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VARIATION OF DISCRETE SPECTRA OF NON-NEGATIVE OPERATORS IN KREIN SPACES

JUSSI BEHRNDT, LESLIE LEBEN, and FRIEDRICH PHILIPP

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ABSTRACT. We study the variation of the discrete spectrum of a bounded non-negative operator in a Krein space under a non-negative Schatten class perturbation of order p. It turns out that there exist so-called extended enumerations of discrete eigenvalues of the unperturbed and perturbed operator, respectively, whose difference is an ℓ^p-sequence. This result is a Krein space version of a theorem by T. Kato for selfadjoint operators in Hilbert spaces.

KEYWORDS: *Krein space, discrete spectrum, analytic perturbation theory, Schatten–von Neumann ideal.*

MSC (2010): 47A11, 47A55, 47B50.

1. INTRODUCTION

In this note we prove a Krein space version of a result by T. Kato from [22] on the variation of the discrete spectra of selfadjoint operators in Hilbert spaces under additive perturbations from the Schatten–von Neumann ideals $\mathfrak{S}_p$. Although perturbation theory for selfadjoint operators in Krein spaces is a well developed field, and compact, finite rank, as well as bounded perturbations have been studied extensively, only very few results exist that take into account the particular $\mathfrak{S}_p$-character of perturbations. To give an impression of the variety of perturbation results for various classes of selfadjoint operators in Krein spaces we refer the reader to [7], [11], [15], [16], [17], [18], [26] for compact perturbations, to [5], [6], [10], [20], [21] for finite rank perturbations, and to [1], [2], [4], [8], [19], [24], [27], [28] for (relatively) bounded and small perturbations.

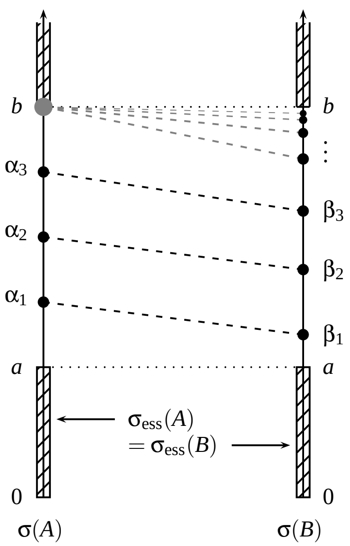
Here we consider a bounded operator A in a Krein space $(\mathcal{K}, [\cdot\,,\cdot])$ which is assumed to be non-negative with respect to the indefinite inner product $[\cdot\,,\cdot]$, and an additive perturbation C which is also non-negative and belongs to some Schatten–von Neumann ideal $\mathfrak{S}_p$, that is, C is compact and its singular values form a sequence in ℓ^p, see, e.g. [14]. Recall that the spectrum of a bounded non-negative operator in $(\mathcal{K}, [\cdot\,,\cdot])$ is real. We also assume that 0 is not a singular critical point of the perturbation C, which is a typical assumption in perturbation theory for selfadjoint operators in Krein spaces; cf. Section 2 for a precise definition. Clearly, the non-negativity and compactness of C imply that the bounded operator

$$B := A + C$$

is also non-negative in $(\mathcal{K}, [\cdot\,,\cdot])$ and its essential spectrum coincides with that of A, whereas the discrete eigenvalues of A and their multiplicity are in general not stable under the perturbation C. Hence, it is particularly interesting to prove qualitative and quantitative results on the discrete spectrum. Our main objective here is to compare the discrete spectra of A and B. For that we make use of the following notion from [22]: Let $\Delta \subset \mathbb{R}$ be a finite union of open intervals. A sequence (α_n) is said to be an *extended enumeration of discrete eigenvalues of A in Δ* if every discrete eigenvalue of A in Δ with multiplicity m appears exactly m-times in the values of (α_n) and all other values α_n are boundary points of the essential spectrum of A in $\overline{\Delta} \subset \mathbb{R}$. An extended enumeration of discrete eigenvalues of B in Δ is defined analogously. The following theorem is the main result of this note.

THEOREM 1.1. *Let A and B be bounded non-negative operators in a Krein space $(\mathcal{K}, [\cdot\,,\cdot])$ such that $B = A + C$, where $C \in \mathfrak{S}_p(\mathcal{K})$ is non-negative, 0 is not a singular critical point of C and $\ker C = \ker C^2$. Then for each finite union of open intervals Δ with $0 \notin \overline{\Delta}$ there exist extended enumerations (α_n) and (β_n) of the discrete eigenvalues of A and B in Δ, respectively, such that*

$$(\beta_n - \alpha_n) \in \ell^p.$$



The adjacent figure illustrates the role of extended enumerations in Theorem 1.1: We consider a gap $(a, b) \subset \mathbb{R}$ in the essential spectrum and compare the discrete spectra of A and B therein. Here the discrete spectrum of the unperturbed operator A in (a, b) consists of the (simple) eigenvalues $\alpha_1, \alpha_2, \alpha_3$, and the eigenvalues β_n, $n = 1, 2, \ldots$, of the perturbed operator B accumulate to the boundary point $b \in \partial\sigma_{\text{ess}}(A)$. Therefore, in the situation of Theorem 1.1 the value b is contained (infinitely many times) in the extended enumeration (α_n) of the discrete eigenvalues of A in (a, b).

For selfadjoint operators A and B in a Hilbert space and an $\mathfrak{S}_p$-perturbation C Theorem 1.1 was proved by T. Kato in [22]. The original proof is based on methods from analytic perturbation theory, in particular, on the properties of a family of real-analytic functions describing the discrete eigenvalues and eigenprojections of the operators $A(t) = A + tC$, $t \in \mathbb{R}$; note that $A(1) = B$ holds. Our proof follows the lines of Kato's proof, but in the Krein space situation some non-trivial additional arguments and adaptions are necessary. In particular, we apply methods from [26] to show that the non-negativity assumptions on A and C yield uniform boundedness of the spectral projections of $A(t)$, $t \in [0,1]$, corresponding to positive and negative intervals, respectively. The non-negativity assumptions on A and C also enter in the construction and properties of the real-analytic functions associated with the discrete eigenvalues of $A(t)$.

