J. OPERATOR THEORY
71:1(2014), 63–83
doi: 10.7900/jot.2011oct27.1994

ADDITIVE PRESERVERS OF THE ASCENT, DESCENT AND RELATED SUBSETS

MOSTAFA MBEKHTA, VLADIMIR MÜLLER, and MOURAD OUDGHIRI

Communicated by Şerban Strătilă

ABSTRACT. In this paper we completely describe additive surjective continuous maps in the algebra of all bounded linear operators acting on a complex separable infinite-dimensional Hilbert space, preserving the operators of finite ascent, the operators of finite descent, Drazin invertible operators, upper semi-Browder operators, lower semi-Browder operators or Browder operators in both directions.

KEYWORDS: *Additive preservers, ascent, descent, Browder.*

MSC (2010): 47B49, 47L99, 47A55, 47B37.

1. INTRODUCTION

Let X be a complex Banach space and H a separable complex infinite dimensional Hilbert space. The algebra of all bounded linear operators acting on X is denoted by $\mathcal{L}(X)$.

In the last decades there has been a remarkable interest in the so-called linear preserver problems which concern the characterization of linear, or additive, maps on Banach algebras that leave invariant a certain subset. One of the most famous problems in this direction is Kaplansky's problem asking whether bijective unital linear maps, between semi-simple Banach algebras, preserving invertibility in both directions are Jordan isomorphisms, see [1], [2], [9], [10], [11], [18], [25], [27].

Most of the linear preserver problems were solved in the finite dimensional context, and extended later to the infinite dimensional one. We refer the interested reader to [2], [6], [26] for the invertibility preservers, and [23] for the idempotents preservers.

Recently, in [15], [16], [17], the authors studied linear maps on $\mathcal{L}(H)$ preserving generalized invertible operators, semi-Fredholm operators or Fredholm operators in both direction. Observe that the problem makes sense only in the infinite dimensional case. In fact, every complex matrix is Fredholm and generalized invertible, and consequently, every map preserves such subsets. Also, it should be mentioned that these subsets are invariant under finite rank perturbations. This constrains to search information on these maps in the Calkin algebra. More precisely, it is shown that such maps preserve the ideal of compact operators in both direction and their induced maps on the Calkin algebra are Jordan automorphism.

In the present paper, we consider another linear preserver problem that is trivial in the finite dimension case, but the related subsets are not stable under finite rank perturbations. In this new context, a new approach is developed and a complete description of additive preserver maps is provided.

For an operator $T \in \mathcal{L}(X)$, write T' for its adjoint, $\mathrm{N}(T)$ for its kernel and $\mathrm{R}(T)$ for its range. The *ascent* $\mathrm{a}(T)$ and *descent* $\mathrm{d}(T)$ of $T \in \mathcal{L}(X)$ are defined by

$$\mathrm{a}(T)=\min\{n\geqslant 0\colon \mathrm{N}(T^n)=\mathrm{N}(T^{n+1})\}, \quad \mathrm{d}(T)=\min\{n\geqslant 0\colon \mathrm{R}(T^n)=\mathrm{R}(T^{n+1})\},$$

where the minimum over the empty set is taken to be infinite, see [19], [28].

An operator $T \in \mathcal{L}(X)$ is said to have a *Drazin inverse*, or to be *Drazin invertible*, if there exists $S \in \mathcal{L}(X)$ and a non-negative integer n such that

$$STS = S, \quad TS = ST \quad \text{and} \quad T^{n+1}S = T^n. \tag{1.1}$$

Note that if T possesses a Drazin inverse, then it is unique and the smallest non-negative integer n in (1.1) is denoted by $\mathrm{i}(T)$. It is well known that T is Drazin invertible if and only if it has finite ascent and descent, and in this case $\mathrm{a}(T) = \mathrm{d}(T) = \mathrm{i}(T)$.

Recall also that an operator $T \in \mathcal{L}(X)$ is called *upper* (respectively *lower*) *semi-Fredholm* if $\mathrm{R}(T)$ is closed and $\dim \mathrm{N}(T)$ (respectively $\operatorname{codim} \mathrm{R}(T)$) is finite. The set of such operators is denoted by $\mathcal{F}_+(X)$ (respectively $\mathcal{F}_-(X)$). The class of *Fredholm* operators is defined by $\mathcal{F}(X) := \mathcal{F}_+(X) \cap \mathcal{F}_-(X)$. Let us introduce the following subsets:

(i) $\mathcal{A}(X) := \{T \in \mathcal{L}(X) : \mathrm{a}(T) < \infty\}$ the set of finite ascent operators,
(ii) $\mathcal{D}(X) := \{T \in \mathcal{L}(X) : \mathrm{d}(T) < \infty\}$ the set of finite descent operators,
(iii) $\mathcal{D}^r(X) := \mathcal{A}(X) \cap \mathcal{D}(X)$ the set of Drazin invertible operators,
(iv) $\mathcal{B}_+(X) := \mathcal{F}_+(X) \cap \mathcal{A}(X)$ the set of *upper semi-Browder* operators,
(v) $\mathcal{B}_-(X) := \mathcal{F}_-(X) \cap \mathcal{D}(X)$ the set of *lower semi-Browder* operators,
(vi) $\mathcal{B}(X) := \mathcal{B}_+(X) \cap \mathcal{B}_-(X)$ the set of *Browder* operators.

We refer to [19] for more information about semi-Fredholm, Fredholm, semi-Browder and Browder operators.

Let $\mathcal{S}$ denote any of the subsets (i)–(vi). A surjective additive map $\Phi : \mathcal{L}(X) \to \mathcal{L}(X)$ is said to *preserve* $\mathcal{S}$ in both directions if $T \in \mathcal{S} \Leftrightarrow \Phi(T) \in \mathcal{S}$.

The main results of this paper are the following two theorems, which characterize all surjective additive continuous maps that preserve the finiteness of ascent, finiteness of descent, upper semi-Browder operators, lower semi-Browder operators, Drazin invertible operators or Browder operators.

THEOREM A. *Let H be a separable infinite-dimensional Hilbert space, let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. Then the following assertions are equivalent:*

(i) *Φ preserves in both directions $\mathcal{A}$;*

(ii) *Φ preserves in both directions $\mathcal{D}$;*

(iii) *Φ preserves in both directions $\mathcal{B}_+$;*

(iv) *Φ preserves in both directions $\mathcal{B}_-$;*

(v) *there exists an invertible bounded linear, or conjugate linear, operator $A : H \to H$ and a non-zero complex number c such that $\Phi(S) = cASA^{-1}$ for all $S \in \mathcal{L}(H)$.*

THEOREM B. *Let H be a separable infinite-dimensional Hilbert space, let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. Then the following assertions are equivalent :*

(i) *Φ preserves in both directions $\mathcal{D}^r$;*

(ii) *Φ preserves in both directions $\mathcal{B}$;*

(iii) *there exists an invertible bounded linear, or conjugate linear, operator $A : H \to H$ and a non-zero complex number c such that either $\Phi(S) = cASA^{-1}$ for all $S \in \mathcal{L}(H)$, or $\Phi(S) = cAS^*A^{-1}$ for all $S \in \mathcal{L}(H)$.*

The paper is organized as follows. In Section 2 we establish some useful results on the perturbation of the ascent which are needed for the proof of our main results in Section 3. We give necessary and sufficient conditions for an operator $T \in \mathcal{L}(X)$ with finite ascent, $x \in X$ and $f \in X'$ when $T + x \otimes f$ has infinite ascent. It is shown also that for each non-zero $F \in \mathcal{L}(H)$, there exists an invertible operator T such that $\mathrm{a}(T + F) = \infty$, and when $\dim \mathrm{R}(T) \geqslant 2$, then we can assume in addition that $\mathrm{a}(T - F) = \infty$. We provide also an interesting characterization of upper semi-Browder operators via the ascent. Analogous results for the descent are also derived.

In Section 3 we prove the main results of the paper: Theorems A and B.

2. ASCENT AND RANK ONE PERTURBATION

Recall that the *hyper-range* and the *hyper-kernel* of an operator $T \in \mathcal{L}(X)$ are respectively the subspaces $\mathcal{R}^\infty(T) := \bigcap\limits_n \mathrm{R}(T^n)$ and $\mathcal{N}^\infty(T) := \bigcup\limits_n \mathrm{N}(T^n)$.

Let $z \in X$ and let f be in the topological dual space X' of X. We denote, as usual, by $z \otimes f$ the rank one operator given by $(z \otimes f)(x) = \langle x, f\rangle z$ for all $x \in X$. Note that every rank one operator in $\mathcal{L}(X)$ can be written in this form.

THEOREM 2.1. *Let $T \in \mathcal{L}(X)$ be an operator with finite ascent p. Then $T|\mathcal{R}^\infty(T)$ is bijective. Moreover, if $z \in X$ and $f \in X'$, then $T + z \otimes f$ has infinite ascent if and only if the following assertions hold:*

(i) *$z = a + z_0$ where $a \in \mathrm{N}(T^p)$ and $z_0 \in \mathcal{R}^\infty(T)$;*

(ii) *$\langle z_1, f\rangle = -1$ and $\langle z_i, f\rangle = 0$ for all $i \geqslant 2$, where $z_i = (T|\mathcal{R}^\infty(T))^{-i} z_0$;*

(iii) $\langle T^i a, f\rangle = 0$ *for all* $i \geqslant 0$.

Moreover, in this case $\{z_i\}_{i\geqslant 0}$ *is a linearly independent set.*

Before presenting the proof of Theorem 2.1, we establish the following two lemmas.

LEMMA 2.2. *Let* $T \in \mathcal{L}(X)$ *be an operator of finite ascent* p, $F = z \otimes f$ *where* $z \in X$ *and* $f \in X'$, *and let* n *be an integer such that* $p < n < \mathrm{a}(T+F)$. *Then there exist linearly independent vectors* x_i, $0 \leqslant i \leqslant n$, *and an integer* $j \leqslant p$ *such that* $(T+F)x_0 = 0$, $(T+F)x_i = x_{i-1}$ *for* $1 \leqslant i \leqslant n$ *and* $\langle x_i, f\rangle = \delta_{ij}$ *for* $0 \leqslant i \leqslant n$.

