $\mathcal{B}(\mathcal{H})$ of the algebra generated by $W_1^f, \dots, W_n^f$. The fundamental property of this algebra is that:

PROPOSITION 1.1 (Popescu). *Let $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$ be a regular positive free power series in n indeterminates. Let $\mathcal{H}$ be a Hilbert space. Let $(T_1, \dots, T_n) \in \mathcal{D}_f(\mathcal{H})$. Then there exists a (necessarily unique) completely contractive unital algebra morphism Φ : $\mathcal{A}(\mathcal{D}_F) \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\Phi(W_j^f) = T_j$ for $j = 1, \dots, n$.*

The algebra $\mathcal{A}(\mathcal{D}_f)$ is the *noncommutative domain algebra of symbol f*. When $f = X_1 + \cdots + X_n$, the algebra $\mathcal{A}(\mathcal{D}_f)$ is the disk algebra in n-generators. In [3], we used techniques of complex analysis on domains in $\mathbb{C}^n$ to study the isomorphism problem for noncommutative domain algebras. In our context, the category NCD of noncommutative domain algebras consists of the algebras $\mathcal{A}(\mathcal{D}_f)$ for all positive, regular n-free formal power series f for objects, and *completely isometric* unital algebra isomorphisms for arrows. We then constructed for each $k \in \mathbb{N}$ a contravariant functor $\mathbb{D}^k$ from NCD to the category HD_{kn^2} of connected open subsets of $\mathbb{C}^{kn^2}$ with holomorphic maps. Given a regular positive formal n-free formal power series f, the functor $\mathbb{D}^k$ associates to $\mathcal{A}(\mathcal{D}_f)$ the domain $\mathbb{D}_f^k$. The key observation of [3] is that any isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ in NCD gives rise to a biholomophic map $\widehat{\Phi}_k = \mathbb{D}^k(\Phi)$ from $\mathbb{D}_g^k$ onto $\mathbb{D}_f^k$, and this construction is functorial (contravariant). Thus, $\mathbb{D}^k$ give fundamental invariants of noncommutative domain algebras. We showed in Theorem 3.18 of [3] that if $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ is an isomorphism in NCD such that $\widehat{\Phi}_1(0) = 0$ then there is an invertible matrix $M \in M_{n\times n}(\mathbb{C})$ such that $\begin{bmatrix} W_1^g & \cdots & W_n^g \end{bmatrix}^{\mathrm{t}} = (M \otimes 1_{\ell(\mathbb{F}_n^+)}) \begin{bmatrix} W_1^f & \cdots & W_n^f \end{bmatrix}^{\mathrm{t}}$. We could then use this result to characterize the disk algebra among all noncommutative domain algebras ([3], Section 4.3): $\mathcal{A}(\mathcal{D}_f)$ is isomorphic to the n-disk algebra in NCD if and only if $f = X_1 + \cdots + X_n$ after possible rescaling (i.e. replacing each indeterminate with a multiple of itself). We could also distinguish between the instructive examples $f = X_1 + X_2 + X_1X_2$ and $g = X_1 + X_2 + \frac{1}{2}X_1X_2 + \frac{1}{2}X_2X_1$ in NCD ([3], Section 4.2).

Though the noncommutative disks were characterized by their symbols, we lacked a more general relation between symbols and isomorphism classes of noncommutative domain algebras. For instance, if $f = X_1 + X_2 + X_1X_2$ and if $\mathcal{A}(\mathcal{D}_f)$ is isomorphic in NCD to $\mathcal{A}(\mathcal{D}_g)$ for some symbol g, what is the relation between f and g? This paper answers this question in general if f is any polynomial such that $\{(z_1,\ldots,z_n) : f(z_1,\ldots,z_n) \leqslant 1\}$ is not a ball in $\mathbb{C}^n$ for the usual hermitian metric.

To do so, we proceed as follows. Given a regular positive free formal polynomial f, we establish that $\mathbb{D}^1_f$ is not a product of proper Reinhardt domains. We then show that the geometry of the boundary of $\mathbb{D}^1_f$ prevents this domain to be a generalized Thullen domain. Thus, by Sunada's classification of Reinhardt domains [10], we conclude that $\mathbb{D}^1_f$ is either a ball, or its automorphism group fixes the origin. This, in turns, implies that isomorphisms in NCD induces biholomorphic maps which fixes 0 for aspherical domains. Thus, [3] shows that isomorphisms between aspherical domain algebras are linear in the canonical generators. We present a new combinatorial argument to conclude that if f is aspherical and $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic in NCD then f and g are in fact obtained from each others by permuting and rescaling their indeterminates.

In the last section of our paper, we show that the techniques developed in this paper can be applied to prove a Cartan's lemma for noncommutative domain algebras. Such a result was proven using a different method in [8], [9] for general symbols. We present here a different, shorter proof for polynomial symbols, based on finite dimensional representations and our combinatorial methods. This approach to generalizing results from several complex variables to noncommutative domain algebras is likely to apply to other results.

In Theorem 4.5 of [9], Popescu proved that two noncommutative domain algebras are isomorphic in NCD if and only if their corresponding noncommutative domains are biholomorphic, in the generalized sense of Popescu. Thus, our result extends to the biholomorphic classification of noncommutative domains in terms of their defining symbols. We note that Popescu's result applies to a larger class of domain algebras and noncommutative domains associated with the Berezin transform. We refer to Theorem 4.5 of [9] for details.

2. ASPHERICAL NONCOMMUTATIVE DOMAIN ALGEBRAS

This first section applies Sunada's classification of Reinhardt domains to the classification of noncommutative domain algebras. In [3], we showed how to classify all noncommutative domain algebras isomorphic to the disk algebras. In this paper, we will provide a complete classification for the class of polynomial aspherical domains as defined below. These two classes are disjoint from each other, and together these two classication results cover most cases of noncommutative domain algebras.

DEFINITION 2.1. Let f, g be two regular positive free power series in n indeterminates. We say that f and g are *permutation-rescaling equivalent* if there exists a permutation σ of $\{1,\ldots,n\}$ and a nonzero positive numbers $\lambda_1,\ldots,\lambda_n$ such that $g(\lambda_1 X_{\sigma(1)}, \ldots, \lambda_n X_{\sigma(n)}) = f(X_1, \ldots, X_n)$ where $X_1, \ldots, X_n$ are the indeterminates.

One checks easily that permutation-rescaling equivalence is an equivalence relation of regular positive free power series. It was shown in Lemma 4.4 of [3] that if f and g are regular positive free formal power series in n indeterminates, and f and g are permutation-rescaling equivalent, then $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic.