Besides the introduction this note consists of three further sections. In Section 2 we recall some definitions and spectral properties of non-negative operators in Krein spaces. Section 3 contains the proof of our main result Theorem 1.1. As a preparation, we discuss the properties of the family of real-analytic functions describing the eigenvalues and eigenspaces of $A(t)$ in Lemma 3.1 and show a result on the uniform definiteness of certain spectral subspaces of $A(t)$ in Lemma 3.2. Afterwards, by modifying and following some of the arguments and estimates in [22] we complete the proof of our main result. Finally, in Section 4 we illustrate Theorem 1.1 with a multiplication operator A and an integral operator C in a weighted L^2-space.

2. PRELIMINARIES ON NON-NEGATIVE OPERATORS IN KREIN SPACES

Throughout this paper let $(\mathcal{K}, [\cdot\,,\cdot])$ be a Krein space. For a detailed study of Krein spaces and operators therein we refer to the monographs [3] and [12]. For the rest of this section let $\|\cdot\|$ be a Banach space norm with respect to which the inner product $[\cdot\,,\cdot]$ is continuous. All such norms are equivalent, see [3]. For closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $\mathcal{K}$ we denote by $L(\mathcal{M},\mathcal{N})$ the set of all bounded and everywhere defined linear operators from $\mathcal{M}$ to $\mathcal{N}$. As usual, we write $L(\mathcal{M}) := L(\mathcal{M},\mathcal{M})$.

Let $T \in L(\mathcal{K})$. The adjoint of T, denoted by T^+, is defined by

$$[Tx, y] = [x, T^+y] \quad \text{for all } x, y \in \mathcal{K}.$$

The operator T is called *selfadjoint* in $(\mathcal{K}, [\cdot\,,\cdot])$ (or $[\cdot\,,\cdot]$-*selfadjoint*) if $T = T^+$. Equivalently, $[Tx, x] \in \mathbb{R}$ for all $x \in \mathcal{K}$. We mention that the spectrum of a selfadjoint operator in a Krein space is symmetric with respect to the real axis but in general not contained in $\mathbb{R}$.

The following definition of spectral points of positive and negative type is from [26].

DEFINITION 2.1. Let $A \in L(\mathcal{K})$ be a selfadjoint operator in the Krein space $\mathcal{K}$. A point $\lambda \in \sigma(A) \cap \mathbb{R}$ is called a *spectral point of positive type* (*negative type*) *of* A if for each sequence $(x_n) \subset \mathcal{K}$ with $\|x_n\| = 1$, $n \in \mathbb{N}$, and $(A - \lambda)x_n \to 0$ as

$n \to \infty$ we have

$$\liminf_{n\to\infty}[x_n, x_n] > 0 \quad (\limsup_{n\to\infty}[x_n, x_n] < 0, \text{ respectively}).$$

The set of all spectral points of positive (negative) type of A is denoted by $\sigma_+(A)$ ($\sigma_-(A)$, respectively). A set $\Delta \subset \mathbb{R}$ is said to be of *positive type* (*negative type*) with respect to A if each spectral point of A in Δ is of positive type (negative type, respectively).

A closed subspace $\mathcal{M} \subset \mathcal{K}$ is called *uniformly positive* (*uniformly negative*) if there exists $\delta > 0$ such that $[x, x] \geqslant \delta\|x\|^2$ ($[x, x] \leqslant -\delta\|x\|^2$, respectively) holds for all $x \in \mathcal{M}$. Equivalently, $(\mathcal{M}, [\cdot\,,\cdot])$ ($(\mathcal{M}, -[\cdot\,,\cdot])$, respectively) is a Hilbert space. For a bounded selfadjoint operator A in $\mathcal{K}$ it follows directly from the definition of $\sigma_+(A)$ and $\sigma_-(A)$ that an isolated eigenvalue $\lambda_0 \in \mathbb{R}$ of A is of positive type (negative type) if and only if $\ker(A - \lambda_0)$ is uniformly positive (uniformly negative, respectively).

A selfadjoint operator $A \in L(\mathcal{K})$ is called *non-negative* if

$$[Ax, x] \geqslant 0 \quad \text{for all } x \in \mathcal{K}.$$

The spectrum of a bounded non-negative operator A is a compact subset of $\mathbb{R}$ and

$$\sigma(A) \cap \mathbb{R}^{\pm} \subset \sigma_{\pm}(A) \tag{2.1}$$

holds, see [25]. The *discrete spectrum* $\sigma_{\mathrm{d}}(A)$ of A consists of the isolated eigenvalues of A with finite multiplicity. The remaining part of $\sigma(A)$ is the *essential spectrum* of the non-negative operator A and is denoted by $\sigma_{\mathrm{ess}}(A)$. Observe that $\sigma_{\mathrm{ess}}(A)$ coincides with the set of λ such that $A - \lambda$ is not a semi-Fredholm operator. Recall that the non-negative operator A admits a spectral function E on $\mathbb{R}$ with a possible singularity at zero, see [25]. The spectral projection $E(\Delta)$ is defined for all Borel sets $\Delta \subset \mathbb{R}$ with $0 \notin \partial\Delta$ and is selfadjoint in $\mathcal{K}$. Hence,

$$\mathcal{K} = E(\Delta)\mathcal{K}[\dotplus](I - E(\Delta))\mathcal{K},$$

which implies that $(E(\Delta)\mathcal{K}, [\cdot\,,\cdot])$ is itself a Krein space. For $\Delta \subset \mathbb{R}^{\pm}$, $0 \notin \overline{\Delta}$, the spectral subspace $(E(\Delta)\mathcal{K}, \pm[\cdot\,,\cdot])$ is a Hilbert space; cf. [25], [26] and (2.1). Note that this implies that every non-zero isolated spectral point of A is necessarily an eigenvalue.