Proof. Note that since $\mathrm{a}(T+F) > n$, there exist linearly independent vectors u_i, $0 \leqslant i \leqslant n$, satisfying $(T+F)u_0 = 0$ and $(T+F)u_i = u_{i-1}$ for $1 \leqslant i \leqslant n$. Let j be the smallest integer such that $\langle u_j, f\rangle \neq 0$. Such a j exists and $j \leqslant p$, since otherwise

$$T^{p+1}u_p = T^p u_{p-1} = \cdots = Tu_0 = 0,$$

a contradiction with the assumption $\mathrm{a}(T) = p$. Let $c_i = \langle u_i, f\rangle$, $0 \leqslant i \leqslant n$. Without loss of generality we can suppose that $c_j = 1$. Set $\alpha_n = 1$. Consider the complex numbers $\alpha_{n-1}, \alpha_{n-2}, \ldots, \alpha_j$ defined inductively by

$$\alpha_{n-1} = -\alpha_n c_{j+1} \quad \cdots \quad \alpha_k = -\sum_{r=1}^{n-k} \alpha_{k+r}c_{j+r} \quad \cdots \quad \alpha_j = -\sum_{r=1}^{n-j} \alpha_{j+r}c_{j+r}.$$

This means that we have

$$\sum_{r=0}^{n-k} \alpha_{k+r}c_{j+r} = 0 \quad (j \leqslant k \leqslant n-1).$$

Let $x_n = u_n + \sum\limits_{s=j}^{n-1} \alpha_s u_s$ and

$$x_i = (T+F)^{n-i}x_n = u_i + \sum_{s=j}^{n-1} \alpha_s u_{s-n+i} \quad \text{for } 0 \leqslant i \leqslant n-1,$$

where we set formally $u_s = 0$ for $s < 0$. Clearly, the vectors x_i, $0 \leqslant i \leqslant n$, are linearly independent. We have $\langle x_i, f\rangle = 0$ for all $i < j$ and $\langle x_j, f\rangle = \langle u_j, f\rangle = 1$. For $j+1 \leqslant i \leqslant n$ we have

$$\begin{aligned}\langle x_i, f\rangle &= c_i + \sum_{s=j}^{n-1} \alpha_s \langle u_{s-n+i}, f\rangle = c_i + \sum_{s=j+n-i}^{n-1} \alpha_s c_{s-n+i} = \sum_{s=j+n-i}^{n} \alpha_s c_{s-n+i} \\ &= \sum_{r=0}^{i-j} \alpha_{r+j+n-i}c_{r+j} = \sum_{r=0}^{n-k} \alpha_{r+k}c_{r+j} = 0\end{aligned}$$

(for $r = s-j-n+i$ and $k = j+n-i$). This completes the proof. ∎

LEMMA 2.3. *Let* $T \in \mathcal{L}(X)$ *be an operator with finite ascent* p. *If* $x \in X$ *satisfies* $Tx \in \mathcal{R}^\infty(T)$ *and* $x \in \mathrm{R}(T^p)$, *then* $x \in \mathcal{R}^\infty(T)$.

Proof. Let $u \in X$ satisfy $T^p u = x$. Let n be an integer such that $n > p$. Since $Tx \in \mathrm{R}(T^n)$, there exists $y \in X$ with $T^n y = Tx$, and so $T^{p+1}(u - T^{n-p-1}y) = 0$. This implies that $x - T^{n-1}y = T^p(u - T^{n-p-1}y) = 0$ because $\mathrm{a}(T) = p$, and consequently $x = T^{n-1}y \in \mathrm{R}(T^{n-1})$. The integer n was arbitrary, therefore $x \in \mathcal{R}^\infty(T)$. ∎

Proof of Theorem 2.1. Since $\mathrm{a}(T) = p < \infty$, we have $\mathrm{N}(T) \cap \mathrm{R}(T^p) = \{0\}$. Hence, the restriction of T to $\mathcal{R}^\infty(T)$ is injective. We show that $T\mathcal{R}^\infty(T) = \mathcal{R}^\infty(T)$. Let $x \in \mathcal{R}^\infty(T)$. Then there exists a vector $u \in X$ such that $T^{p+1}u = x$. Let $v = T^p u$. So $Tv = x$. By Lemma 2.3, $v \in \mathcal{R}^\infty(T)$. Hence, $T|\mathcal{R}^\infty(T)$ is invertible.

Suppose that $z \in X$ and $f \in X'$ satisfy (i)–(iii). Write $S = T + z \otimes f$. For $i \in \mathbb{N}$, let $z_i = (T|\mathcal{R}^\infty(T))^{-i} z_0$. Then $Tz_i = z_{i-1}$ for all $i \geqslant 1$.

We have $Sz_1 = Tz_1 + (z \otimes f)z_1 = z_0 - z = -a$. By induction we can prove easily that $S^{i+1}z_1 = -S^i a = -T^i a$. In particular, $S^{p+1}z_1 = -T^p a = 0$. So $z_1 \in \mathrm{N}(S^{p+1})$. For $i \geqslant 2$, we have $Sz_i = Tz_i + (z \otimes f)z_i = z_{i-1}$. So $\mathrm{a}(S) = \infty$.

Suppose that $S = T + z \otimes f$ has infinite ascent, and consider an arbitrary integer $k > p$. For $n = 2p + k$, Lemma 2.2 ensures the existence of a sequence $\{x_i\}_{i=0}^n$ of linearly independent vectors and an integer $j \leqslant p$ such that $(T + z \otimes f)x_0 = 0$, $(T + z \otimes f)x_i = x_{i-1}$ for $1 \leqslant i \leqslant n$, and $\langle x_i, f \rangle = \delta_{ij}$ for $0 \leqslant i \leqslant n$. Therefore, $Tx_i = x_{i-1}$ for all $i \neq j$ and $1 \leqslant i \leqslant n$, and

$$Tx_j + z = Sx_j = x_{j-1} \in \mathrm{N}(T^j), \tag{2.1}$$

where we set formally $x_i = 0$ for $i < 0$. We have $Tx_j = T^{n-j+1}x_n \in \mathrm{R}(T^{n-j+1})$. Hence $z = x_{j-1} - Tx_j \in \mathrm{N}(T^p) + \mathrm{R}(T^{n-j+1}) \subset \mathrm{N}(T^p) + \mathrm{R}(T^p)$. Since $\mathrm{N}(T^p) \cap \mathrm{R}(T^p) = \{0\}$, this decomposition is unique and independent of n. Hence $z = a + z_0$ where $a = x_{j-1} \in \mathrm{N}(T^p)$ and $z_0 = -Tx_j \in \mathcal{R}^\infty(T)$.

For $i \geqslant 0$ we have $\langle T^i a, f \rangle = \langle T^i x_{j-1}, f \rangle = \langle x_{j-i-1}, f \rangle = 0$. Since $Tx_j = -z_0 \in \mathcal{R}^\infty(T)$ and $x_j \in \mathrm{R}(T^p)$, by Lemma 2.3 we have $x_j \in \mathcal{R}^\infty(T)$. Similarly, $x_{k+p}, \ldots, x_j \in \mathcal{R}^\infty(T)$. Hence $\langle (T|\mathcal{R}^\infty(T))^{-1} z_0, f \rangle = \langle -x_j, f \rangle = -1$. Similarly $\langle (T|\mathcal{R}^\infty(T))^{-k} z_0, f \rangle = \langle -x_{k+j-1}, f \rangle = 0$.

Suppose on the contrary that the vectors z_i, $i \geqslant 1$, are linearly dependent. Let $m \in \mathbb{N}$ and $\sum\limits_{i=1}^m \beta_i z_i = 0$ for some nontrivial complex coefficients β_i. Let s be the smallest integer such that $\beta_s \neq 0$. Then

$$0 = T^{s-1} \sum_{i=1}^m \beta_i z_i = \sum_{i=s}^m \beta_i z_{i-s+1}.$$

So $\beta_s = \left\langle \sum\limits_{i=s}^m \beta_i z_{i-s+1}, f \right\rangle = 0$, a contradiction. Finally, since $T_{|\mathcal{R}^\infty(T)}$ is invertible, we get that $\{z_i\}_{i \geqslant 0} = T_{|\mathcal{R}^\infty(T)} \{z_i\}_{i \geqslant 1}$ is linearly independent. ∎

For subspaces M and M' of X we write $M \overset{\mathrm{e}}{=} M'$ if there exist finite dimensional subspaces L and L' such that $M \subseteq M' + L'$ and $M' \subseteq M + L$. One can easily verify that if $T \in \mathcal{L}(X)$ is surjective and $M \overset{\mathrm{e}}{=} X$, then $TM \overset{\mathrm{e}}{=} X$.

LEMMA 2.4. *Let $T \in B(X)$ be an operator with finite descent p, and let $S = T + z \otimes f$ where $z \in X$ and $f \in X'$. If $\mathrm{d}(S) = \infty$ then $\mathrm{a}(S') = \infty$.*

Proof. Suppose on the contrary that S' has finite ascent q. It follows in particular that $\overline{\mathrm{R}(S^{q+1})} = \overline{\mathrm{R}(S^q)}$. Let $M = \mathrm{R}(T^p)$ and $M' = \mathrm{R}(S^p)$. Then $TM = M$, and since $S^p - T^p$ is a finite rank operator, we obtain that $M \overset{\mathrm{e}}{=} M'$. Hence, if we let $Y = \mathrm{N}(T - S) = \mathrm{N}(f)$, we get that $M \overset{\mathrm{e}}{=} T(M \cap M' \cap Y) = S(M \cap M' \cap Y)$ because $M \overset{\mathrm{e}}{=} M \cap M' \cap Y$ and $TM = M$. Define a new norm on M' by $\|x\|_1 = \inf\{\|y\| : S^p y = x\}$. Clearly, equipped with this norm, M' is a Banach space isometrically isomorphic to $X/\mathrm{N}(S^p)$ and $S|M'$ is a lower semi-Fredholm operator. So for each n, $\mathrm{R}(S^n|M') = \mathrm{R}(S^{n+p})$ is closed in $(M', \|\cdot\|_1)$. Therefore, $S^{-p}S^n M' = \mathrm{R}(S^n) + \mathrm{N}(S^p)$ is closed in X. Finally, since $\mathrm{R}(S^q) \subset \overline{\mathrm{R}(S^{q+1})}$, we obtain that $\mathrm{R}(S^q) \subset \mathrm{R}(S^{q+1}) + \mathrm{N}(S^p)$, and so $\mathrm{R}(S^{q+p}) \subset \mathrm{R}(S^{q+p+1})$, a contradiction with the assumption that $\mathrm{d}(S) = \infty$. ∎

Analogously, for the descent, we have the following characterization:

THEOREM 2.5. *Let $T \in \mathcal{L}(X)$ be an operator with finite descent $\mathrm{d}(T) = q < \infty$, let $z \in X$ and $f \in X'$. Then $T + z \otimes f$ has infinite descent if and only if there exists a sequence of linearly independent forms $\{f_i\}_{i \geqslant 0}$ in X' such that :*

(i) *$f = g + f_0$ where $g \in \mathrm{N}(T'^q)$ and $T' f_{i+1} = f_i$ for all $i \geqslant 0$;*
(ii) *$\langle z, f_1 \rangle = -1$ and $\langle z, f_i \rangle = 0$ for all $i \geqslant 2$;*
(iii) *$\langle T^i z, g \rangle = 0$ for all $i \geqslant 0$.*

Proof. Suppose that $\mathrm{d}(T + z \otimes f) = \infty$. Then $\mathrm{a}(T') \leqslant q < \infty$ and, by Lemma 2.4, $\mathrm{a}(T' + f \otimes Jz) = \infty$ where $J : X \to X''$ denotes the canonical embedding. Hence, from Theorem 2.1, it follows that there exist a linearly independent sequence $\{f_i\}_{i \geqslant 1}$ in X' and $g \in \mathrm{N}(T'^q)$ such that $f = g + f_0$, $T' f_{i+1} = f_i$ for all $i \geqslant 0$. Furthermore, the assertions (ii) and (iii) in Theorem 2.1 imply that $\langle z, f_1 \rangle = \langle f_1, Jz \rangle = -1$, $\langle z, f_i \rangle = 0$ for all $i \geqslant 2$ and $\langle T^i z, g \rangle = \langle T'^i g, Jz \rangle = 0$ for all $i \geqslant 0$.