In Theorem 4.7 of [3], we showed that it is possible to characterize the noncommutative disk algebras $\mathcal{A}_n$ (also known as noncommutative analytic Toeplitz algebras in [4]) with their symbols:

THEOREM 2.2. *Let f be a free formal polynomial in n indeterminates $X_1, \ldots, X_n$. The noncommutative algebra $\mathcal{A}(\mathcal{D}_f)$ is isomorphic in* NCD *to $\mathcal{A}_n$ if and only if f is permuation-rescaling equivalent to $X_1 + X_2 + \cdots + X_n$.*

We also proved that a few other noncommutative domain algebras, with distinct symbols, were not isomorphic in NCD. We now propose to completely classify by their symbol a very large class of noncommutative domain algebras, which is disjoint from the noncommutative analytic Toeplitz algebras class. The fundamental property of the algebras of this new class is that their spectra $\mathbb{D}^1$ are not given by (hermitian) balls (note that all the spectra, in the sense of [3], of noncommutative analytic Toeplitz algebras, are in fact unit balls in hermitian spaces). We formally introduce this class in the next few definitions.

DEFINITION 2.3. Let f be a positive regular n-free formal power series. Then f is *spherical* if it is permutation-rescaling to some g such that:

$$\mathbb{D}_g^1 = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

DEFINITION 2.4. Let f be a positive regular n-free formal power series. Then f is *aspherical* if f is not spherical.

DEFINITION 2.5. A free power series f is a *free polynomial* if, writing $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$, the set $\{\alpha \in \mathbb{F}_n^+ : a_\alpha \neq 0\}$ is finite. We shall write n-free polynomial for a free polynomial in n-indeterminates.

DEFINITION 2.6. The noncommutative domain $\mathcal{A}(\mathcal{D}_f)$ is *aspherical* when f is aspherical, and *polynomial* when f is a free polynomial.

It should be noted that there are domains which are not aspherical but not isomorphic to the disk algebras. Let us give a few examples to illustrate our definition.

EXAMPLE 2.7. Let $f = \frac{1}{2}X_1 + \frac{1}{2}X_2 + \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + X_1X_2$. Then $\mathbb{D}_f^1$ is the open unit ball of $\mathbb{C}^2$. Yet, by Theorem 4.7 of [3], we know that $\mathcal{A}(\mathcal{D}_f)$ is not a disk algebra.

EXAMPLE 2.8. Let $f = \frac{1}{2}X_1 + \frac{1}{2}X_2 + \frac{1}{2}(X_1 + X_2)^2$. Using Theorem 4.7 of [3], we see again that $\mathcal{A}(\mathcal{D}_f)$ is not isomorphic to a disk algebra in NCD. However, there is a completely bounded isomorphism from the disk algebra onto $\mathcal{A}(\mathcal{D}_f)$. This illustrates the importance of our choice of isomorphisms as *completely isometric* unital algebra isomorphisms.

EXAMPLE 2.9. Let $f = X_1 + X_2 + 3X_1X_2$, $g = 2X_1 + X_2 + 6X_2X_1$ and $h = X_1 + 2X_2 + X_1^2$. All three symbols are polynomial and aspherical. We will show in section 3 using Theorem 3.2 that $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic, but $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_h)$ are not.

It is of course quite easy to produce examples of polynomial aspherical symbols, so our work will apply to a large class of examples.

It is immediate, by definition, that:

OBSERVATION 2.10. Let f be a positive regular free formal power series in n indeterminates. Then $\mathbb{D}_f^1$ is a bounded Reinhardt domain in $\mathbb{C}^{n^2}$.

Now, Sunada in Theorem A of [10] shows that up to rescaling-permutation of the canonical basis vectors of $\mathbb{C}^n$, all bounded Reinhardt domains can be written in a normalized form. We cite this theorem with the necessary notations added in the statement.

THEOREM 2.11 ([10], Theorem A). *Let D be a bounded Reinhardt domain in $\mathbb{C}^n$. Up to applying to D a map of the form $(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto (\lambda_1 z_{\sigma(1)}, \ldots, \lambda_n z_{\sigma(n)})$ with σ some permutation of $\{1, \ldots, n\}$ and $\lambda_1, \ldots, \lambda_n$ some nonzero complex numbers, D can be described as follows. There exists integers $0 = n_0 < n_1 < \cdots < n_s = n$, integers $p, r \in \{1, \ldots, n\}$ with $n_r = p$ and a bounded Reinhard domain D_1 in $\mathbb{C}^{n-p}$ such that, if for any $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $\mathbf{z}_j = (z_{n_{j-1}+1}, \ldots, z_{n_j})$ and $r = n - p$, then:*

(i) $D \cap (\mathbb{C}^p \times \{0\}) = \{(\mathbf{z}_1, \ldots, \mathbf{z}_s) \in \mathbb{C}^n : |\mathbf{z}_1| < 1 \wedge \cdots \wedge |\mathbf{z}_r| < 1 \wedge (\mathbf{z}_{r+1}, \ldots, \mathbf{z}_s) = (0, \ldots, 0)\}$.

(ii) $\{0\} \times D_1 = D \cap (\{0\} \times \mathbb{C}^{n-p})$.

(iii) *We have:*

$$D = \Big\{(\mathbf{z}_1, \ldots, \mathbf{z}_s) : |\mathbf{z}_1| < 1 \wedge \cdots \wedge |\mathbf{z}_r| < 1 \wedge \\ \cdot \Big(\frac{\mathbf{z}_{r+1}}{\prod_{j=1}^r (1 - |\mathbf{z}_j|^2)^{q_{r+1,j}}}, \ldots, \frac{\mathbf{z}_s}{\prod_{j=1}^r (1 - |\mathbf{z}_j|^2)^{q_{s,j}}}\Big) \in D_1\Big\}$$

where $q_{r+1,1}, \ldots, q_{s,r}$ are positive nonzero real numbers distinct from 1.

We also observe that if f is a regular positive n-free power series, then $\mathbb{D}_f^1 \cap (\mathbb{C}^p \times \{0\}) = \mathbb{D}_g^1$ for the power series g obtained by evaluating $X_{p+1}, \ldots, X_n$ to 0. One readily checks that g is a regular positive p-free power series. This simple observation will prove useful.

Our strategy to prove our main theorem of this section, Theorem 2.15, is as follows. Let f be a regular positive aspherical n-free polynomial.

(i) Since f is aspherical, $n_1 < n$ and thus $r < s$ in Theorem 2.11 for $\mathbb{D}_f^1$.

(ii) We first show that $\mathbb{D}_f^1$ can not be a product of bounded Reinhardt domains. This implies that $r \in \{0,1\}$ in Theorem 2.11 for $\mathbb{D}_f^1$.

(iii) We then show that if $n=2$, i.e. f has only two indeterminates, then $\mathbb{D}_f^1$ can not be a Thullen domain [11]. This implies that $r=0$ in Theorem 2.11 for $\mathbb{D}_f^1$ for an arbitrary n.

(iv) We then conclude, using Corollary 2, Section 6, p. 126 of [10], that the automorphism group of $\mathbb{D}_f^1$ fixes the origin.

To implement this approach, we start with a few lemmas.