The point zero is called a *critical point* of a non-negative operator $A \in L(\mathcal{K})$ if $0 \in \sigma(A)$ is neither of positive nor negative type. If zero is a critical point of A, it is called *regular* if $\|E([-\frac{1}{n}, \frac{1}{n}])\|$, $n \in \mathbb{N}$, is uniformly bounded, i.e. if zero is not a singularity of the spectral function E. Otherwise, the critical point zero is called *singular*. It should be noted that the non-negative operator $A \in L(\mathcal{K})$ is (similar to) a selfadjoint operator in a Hilbert space if and only if zero is not a singular critical point of A and $\ker A^2 = \ker A$.

3. PROOF OF THEOREM 1.1

Throughout this section let A, B and C be bounded non-negative operators in the Krein space $(\mathcal{K}, [\cdot\,,\cdot])$ as in Theorem 1.1. By assumption 0 is not a singular critical point of C and $C \in \mathfrak{S}_p(\mathcal{K})$. In order to prove Theorem 1.1 we consider the analytic operator function

$$A(z) := A + zC, \quad z \in \mathbb{C}.$$

Note that $A(t)$ is non-negative for $t \geqslant 0$ and $A(1) = B$ holds. Moreover, since C is compact, the essential spectrum of $A(z)$ does not depend on z and hence

$$\sigma_{\text{ess}}(A) = \sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A(z)), \quad z \in \mathbb{C}. \tag{3.1}$$

The following lemma describes the evolution of the *discrete* spectra of the operators $A(t)$, $t \geqslant 0$.

LEMMA 3.1. *Assume that $\sigma_{\mathrm{d}}(A(t_0)) \neq \varnothing$ for some $t_0 \geqslant 0$. Then there exist intervals $\Delta_j \subset \mathbb{R}_0^+$, $j = 1, \dots, m$ or $j \in \mathbb{N}$, and real-analytic functions*

$$\lambda_j(\cdot) : \Delta_j \to \mathbb{R}_0^+ \quad \text{and} \quad E_j(\cdot) : \Delta_j \to L(\mathcal{K}),$$

such that the following holds:

(i) *The sets Δ_j are $\mathbb{R}_0^+$-open intervals which are maximal with respect to* (ii)–(vi) *below.*

(ii) *For each $t \geqslant 0$ we have*

$$\sigma_{\mathrm{d}}(A(t)) \cap \mathbb{R}^+ = \{\lambda_j(t) : j \in \mathbb{N} \text{ such that } t \in \Delta_j \text{ and } \lambda_j(t) \neq 0\}.$$

(iii) *For all j and $t \in \Delta_j$ the set $\{k \in \mathbb{N} : \lambda_k(t) = \lambda_j(t)\}$ is finite and*

$$\sum_{k:\lambda_k(t)=\lambda_j(t)} E_k(t)$$

is the $[\cdot\,,\cdot]$-selfadjoint projection onto $\ker(A(t) - \lambda_j(t))$.

(iv) *For all j the value*

$$m_j := \dim E_j(t)\mathcal{K}, \quad t \in \Delta_j,$$

is constant.

(v) *For all j and $t \in \Delta_j$ there exists an orthonormal basis $\{x_i^j(t)\}_{i=1}^{m_j}$ of the Hilbert space $(E_j(t)\mathcal{K}, [\cdot\,,\cdot])$, such that the functions $x_i^j(\cdot) : \Delta_j \to \mathcal{K}$ are real-analytic and the differential equation*

$$\lambda_j'(t) = \frac{1}{m_j} \sum_{k=1}^{m_j} [Cx_k^j(t), x_k^j(t)] \geqslant 0 \tag{3.2}$$

holds. In particular, $\lambda_j'(t) = 0$ implies $E_j(t)\mathcal{K} \subset \ker C$.

(vi) *Let* $\mathbb{R}^+ \setminus \sigma_{\rm ess}(A) = \dot{\bigcup\limits_n} \mathcal{U}_n$ *with mutually disjoint open intervals* $\mathcal{U}_n \subset \mathbb{R}^+$. *For every* j *there exists* $n \in \mathbb{N}$ *such that*

$$\lambda_j(t) \in \mathcal{U}_n \text{ for all } t \in \Delta_j \quad \text{if } 0 \notin \partial\mathcal{U}_n,$$
$$\lambda_j(t) \in \mathcal{U}_n \cup \{0\} \text{ for all } t \in \Delta_j \quad \text{if } 0 \in \partial\mathcal{U}_n.$$

If $\sup \Delta_j < \infty$ *then* $\sup \mathcal{U}_n < \infty$ *and* $\lim\limits_{t\uparrow \sup \Delta_j} \lambda_j(t) = \sup \mathcal{U}_n$. *Moreover,*

$$\lim_{t\downarrow \inf \Delta_j} \lambda_j(t) = \inf \mathcal{U}_n \quad \text{if } \Delta_j \text{ is open},$$
$$\lim_{t\downarrow 0} \lambda_j(t) \in \mathcal{U}_n \cup \{\inf \mathcal{U}_n\} \quad \text{if } \Delta_j = [0, \sup \Delta_j).$$

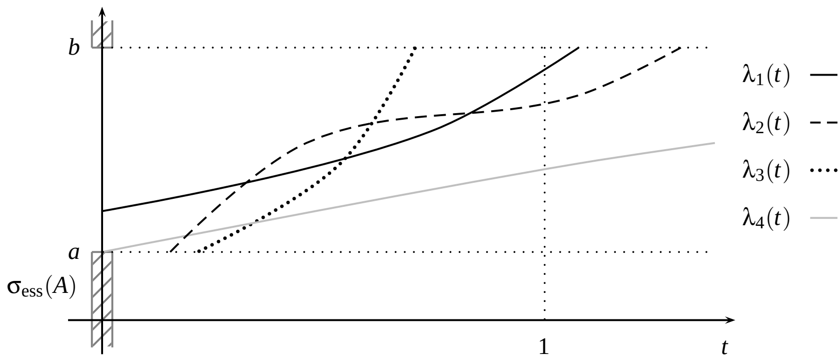


FIGURE 1. Typical situation for the evolution of the discrete eigenvalues of the operator function $A(\cdot)$ in a gap $(a, b) \subset \mathbb{R}$ of the essential spectrum.