Conversely, let $z \in X$ and $f \in X'$ satisfy (i)–(iii). Then f and Jz satisfy the conditions of Theorem 2.1 for T'. Hence $\mathrm{a}(T' + f \otimes Jz) = \infty$, and so $\mathrm{d}(T + z \otimes f) = \infty$. ∎

As an immediate consequence of Theorems 2.1 and 2.5, we derive the following corollary.

COROLLARY 2.6. *Let T be a bounded operator on X. Then:*

(i) *if $\mathrm{a}(T)$ and $\dim \mathcal{R}^\infty(T)$ are both finite, then so is $\mathrm{a}(T + F)$ for all rank one operators $F \in \mathcal{L}(X)$;*

(ii) *if* $\mathrm{d}(T)$ *and* $\dim \mathcal{R}^\infty(T')$ *are both finite, then so is* $\mathrm{d}(T+F)$ *for all rank one operators* $F \in \mathcal{L}(X)$.

Notice that for $T \in \mathcal{L}(X)$ of finite ascent and $F \in \mathcal{L}(X)$ of rank one, Theorem 2.1 ensures that either $T+F$ or $T-F$ has finite ascent. In the following proposition we give an estimate of the ascent of such perturbations.

PROPOSITION 2.7. *Let* $T \in \mathcal{L}(X)$ *be an operator of ascent* p. *Then for every rank one operator* $F \in \mathcal{L}(X)$, *either* $\mathrm{a}(T+F) \leqslant 2p$ *or* $\mathrm{a}(T-F) \leqslant 2p$.

Proof. Suppose on the contrary that $\mathrm{a}(T+F) \geqslant 2p+1$ and $\mathrm{a}(T-F) \geqslant 2p+1$, and write $F = z \otimes f$ with $x \in X$, $f \in X'$. Then, by Lemma 2.2, there exist two sequences $\{x_k\}_{k=0}^{2p}$ and $\{y_k\}_{k=0}^{2p}$ of a linearly independent vectors and an integers $i, j \leqslant p$ such that

$$\begin{cases} (T+F)x_0 = (T-F)y_0 = 0; \\ (T+F)x_k = x_{k-1} \text{ and } (T-F)y_k = y_{k-1} \text{ for } 1 \leqslant k \leqslant 2p; \\ \langle x_k, f\rangle = \delta_{ki} \text{ and } \langle y_k, f\rangle = \delta_{kj} \text{ for } 0 \leqslant k \leqslant 2p. \end{cases}$$

We may assume that $j \leqslant i$. Let $z = x_{2p} + y_{2p-i+j}$. Then it follows that

$$\begin{aligned} T^{2p-i}z &= T^{2p-i}x_{2p} + T^{2p-i}y_{2p-i+j} \\ &= (T+F)^{2p-i}x_{2p} + (T-F)^{2p-i}y_{2p-i+j} = x_i + y_j. \end{aligned}$$

Now, $T^{2p-i+1}z = Tx_i + Ty_j = (T+F)x_i + (T-F)y_j = x_{i-1} + y_{j-1}$, and hence $T^{2p}z = x_0 + y_{j-i}$, where we set $y_s = 0$ for $s < 0$. Thus, $T^{2p+1}z = 0$, and since $\mathrm{a}(T) = p$, we obtain that $T^{2p-i}z = x_i + y_j = 0$. This leads to a contradiction because $\langle x_i + y_j, f\rangle = \langle x_i, f\rangle + \langle y_j, f\rangle = 2$. ∎

REMARK 2.8. Notice that this inequality is optimal. Indeed, if we define

$$T = \begin{pmatrix} 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&1&0 \end{pmatrix} \quad \text{and} \quad F = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&1&0&0 \\ 0&0&0&0 \end{pmatrix},$$

then we get easily that $\mathrm{a}(T) = 2$ and $\mathrm{a}(T+F) = \mathrm{a}(T-F) = 4$.

The following result [8] will be used in the sequel.

PROPOSITION 2.9. *Let* $T \in \mathcal{L}(X)$ *satisfy* $\min\{\dim \mathrm{N}(T), \operatorname{codim} \mathrm{R}(T)\} < \infty$. *Then*

$$\mathrm{a}(T) < \infty \Rightarrow \operatorname{codim} \mathrm{R}(T) \geqslant \dim \mathrm{N}(T)$$

and

$$\mathrm{d}(T) < \infty \Rightarrow \operatorname{codim} \mathrm{R}(T) \leqslant \dim \mathrm{N}(T).$$

Moreover, if $\dim \mathrm{N}(T) = \operatorname{codim} \mathrm{R}(T) < \infty$ *then* $\mathrm{a}(T) = \mathrm{d}(T)$.

COROLLARY 2.10. *Let* Λ *denote any of the subsets* $\mathcal{B}_+(X)$, $\mathcal{B}_-(X)$ *and* $\mathcal{B}(X)$. *If* $T \in \Lambda$ *then for every rank one operator* F, *either* $T+F \in \Lambda$ *or* $T-F \in \Lambda$.

Proof. Let $T \in \Lambda$. It follows that both $T+F$ and $T-F$ are semi-Fredholm, and $\operatorname{ind}(T+F) = \operatorname{ind}(T-F) = \operatorname{ind}(T)$.

(i) If $\Lambda = \mathcal{B}_+(X)$ then Proposition 2.7 implies that either $T+F \in \mathcal{B}_+(X)$ or $T-F \in \mathcal{B}_+(X)$.

(ii) The case when $\Lambda = \mathcal{B}_-(X)$ follows by duality from (i).

(iii) Let $\Lambda = \mathcal{B}(X)$. By (i), either $\mathrm{a}(T+F) < \infty$ or $\mathrm{a}(T-F) < \infty$. Without loss of generality we may assume that $\mathrm{a}(T+F) < \infty$. Moreover, $T+F$ is Fredholm and $\operatorname{ind}(T+F) = 0$. By the previous proposition, $\mathrm{a}(T+F) = \mathrm{d}(T+F) < \infty$, and consequently $T+F \in \mathcal{B}(X)$. ∎

PROPOSITION 2.11. *If $T \in \mathcal{L}(X)$ is Drazin invertible, then for every rank one operator F, either $T+F$ is Drazin invertible or $T-F$ is Drazin invertible.*

Proof. Let $p = \mathrm{a}(T) = \mathrm{d}(T)$. Then $\mathrm{R}(T^k)$ is closed for all $k \geqslant p$, and $\mathrm{R}(T'^p) = \mathrm{R}(T'^{p+1})$. Hence it follows by Proposition 2.7 that either $\mathrm{a}(T+F) \leqslant 2p$ or $\mathrm{a}(T-F) \leqslant 2p$, and either $\mathrm{a}(T'+F') \leqslant 2p$ or $\mathrm{a}(T'-F') \leqslant 2p$. Moreover, since $\mathrm{R}(T+F)^{2p} \overset{\mathrm{e}}{=} \mathrm{R}(T^{2p})$, we obtain that $\mathrm{R}(T+F)^{2p}$ is closed, and so either $\mathrm{d}(T+F) \leqslant 2p$ or $\mathrm{d}(T-F) \leqslant 2p$.

Now, suppose that neither $T+F$ nor $T-F$ is Drazin invertible. It follows that either $\mathrm{a}(T+F) = \mathrm{d}(T-F) = \infty$ or $\mathrm{a}(T-F) = \mathrm{d}(T+F) = \infty$. Assume that $\mathrm{a}(T+F) = \mathrm{d}(T-F) = \infty$. Write $F = z \otimes f$ with $z \in X$ and $f \in X'$. Write $z = a + z_0$ and $-f = g + f_0$ as in Theorems 2.1 and 2.5. We have

$$\begin{aligned}
-1 = \langle z, f_1\rangle &= \langle a, f_1\rangle + \langle z_0, f_1\rangle = \langle a, T'^{p} f_{p+1}\rangle + \langle Tz_1, f_1\rangle \\
&= \langle T^p a, f_{p+1}\rangle + \langle z_1, f_0\rangle = \langle z_1, -g\rangle + \langle z_1, -f\rangle \\
&= \langle T^p z_{p+1}, -g\rangle + 1 = 1,
\end{aligned}$$

the desired contradiction.

The case $\mathrm{a}(T-F) = \mathrm{d}(T+F) = \infty$ can be treated similarly. ∎

Throughout the sequel, H will denote a separable infinite-dimensional complex Hilbert space.

PROPOSITION 2.12. *Let T be a bounded operator on H. Then the following assertions are equivalent :*

(i) *T is upper (respectively lower) semi-Browder;*

(ii) *for every $S \in \mathcal{L}(H)$ there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S$ has finite ascent (respectively descent) for all $\varepsilon < \varepsilon_0$.*

Proof. (i) $\Rightarrow$ (ii) follows from the openness of the sets $\mathcal{B}_+(H)$ and $\mathcal{B}_-(H)$.