LEMMA 2.12. *Let f be a positive regular n-free formal polynomial. Then $\mathcal{D}_f(\mathbb{C})$ is not a Cartesian product of bounded domains.*

Proof. Up to rescaling, we assume that $a_\alpha = 1$ for all words α of length 1. Assume that there are two domains D and D', respectively in $\mathbb{C}^p$ and $\mathbb{C}^q$ with $p+q=n$, $p,q>0$ and such that:

$$\mathbb{D}_f^1 = D \times D'.$$

Let $(z_1,\dots,z_p) \in D$ be a boundary point. Then $(z_1,\dots,z_p,0,\dots,0) \in \mathbb{C}^n$ is also a boundary point of $\mathbb{D}_f^1$. Hence:

$$\sum_{\alpha\in\mathbb{F}_p^+} a_\alpha |z_\alpha|^2 = 1.$$

Let $(w_1,\dots,w_q) \in D'\backslash\{0\}$. Then $(z_1,\dots,z_p,w_1,\dots,w_q) \in \mathbb{D}_f^1$, so (if we let $z_{p+1} = w_1,\dots,z_n = w_q$):

$$1 \geqslant \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha |z_\alpha|^2 \geqslant \sum_{\alpha\in\mathbb{F}_p^+} a_\alpha |z_\alpha|^2 + |w_1|^2 + \cdots + |w_q|^2 \geqslant 1 + |w_1|^2 + \cdots + |w_q|^2 > 1$$

which is a contradiction. Hence $\mathbb{D}_f^1$ is not a product of domains in proper subspaces of $\mathbb{C}^n$. ∎

Thullen [11] proved that all bounded Reinhardt domains containing the origin in $\mathbb{C}^2$ are biholomorphic to either the unit ball, the polydisk, a domain whose automorphism group fixes 0, or to a Thullen domain:

$$\{(z,w) \in \mathbb{C}^2 : |z|^2 + |w|^{2q} \leqslant 1\}$$

for some $q \in (0,1) \cup (1,\infty)$. This result, of course, is generalized in Theorem 2.11. We will start by showing that in two dimensions, Thullen domains are not in the range of the object map of the functors $\mathbb{D}^1$.

LEMMA 2.13. *Let f be an aspherical positive regular* 2*-free formal polynomial. Then $\mathcal{D}_f(\mathbb{C})$ is not a Thullen domain* [11].

Proof. Assume $\mathcal{D}_f(\mathbb{C})$ is a Thullen domain [11]. Up to applying a rescaling and permutation of the free variables of f ([3], Lemma 4.4), there exists $q \in (0,1) \cup (1,\infty)$ such that, writing $\tau : (z_1, z_2) \mapsto |z_1|^2 + |z_2|^{2q} - 1$, we have:

$$\mathbb{D}^1_f = \{(z,w) \in \mathbb{C}^2 : \tau(z,w) < 0\}$$

and the boundary $\partial\mathbb{D}^1_f$ of $\mathbb{D}^1_f$ is then

$$\{(z_1,z_2) \in \mathbb{C}^2 : \tau(z_1,z_2) = 0\}.$$

We shall adopt the standard notation that given $(z_1,z_2) \in \mathbb{C}^2$, we have $z_j = x_j + iy_j$ for $x_j, y_j \in \mathbb{R}$. We identity $\mathbb{C}^2$ with $\mathbb{R}^4$. Then:

$$\tau : (x_1,y_1,x_2,y_2) \mapsto x_1^2 + y_1^2 + (x_2^2 + y_2^2)^q - 1.$$

Let ∇ denote the gradient operator on $\mathbb{R}^4$ (i.e., with usual abuse of notations, $\nabla = \begin{bmatrix} \frac{\partial}{\partial x_1} & \frac{\partial}{\partial y_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial y_2} \end{bmatrix}^t$). Then for all $(z_1,z_2) \in \partial\mathbb{D}^1_f$):

$$\nabla\tau(z_1,z_2) = \begin{bmatrix} 2x_1 \\ 2y_1 \\ 2qx_2(x_2^2+y_2^2)^{q-1} \\ 2qy_2(x_2^2+y_2^2)^{q-1} \end{bmatrix}.$$

On the other hand, by definition, the boundary of $\mathbb{D}^1_f$ is:

$$\left\{(z_1,z_2) \in \mathbb{C}^2 : \sum a_\alpha |z_\alpha|^2 - 1 = 0\right\}.$$

To simplify notations, we shall introduce the coefficients $(c_{n,m})_{n,m\in\mathbb{N}}$ as follows:

$$c_{n,m} = \sum\{a_\alpha : \alpha \in \mathbb{F}_2^+,\ |\alpha|_1 = n \wedge |\alpha|_2 = m\}$$

where $|\alpha|_i$ is the number of times the generator g_i appears in the word α ($i = 1,2$). Thus we can write for all $(z_1,z_2) \in \mathbb{C}^2$:

$$\sum_{\alpha\in\mathbb{F}_2^+} a_\alpha |z_\alpha|^2 = \sum_{n,m\in\mathbb{N}} c_{n,m} |z_1|^{2n} |z_2|^{2m}.$$

Now, let:

$$\rho : (z_1,z_2) \in \mathbb{C}^2 \mapsto \sum_{n,m\in\mathbb{N}} c_{n,m} |z_1|^{2n} |z_2|^{2m} - 1.$$

We thus have:

$$\rho(x_1,y_1,x_2,y_2) = \sum_{n,m\in\mathbb{N}} c_{n,m} (x_1^2+y_1^2)^n (x_2^2+y_2^2)^m - 1.$$

Then for all $(z_1, z_2) \in \partial \mathbb{D}_f^1$:

$$\nabla\rho(z_1,z_2) = \begin{bmatrix} \sum_{n,m\in\mathbb{N},\,(n,m)\neq(0,0)} c_{n,m} 2n x_1 (x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m \\ \sum_{n,m\in\mathbb{N},\,(n,m)\neq(0,0)} c_{n,m} 2n y_1 (x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m \\ \sum_{n,m\in\mathbb{N},\,(n,m)\neq(0,0)} c_{n,m} 2m x_2 (x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1} \\ \sum_{n,m\in\mathbb{N},\,(n,m)\neq(0,0)} c_{n,m} 2m y_2 (x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1} \end{bmatrix}$$

$$= \begin{bmatrix} 2x_1\left(c_{1,0} + \sum_{n,m\in\mathbb{N},\,n>1} n c_{n,m} (x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m\right) \\ 2y_1\left(c_{1,0} + \sum_{n,m\in\mathbb{N},\,n>1} n c_{n,m} (x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m\right) \\ 2x_2\left(c_{0,1} + \sum_{n,m\in\mathbb{N},\,m>1} m c_{n,m} (x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1}\right) \\ 2y_2\left(c_{0,1} + \sum_{n,m\in\mathbb{N},\,m>1} m c_{n,m} (x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1}\right) \end{bmatrix}.$$