Proof. The proof is based on analytic perturbation theory of the discrete eigenvalues; cf. Chapter II and VII of [23], [9] and [22]. We fix some $t_0 \geqslant 0$ for which an eigenvalue $\lambda_0 \in \sigma_{\rm d}(A(t_0)) \cap \mathbb{R}^+$ exists and set $M(t_0) := \ker(A(t_0) - \lambda_0)$. Due to the non-negativity of A and C and since $\lambda_0 > 0$, the inner product space $(M(t_0), [\cdot\,,\cdot])$ is a (finite-dimensional) Hilbert space; cf. (2.1). Therefore, the decomposition

$$\mathcal{K} = M(t_0)[\dot{+}]M(t_0)^{[\perp]}$$

reduces the operator $A(t_0)$. As in Chapter VII, Section 3.1 of [23] one shows that for z in an $\mathbb{R}$-symmetric neighbourhood $\mathcal{D} \subset \mathbb{C}$ of t_0 there exists an analytic operator function $U(\cdot) : \mathcal{D} \to L(\mathcal{K})$ with $U(z)^{-1} = U(\overline{z})^+$, $U(t_0) = I$ and such that $M(t_0)$ is $U(z)^{-1}A(z)U(z)$-invariant, $z \in \mathcal{D}$. Hence, there exist a finite number of (possibly multivalued) analytic functions $\lambda_k(\cdot)$ describing the eigenvalues of the restricted operators $B(z) := U(z)^{-1}A(z)U(z)|M(t_0)$ for $z \in \mathcal{D}$, see, e.g., [9]. Since for real $t \in \mathcal{D}$ the operator $B(t)$ is selfadjoint in the Hilbert space $(M(t_0), [\cdot\,,\cdot])$ it follows from Chapter II, Theorem 1.10 of [23] that the functions $\lambda_k(\cdot)$ are in fact single-valued. The same is true for the eigenprojection functions $E_k(\cdot)$,

$$E_k(z) = -\frac{1}{2\pi \mathrm{i}} \int_{\Gamma_k(z)} (A(z) - \lambda)^{-1} \mathrm{d}\lambda, \quad z \in \mathcal{D},$$

where $\Gamma_k(z)$ is a small circle with center $\lambda_k(z)$. Now a continuation argument implies that there exist functions $\lambda_j(\cdot)$, $E_j(\cdot)$ with the properties (i)–(iv) and (vi); cf. [22].

It remains to prove (v). For this fix $j \in \mathbb{N}$ and $t_0 \in \Delta_j$. Similarly as above there exists a function $U_j(\cdot) : \Delta_j \to E_j(t_0)\mathcal{K}$ with $U_j(t)^+ = U_j(t)^{-1}$, $U_j(t_0) = I$, and $E_j(t) = U_j(t)^+ E_j(t_0) U_j(t)$ for every $t \in \Delta_j$. We choose an orthonormal basis $\{x_1, \ldots, x_{m_j}\}$ of the m_j-dimensional Hilbert space $(E_j(t_0)\mathcal{K}, [\cdot\,,\cdot])$ and define

$$x_k(t) := U_j(t) x_k, \quad t \in \Delta_j, k = 1, \ldots, m_j.$$

For every $t \in \Delta_j$, the set $\{x_1(t), \ldots, x_{m_j}(t)\}$ forms an orthonormal basis of the subspace $(E_j(t)\mathcal{K}, [\cdot\,,\cdot])$, since for $k, l \in \{1, \ldots, m_j\}$ we have

$$[x_k(t), x_l(t)] = [U_j(t)x_k, U_j(t)x_l] = [x_k, x_l] = \delta_{kl}.$$

Let $k \in \{1, \ldots, m_j\}$. Then

$$[x_k'(t), x_k(t)] + [x_k(t), x_k'(t)] = \frac{\mathrm{d}}{\mathrm{d}t}[x_k(t), x_k(t)] = 0$$

and hence

$$\begin{aligned} \lambda_j'(t) &= \frac{\mathrm{d}}{\mathrm{d}t}[\lambda_j(t)x_k(t), x_k(t)] = \frac{\mathrm{d}}{\mathrm{d}t}[A(t)x_k(t), x_k(t)] \\ &= [Cx_k(t), x_k(t)] + [A(t)x_k'(t), x_k(t)] + [A(t)x_k(t), x_k'(t)] \\ &= [Cx_k(t), x_k(t)] + \lambda_j(t)[x_k'(t), x_k(t)] + \lambda_j(t)[x_k(t), x_k'(t)] \\ &= [Cx_k(t), x_k(t)] \geqslant 0. \end{aligned}$$

This yields (3.2). Finally if we have $\lambda_j'(t) = 0$ then $[Cx_k(t), x_k(t)] = 0$ holds for $k = 1, \ldots, m_j$. Since C is non-negative, the Cauchy–Schwarz inequality applied to the non-negative inner product $[C\cdot, \cdot]$ yields

$$\|Cx_k(t)\|^2 = [Cx_k(t), JCx_k(t)] \leqslant [Cx_k(t), x_k(t)]^{1/2}[CJCx_k(t), JCx_k(t)]^{1/2} = 0$$

for every $k \in \{1, \ldots, m_j\}$. This shows $E_j(t)\mathcal{K} \subset \ker C$. ∎

In the proof of the following lemma we make use of methods from [26] in order to show the uniform definiteness of a family of spectral subspaces of $A(t)$.

LEMMA 3.2. *Let $E_{A(t)}$ be the spectral function of the non-negative operator $A(t)$, $t \geqslant 0$, and let $a > 0$. Then there exists $\delta > 0$ such that for all $t \in [0, 1]$ and all $x \in E_{A(t)}([a, \infty))\mathcal{K}$ we have*

$$[x, x] \geqslant \delta \|x\|^2. \tag{3.3}$$

Proof. Since $\max \sigma(A(t)) \leqslant b := \|A\| + \|C\|$ for all $t \in [0,1]$, it is sufficient to prove (3.3) only for $x \in E_{A(t)}([a,b])$. The proof is divided into four steps.