(ii) $\Rightarrow$ (i) Suppose that for every $S \in \mathcal{L}(H)$, there exists $\varepsilon_0 > 0$ such that $\mathrm{a}(T+\varepsilon S) < \infty$ for all $\varepsilon < \varepsilon_0$ and that $T \notin \mathcal{B}_+(H)$. It follows that $\mathrm{a}(T)$ is finite and T is not upper semi-Fredholm. So either $\dim \mathrm{N}(T) = \infty$ or $\mathrm{R}(T)$ is not closed. In both cases, for every finite-codimensional subspace $H_0 \subset H$ the restriction $T|H_0$ is not bounded below. So for each $\varepsilon > 0$ there exists $x \in H_0$ such that $\|x\| = 1$ and

$\|Tx\| < \varepsilon$. Hence, we can find inductively an orthonormal system $x_{n,k}$, $n,k \in \mathbb{N}$, such that $\|Tx_{n,k}\| < 4^{-n}2^{-k}$. Let $M = \mathrm{Span}\{x_{n,k} : n,k \in \mathbb{N}\}$. Define $S \in B(H)$ by $S|M^{\perp} = 0$, and

$$\begin{cases} Sx_{n,1} = -2^n Tx_{n,1} & \text{for all } n \geqslant 0; \\ Sx_{n,k} = -2^n Tx_{n,k} + 2^{-(n+k)}x_{n,k-1} & \text{for all } n \geqslant 0 \text{ and } k \geqslant 2. \end{cases}$$

Since $\sum_{n,k} \|Sx_{n,k}\| \leqslant 2\sum_{n,k} 2^{-(n+k)} < \infty$, the operator S is bounded. Moreover, for each $n \in \mathbb{N}$ we have $(T + 2^{-n}S)x_{n,1} = 0$ and $(T + 2^{-n}S)x_{n,k} = 2^{-k}4^{-n}x_{n,k-1}$ for all $k \geqslant 2$. Hence $\mathrm{a}(T + 2^{-n}S) = \infty$ for each n, the desired contradiction.

Suppose now that for each $S \in \mathcal{L}(H)$ there exists $\varepsilon_0 > 0$ such that $\mathrm{d}(T + \varepsilon S) < \infty$ for all $\varepsilon < \varepsilon_0$. By duality, we conclude that $T^* \in \mathcal{B}_+(H)$. Thus, $T \in \mathcal{B}_-(H)$. ∎

Let $T \in \mathcal{L}(H)$ be a semi-Fredholm operator. By Proposition 2.9, T is upper (respectively lower) semi-Browder if and only if T has finite ascent (respectively descent), i.e.

$$\mathcal{B}_+(H) = \mathcal{F}_\pm(H) \cap \mathcal{A}(H) \quad \text{and} \quad \mathcal{B}_-(H) = \mathcal{F}_\pm(H) \cap \mathcal{D}(H). \tag{2.2}$$

As an immediate consequence of Proposition 2.12 we derive the following result:

COROLLARY 2.13. *Let T be a bounded operator on H. Then following assertions are equivalent:*

(i) *T is a Browder operator;*

(ii) *for every $S \in \mathcal{L}(H)$ there exists $\varepsilon_0 > 0$ such that $T + \varepsilon S$ is Drazin invertible for all $\varepsilon < \varepsilon_0$.*

For a subset $\Gamma \subseteq \mathcal{L}(H)$, we write $\mathrm{Int}(\Gamma)$ for its interior.

COROLLARY 2.14. *The following assertions hold:*

(i) $\mathrm{Int}(\mathcal{A}(H)) = \mathcal{B}_+(H)$;

(ii) $\mathrm{Int}(\mathcal{D}(H)) = \mathcal{B}_-(H)$;

(iii) $\mathrm{Int}(\mathcal{D}^r(H)) = \mathcal{B}(H)$.

Proof. (i) Since $\mathcal{B}_+(H)$ is an open subset contained in $\mathcal{A}(H)$, it suffices to show that $\mathrm{Int}(\mathcal{A}(H)) \subseteq \mathcal{B}_+(H)$. Let $T \notin \mathcal{B}_+(H)$. Then, using Proposition 2.12, there exists $S \in \mathcal{L}(H)$ and a sequence (ε_n) that converges to zero and for which $\mathrm{a}(T + \varepsilon_n S) = \infty$. This implies that $T \notin \mathrm{Int}(\mathcal{A}(H))$.

(ii) and (iii) can be proved in a similar way. ∎

The following theorem, which is interesting in itself, will play a crucial role in the next section.

THEOREM 2.15. *Let $T \in \mathcal{L}(H)$ be a non-zero operator. The following assertions hold:*

(i) *there exists an invertible operator* $S \in \mathcal{L}(H)$ *such that* $\mathrm{a}(S+T) = \infty$;

(ii) *if* $\dim \mathrm{R}(T) \geqslant 2$ *then there exists an invertible operator* $S \in \mathcal{L}(H)$ *such that* $\mathrm{a}(S+T) = \mathrm{a}(S-T) = \infty$.

The following lemma is a special case of Theorem 2.15, and it will be required for proving that theorem.

LEMMA 2.16. *Let* $T \in \mathcal{L}(H)$ *be a non-zero operator such that* $\dim \mathrm{N}(T) = \infty$. *The following assertions hold:*

(i) *there exists an invertible operator* $S \in \mathcal{L}(H)$ *such that* $\mathrm{a}(S+T) = \infty$;

(ii) *if* $\dim \mathrm{R}(T) \geqslant 2$ *then there exists an invertible operator* $S \in \mathcal{L}(H)$ *such that* $\mathrm{a}(S+T) = \mathrm{a}(S-T) = \infty$.

Proof. (i) Note that T is not a scalar multiple of the identity. Consider an $x_0 \in H$ such that x_0 and Tx_0 are linearly independent. Since $\mathrm{N}(T)$ is infinite-dimensional, then so is $\{x_0, Tx_0\}^\perp \cap \mathrm{N}(T)$, and consequently it contains an orthonormal subset $\{x_i : i \geqslant 1\}$ with an infinite-dimensional orthogonal complement. Let $H_0 = \mathrm{Span}\{x_i : i \geqslant 0\}$ and $H_1 = \mathrm{Span}(\{Tx_0\} \cup \{x_i : i \geqslant 0\})$. Consider an operator $S \in \mathcal{L}(H)$ such that $S : H_0^\perp \to H_1^\perp$ is invertible, $Sx_0 = -Tx_0$ and $Sx_i = x_{i-1}$ for all $i \geqslant 1$. Then S is invertible. Moreover, $(S+T)^i(x_i) = x_0 \in \mathrm{N}(S+T)$ for all $i \geqslant 0$, and therefore $\mathrm{a}(S+T) = \infty$.

(ii) Suppose that $\dim \mathrm{R}(T) \geqslant 2$. Find vectors x_0, u_0 such that $\{Tx_0, Tu_0\}$ are linearly independent. Perturbing x_0, u_0 by suitable elements of $\mathrm{N}(T)$ we may assume that the vectors $\{x_0, y_0, Tx_0, Ty_0\}$ are linearly independent. Then there exists an orthonormal subset

$$\{x_i, y_i : i \geqslant 1\} \subseteq \{x_0, y_0, Tx_0, Ty_0\}^\perp \cap \mathrm{N}(T)$$

with an infinite-dimensional orthogonal complement. Let $H_0 = \mathrm{Span}\{x_i, y_i : i \geqslant 0\}$ and $H_1 = \mathrm{Span}(\{Tx_0, Ty_0\} \cup \{x_i, y_i : i \geqslant 0\})$, and consider an operator $S \in \mathcal{L}(H)$ such that $S : H_0^\perp \to H_1^\perp$ is invertible, $Sx_0 = Tx_0$, $Sy_0 = -Ty_0$, $Sx_i = x_{i-1}$ and $Sy_i = y_{i-1}$ for $i \geqslant 1$. It follows that S is invertible and that $(S-T)x_0 = (S+T)y_0 = 0$, $(S+T)^i(x_i) = x_0$ and $(S-T)^i y_i = y_0$ for all $i \geqslant 0$. This shows that both $S+T$ and $S-T$ are of infinite ascent. ∎

LEMMA 2.17. *Let* H *be an infinite-dimensional Hilbert space, and let* $T = cI + F$ *where* c *is a non-zero complex number and* $F \in \mathcal{L}(H)$ *is a finite rank operator. Then there exists an invertible operator* $S \in B(H)$ *such that* $\mathrm{a}(S+T) = \mathrm{a}(S-T) = \infty$.

Proof. Find infinite-dimensional subspaces $H_1, H_2 \subset \mathrm{N}(F)$ such that $H_1 \perp H_2$. Define $S \in B(H)$ by $S|(H_1 \oplus H_2)^\perp = I$, $S|H_1 = -cI_{H_1} + (c/2)B_1$ and $S|H_2 = cI_{H_2} + (c/2)B_2$, where B_1, B_2 are backward shifts in H_1 and H_2, respectively. Then $S|(H_1 \oplus H_2) : H_1 \oplus H_2 \to H_1 \oplus H_2$ is invertible, and so S is invertible. Furthermore, $(S+T)H_1 \subset H_1$, $(S+T)|H_1 = (c/2)B_1$. So $\mathrm{a}(S+T) = \infty$ and similarly, $\mathrm{a}(S-T) = \infty$. ∎

Proof of Theorem 2.15. We may assume that $\dim \mathrm{N}(T) < \infty$ and T is not of the form $T = cI + F$ with $c \neq 0$ and $\dim \mathrm{R}(F) < \infty$. This means that for each subspace $M \subset H$ of finite codimension there exists $w \in M \cap T^{-1}M$ such that the vectors w, Tw are linearly independent. Without loss of generality we may assume that $\| T \| \leqslant 1/2$. Note that $\dim \mathrm{R}(T) \geqslant 2$.

We claim that there exists a sequence $\{x_i\}_{i \geqslant 0}$ such that $\{x_i, Tx_i\}$ is linearly independent and

$$\{x_{i+1}, Tx_{i+1}\} \perp \{x_k, Tx_k : k = 0, \cdots, i\} \quad \text{for all } i \geqslant 0.$$

Find x_0 such that $\{x_0, Tx_0\}$ is linearly independent. Let $H_0 = \{x_0, Tx_0\}^{\perp}$, then there exists $x_1 \in H_0 \cap T^{-1}H_0$ such that $\{x_1, Tx_1\}$ is linearly independent. By repeating the same argument, we construct the sequence $\{x_i\}_{i \geqslant 0}$. Further, we can assume the orthogonal complement of $\mathrm{Span}\{Tx_i, x_i : i \geqslant 0\}$ is infinite-dimensional, because otherwise we can replace $\{x_i\}_{i \geqslant 0}$ by $\{x_{2i}\}_{i \geqslant 0}$. Let $\{y_i\}_{i \geqslant 0}$ be an orthonormal basis of the orthogonal complement of $\mathrm{Span}\{x_i, Tx_i : i \geqslant 0\}$.