The tangent plane in $\mathbb{R}^4$ to a boundary point (z_1, z_2) (where $z_2 \neq 0$ so that we work at a regular point for τ) of $\mathbb{D}_f^1$ is the orthogonal space to any normal vector to the boundary of $\mathbb{D}_f^1$ at that point, namely it is the orthogonal of $\nabla\rho(z_1, z_2)$, as well as the orthogonal space to $\nabla\tau(z_1, z_2)$ (see, for instance, Chapter 3 of [6]). Thus, these two vectors must be co-linear. In particular, let us focus on the first and third coordinates. It is therefore necessary that if $x_1 \neq 0, x_2 \neq 0$:

$$\begin{aligned}\Big(c_{1,0} + \sum_{n,m\in\mathbb{N},\,n>1} n c_{n,m} (x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m\Big) q (x_2^2+y_2^2)^{q-1} \\ = c_{0,1} + \sum_{n,m\in\mathbb{N},\,m>1} m c_{n,m} (x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1},\end{aligned}$$

which is equivalent to:

$$\Big(c_{1,0} + \sum_{n,m\in\mathbb{N},\,n>1} n c_{n,m} |z_1|^{2n-2}|z_2|^{2m}\Big) q|z_2|^{2q-2} = c_{0,1} + \sum_{n,m\in\mathbb{N},\,m>1} m c_{n,m} |z_1|^{2n}|z_2|^{2m-2}.$$

Now, since (z_1, z_2) is on the boundary of the Thullen domain $\mathbb{D}_f^1$, we have $\tau(z_1, z_2) = 0$ i.e. $|z_1|^2 = 1 - |z_2|^{2q}$. Hence:

$$\begin{aligned}\Big(c_{1,0} + \sum_{n,m\in\mathbb{N},\,n>1} n c_{n,m} (1-|z_2|^{2q})^{n-1}|z_2|^{2m}\Big) q|z_2|^{2q-2} \\ = c_{0,1} + \sum_{n,m\in\mathbb{N},m>1} m c_{n,m} (1-|z_2|^{2q})^{n}|z_2|^{2m-2}.\end{aligned}$$

This identity must be valid for all (z_1, z_2) on the boundary of $\mathbb{D}_f^1$ except when $z_2 = 0$. Thus it is true on a neighborhood of 0, except at 0. Hence both sides of this identity must be continuous at 0 since the right hand side is as a polynomial in $\mathbb{R}^2$. This precludes that $q < 1$. Hence $q > 1$ and we get at the limit when $z_2 \to 0$

that $0 = c_{0,1}$, which contradicts the definition of f regular positive. So $\mathcal{D}_f(\mathbb{C})$ is not a Thullen domain. ∎

REMARK 2.14. An important observation is that Thullen domains are a special case of the domains described in Theorem 2.11. Indeed, assume, in the notations of Theorem 2.11, that $n = 2$, $s = 2$, $r = 1$. Then D_1, which is a bounded Reinhard domain in $\mathbb{C}$ which we can put in standard form, is just the unit disk in $\mathbb{C}$, so:

$$\begin{aligned} D &= \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \wedge |z_2| < (1 - |z_1|^2)^q\} \\ &= \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2/q} < 1\} \end{aligned}$$

which is to say that D is a Thullen domain [11]. In particular, if n is now arbitrary, $s > r > 1$, then the intersection of D with the plane spanned by the first and $(p+1)^{\text{th}}$ canonical basis vectors of $\mathbb{C}^n$ is a Thullen domain.

THEOREM 2.15. *Let f be a aspherical regular positive n-free polynomial. Then the automorphism group of $\mathcal{D}_f(\mathbb{C})$ fixes* 0.

Proof. By Proposition 3.11 of [3], $\mathbb{D}_f^1$ is a bounded Reinhardt domain in $\mathbb{C}^n$.

Up to replacing f by a permutation-scaling equivalent symbol, we can assume that $\mathbb{D}_f^1$ is in standard form, i.e. of the form given in Theorem 2.11. We recall from Lemma 4.4 of [3] that if two symbols are permutation-scaling equivalent, then their associated noncommutative domain algebras are isomorphic.

We shall now use the notations of Theorem 2.11 applied to $\mathbb{D}_f^1$.

By definition of aspherical, $\mathbb{D}_f^1$ is not the open unit ball of $\mathbb{C}^n$, so $n_1 < n$, $r < s$ and $p < n$.

Assume now that $r \geqslant 1$. Then $\mathbb{D}_f^1 \cap (\mathbb{C}^p \times \{0\})$ is a product of open unit balls. Yet, if g is obtained from f by mapping $X_{p+1}, \ldots, X_n$ by 0, then g is a regular positive p-free polynomial such that $\mathbb{D}_g^1 \times 0 = \mathbb{D}_f^1 \cap (\mathbb{C}^p \times \{0\})$ by construction. By Lemma 2.12, $\mathbb{D}_g^1$ is not a proper product, so it must be at most one unit ball. Hence, $r = 1$.

Therefore, in general, $r \in \{0, 1\}$. Assume that $r = 1$, so $1 \leqslant p < n$ (since $r < s$ as well). Let h be the regular formal 2-free polynomial obtained from f by mapping $X_2, \ldots, X_p, X_{p+2}, \ldots, X_n$ to 0 (once again it is immediate to check that indeed h is a regular positive polynomial in indeterminates X_1, X_{p+1}). In particular, observe that $\mathbb{D}_h^1$ is the intersection of $\mathbb{D}_f^1$ by the plane spanned by the first and $(p+1)^{th}$ canonical basis vector in $\mathbb{C}^n$. By Remark 2.14, $\mathbb{D}_h^1$ is a Thullen domain. By Lemma 2.13, this is impossible. So we have reached a contradiction.

Hence $r = 0$. By Corollary 2 of Theorem B, Section 6, p. 126 of [10], all automorphisms of $\mathbb{D}_f^1$ fix 0. Our theorem is proven. ∎

Applying Theorem 3.18 of [3], we get the following important result:

THEOREM 2.16. *Let f and g be regular positive n-free polynomials, with f aspherical. Then if $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ is an isomorphism, then $\widehat{\Phi}(0) = 0$ and therefore, there exists $M \in M_{n\times n}(\mathbb{C})$ such that:*

$$\left[\begin{array}{ccc} \Phi(W_1^f) & \cdots & \Phi(W_n^f) \end{array}\right]^{\mathrm{t}} = (M \otimes 1_{\ell^2(\mathbb{F}_n)}) \left[\begin{array}{ccc} W_1^g & \cdots & W_n^g \end{array}\right]^{\mathrm{t}}.$$