Step 1. In this step we show that there exist $\varepsilon > 0$ and an open neighbourhood $\mathcal{U}$ of $[a,b]$ in $\mathbb{C}$ such that for all $t \in [0,1]$, all $\lambda \in \mathcal{U}$ and all $x \in \mathcal{K}$ we have

$$\|(A(t) - \lambda)x\| \leqslant \varepsilon\|x\| \quad \Longrightarrow \quad [x,x] \geqslant \varepsilon\|x\|^2. \tag{3.4}$$

Assume that ε and $\mathcal{U}$ as above do not exist. Then there exist sequences $(t_n) \subset [0,1]$, $(\lambda_n) \subset \mathbb{C}$ and $(x_n) \subset \mathcal{K}$ with $\|x_n\| = 1$ and $\operatorname{dist}(\lambda_n, [a,b]) < 1/n$ for all $n \in \mathbb{N}$, such that $\|(A(t_n) - \lambda_n)x_n\| \leqslant 1/n$ and $[x_n, x_n] \leqslant 1/n$. It is no restriction to assume that $\lambda_n \to \lambda_0 \in [a,b]$ and $t_n \to t_0 \in [0,1]$ as $n \to \infty$. Therefore,

$$(A(t_0) - \lambda_0)x_n = (t_0 - t_n)Cx_n + (A(t_n) - \lambda_n)x_n + (\lambda_n - \lambda_0)x_n$$

tends to zero as $n \to \infty$. But by (2.1) we have $\lambda_0 \in \sigma_+(A(t_0))$ which implies $\liminf_{n\to\infty}[x_n, x_n] > 0$, contradicting $[x_n, x_n] < 1/n$, $n \in \mathbb{N}$.

Step 2. In the following $\varepsilon > 0$ and $\mathcal{U}$ are fixed such that (3.4) holds, and, in addition, we assume that $|\operatorname{Im}\lambda| < 1$ holds for all $\lambda \in \mathcal{U}$. Next, we verify that for all $t \in [0,1]$

$$\|(A(t) - \lambda)^{-1}\| \leqslant \frac{\varepsilon^{-1}}{|\operatorname{Im}\lambda|}, \quad \lambda \in \mathcal{U} \setminus \mathbb{R}, \tag{3.5}$$

holds. Indeed, for all $t \in [0,1]$, all $\lambda \in \mathcal{U}$ and all $x \in \mathcal{K}$ we either have

$$\|(A(t) - \lambda)x\| > \varepsilon\|x\|$$

or, by (3.4),

$$\varepsilon|\operatorname{Im}\lambda|\|x\|^2 \leqslant |\operatorname{Im}\lambda[x,x]| = |\operatorname{Im}[(A(t) - \lambda)x, x]| \leqslant \|(A(t) - \lambda)x\|\|x\|.$$

Hence, it follows that for all $t \in [0,1]$, all $\lambda \in \mathcal{U}$ and all $x \in \mathcal{K}$ we have

$$\|(A(t) - \lambda)x\| \geqslant \varepsilon|\operatorname{Im}\lambda|\|x\|,$$

which implies (3.5).

Step 3. In the remainder of this proof we set

$$d := \operatorname{dist}([a,b], \partial\mathcal{U}) \quad \text{and} \quad \tau_0 := \min\{\varepsilon^2, \tfrac{d}{2}\}.$$

Let $\Delta \subset [a,b]$ be an interval of length $R \leqslant \tau_0$ and let μ_0 be the center of Δ. We show that for all $t \in [0,1]$ the estimate

$$\|(A(t)|E_t(\Delta)\mathcal{K}) - \mu_0\| \leqslant \varepsilon \tag{3.6}$$

holds. For this let $B(t) := (A(t)|E_t(\Delta)\mathcal{K}) - \mu_0$, $t \in [0,1]$, and note that

$$\sigma(B(t)) \subset [-\tfrac{R}{2}, \tfrac{R}{2}] \subset (-R, R). \tag{3.7}$$

As $R < d$, for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$ with $|\lambda| < R$ we have $\mu_0 + \lambda \in \mathcal{U} \setminus \mathbb{R}$ and hence

$$\|(B(t) - \lambda)^{-1}\| \leqslant \|(A(t) - (\mu_0 + \lambda))^{-1}\| \leqslant \frac{\varepsilon^{-1}}{|\operatorname{Im}\lambda|}$$

by (3.5). From Section 2(b) of [26] we now obtain $\|B(t)\| \leqslant 2\varepsilon^{-1} r(B(t))$, where $r(B(t))$ denotes the spectral radius of $B(t)$. Now (3.6) follows from (3.7) and $R \leqslant \tau_0 \leqslant \varepsilon^2$.

Step 4. We cover the interval $[a,b]$ with mutually disjoint intervals $\Delta_1, \dots, \Delta_n$ of length $< \tau_0$. Let μ_j be the center of the interval Δ_j, $j = 1, \dots, n$. From Step 3 we obtain for all $t \in [0,1]$:

$$\|(A(t)|E_{A(t)}(\Delta_j)\mathcal{K}) - \mu_j\| \leqslant \varepsilon.$$

Hence, by Step 1 of the proof $[x_j, x_j] \geqslant \varepsilon \|x_j\|^2$ for $x_j \in E_{A(t)}(\Delta_j)$, $j = 1, \dots, n$, and $t \in [0,1]$. But

$$E_{A(t)}([a,b]) = E_{A(t)}(\Delta_1)[\dot{+}] \dots [\dot{+}] E_{A(t)}(\Delta_n),$$

and therefore with $x_j := E_{A(t)}(\Delta_j)x$, $j = 1, \dots, n$, we find that

$$[x,x] \geqslant \varepsilon(\|x_1\|^2 + \cdots + \|x_n\|^2) \geqslant \frac{\varepsilon}{2^{n-1}} \|x_1 + \cdots + x_n\|^2 = \frac{\varepsilon}{2^{n-1}} \|x\|^2$$

holds for all $x \in E_{A(t)}([a,b])$ and $t \in [0,1]$, i.e. (3.3) holds with $\delta := \varepsilon / 2^{n-1}$. ∎

Proof of Theorem 1.1. It suffices to prove the theorem for the case that Δ is an open interval (a,b) with $a > 0$. In the case $b < 0$ consider the non-negative operators $-A$, $-B$ and $-C$ in the Krein space $(\mathcal{K}, -[\cdot\,,\cdot])$.