For each $i \geqslant 0$, let $w_i \in \mathrm{Span}\{x_i, Tx_i\}$ be such that $\| w_i \| = 1$ and $w_i \perp x_i$, then $Tx_i = \alpha_i x_i + \beta_i w_i$ for some complex α_i, β_i with $| \alpha_i | \leqslant 1/2$ and $0 < | \beta_i | \leqslant 1/2$. Consider the bounded operator S given by

$$\begin{cases} Sx_0 & = -Tx_0 \text{ and } Sx_1 = Tx_1, \\ Sw_0 & = y_0 \text{ and } Sw_1 = y_1, \\ Sx_i & = x_{i-2} - (-1)^i Tx_i \text{ for all } i \geqslant 2, \\ Sy_i & = y_{i+2} \text{ and } Sw_{i+2} = w_{i+2} \text{ for all } i \geqslant 0. \end{cases}$$

Let $L_1 = \mathrm{Span}\{Tx_0, Tx_1, y_i, w_{i+2} : i \geqslant 0\}$ and $L_2 = \mathrm{Span}\{x_i : i \geqslant 0\}$. We claim that S is invertible. In fact, one can easily show that S is injective and $L_1 \subseteq \mathrm{R}(S)$. Hence, it remains only to show that $PS|L_2$ is onto, where P is the projection on L_2 relatively to the decomposition $H = L_1 \oplus L_2$. Write $PS|L_2 = V_1 + V_2$ where $V_1 x_0 = V_1 x_1 = 0$, $V_1 x_{i+2} = x_i$ and $V_2 x_i = (-1)^{i+1}\alpha_i x_i$ for all $i \geqslant 2$. Since the surjectivity modulus of V_1 equals 1 and $\| V_2 \| \leqslant 1/2$, $V_1 + V_2$ is surjective, see [19]. Finally, because $(S+T)x_0 = (S-T)x_1 = 0$, $(S+T)^i x_{2i} = x_0$ and $(S-T)^i x_{2i+1} = x_1$ for all $i \geqslant 0$, we get that $S+T$ and $S-T$ are both of infinite ascent. ∎

3. PROOF OF MAIN RESULTS

In this section we prove the main results of the paper : Theorems A and B.

First we characterize additive mappings preserving either the class of upper semi-Browder operators or the class of Browder operators. Since the proofs are parallel we will do it simultaneously. The full Theorems A and B will be proved at the end of the section.

Throughout this section, H will denote an infinite-dimensional separable complex Hilbert spaces. Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective continuous map which is additive (i.e., $\Phi(T_1 + T_2) = \Phi(T_1) + \Phi(T_2)$ for all $T_1, T_2 \in \mathcal{L}(H)$).

LEMMA 3.1. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map, and let Λ denote any one of the sets $\mathcal{B}$ or $\mathcal{B}_+$. If ϕ preserves Λ in both directions then Φ is injective and preserves the set of rank one operators in both directions.*

Proof. Suppose on the contrary that there exists $F \neq 0$ such that $\Phi(F) = 0$. Then, by Theorem 2.15, there exists an invertible operator $S \in \mathcal{L}(H)$ such that $\mathrm{a}(S + F) = \infty$. Hence, $S + F \notin \Lambda$ and $\Phi(S + F) = \Phi(S) \in \Lambda$, the desired contradiction.

Now, let $T \in \mathcal{L}(H)$ be such that $\dim \mathrm{R}(T) \geqslant 2$. Then, again by Theorem 2.15, there exists an invertible operator $R \in \mathcal{L}(H)$ such that $\mathrm{a}(R + T) = \mathrm{a}(R - T) = \infty$. It follows, in particular, that $R + T$ and $R - T$ do not belong to Λ, and hence so do not $\Phi(R + T)$ and $\Phi(R - T)$. Consequently, by Corollary 2.10, $\dim \mathrm{R}(\Phi(T)) \geqslant 2$. Since Φ is bijective and Φ^{-1} satisfies the same properties as Φ, we obtain that Φ preserves the set of rank one operators in both directions. This completes the proof. ∎

Let τ be a field automorphism of $\mathbb{C}$. An additive map $A : H \to H$ will be called τ-semi linear if $A(\lambda x) = \tau(\lambda)Ax$ holds for all $\lambda \in \mathbb{C}$ and $x \in H$. Notice that if A is bounded, then so is τ, and consequently, τ is either the identity or the complex conjugation, see [13].

Moreover, in this case, the adjoint operator $A' : H' \to H'$ defined by the equation $\langle x, A'y' \rangle = \tau(\langle Ax, y' \rangle)$ for all $x \in H, y' \in H'$, is again τ-semi linear.

Note that we do not identify H with its dual H'. Let $J : H \to H'$ be the natural conjugate linear mapping defined by $\langle u, Jx \rangle = \langle u, x \rangle$ $(x, u \in H)$.

For $A \in \mathcal{L}(H)$, let $A^* : H \to H$ be the Hilbert space adjoint. We have $A^* = J^{-1}A'J$.

LEMMA 3.2. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map, and let Λ denote any of the sets $\mathcal{B}$ and $\mathcal{B}_+$. If Φ preserves Λ in both directions, then:*

either there exist continuous bijective mappings $A, B : H \to H$, either both linear or both conjugate linear, such that

$$\Phi(F) = AFB \text{ for all finite rank operators } F \in \mathcal{L}(H), \tag{3.1}$$

or there exist continuous bijective mappings $C, D : H \to H$, either both linear or both conjugate linear, such that

$$\Phi(F) = CF^*D \text{ for all finite rank operators } F \in \mathcal{L}(H). \tag{3.2}$$

Proof. From the previous Lemma 3.1 and Theorem 3.3 of [21], there exists a ring automorphism τ of $\mathbb{C}$, and either τ-semi linear bijective maps $A : H \to H$ and $E : H' \to H'$, such that

$$\Phi(x \otimes f) = Ax \otimes Ef \quad \text{for all } x \in H \text{ and } f \in H', \tag{3.3}$$

or τ-semi linear bijective maps $R : H \to H'$ and $G : H' \to H$ such that

$$\Phi(x \otimes f) = Gf \otimes Rx \quad \text{for all } x \in H \text{ and } f \in H'. \tag{3.4}$$

Since Φ is continuous, then so are τ, A, E, R and G.

In the first case set $B = E'$. It is easy to verify that $\Phi(F) = AFB$ for all rank one operators and, by additivity of Φ, for all finite rank operators $F \in \mathcal{L}(H)$.

In the second case set $C = GJ$ and $D = J^{-1}R'$. Again it is easy to verify that $\Phi(F) = CF^*D$ for all finite rank operators $F \in \mathcal{L}(H)$. ∎

If we replace Φ by $\Psi : \mathcal{L}(H) \to \mathcal{L}(H)$ defined by $\Psi(T) = A^{-1}\Phi(T)A$ in the first case (by $\Psi(T) = C^{-1}\Phi(T)C$ in the second case, respectively), we can assume that A (respectively C) is the identity mapping. Note that in this case B (respectively D) is a linear mapping.

For $T \in \mathcal{L}(H)$, we write

$$\mathrm{M}(T) := \{x \in H : \text{ there exists } y \in H \text{ such that } \mathrm{a}(T + x \otimes y) = \infty\}.$$

Note that if T is invertible, then by Theorem 2.1 we have $\mathrm{M}(T) = \{z : z \notin \operatorname{Span}\{T^{-i}z : i \geqslant 1\}\}$, and in this case $T^n z \in \mathrm{M}(T)$, for all $n \in \mathbb{Z}$, whenever $z \in \mathrm{M}(T)$.

LEMMA 3.3. *Let $T \in \mathcal{L}(H)$ be an operator of finite ascent. Then*
(i) $\mathrm{M}(T) \subseteq \mathcal{R}^\infty(T) + \mathcal{N}^\infty(T)$,
(ii) *if $\mathcal{R}^\infty(T)$ is closed then* $\mathrm{M}(T) = \mathrm{M}(T_{|\mathcal{R}^\infty(T)}) + \mathcal{N}^\infty(T) = \mathrm{M}(T) + \mathcal{N}^\infty(T)$.

Proof. (i) follows immediately from Theorem 2.1.

(ii) Suppose that $\mathcal{R}^\infty(T)$ is closed, and let T_o be the restriction of T to $\mathcal{R}^\infty(T)$. It follows easily from Theorem 2.1 that $\mathrm{M}(T) \subseteq \mathrm{M}(T_o) + \mathcal{N}^\infty(T)$. Let $z \in \mathrm{M}(T_o)$ and $a \in \mathcal{N}^\infty(T)$. Then we get that $T_o^{-1}z \in \mathcal{R}^\infty(T) \setminus \operatorname{Span}\{T_o^{-i}z : i \geqslant 2\}$. But, since T has finite ascent, $\mathcal{R}^\infty(T) \cap \mathcal{N}^\infty(T) = \{0\}$, and consequently

$$T_o^{-1}z \notin \operatorname{Span}\{T_o^{-i}z : i \geqslant 2\} + \operatorname{Span}\{T^i a : i \geqslant 0\}.$$

Thus there exists $y \in H$ such that $\langle T_o^{-i}z, y\rangle = -1$ and $\langle T_o^{-i}z, y\rangle = \langle T^j a, y\rangle = 0$ for all $i \geqslant 2$ and $j \geqslant 0$. This shows that $z + a \in \mathrm{M}(T)$, and so $\mathrm{M}(T) = \mathrm{M}(T_o) + \mathcal{N}^\infty(T)$. The second equality follows from the first. ∎

LEMMA 3.4. *Let $H = H_1 \oplus H_2$ and $T = T_1 \oplus T_2$ where $T_1 \in \mathcal{L}(H_1)$ and $T_2 \in \mathcal{L}(H_2)$ are two invertible operators. Then*

$$(\mathrm{M}(T_1) + H_2) \cup (H_1 + \mathrm{M}(T_2)) \subseteq \mathrm{M}(T).$$

Proof. Let $z = z_1 + z_2$ where $z_1 \in H_1$ and $z_2 \in H_2$. If $z \notin \mathrm{M}(T)$ then $T_1^{-1}z_1 + T_2^{-1}z_2$ belongs to

$$\operatorname{Span}\{T^{-i}z : i \geqslant 2\} \subseteq \operatorname{Span}\{T_1^{-i}z_1 : i \geqslant 2\} \oplus \operatorname{Span}\{T_2^{-i}z_2 : i \geqslant 2\}.$$

Hence, $z_1 \notin \mathrm{M}(T_1)$ and $z_2 \notin \mathrm{M}(T_2)$, as desired. ∎

Recall that an operator $U \in \mathcal{L}(H)$ is called a bilateral shift of multiplicity p if there are pairwise orthogonal subspaces H_n, $n \in \mathbb{Z}$, of dimension p such that $H = \bigoplus_n H_n$ and U maps isometrically H_n onto H_{n+1} for all $n \in \mathbb{Z}$. By a *bilateral shift* we mean the bilateral shift of multiplicity 1.