Proof. Let $\lambda = \widehat{\Phi}(0)$. Assume $\lambda \neq 0$. Let $j \in \{1, \ldots, n\}$ such that $\lambda_j \neq 0$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let ρ be the linear transformation of $\mathbb{C}^n$ given by the matrix $\begin{bmatrix} 1_{j-1} & & \\ & -1 & \\ & & 1_{n-j} \end{bmatrix}$ in the canonical basis of $\mathbb{C}^n$ (where 1_k is the identity of order k for any $k \in \mathbb{N}$). Then $\rho(\lambda) \neq \lambda$. By Lemma 4.4 of [3], there exists a unique automorphism $\widetilde{\rho}$ of $\mathcal{A}(\mathcal{D}_g)$ such that $\mathbb{D}^1(\widetilde{\rho}) = \rho$ since ρ only rescale the coordinates. Let $\tau = \Phi^{-1} \circ \widetilde{\rho} \circ \Phi$: by construction, τ is an automorphism of $\mathcal{A}(\mathcal{D}_f)$, and moreover $\widehat{\tau}(0) = \widehat{\Phi}^{-1}(\rho(\lambda)) \neq 0$ since $\widehat{\Phi}^{-1}$ is injective. Thus $\widehat{\tau}$ is an automorphism of $\mathcal{D}_f(\mathbb{C})$ which does not fix 0. By Theorem 2.15, since f is aspherical, this is a contradiction. Hence $\widehat{\Phi}(0) = 0$. The theorem follows from Theorem 3.18 of [3]. ∎

COROLLARY 2.17. *Let f, g be aspherical regular positive n-free polynomials. Then there exist an isomorphism Φ from $\mathcal{A}(\mathcal{D}_f)$ onto $\mathcal{A}(\mathcal{D}_g)$ if and only if there exists an invertible matrix $M \in M_{n\times n}(\mathbb{C})$ such that:*

$$\left[\begin{array}{ccc} W_1^f & \cdots & W_n^f \end{array}\right]^{\mathrm{t}} = (M \otimes 1_{\ell^2(\mathbb{F}_n)}) \left[\begin{array}{ccc} W_1^g & \cdots & W_n^g \end{array}\right]^{\mathrm{t}}.$$

In particular, the automorphism group of aspherical polynomial noncommutative domain algebras consists only of invertible linear transformation on the generators, in contrast with the disk algebras which have the full automorphism group of the unit ball as a normal subgroup of automorphism [4]. We also contrast this result to the computation of other nontrivial automorphism groups associated to the disk algebras in [2].

COROLLARY 2.18. *Let f be an aspherical regular positive n-free polynomial. The automorphism group of $\mathcal{A}(\mathcal{D}_f)$ is a subgroup of the unitary group $U(n)$.*

3. CLASSIFICATION OF POLYNOMIAL ASPHERICAL NONCOMMUTATIVE DOMAIN ALGEBRAS

This section establishes an explicit equivalence on aspherical regular positive n-free polynomials which corresponds to isomorphism of the associated noncommutative domain algebra. This result generalizes Section 4 of [3].

We defined in [3] dual maps associated to an isomorphism in NCD as follows. Fix f and g two regular, positive n-free formal power series, and let $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ be an isomorphism in NCD. Let $k \in \mathbb{N}$, $k > 0$. For any

$\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{D}_f^k \subseteq (M_{k\times k}(\mathbb{C}))^n$, there exists a unique completely contractive representation $\pi_\lambda : \mathcal{A}(\mathcal{D}_g) \to M_{n\times n}(\mathbb{C})$ such that $\pi(W_j^g) = \lambda_j$ for $j = 1, \ldots, n$. Now, $\rho = \pi_\lambda \circ \Phi$ is a completely contractive representation of $\mathcal{A}(\mathcal{D}_f)$ on $M_{n\times n}(\mathbb{C})$, so there exists $(\mu_1, \ldots, \mu_n) \in \mathbb{D}_f^k$ such that $\rho(W_j^f) = \mu_j$ for $j = 1, \ldots, n$. We set $\widehat{\Phi}_k(\lambda_1, \ldots, \lambda_n) = (\mu_1, \ldots, \mu_n)$. We showed in [3] that these maps are biholomorphic and that this construction is functorial and contravariant.

We start with a combinatorial result.

PROPOSITION 3.1. *Let* $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^f X_\alpha$ *and* $g = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^g X_\alpha$ *be regular positive n-free polynomials. Up to rescaling, we assume* $a_\alpha^f = a_\alpha^g = 1$ *for all words* α *of length* 1. *Let* $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ *be an isomorphism such that* $\widehat{\Phi}(0) = 0$. *Let* $U = [u_{ij}]_{1\leqslant i,j\leqslant n} \in M_{n\times n}(\mathbb{C})$ *be the unitary matrix such that for all* $i \in \{1, \ldots, n\}$:

$$\Phi(W_i^f) = \sum_{j=1}^{n} u_{ij} W_j^g. \tag{3.1}$$

We define the support function s_Φ *of* Φ *by setting, for any subset* A *of* $\{1, \ldots, n\}$:

$$s_\Phi(A) = \{j \in \{1, \ldots, n\} : \exists i \in A \quad u_{ij} \neq 0\}.$$

Then there exists partitions $\{\sigma_1, \ldots, \sigma_p\}$ *and* $\{\psi_1, \ldots, \psi_p\}$ *of* $\{1, \ldots, n\}$ *such that for all* $i \in \{1, \ldots, p\}$ *we have* $s_\Phi(\sigma_i) = \psi_i$, $s_{\Phi^{-1}}(\psi_i) = \sigma_i$ *and* $|\sigma_i| = |\psi_i|$. *Moreover, if* $A \subseteq \{1, \ldots, n\}$ *satisfies* $s_{\Phi^{-1}} \circ s_\Phi(A) = A$, *then:*

$$A = \bigcup \{\sigma_i : i \in \{1, \ldots, p\} \wedge \sigma_i \subseteq A\}.$$

Proof. Since Φ is an isomorphism such that $\widehat{\Phi}(0) = 0$, we conclude by Theorem 3.18 of [3] that there exists a unitary $U = [u_{ij}]_{1\leqslant i,j\leqslant n} \in M_{n\times n}(\mathbb{C})$ such that equality (3.1) holds for $i \in \{1, \ldots, n\}$. Note that by definition $s_\Phi(A) = \bigcup_{i\in A} s_\Phi(\{i\})$. Moreover, for the same reason, there exists $V = [v_{ij}]_{1\leqslant i,j\leqslant n} \in M_{n\times n}(\mathbb{C})$ such that $\Phi^{-1}(W_i^g) = \sum_{j=1}^{n} v_{ij} W_j^f$ and it is immediate that $V = U^*$ i.e.:

$$\Phi^{-1}(W_i^g) = \sum \overline{u}_{ji} W_j^f.$$

This implies that for $A \subseteq \{1, \ldots, n\}$ we have:

$$\begin{aligned} s_{\Phi^{-1}}(A) &= \{j \in \{1, \ldots, n\} : \exists i \in A \quad v_{ij} \neq 0\} \quad \text{(by definition)}, \\ &= \{j \in \{1, \ldots, n\} : \exists i \in A \quad u_{ji} \neq 0\} \quad \text{(since } V = U^*). \end{aligned}$$

In particular:

$$k \in s_\Phi(\{i\}) \iff i \in s_{\Phi^{-1}}(\{k\}).$$

Thus $i \in s_{\Phi^{-1}} \circ s_\Phi(\{i\})$.