Suppose that for some $t_0 \in [0,1]$ we have $\sigma_{\mathrm{d}}(A(t_0)) \neq \varnothing$, otherwise the theorem is obviously true. Then it follows that there exist

$$\Delta_j, \lambda_j(\cdot), E_j(\cdot) \quad \text{and} \quad x_k^j(\cdot)$$

as in Lemma 3.1 such that $\Delta_j \cap [0,1] \neq \varnothing$ for some $j \in \mathbb{N}$. By $\mathfrak{K}$ denote the set of all j such that $\lambda_j(t) \in (a,b)$ for some $t \in \Delta_j \cap [0,1]$ and for $j \in \mathfrak{K}$ define

$$\widetilde{\Delta}_j := \{t \in \Delta_j \cap [0,1] : \lambda_j(t) \in (a,b)\} = \lambda_j^{-1}((a,b)) \cap [0,1].$$

Due to (3.2) and the continuity of $\lambda_j(\cdot)$ the set $\widetilde{\Delta}_j$ is a (non-empty) subinterval of Δ_j which is open in $[0,1]$. For $j \in \mathfrak{K}$, $t \in [0,1]$ and $k \in \{1, \dots, m_j\}$ we set

$$\widetilde{\lambda}_j(t) := \begin{cases} \lim\limits_{s \downarrow \inf \widetilde{\Delta}_j} \lambda_j(s) & 0 \leqslant t \leqslant \inf \widetilde{\Delta}_j, \\ \lambda_j(t) & t \in \widetilde{\Delta}_j, \\ \lim\limits_{s \uparrow \sup \widetilde{\Delta}_j} \lambda_j(s) & \sup \widetilde{\Delta}_j \leqslant t \leqslant 1, \end{cases} \tag{3.8}$$

$$\widetilde{E}_j(t) := \begin{cases} E_j(t) & t \in \widetilde{\Delta}_j, \\ 0 & t \in [0,1] \setminus \widetilde{\Delta}_j, \end{cases} \quad \text{and} \quad \widetilde{x}_k^j(t) := \begin{cases} x_k^j(t) & t \in \widetilde{\Delta}_j, \\ 0 & t \in [0,1] \setminus \widetilde{\Delta}_j. \end{cases}$$

The functions $\widetilde{\lambda}_j(\cdot)$, $\widetilde{E}_j(\cdot)$, and $\widetilde{x}_k^j(\cdot)$ are differentiable in all but at most two points $t \in [0,1]$ and for each $j \in \mathfrak{K}$ the differential equation

$$\widetilde{\lambda}_j'(t) = \frac{1}{m_j} \sum_{k=1}^{m_j} [C\widetilde{x}_k^j(t), \widetilde{x}_k^j(t)] \geqslant 0 \tag{3.9}$$

holds in all but at most two points $t \in [0,1]$; cf. (3.2). In addition, the projections $\widetilde{E}_j(t)$ are $[\cdot\,,\cdot]$-selfadjoint for every $t \in [0,1]$. The rest of this proof is divided into several steps.

3.1. BASIS REPRESENTATIONS. By E_C denote the spectral function of the non-negative operator C. Since 0 is not a singular critical point of C, the spectral projections $E_C(\mathbb{R}^+)$, $E_C(\mathbb{R}^-)$ and $E_C(\{0\})$ exist. In particular, $E_C(\{0\})\mathcal{K} = \ker C^2 = \ker C$ is a Krein space. Let

$$\ker C = \mathcal{H}_+[\dot{+}]\mathcal{H}_-$$

be an arbitrary fundamental decomposition of $\ker C$. Then with the definition $\mathcal{K}_\pm := \mathcal{H}_\pm[\dot{+}]E_C(\mathbb{R}^\pm)\mathcal{K}$ we obtain a fundamental decomposition

$$\mathcal{K} = \mathcal{K}_+[\dot{+}]\mathcal{K}_-$$

of $\mathcal{K}$. By J denote the fundamental symmetry associated with this fundamental decomposition and set $(\cdot\,,\cdot) := [J\cdot,\cdot]$. Then $(\cdot\,,\cdot)$ is a Hilbert space scalar product on $\mathcal{K}$, and C is a selfadjoint operator in the Hilbert space $(\mathcal{K}, (\cdot\,,\cdot))$. By $\|\cdot\|$ denote the norm induced by $(\cdot\,,\cdot)$. Let (γ_l) be an enumeration of the non-zero eigenvalues of C (counting multiplicities). Since $C \in \mathfrak{S}_p(\mathcal{K})$, we have

$$(\gamma_l) \in \ell^p. \tag{3.10}$$

Let $\{\varphi_l\}_l$ be an $(\cdot\,,\cdot)$-orthonormal basis of $\overline{\operatorname{ran} C}$ such that φ_l is an eigenvector of C corresponding to the eigenvalue γ_l. Then we have $|[\varphi_l, \varphi_i]| = \delta_{li}$. In the following we do not distinguish the cases $\dim \operatorname{ran} C < \infty$ and $\dim \operatorname{ran} C = \infty$, that is, $l = 1, \ldots, m$ for some $m \in \mathbb{N}$ and $l \in \mathbb{N}$, respectively.