Let $\mathbb{T}$ be the unit circle of $\mathbb{C}$ and $\mathrm{L}^2(\mathbb{T})$ be the Hilbert space of square integrable functions with respect to the normalized measure μ on $\mathbb{T}$. For $f \in \mathrm{L}^2(\mathbb{T})$, it follows by the Szegö–Kolmogorov theorem, see [20], that

$$\inf_{p\in\mathcal{A}} \int_{\mathbb{T}} |f - pf|^2 \mathrm{d}\mu = \exp\Big(\int_{\mathbb{T}} \log(|f|)\mathrm{d}\mu\Big) \tag{3.5}$$

where $\mathcal{A}$ denotes the set of all polynomials in z vanishing at the origin.

Let U_z denote the bilateral shift defined by $U_z(f) = zf$ for all $f \in \mathrm{L}^2(\mathbb{T})$. Note that $U_z^{-1} = U_{\overline{z}}$, and using (3.5) one can easily get that

$$\{z^n : n\in\mathbb{Z}\} \subseteq \mathrm{M}(U_z^{-1}) = \mathrm{M}(U_z) = \Big\{f \in \mathrm{L}^2(\mathbb{T}) : \int_{\mathbb{T}} \log(|f|)\mathrm{d}\mu > -\infty\Big\} \subsetneq \mathrm{L}^2(\mathbb{T}). \tag{3.6}$$

LEMMA 3.5. *Let $H = \mathrm{L}^2(\mathbb{T}) \oplus \mathrm{L}^2(\mathbb{T})$ and $U = U_z \oplus U_z$. The following assertions hold:*

(i) *if $f \notin \mathrm{M}(U_z)$ then $f \oplus f \notin \mathrm{M}(U)$;*
(ii) *$H = \mathrm{M}(U) + \mathrm{M}(U)$;*
(iii) *if $f + \mathrm{M}(U) \subseteq \mathrm{M}(U)$ then $f = 0$.*

Proof. (i) We have $f \in \mathrm{Span}\{U_z^{-i}f : i \geqslant 1\}$. Hence $f \oplus f \in \mathrm{Span}\{U_z^{-i}f \oplus U_z^{-i}f : i \geqslant 1\}$, and so $f \oplus f \notin \mathrm{M}(U)$.

(ii) Let $f = g \oplus h$ where $g, h \in \mathrm{L}^2(\mathbb{T})$, and let $A = \{z \in \mathbb{T} : |g(z)| \geqslant 1\}$. Define

$$g_1(z) = \begin{cases} g(z)/2 & \text{if } z \in A, \\ 1 + g(z)/2 & \text{if } z \in \mathbb{T} \setminus A, \end{cases} \quad \text{and} \quad g_2(z) = \begin{cases} g(z)/2 & \text{if } z \in A, \\ -1 + g(z)/2 & \text{if } z \in \mathbb{T} \setminus A. \end{cases}$$

Clearly, g_1 and g_2 belong to $\mathrm{M}(U_z)$ and $g = g_1 + g_2$. Hence, by Lemma 3.4 we obtain that $g \oplus h = g_1 \oplus h + g_2 \oplus 0 \in \mathrm{M}(U)$.

(iii) Suppose that $f = g \oplus h$ and $g \neq 0$. Then there is $\varepsilon > 0$ such that $B = \{z \in \mathbb{T} : |g(z)| \geqslant \varepsilon\}$ has a positive measure. Consider

$$k(z) = \begin{cases} 0 & \text{if } z \in B, \\ 1 + g(z) & \text{if } z \in \mathbb{T} \setminus B. \end{cases}$$

By the equality (3.6), it follows that $k \notin \mathrm{M}(U_z)$, and so $k \oplus k \notin \mathrm{M}(U)$. On the other hand, $k - g \in \mathrm{M}(U_z)$, and Lemma 3.4 implies that $k \oplus k - g \oplus h = (k - g) \oplus (k - h) \in \mathrm{M}(U)$. Consequently, $k \oplus k = f + k \oplus k - g \oplus h \in \mathrm{M}(U)$, a contradiction. Similarly we show that $h = 0$. ∎

Clearly the assertions (ii) and (iii) of the previous lemma are true for any bilateral shift operator of multiplicity 2.

LEMMA 3.6. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive map that preserves either $\mathcal{B}_+(H)$, or $\mathcal{B}(H)$, in both directions, and let $S \in \mathcal{L}(H)$ be an invertible operator. The following assertions hold:*

(i) *if there exists an invertible operator $B \in \mathcal{L}(H)$ for which $\Phi(F) = FB$ for all finite rank operators $F \in \mathcal{L}(H)$, then* $\mathrm{M}(\Phi(S)) = \mathrm{M}(S)$*;*

(ii) *if there exists an invertible operator $D \in \mathcal{L}(H)$ for which $\Phi(F) = F^*D$ for all finite rank operators $F \in \mathcal{L}(H)$, then* $\mathrm{M}(\Phi(S)) = \mathrm{M}(S^*)$.

Proof. First observe that if $T \in \mathcal{B}(H)$, then for every rank one operator F, $T + F \in \mathcal{B}(H)$ if and only if $T + F \in \mathcal{B}_+(H)$ if and only if $\mathrm{a}(T + F)$ is finite.

(i) A vector $x \in H$ belongs to $\mathrm{M}(S)$ if and only if there exists $y \in H$ such that $\mathrm{a}(S + x \otimes y) = \infty$, which is equivalent to $\mathrm{a}(\Phi(S) + x \otimes B^*y) = \infty$. Since B is invertible, this is equivalent to $x \in \mathrm{M}(\Phi(S))$.

(ii) Let $x, y \in H$. Then

$$\begin{aligned} \mathrm{a}(\Phi(S) + x \otimes y) = \infty &\Leftrightarrow \mathrm{a}(\Phi(S + D^{*-1}y \otimes x)) = \infty \\ &\Leftrightarrow \mathrm{a}(S + D^{*-1}y \otimes x) = \infty \\ &\Leftrightarrow \mathrm{a}(S^* + x \otimes D^{*-1}y) = \infty \end{aligned}$$

and so $\mathrm{M}(\Phi(S)) = \mathrm{M}(S^*)$. ∎

LEMMA 3.7. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive map that preserves either $\mathcal{B}_+(H)$ or $\mathcal{B}(H)$ in both directions. If $\Phi(F) = FB$ (respectively $\Phi(F) = F^*D$) for all finite rank operators F, then $\Phi(U)$ is invertible whenever U is a bilateral shift of multiplicity* 2.

Proof. Let H_1, H_2 be an infinite dimensional subspaces of H such that $H = H_1 \oplus H_2$, and let $U_1 \in \mathcal{L}(H_1)$ and $U_2 \in \mathcal{L}(H_2)$ be two bilateral shifts such that $U = U_1 \oplus U_2$.

Suppose that $\Phi(F) = FB$ for all finite rank operators F. It follows that $T = \Phi(U)$ has finite ascent p, and $\mathrm{M}(T) \subset \mathcal{R}^\infty(T) \oplus \mathrm{N}(T^p)$ by Lemma 3.3. Hence, using Lemmas 3.5(ii), we get that

$$H = (\mathrm{M}(U) + \mathrm{M}(U)) = \mathrm{M}(T) + \mathrm{M}(T) \subseteq \mathcal{R}^\infty(T) \oplus \mathrm{N}(T^p).$$

Consequently, $\mathcal{R}^\infty(T) = \mathrm{R}(T^p)$ is closed (see Theorem 5.10 of [28]). Let $z_1 \in H_1$ and $z_2 \in H_2$ be such that $z_1 + z_2 \in \mathcal{N}^\infty(T)$. Then, by Lemma 3.3, $z_1 + z_2 + \mathrm{M}(T) \subseteq \mathrm{M}(T)$, and so $z_1 + z_2 + \mathrm{M}(U) \subseteq \mathrm{M}(U)$. Therefore, by Lemma 3.5(ii), we obtain that $z_1 = z_2 = 0$. This shows that T is invertible.

If $\Phi(F) = F^*D$ for all finite rank operators F, then using the fact that $\mathrm{M}(T) = \mathrm{M}(U^*) = \mathrm{M}(U)$, we obtain in the same way that T is invertible. ∎

Let $S \in \mathcal{L}(H)$. We associate for each $x \in \mathrm{M}(S)$ the following subsets

$$\begin{aligned} \mathrm{M}_x(S) &:= \{y \in H : \mathrm{a}(S + x \otimes y) = \infty\} \quad \text{and} \\ \mathrm{L}_x(S) &:= \{y \in H : y + \mathrm{M}_x(S) = \mathrm{M}_x(S)\}. \end{aligned}$$

Notice that if S is invertible, then it follows by Theorem 2.1 that $y \in \mathrm{M}_x(S)$ if and only if $y \in \mathrm{Span}\{T^{-i}x : i \geqslant 2\}^{\perp}$ and $\langle T^{-1}x, y\rangle = -1$; consequently $\mathrm{L}_x(T) = \mathrm{Span}\{T^{-i}x : i \geqslant 1\}^{\perp}$.

REMARK 3.8. Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive map that preserves $\mathcal{B}_+(H)$, or $\mathcal{B}(H)$, in both directions, and let $S \in \mathcal{L}(H)$ be an invertible operator. By the proof of Lemma 3.6, we have:

(i) if $B \in \mathcal{L}(H)$ is invertible and $\Phi(F) = FB$ for all finite rank operators F, then $\mathrm{M}_x(\Phi(S)) = B^*\mathrm{M}_x(S)$ and $\mathrm{L}_x(\Phi(S)) = B^*\mathrm{L}_x(S)$;

(ii) if $D \in \mathcal{L}(H)$ is invertible and $\Phi(F) = F^*D$ for all finite rank operators F, then $\mathrm{M}_x(\Phi(S)) = D^*\mathrm{M}_x(S^*)$ and $\mathrm{L}_x(\Phi(S)) = D^*\mathrm{L}_x(S^*)$.

LEMMA 3.9. *Let $H = H_1 \oplus H_2$ and $U = U_1 \oplus U_2$ where $U_1 \in \mathcal{L}(H_1)$ and $U_2 \in \mathcal{L}(H_2)$ are two bilateral shifts, and let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map that preserves $\mathcal{B}_+(H)$, or $\mathcal{B}(H)$, in both directions.*

(i) *If $\Phi(F) = FB$ for all finite rank operators F, then $\Phi(U)B^{-1}H_i = H_i$, for $i = 1, 2$, and U commutes with $\Phi(U)B^{-1}$.*

(ii) *If $\Phi(F) = F^*D$ for all finite rank operators F, then $\Phi(U)D^{-1}H_i = H_i$, for $i = 1, 2$, and U^* commutes with $\Phi(U)D^{-1}$.*

Proof. By Lemma 3.7, it follows that $T = \Phi(U)$ is invertible. Let $\{x_k\}_{k\in\mathbb{Z}}$ and $\{y_k\}_{k\in\mathbb{Z}}$ be respectively orthonormal bases in H_1 and H_2 such that $U_1x_k = x_{k+1}$ and $U_2y_k = y_{k+1}$ for all $k \in \mathbb{Z}$.