We now observe that given any partition $\{\sigma_1,\ldots,\sigma_n\}$, if we set $\psi_i = s_\Phi(\sigma_i)$ for all $i \in \{1,\ldots,n\}$, then $|\psi_i| \geqslant |\sigma_i|$ for all $i \in \{1,\ldots,n\}$. Indeed, fix $i \in \{1,\ldots,n\}$. Let $j \in \sigma_i$. By definition:

$$\Phi(W_j^f) = \sum_{k\in\psi_i} u_{jk} W_k^g.$$

Since Φ is injective and $\{W_1^f,\ldots,W_n^f\}$ and $\{W_1^g,\ldots,W_n^g\}$ are linearly independent sets, we have:

$$|\sigma_i| = \dim\operatorname{span}\{W_j : j\in\sigma_i\} = \dim\operatorname{span}\{\Phi(W_j^f) : j\in\sigma_i\} \leqslant \dim\operatorname{span}\{W_k^g : k\in\psi_i\} = |\psi_i|.$$

If, moreover, $\sigma_i = s_{\Phi^{-1}}(\psi_i)$ for all $i \in \{1,\ldots,n\}$, then one gets $|\sigma_i| = |\psi_i|$ for $i \in \{1,\ldots,n\}$.

Now, we turn to the construction of a partition $\{\sigma_1,\ldots,\sigma_n\}$ of $\{1,\ldots,n\}$ such that $s_{\Phi^{-1}} \circ s_\Phi(\sigma_i) = \sigma_i$ for all $i \in \{1,\ldots,n\}$. Let $i \in \{1,\ldots,n\}$. Set $A_0 = \{i\}$ and $A_{m+1} = s_{\Phi^{-1}} \circ s_\Phi(A_m)$ for all $m \in \mathbb{N}$. Let $k \in A_m$ for some $m \in \mathbb{N}$. Then there exists $j \in s_\Phi(A_m)$ such that $u_{kj} \neq 0$. Since $j \in s_\Phi(\{k\})$ we have $k \in s_{\Phi^{-1}}(\{j\}) \subseteq s_{\Phi^{-1}} \circ s_\Phi(A_m)$. Hence $k \in A_{m+1}$. So $(A_m)_{m\in\mathbb{N}}$ is a sequence of subsets of $\{1,\ldots,n\}$ increasing for the inclusion. Since $\{1,\ldots,n\}$ is finite, there exists $N \in \mathbb{N}$ such that $A_N = A_{N+1}$. We define $\operatorname{cl}(i) = A_N$.

We thus construct subsets $\operatorname{cl}(1),\ldots,\operatorname{cl}(n)$ of $\{1,\ldots,n\}$ such that $i \in \operatorname{cl}(i)$ for all $i \in \{1,\ldots,n\}$, so $\bigcup_{i\in\{1,\ldots,n\}} \operatorname{cl}(i) = \{1,\ldots,n\}$. To show $\{\operatorname{cl}(1),\ldots,\operatorname{cl}(n)\}$ is a partition, it is thus sufficient to show that if, for some $i,j \in \{1,\ldots,n\}$ $\operatorname{cl}(i) \cap \operatorname{cl}(j) \neq \emptyset$ then $\operatorname{cl}(i) = \operatorname{cl}(j)$.

To do so, let us assume that $k \in \operatorname{cl}(i)$. Since $s_{\Phi^{-1}} \circ s_\Phi(\operatorname{cl}(i)) = \operatorname{cl}(i)$ by definition, we conclude that $\operatorname{cl}(k) \subseteq \operatorname{cl}(i)$. On the other hand, by construction, there exists $j_1,\ldots,j_q$ (for $q \leqslant n$) such that $j_1 = i$, $j_q = k$ and $j_{m+1} \in s_{\Phi^{-1}} \circ s_\Phi(\{j_m\})$. Fix $m \in \{1,\ldots,q\}$. Since $j_{m+1} \in s_{\Phi^{-1}} \circ s_\Phi(\{j_m\})$ there exists $r_m \in s_\Phi(\{j_m\})$ such that $j_{m+1} \in s_{\Phi^{-1}}(\{r_m\})$ so $r_m \in s_\Phi(\{j_{m+1}\})$. Since $r_m \in s_\Phi(\{j_m\})$ we have $j_m \in s_{\Phi^{-1}}(\{r_m\})$ and therefore $j_m \in s_{\Phi^{-1}} \circ s_\Phi(j_{m+1})$. Hence, $i \in \operatorname{cl}(k)$. Therefore, $\operatorname{cl}(k) = \operatorname{cl}(i)$.

Assume now that $\operatorname{cl}(i) \cap \operatorname{cl}(j) \neq \emptyset$ for some $i,j \in \{1,\ldots,n\}$. Let $k \in \operatorname{cl}(i) \cap \operatorname{cl}(j)$. We have shown that $\operatorname{cl}(i) = \operatorname{cl}(k) = \operatorname{cl}(j)$. Therefore, $\{\operatorname{cl}(1),\ldots,\operatorname{cl}(n)\}$ is a partition of $\{1,\ldots,n\}$ such that $s_{\Phi^{-1}} \circ s_\Phi(\operatorname{cl}(i)) = \operatorname{cl}(i)$ for $i = 1,\ldots,n$. We rewrite this partition as $\{\sigma_1,\ldots,\sigma_p\}$ (the order is unimportant). Setting $\psi_i = s_\Phi(\sigma_i)$ for $i = 1,\ldots,p$, we have found a partition satisfying our theorem.

Note, at last, that if $A \subseteq \{1,\ldots,n\}$ such that $s_{\Phi^{-1}} \circ s_\Phi(A) = A$ then for $i \in A$ then

$$s_{\Phi^{-1}} \circ s_\Phi(i) \subseteq s_{\Phi^{-1}} \circ s_\Phi(A) = A$$

and thus $\operatorname{cl}(i) \subseteq A$, as desired. ∎

We now show that the permutation-rescaling equivalence on symbols corresponds precisely to isomorphism of polynomial aspherical noncommutative domain algebras. To this end, we shall recall the following observations. Let $f = \sum_{\alpha} a_\alpha X_\alpha$ be a regular positive n-free formal power series.

(i) There exists positive real numbers b_α for all $\alpha \in \mathbb{F}_n^+$ such that W_j maps the canonical basis vector δ_α at $\alpha \in \mathbf{F}_n^+$ to $\sqrt{\frac{b_\alpha}{b_{g_j\alpha}}}\delta_{g_j\alpha}$ for $j = 1, \ldots, n$ [7].

(ii) The correspondence between the coefficients $(a_\alpha)_{\alpha\in\mathbb{F}_n^+}$ and $(b_\alpha)_{\alpha\in\mathbb{F}_n^+}$ is bijective [7].

(iii) For any $\alpha \in \mathbb{F}_n^+$ we have $\|W_\alpha^f\| = \frac{1}{b_\alpha^f}$ [7].