Consider the basis representation of $v \in \overline{\operatorname{ran} C}$ with respect to $\{\varphi_l\}_l$. There exist $\alpha_l \in \mathbb{C}$ such that $v = \sum\limits_l \alpha_l \varphi_l$. Therefore

$$[v, \varphi_k] = \sum_l \alpha_l [\varphi_l, \varphi_k] = \alpha_k [\varphi_k, \varphi_k] \quad \text{and} \quad v = \sum_l \frac{[v, \varphi_l]}{[\varphi_l, \varphi_l]} \varphi_l.$$

Consequently, for $x = u + v$, $u \in \ker C$, $v \in \overline{\operatorname{ran} C}$, we have $[x, \varphi_l] = [v, \varphi_l]$ and

$$\begin{aligned} [Cx, x] = [Cx, v] &= \Big[Cx, \sum_l \frac{[x, \varphi_l]}{[\varphi_l, \varphi_l]} \varphi_l\Big] = \sum_l [Cx, \varphi_l] \frac{[\varphi_l, x]}{[\varphi_l, \varphi_l]} \\ &= \sum_l [x, C\varphi_l] \frac{[\varphi_l, x]}{[\varphi_l, \varphi_l]} = \sum_l [x, \gamma_l \varphi_l] \frac{[\varphi_l, x]}{[\varphi_l, \varphi_l]} \\ &= \sum_l \frac{\gamma_l}{[\varphi_l, \varphi_l]} |[x, \varphi_l]|^2 = \sum_l |\gamma_l| |[x, \varphi_l]|^2, \end{aligned} \tag{3.11}$$

where the non-negativity of C was used in the last equality; cf. (2.1). Let $j \in \mathfrak{K}$ be fixed, $t \in \widetilde{\Delta}_j$ and $x \in \mathcal{K}$. Then

$$E_j(t)x = \sum_{k=1}^{m_j} [E_j(t)x, x_k^j(t)]x_k^j(t) = \sum_{k=1}^{m_j} [x, E_j(t)x_k^j(t)]x_k^j(t) = \sum_{k=1}^{m_j} [x, x_k^j(t)]x_k^j(t).$$

If $t \in [0,1] \setminus \widetilde{\Delta}_j$ then $\widetilde{E}_j(t) = 0$ and $\widetilde{x}_k^j(t) = 0$, $k = 1, \ldots, m_j$. Hence

$$\widetilde{E}_j(t)x = \sum_{k=1}^{m_j} [x, \widetilde{x}_k^j(t)]\widetilde{x}_k^j(t) \tag{3.12}$$

holds for all $t \in [0,1]$ and all $x \in \mathcal{K}$.

3.2. NORM BOUNDS. In the following we prove that the projections $\widetilde{E}_j(t)$ are uniformly bounded in $j \in \mathfrak{K}$ and $t \in [0,1]$. For $x \in \mathcal{K}$ we have $\widetilde{E}_j(t)x \in E_{A(t)}([a,b])\mathcal{K}$, and with Lemma 3.2 we obtain

$$\begin{aligned} \|J\widetilde{E}_j(t)x\|\|x\| &\geqslant (J\widetilde{E}_j(t)x, x) = [\widetilde{E}_j(t)x, x] = [\widetilde{E}_j(t)x, \widetilde{E}_j(t)x] \\ &\geqslant \delta\|\widetilde{E}_j(t)x\|^2 = \delta\|J\widetilde{E}_j(t)x\|^2. \end{aligned}$$

This implies

$$\|J\widetilde{E}_j(t)\| \leqslant \frac{1}{\delta}. \tag{3.13}$$

Similarly, $\|E_{A(t)}(\mathfrak{b})\| \leqslant 1/\delta$ is shown to hold for $t \in [0,1]$ and every Borel set $\mathfrak{B} \subseteq (a,b)$. Consequently, the eigenvalues of $J\widetilde{E}_j(t)$ do not exceed $1/\delta$, and from $\dim J\widetilde{E}_j(t)\mathcal{K} \leqslant m_j$ it follows that the $(\cdot,\cdot)$-selfadjoint operator $J\widetilde{E}_j(t)$ has at most m_j non-zero eigenvalues. Hence, its trace $\operatorname{tr}(J\widetilde{E}_j(t))$ satisfies

$$\operatorname{tr}(J\widetilde{E}_j(t)) \leqslant \frac{m_j}{\delta}.$$

3.3. THE MAIN ESTIMATE. Let $j \in \mathfrak{K}$. For $t \in [0,1]$ we have

$$\{\widetilde{\lambda}_j(t) : j \in \mathfrak{K}, \widetilde{\Delta}_j \ni t\} = (a,b) \cap \sigma_{\mathrm{d}}(A(t)) =: \Xi(t),$$

and it follows from the (strong) σ-additivity of the spectral function $E_{A(t)}$ (see, e.g., [26]) that for every $x \in \mathcal{K}$

$$\sum_{j \in \mathfrak{K}} \widetilde{E}_j(t)x = \sum_{j \in \mathfrak{K}, t \in \widetilde{\Delta}_j} E_j(t)x = \sum_{\lambda \in \Xi(t)} E_{A(t)}(\{\lambda\})x = E_{A(t)}(\Xi(t))x. \tag{3.14}$$

From the differential equation (3.9) we obtain for $j \in \mathfrak{K}$

$$\begin{aligned} \widetilde{\lambda}_j(1) - \widetilde{\lambda}_j(0) &= \frac{1}{m_j} \int_0^1 \sum_{k=1}^{m_j} [C\widetilde{x}_k^j(t), \widetilde{x}_k^j(t)]\mathrm{d}t \overset{(3.11)}{=} \frac{1}{m_j} \int_0^1 \sum_{k=1}^{m_j} \sum_l |\gamma_l| |[\widetilde{x}_k^j(t), \varphi_l]|^2 \mathrm{d}t \\ &= \sum_l \frac{|\gamma_l|}{m_j} \int_0^1 \Big[\sum_{k=1}^{m_j} [\varphi_l, \widetilde{x}_k^j(t)]\widetilde{x}_k^j(t), \varphi_l \Big] \mathrm{d}t \overset{(3.12)}{=} \sum_l \frac{|\gamma_l|}{m_j} \int_0^1 [\widetilde{E}_j(t)\varphi_l, \varphi_l]\mathrm{d}t. \end{aligned} \tag{3.15}$$

For $j \in \mathfrak{K}$ and l we set

$$\sigma_{jl} := \frac{1}{m_j} \int_0^1 [\widetilde{E}_j(t)\varphi_l, \varphi_l]\mathrm{d}t \quad \text{and} \quad \sigma_j := \sum_l \sigma_{jl}.$$