(i) For $z \in \mathrm{M}(U)$, we have, via Remarks 3.8(i),

$$\begin{aligned} y \in \mathrm{Span}\{T^{-i}z : i \geqslant 1\} &\Leftrightarrow \langle y, v\rangle = 0 \text{ for all } v \in \mathrm{L}_z(T) \\ &\Leftrightarrow \langle y, B^*u\rangle = 0 \text{ for all } u \in \mathrm{L}_z(U) \\ &\Leftrightarrow \langle By, u\rangle = 0 \text{ for all } u \in \mathrm{L}_z(U) \\ &\Leftrightarrow By \in \mathrm{Span}\{U^{-i}z : i \geqslant 1\}. \end{aligned}$$

This shows that

$$B\,\mathrm{Span}\{T^{-i}z : i \geqslant 1\} = \mathrm{Span}\{U^{-i}z : i \geqslant 1\} \quad \text{for all } z \in \mathrm{M}(U). \tag{3.7}$$

On the other hand, for each $k \in \mathbb{Z}$, Lemma 3.4 implies that $x_k \in \mathrm{M}(U)$, and consequently, it follows by (3.7) that $BT^{-1}x_k \in H_1$. Therefore, $BT^{-1}H_1 \subseteq H_1$. Similarly, we obtain that $BT^{-1}H_2 \subseteq H_2$, and hence $\Phi(U)B^{-1}H_i = H_i$, for $i = 1, 2$, because $\Phi(U)B^{-1}$ is invertible. On the other hand, it follows from (3.7) that every closed subspace invariant for U_1^{-1} is also invariant for BT^{-1}. Moreover, since U_1^{-1} is a reflexive operator, we get that the restriction of BT^{-1} to H_1 is a SOT-limit of polynomials of U_1^{-1}, see [5], and consequently it commutes with U_1^{-1}. Analogously, we show that the restriction of BT^{-1} to H_2 commutes with U_2^{-1}.

(ii) In similar way, we establish, using Remark 3.8(ii), that

$$D(\mathrm{Span}\{T^{-i}z : i \geqslant 1\}) = \mathrm{Span}\{U^{i}z : i \geqslant 1\} \quad \text{for all } z \in \mathrm{M}(U), \tag{3.8}$$

and that DT^{-1} leaves invariant H_1 and H_2. Moreover, since U_1 is a reflexive operator, then using (3.8), we obtain that U_1 commutes with the restriction of DT^{-1} to H_1. This completes the proof. ∎

LEMMA 3.10. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map that preserves $\mathcal{B}_+(H)$, or $\mathcal{B}(H)$, in both directions. Then:*

(i) *if $\Phi(F) = FB$ for all finite rank operator F, then there exists a linear functional ϕ such that $\Phi(S) = SB + \varphi(S)B$ for all $S \in \mathcal{L}(H)$;*

(ii) *if $\Phi(F) = F^*D$ for all finite rank operator F, then there exists a conjugate linear functional ψ such that $\Phi(S) = S^*D + \psi(S)D$ for all $S \in \mathcal{L}(H)$.*

Proof. Let H_1 and H_2 be two closed subspaces satisfying $H = H_1 \oplus H_2$. Consider also an arbitrary bilateral shifts operators $U_1 \in \mathcal{L}(H_1)$ and $U_2 \in \mathcal{L}(H_2)$ and an orthonormal basis $\{x_k\}_{k\in\mathbb{Z}}$ and $\{y_k\}_{k\in\mathbb{Z}}$ of H_1 and H_2, respectively, such that $U_1x_k = x_{k+1}$ and $U_2y_k = y_{k+1}$ for all $k \in \mathbb{Z}$. Then $U = U_1 \oplus U_2$ is a bilateral shift of multiplicity 2.

(i) For $k \in \mathbb{Z}$ and $F = (y_k - x_k) \otimes (x_{k-1} - y_{k-1})$, it follows that $U + F$ is again a bilateral shift of multiplicity 2. Hence, using Lemma 3.9, we get that U and $U + F$ commute respectively with $\Phi(U)B^{-1}$ and $\Phi(U + F)B^{-1} = \Phi(U)B^{-1} + F$, and consequently $(U - TB^{-1})F = F(U - TB^{-1})$ where $T = \Phi(U)$. Hence, for all $k \in \mathbb{Z}$, there exists a complex number c_k such that $(U - TB^{-1})(y_k - x_k) = c_k(y_k - x_k)$. On the other hand,

$$\begin{aligned} c_{k+1}(y_{k+1} - x_{k+1}) &= (U - TB^{-1})(y_{k+1} - x_{k+1}) = U(U - TB^{-1})(y_k - x_k) \\ &= c_kU(y_k - x_k) = c_k(y_{k+1} - x_{k+1}). \end{aligned}$$

Let $c = c_k$ for $k \in \mathbb{Z}$. Then it follows by Lemma 3.9 that $(U - TB^{-1})x_k = cx_k$ and $(U - TB^{-1})y_k = cy_k$ for all $k \in \mathbb{Z}$. This shows that $\Phi(U) = UB - cB$, and since the complex number c is uniquely determined, we set $\varphi(U) = -c$. Let $\widetilde{U} = U_1 \oplus (-U_2)$. Then

$$\Phi(U_1\oplus 0)=\frac{1}{2}(\Phi(U)+\Phi(\widetilde{U}))=(U_1\oplus 0)B+\frac{\varphi(U)+\varphi(\widetilde{U})}{2}B=(U_1\oplus 0)B+\varphi(U_1\oplus 0)B,$$

where we set $\varphi(U_1 \oplus 0) = 2^{-1}(\varphi(U) + \varphi(\widetilde{U}))$. Moreover, since every operator $S \in \mathcal{L}(H_1)$ can be written as a sum of finite number of bilateral shifts, we obtain that

$$\Phi(S \oplus 0) = (S \oplus 0)B + \varphi(S \oplus 0)B \tag{3.9}$$

for some $\varphi(S \oplus 0) \in \mathbb{C}$.

Let $S \in B(H)$ be an arbitrary operator. Let $H = L_1 \oplus L_2 \oplus L_3$, where L_1, L_2, L_3 are infinite dimensional subspaces, and write

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix} = \begin{pmatrix} S_{11} & S_{12} & 0 \\ S_{21} & S_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & S_{23} \\ 0 & S_{32} & S_{33} \end{pmatrix} + \begin{pmatrix} 0 & 0 & S_{13} \\ 0 & 0 & 0 \\ S_{31} & 0 & 0 \end{pmatrix}.$$

Now using (3.9) for each of these summands, we get that $\Phi(S) = SB + \varphi(S)B$ for some $\varphi(S) \in \mathbb{C}$. Clearly φ is a linear functional.

(ii) Consider the operator F defined in (i). It follows by Lemma 3.9 that U^* and $(U+F)^*$ commute respectively with $\Phi(U)D^{-1}$ and $\Phi(U+F)D^{-1}$, and hence $(\Phi(U)D^{-1} - U^*)F^* = F^*(\Phi(U)D^{-1} - U^*)$. This implies that there exists a complex number $\psi(U)$ for which $\Phi(U) = U^*D + \psi(U)D$. Arguing as above, we obtain that $\Phi(S) = S^*D + \psi(S)D$ for all $S \in \mathcal{L}(H)$, and ψ is a conjugate linear form. ∎

LEMMA 3.11. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map, and Λ denote any of the sets $\mathcal{B}_+$ and $\mathcal{B}$. Suppose that Φ preserves Λ, in both directions. Then:*

(i) *if $\Phi(F) = FB$ for all finite rank operator F, then there exists a non-zero $c \in \mathbb{C}$ such that $\Phi(S) = cS$ for all $S \in \mathcal{L}(H)$;*

(ii) *if $\Lambda = \mathcal{B}$ and $\Phi(F) = F^*D$ for all finite rank operator F, then there exists a non-zero $c \in \mathbb{C}$ such that $\Phi(S) = cS^*$ for all $S \in \mathcal{L}(H)$;*

(iii) *if $\Lambda = \mathcal{B}_+$ then Φ can not take the form $\Phi(F) = F^*D$ for all finite rank operator F.*

Proof. (i) By Lemma 3.10, $\Phi(S) = SB + \varphi(S)B$ for all $S \in \mathcal{L}(H)$. Let $Q \in \mathcal{L}(H)$ be a quasinilpotent operator. Then Q, and hence $\Phi(Q)$, does not belong to Λ. If $\varphi(Q) \neq 0$, we get that $\Phi(Q) = (Q - \varphi(Q)I)B$ is invertible and so belongs to Λ, a contradiction. Thus, $\varphi(Q) = 0$. On the other hand, since every operator can be written as a finite sum of quasi-nilpotent operators, see [22], we obtain that $\varphi = 0$. Now, let us show that there exists a $c \in \mathbb{C}$ such that $B = cI$. Suppose on the contrary that there exist two linearly independent vectors $\{u, v\}$ such that $Bu = v$. Let $\{x_i, y_i : i \geqslant 1\}$ be an orthonormal basis of $H_0 = \{u, v\}^\perp$, and consider the operator $T \in \mathcal{L}(H)$ given by

$$\begin{cases} Tx_i = x_{i-1} & \text{and} \quad Ty_{i-1} = y_i \quad \text{for all } i \geqslant 2, \\ Tx_1 = v, Tu = y_1 & \text{and} \quad Tv = 0. \end{cases}$$

It follows that $\mathrm{R}(T) = H_0 \oplus \mathrm{Span}\{v\}$ and $\mathrm{N}(T) = \mathrm{Span}\{v\}$. Hence, T, and so TB, is Fredholm of index zero, and since $T^i x_i = v \in \mathrm{N}(T)$ for all $i \geqslant 1$, T has infinite ascent. Consequently, $T \notin \Lambda$. But, since

$$\mathrm{R}(TB) \cap \mathrm{N}(TB) = \mathrm{R}(T) \cap \mathrm{Span}\{u\} = \{0\},$$

we have $\mathrm{a}(TB) = 1$, and therefore $TB \in \Lambda$, a contradiction.

To prove (ii) and (iii), suppose that $\Phi(F) = F^*D$ for all finite rank operators F. Then, by Lemma 3.10, $\Phi(S) = S^*D + \psi(S)D$ for all $S \in \mathcal{L}(H)$, and the same argument used in (i) shows that $\psi = 0$. If D is not of the form cI, where $c \in \mathbb{C}$, then by considering the operator T introduced in (i), we obtain that $T^* \notin \Lambda$ and $\Phi(T^*) = TD \in \Lambda$, a contradiction.