(iv) The Pythagorean identity holds for the weighted shifts W_α^f in the following sense:

$$\Big\| \sum_{j=1}^{k} c_j W_{\alpha_j}^f \Big\|^2 = \sum_{j-1}^{n} |c_j|^2 \|W_{\alpha_j}^f\|^2$$

where $c_1, \ldots, c_k$ are arbitrary complex numbers and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_n^+$ are all words *of the same length* [3].

We now can show:

THEOREM 3.2. *Let f, g be two regular, positive n-free polynomials. There exists an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ such that $\widehat{\Phi}(0) = 0$ if and only if f and g are permutation-rescaling equivalent.*

Proof. Write $f = \sum a_\alpha^f X_\alpha$ and $g = \sum a_\alpha^g X_\alpha$. Up to rescaling (see Lemma 4.4 of [3]), we assume $a_\alpha^f = a_\alpha^g = 1$ for all words α of length 1. We first assume that there exists an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ such that $\widehat{\Phi}(0) = 0$. By Proposition 3.1, we can find a partition $\{\sigma_1, \ldots, \sigma_p\}$ of $\{1, \ldots, n\}$ such that $s_{\Phi^{-1}} \circ s_\Phi(\sigma_i) = \sigma_i$ for all $i \in \{1, \ldots, p\}$, where s_Φ (respectively $s_{\Phi^{-1}}$) is the support function for Φ (respectively Φ^{-1}). Up to permutation of the free variables in g we may assume that $s_\Phi(\sigma_i) = \sigma_i$ for $i \in \{1, \ldots, p\}$.

Let $l \in \mathbb{N}$ with $l > 1$ be given. The finite set $\{b_\alpha^f, b_\alpha^g : \alpha \in \mathbb{F}_n^+ \wedge |\alpha| = l\}$ of real numbers has a smallest number, which we note b_ω^f (if the minimum is reached for a coefficient for g instead, we flip the notations for f and g for this case), with $\omega \in \mathbb{F}_n^+$ and $|\omega| = l$. We write $\omega = g_{i_1} \cdots g_{i_l}$ where $g_1, \ldots, g_n$ are the canonical generators of $\mathbb{F}_n^+$. Now, since Φ is an isometry and is implemented on the generators $W_1^f, \ldots, W_n^f$ by a scalar unitary $[u_{ij}]_{1\leqslant i,j\leqslant n}$ acting on $(W_1^g, \ldots, W_n^g)$ (see Proposition 3.1 for notations), we have:

$$\frac{1}{b_\omega^f} = \|W_\omega^f\|^2 = \|\Phi(W_\omega^f)\|^2 = \Big\| \prod_{k=1}^{l} \Phi(W_{i_k}^f) \Big\|^2 = \Big\| \prod_{k=1}^{l} \sum_{r_k \in s_\Phi(\{i_k\})} u_{i_k r_k} W_{r_k}^g \Big\|^2$$

$$= \Big\| \sum_{r_1 \in s_\Phi(\{i_1\})} \cdots \sum_{r_l \in s_\Phi(\{i_l\})} u_{i_1 r_1} u_{i_2 r_2} \cdots u_{i_l r_l} W_{r_1}^g \cdots W_{r_l}^g \Big\|^2$$

$$= \sum_{r_1 \in s_\Phi(\{i_1\})} \cdots \sum_{r_l \in s_\Phi(\{i_l\})} |u_{i_1 r_1}|^2 |u_{i_2 r_2}|^2 \cdots |u_{i_l r_l}|^2 \frac{1}{b^g_{g_{r_1} \cdots g_{r_l}}}.$$

Thus, since U is unitary, we conclude that $\frac{1}{b^f_\omega}$ is a convex combination of elements in $\left\{ \frac{1}{b^f_\alpha}, \frac{1}{b^g_\alpha} : \alpha \in \mathbb{F}^+_n \wedge |\alpha| = l \right\}$, though it is its maximum. This is only possible if:

$$\forall r_1 \in s_\Phi(\{i_1\}) \quad \cdots \quad \forall r_l \in s_\Phi(\{i_1\}) \ \frac{1}{b^g_{g_{r_1} \cdots g_{r_l}}} = \frac{1}{b^f_\omega}.$$

Let us adopt the following notation. A word α will be of type $\sigma_{i_1} \cdots \sigma_{i_l}$ if $\alpha = g_{r_1} \cdots g_{r_n}$ for $r_j \in \sigma_j$ with $j = 1, \ldots, l$. Let $\sigma_{v_1} \cdots \sigma_{v_l}$ be the type of the ω as above. By repeating the above argument, we can thus show that $b^f_\alpha = b^g_\alpha = b^f_\omega$ for all α of type ω.

We can now repeat the proof above by picking the minimum of

$$\left\{ \frac{1}{b^f_\alpha}, \frac{1}{b^g_\alpha} : \alpha \in \mathbb{F}^+_n \wedge |\alpha| = l \wedge \alpha \text{ is not of the same type as } \omega \right\}$$

and so forth until we exhaust the set $\left\{ \frac{1}{b^f_\alpha}, \frac{1}{b^g_\alpha} : \alpha \in \mathbb{F}^+_n \wedge |\alpha| = l \right\}$ to show that $b^f_\alpha = b^g_\alpha = b^f_\omega$ for all $\alpha \in \mathbb{F}^+_n$ with $|\alpha| = l$. This completes our proof. ∎

COROLLARY 3.3. *Let f, g be two regular positive n-free polynomials, and assume f is aspherical. Then $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic if and only if f and g are scale-permutation equivalent.*

This result follows from Theorem 2.15 and Theorem 3.2.

We can summarize the current understanding of classification for noncommutative domain algebra:

THEOREM 3.4. *Let f be a regular, positive n-free polynomial. Then:*

(i) *If $f = \sum c_i X_i$ for some $c_1, \ldots, c_n \in (0, \infty)$ then $\mathcal{A}(\mathcal{D}_f)$ is isomorphic to the disk algebra $\mathcal{A}_n$.*

(ii) *If f is aspherical, then $\mathcal{A}(\mathcal{D}_g)$ is isomorphic to $\mathcal{A}(\mathcal{D}_f)$ if and only if $f = g$ after rescaling/permutation of the free variables of g.*

This leaves the matter of classifying $\mathcal{A}(\mathcal{D}_f)$ when f is spherical, i.e. $\mathcal{D}_f(\mathbb{C})$ is the closed unit ball of $\mathbb{C}^n$, yet f is not of degree 1.

We can now use Theorem 3.2 to prove the last statements of Example 2.9. Indeed, $M = \left[\begin{smallmatrix} 2 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ implements the isomorphism of $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ since we only rescale X_1 by 2. On the other hand, the second statement fo Theorem 2.16 implies that, since f is aspherical, if $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_h)$ are isomorphic, then $\mathcal{D}_f(\mathbb{C})$ and $\mathcal{D}_h(\mathbb{C})$ are linearly isomorphic. However, this is not the case for f and h of Example 2.9. In fact, f and h are not obtained from each other by rescaling or permuting the indeterminates, so by Theorem 3.2, the domains $\mathcal{D}_f(\mathbb{C})$ and $\mathcal{D}_h(\mathbb{C})$ are not linearly isomorphic, so that the associated noncommutative domain algebras are not isomorphic.