Finally, assume that $\Lambda = \mathcal{B}_+$. Let $U \in \mathcal{L}(H)$ be the backward shift. It follows that $U \notin \mathcal{B}_+(H)$, and $\Phi(U) = cU^* \in \mathcal{B}_+(H)$, a contradiction. ∎

Proof of Theorem A. The implications (i) $\Rightarrow$ (iii) and (ii) $\Rightarrow$ (iv) follow immediately from Proposition 2.12. Moreover, (v) implies the assertions (i) and (ii), and hence it remains to establish the implications (iii) $\Rightarrow$ (v) and (iv) $\Rightarrow$ (v). Suppose that Φ preserves $\mathcal{B}_+$. Then Φ can not take the form (3.2). Indeed, otherwise the maps $\Phi_1(\cdot) = C^{-1}\Phi(\cdot)C$ would preserve $\mathcal{B}_+$ and satisfy $\Phi_1(F) = F^*DC$ for all finite rank one operators F, which contradicts Lemma 3.11(iii). Hence, Φ takes the form (3.1). Clearly, the map $\Phi_2(\cdot) = A^{-1}\Phi(\cdot)A$ preserves $\mathcal{B}_+$, and $\Phi_2(F) = FBA$ for all F of finite rank. Hence, by Lemma 3.11(i), there is a nonzero complex c such that $\Phi_2(S) = cS$, and so $\Phi(S) = cASA^{-1}$, for all $S \in \mathcal{L}(H)$.

The implication (iv) $\Rightarrow$ (v) follows from (iii) $\Rightarrow$ (v) by considering the map $\Psi(T) = \Phi(T^*)^*$ for $T \in \mathcal{L}(H)$. ∎

Proof of Theorem B. The implication (i) $\Rightarrow$ (ii) follows from Corollary 2.13, see also Corollary 2.14(iii). The implication (iii) $\Rightarrow$ (i) is obvious. If we suppose (ii), then arguing as in the proof of Theorem A, we get by Lemmas 3.2 and 3.11 that Φ possesses one of the two forms given in (iii). ∎

Recall that a complex $\lambda \in \mathbb{C}$ is a pole of the resolvent of $T \in \mathcal{L}(H)$ of order $n \geqslant 0$ if and only if $T - \lambda$ is Drazin invertible of index n, which is equivalent to $n = \mathrm{a}(T - \lambda) = \mathrm{d}(T - \lambda)$, see [19], [28]. For $T \in \mathcal{L}(H)$, we denote by $\mathrm{P}(T)$ the set of all the poles of its resolvent.

COROLLARY 3.12. *Let $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ be a surjective additive continuous map. Then $\mathrm{P}(\Phi(T)) = \mathrm{P}(T)$ for all $T \in \mathcal{L}(H)$ if and only if there exists an invertible bounded linear, or conjugate linear, operator $A : H \to H$ such that either $\Phi(S) = ASA^{-1}$ for all $S \in \mathcal{L}(H)$, or $\Phi(S) = AS^*A^{-1}$ for all $S \in \mathcal{L}(H)$.*

Proof. Clearly, Φ preserves in both directions $\mathcal{D}^r$, and so it takes one of the two forms in Theorem B(iii). To show that the constant $c = 1$, consider an arbitrary quasinilpotent operator $Q \in \mathcal{L}(H)$ with infinite ascent, and let $T = Q + I$. It follows that

$$\mathbb{C} \setminus \{1\} = \mathrm{P}(T) = \mathrm{P}(\Phi(T)) = c\mathrm{P}(T) = \mathbb{C} \setminus \{c\}.$$

Hence, $c = 1$. ∎

We end this section by the following remarks:

REMARK 3.13. In [4], the author asked which additive maps preserve the Drazin invertible operators. Theorem B presents a complete answer to this question.

REMARK 3.14. In [3], the authors considered additive maps Φ such that $\mathrm{a}(\Phi(T)) = \mathrm{a}(T)$ for all T or $\mathrm{d}(\Phi(T)) = \mathrm{d}(T)$ for all T. Clearly, such maps preserve injective operators or surjective operators, and their forms are determined in [24].

An additive map Φ between two algebras is called *unital* if $\Phi(1) = 1$.

REMARK 3.15. Let $\mathcal{R}$ be any one of the subsets $\{\mathcal{A}, \mathcal{D}, \mathcal{D}^r, \mathcal{B}_+, \mathcal{B}_-, \mathcal{B}\}$, and define the corresponding spectrum by

$$\sigma_{\mathcal{R}}(T) := \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{R}\}.$$

Using Theorems A and B, the form of unital continuous additive maps $\Phi : \mathcal{L}(H) \to \mathcal{L}(H)$ such that $\sigma_{\mathcal{R}}(\Phi(T)) = \sigma_{\mathcal{R}}(T)$ can be easily determined.

REMARK 3.16. Theorems A and B can be without any change formulated for additive mappings $\Phi : \mathcal{L}(H) \to \mathcal{L}(K)$ preserving any of the classes $\mathcal{A}$, $\mathcal{D}$, $\mathcal{D}^r$, $\mathcal{B}_+$, $\mathcal{B}_-$, $\mathcal{B}$, where H, K are separable infinite-dimensional Hilbert spaces.

Acknowledgements. The first and the third authors were partially supported by I+D MEC project no. MTM 2010-17687 (Spain). The second author was supported by grants 201/09/0473 of GA CR and IAA100190903 of GA AV.

REFERENCES

[1] B. AUPETIT, Sur les transformations qui conservent le spectre, in *Banach Algebra'97*, Walter de Gruyter, Berlin 1998, pp. 55–78.

[2] B. AUPETIT, Spectrum-preserving linear mappings between Banach algebras or Jordan–Banach algebras, *J. London Math. Soc.* **62**(2000), 917–924.

[3] M. BENDAOUD, M. SARIH, Additive maps preserving the ascent and descent of operators, *Linear Algebra Appl.* **431**(2009), 1740–1744.

[4] J. CUI, Additive Drazin inverse preservers, *Linear Algebra Appl.* **426**(2007), 448–453.

[5] L.A. DEDDENS, Every isometry is reflexive, *Proc. Amer. Math. Soc.* **28**(2007), 509–512.

[6] J. DIEUDONNÉ Sur une généralisation du groupe orthogonal à quatre variables, *Arch. Math (Basel)* **1**(1949), 282–287.

[7] C.K. FONG, A.R. SOUROUR, Sums and products of quasi-nilpotent operators, *Proc. Royal Soc. Edinburgh* **99A**(1984), 193–200.

[8] H. HEUSER, Über Operatoren mit endlichen Defecten, Inaug. Diss. Tübingen, Tübingen 1956.

[9] A. JAFARIAN, A.R. SOUROUR, Spectrum preserving linear maps, *J. Funct. Anal.* **66**(1986), 255–261.

[10] C.R. JOHNSON, C.-K. LI, L. RODMAN, I. SPITKOVSKY, S. PIERCE, Linear maps preserving regional eigenvalue location, *Linear Multilinear Algebra* **32**(1992), 253–264.

[11] I. KAPLANSKY, *Algebraic and Analytic Aspects of Operator Algebras*, Amer. Math. Soc., Providence, RI 1970.

[12] H. KROH, P. VOLKMANN, Störungssätze für Semifredholmoperatoren, *Math. Z.* **148**(1976), 295–297.

[13] M. KUCZMA, *An Introduction to the Theory of Functional Equations and Inequalities*, Państwowe Wydawnictwo Naukowe, Warszawa 1985.

[14] L.W. MARCOUX, Sums and products of weighted shifts, *Canad. Math. Bull.* **44**(2001), 469–481.

[15] M. MBEKHTA, Linear maps preserving a set of Fredholm operators, *Proc. Amer. Math. Soc.* **135**(2007), 3613–3619.

[16] M. MBEKHTA, L. RODMAN, P. ŠEMRL, Linear maps preserving generalized invertibility, *Integral Equations Operator Theory* **55**(2006), 93–109.

[17] M. MBEKHTA, P. ŠEMRL, Linear maps preserving semi-Fredholm operators and generalized invertibility, *Linear Multilinear Algebra* **57**(2009), 55–64.

[18] L. MOLNÀR, *Selected Preserver Problems on Algebraic Structures of Linear Operators and on Function Spaces*, Lecture Notes in Math., vol. 1895, Springer-Verlag, Berlin 2007.

[19] V. MÜLLER, *Spectral Theory of Linear Operators and Spectral Systems in Banach Algebras*, Second edition, Oper. Theory Adv. Appl., vol. 139, Birkhäuser Verlag, Basel 2007.

[20] N.K. NIKOL'SKIJ, *Treatise on the Shift Operator,* Springer-Verlag, Berlin 1986.

[21] M. OMLADIČ, P. ŠEMRL, Additive mappings preserving operators of rank one, *Linear Algebra Appl.* **182**(1993), 239–256.

[22] C. PEARCY, D. TOPPING, Sums of small numbers of idempotents, *Michigan Math. J.* **14**(1967), 453–465.

[23] P. ŠEMRL, Mapping which preserve idempotents, local automorphisms, and local derivations, *Canad. J. Math.* **45**(1993), 483–496.

[24] P. ŠEMRL, Two characterizations of automorphisms on $B(X)$, *Studia Math.* **150**(1993), 143–149.

[25] P. ŠEMRL, Maps on idempotent operators, in *Perspectives in Operator Theory*, Banach Center Publ., vol. 75, Polish Acad. Sci., Warsaw 2007, pp. 289–301.

[26] A.R. SOUROUR, *The Gleason–Kahane–Żelazko Theorem and its Generalizations*, Banach Center Publ., vol. 30, Polish Acad. Sci., Warsaw 1994, pp. 327–331.

[27] A.R. SOUROUR, Invertibility preserving linear maps on $\mathcal{L}(X)$, *Trans. Amer. Math. Soc.* **348**(1996), 13–30.

[28] A.E. TAYLOR, D.C. LAY, *Introduction to Functional Analysis*, Wiley, New York-Chichester-Brisbane 1980.

MOSTAFA MBEKHTA, UFR DE MATHÉMATIQUES, UNIVERSITÉ LILLE1, VILLENEUVE CEDEX, 59655, FRANCE
E-mail address: Mostafa.Mbekhta@math.univ-lille1.fr

VLADIMIR MÜLLER, MATHEMATICAL INSTITUTE, CZECH ACADEMY OF SCIENCES ZITNA 25, PRAGUE 1, 11567, CZECH REPUBLIC
E-mail address: muller@math.cas.cz

MOURAD OUDGHIRI, LABORATOIRE LAGA FACULTÉ DES SCIENCES, UNIVERSITÉ MOHAMMED PREMIER, OUJDA, 60000, MOROCCO

Received October 27, 2011; posted on February 17, 2014.