4. CARTAN'S LEMMA

In Theorem 1.4 of [8] and Theorem 4.5 of [9], Popescu establishes a generalization of Cartan's lemma, first in the context of the unit ball of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, then for a large class of noncommutative domains that includes the domains considered in this paper. We propose to illustrate in this section that our methods, as implemented in this paper and in [3], can be used to obtain a simpler proof of these results. We refer to [7], [8], [9] for definitions and the general theory of holomorphic functions in the context of noncommutative domains. We shall only use the following special case of holomorphic functions:

DEFINITION 4.1. Let f be a positive, regular n-free formal power series. A holomorphic map F with domain and codomain $\mathcal{D}_f$ is the given of a family of complex coefficients $(c_\alpha)_{\alpha\in\mathbb{F}_n^+}$ such that, for any Hilbert space $\mathcal{H}$, the function:

$$F_{\mathcal{H}} : (T_1, \ldots, T_n) \in \text{interior}\, \mathcal{D}_f(\mathcal{H}) \longmapsto \Big(\sum_{\alpha\in\mathbb{F}_n^+} c_{1,\alpha} T_\alpha, \ldots, \sum_{\alpha\in\mathbb{F}_n^+} c_{n,\alpha} T_\alpha \Big)$$

is well-defined and its range is a subset of the interior of $\mathcal{D}_f(\mathcal{H})$.

REMARK 4.2. The domain and codomain of a holomorphic function, in this context, is the noncommutative domain $\mathcal{D}_f$ which, itself, is not a set, but a map from separable Hilbert spaces to subsets of the algebra of linear bounded operators on the given Hilbert space. Note moreover that the domain and codomain of the maps induced on various noncommutative domains by holomorphic maps are the *interior* of the noncommutative domains.

DEFINITION 4.3. Let f be a positive, regular, n-free formal power series. A biholomorphic map F on $\mathcal{D}_f$ is a holomorphic map from $\mathcal{D}_f$ to $\mathcal{D}_f$ such that there exists a holomorphic map G from $\mathcal{D}_f$ to $\mathcal{D}_f$ such that $F \circ G$ and $G \circ F$ both induce the identity on $\mathcal{D}_F(\mathcal{H})$ for all Hilbert spaces $\mathcal{H}$.

REMARK 4.4. By definition, the map induced by a holomorphic map on a specific noncommutative domain is a holomorphic map on the interior of this domain.

THEOREM 4.5 (Cartan's lemma). *$\mathcal{D}_f(\mathbb{C})$ and $\mathcal{D}_h(\mathbb{C})$ are linearly isomorphic. However, this is not the case for f and h of Example* 2.9. *The results of Section* 3 *also prove that $\mathcal{D}_f(\mathbb{C})$ and $\mathcal{D}_h(\mathbb{C})$ are not linearly isomorphic since we cannot obtain h by rescaling and permuting the symbols of f.*

Let f be a regular positive n-free formal power series. Let F be a biholomorphic map of D_f such that $F(0) = 0$. Then F is linear.

Proof. By definition, there exists complex coefficients $(c_{j,\alpha})_{j\in\{1,\dots,n\},\alpha\in\mathbb{F}_n^+}$ such that for all Hilbert space $\mathcal{H}$ and all $(T_1,\dots,T_n)\in$ interior $\mathcal{D}_f(\mathcal{H})$ we have:

$$F_{\mathcal{H}}(T_1,\dots,T_n)=\Big(\sum_{\alpha\in\mathbb{F}_n^+}c_{1,\alpha}T_\alpha,\dots,\sum_{\alpha\in\mathbb{F}_n^+}c_{n,\alpha}T_\alpha\Big)\in \text{interior }\mathcal{D}_f(\mathcal{H}).$$

When $\mathcal{H}=\mathbb{C}^k$, we will write F_k for the biholomorphic map induced by F on $\mathbb{D}_f^k$ for $k\in\mathbb{N}$, $k>0$.

Since this proof is essentially the same as Theorem 3.18 of [3], we shall only deal with the case where $\alpha\in\mathbb{F}_n^+$, $|\alpha|\leqslant 2$ and $n=2$.

By definition, F induces a biholomorphic map F_1 on $\mathbb{D}_f^1$. This map fixes 0 and $\mathbb{D}_f^1$ is a circular domain (even Reinhardt) in $\mathbb{C}^n$, Cartan's lemma [6] implies that F_1 is linear.

Now, the j^{th} coordinate of $F_1(z_1,z_2)$ for an arbitrary $(z_1,z_2)\in\mathbb{D}_f^1$ is:

$$c_{j,g_1}z_1+c_{j,g_2}z_2+c_{j,g_1g_1}z_1^2+c_{j,g_2g_2}z_2^2+(c_{j,g_1g_2}+c_{j,g_2g_1})z_1z_2+\cdots$$

so the linearity of F_1 implies that:

$$c_{j,g_1g_1}z_1^2+c_{j,g_2g_2}z_2^2+(c_{j,g_1g_2}+c_{j,g_2g_1})z_1z_2+\cdots=0$$

which in turns shows that $c_{j,g_1g_1}=c_{j,g_2g_2}=c_{j,g_1g_2}+c_{j,g_2g_1}=0$. To show that $c_{j,g_1g_2}=c_{j,g_2g_1}=0$ we go to higher dimensions. Again by definition, F induces a biholomorphic map on $\mathbb{D}_f^2$. The later domain is circular (not Reinhardt in general), and thus again by Cartan's lemma [6], since $F_2(0)=0$ we conclude that F_2 is linear.

Now, a quick computation shows that the $(2,1)$ component of the j^{th} coordinate of $F_2(M,N)$ where $M=\begin{bmatrix}\lambda_1&\lambda_2\\\lambda_3&\lambda_4\end{bmatrix}$ and $N=\begin{bmatrix}\lambda_5&\lambda_6\\\lambda_7&\lambda_8\end{bmatrix}$ are 2×2 complex matrices in $\mathbb{D}_f^2$ is given by:

$$\begin{aligned}c_{j,g_1}\lambda_2+c_{j,g_2}\lambda_6+c_{j,g_1g_2}(\lambda_1\lambda_6+\lambda_2\lambda_8)+c_{j,g_2g_1}(\lambda_2\lambda_5+\lambda_4\lambda_6)\\+\text{ terms of higher degrees in }\lambda_1,\dots,\lambda_8\end{aligned}$$

so linearity of F_2 implies that $c_{j,g_1g_2}=c_{j,g_2g_1}=0$.

The proof for higher terms is similar and undertaken in Theorem 3.18 of [3]. ∎

We expect that the same method of reduction to finite dimension can yield other generalizations of results from the study of domains in complex analysis to the framework of Popescu's noncommutative domains.

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