Then $\sigma_j \geqslant 0$ for all $j \in \mathfrak{K}$, as $\sigma_{jl} \geqslant 0$ for all l. In fact, we have $\sigma_j > 0$ for each $j \in \mathfrak{K}$. Indeed if $\sigma_j = 0$ for some $j \in \mathfrak{K}$ then for every $t \in [0,1]$

$$\operatorname{tr}(J\widetilde{E}_j(t)) = \sum_l (J\widetilde{E}_j(t)\varphi_l, \varphi_l) = \sum_l [\widetilde{E}_j(t)\varphi_l, \varphi_l] = 0,$$

which implies $J\widetilde{E}_j(t) = 0$ (and thus $\widetilde{E}_j(t) = 0$), since the $(\cdot,\cdot)$-selfadjoint operator $J\widetilde{E}_j(t)$ has only non-negative eigenvalues. Therefore, $\widetilde{\Delta}_j = \varnothing$, which is not possible. Moreover,

$$\begin{aligned} \sigma_j &= \frac{1}{m_j} \int_0^1 \sum_l [\widetilde{E}_j(t)\varphi_l, \varphi_l]\mathrm{d}t = \frac{1}{m_j} \int_0^1 \sum_l (J\widetilde{E}_j(t)\varphi_l, \varphi_l)\mathrm{d}t \\ &\leqslant \frac{1}{m_j} \int_0^1 \operatorname{tr}(J\widetilde{E}_j(t))\mathrm{d}t \leqslant \frac{1}{m_j} \int_0^1 \frac{m_j}{\delta}\mathrm{d}t = \frac{1}{\delta}. \end{aligned} \tag{3.16}$$

In addition (cf. (3.13) and (3.14)), for each l we have

$$\begin{aligned} \sum_{j\in\mathfrak{K}} m_j\sigma_{jl} &= \sum_{j\in\mathfrak{K}} \int_0^1 [\widetilde{E}_j(t)\varphi_l, \varphi_l]\mathrm{d}t = \int_0^1 \Big[\sum_{j\in\mathfrak{K}} \widetilde{E}_j(t)\varphi_l, \varphi_l \Big]\mathrm{d}t \\ &= \int_0^1 [E_{A(t)}(\Xi(t))\varphi_l, \varphi_l]\mathrm{d}t \leqslant \int_0^1 \|E_{A(t)}(\Xi(t))\| \|\varphi_l\|^2 \mathrm{d}t \leqslant \frac{1}{\delta}. \end{aligned} \tag{3.17}$$

Let $j \in \mathfrak{K}$. For $n \in \mathbb{N}$ we set $c_n := \sum_{l=1}^{n} \sigma_{jl}/\sigma_j \leqslant 1$. Then the convexity of $x \mapsto |x|^p$, (3.15), and (3.16) imply

$$\begin{aligned} |\widetilde{\lambda}_j(1) - \widetilde{\lambda}_j(0)|^p &= \lim_{n\to\infty} c_n^p \Big(\sum_{l=1}^{n} \frac{\sigma_{jl}}{c_n\sigma_j} \sigma_j |\gamma_l| \Big)^p \leqslant \lim_{n\to\infty} c_n^{p-1} \sum_{l=1}^{n} \frac{\sigma_{jl}}{\sigma_j} \sigma_j^p |\gamma_l|^p \\ &\leqslant \sum_{l=1}^{\infty} \sigma_{jl}\sigma_j^{p-1}|\gamma_l|^p \leqslant \frac{1}{\delta^{p-1}} \sum_{l=1}^{\infty} \sigma_{jl}|\gamma_l|^p \end{aligned}$$

in the case that ran C is infinite dimensional (that is, $l = 1, \dots, \infty$); otherwise the above estimate holds with a finite sum on the right hand side. Hence, (3.17) and (3.10) yield

$$\sum_{j\in\mathfrak{K}} m_j |\widetilde{\lambda}_j(1) - \widetilde{\lambda}_j(0)|^p \leqslant \frac{1}{\delta^{p-1}} \sum_{j\in\mathfrak{K}} \sum_l m_j\sigma_{jl}|\gamma_l|^p \leqslant \frac{1}{\delta^p} \sum_l |\gamma_l|^p < \infty. \tag{3.18}$$

3.4. FINAL CONCLUSION. It suffices to consider the case $[a,b] \cap \sigma_{\mathrm{ess}}(A) \neq \varnothing$, as otherwise $\sigma_p(A) \cap (a,b)$ and $\sigma_p(B) \cap (a,b)$ are finite sets and hence the theorem holds. We consider the following three possibilities separately: $a, b \in \sigma_{\mathrm{ess}}(A)$, exactly one endpoint of (a,b) belongs to $\sigma_{\mathrm{ess}}(A)$, and $a, b \notin \sigma_{\mathrm{ess}}(A)$.

(i) Assume that $a, b \in \sigma_{\mathrm{ess}}(A)$. Then, by Lemma 3.1 and (3.8) for all $j \in \mathfrak{K}$ the values $\widetilde{\lambda}_j(0)$ and $\widetilde{\lambda}_j(1)$ either are boundary points of $\sigma_{\mathrm{ess}}(A) = \sigma_{\mathrm{ess}}(B)$ (see (3.1)) or points in the discrete spectrum of A and B, respectively. Taking into account the multiplicities of the discrete eigenvalues of A and B it is easy to construct sequences

$$(\alpha_n) \subset \{\widetilde{\lambda}_j(0) : j \in \mathfrak{K}\} \quad \text{and} \quad (\beta_n) \subset \{\widetilde{\lambda}_j(1) : j \in \mathfrak{K}\}$$

such that (α_n) and (β_n) are extended enumerations of discrete eigenvalues of A and B in (a,b) and $(\beta_n - \alpha_n) \in \ell^p$ by (3.18).

(ii) Suppose that $a \notin \sigma_{\mathrm{ess}}(A)$ and $b \in \sigma_{\mathrm{ess}}(A)$ (the case $a \in \sigma_{\mathrm{ess}}(A)$ and $b \notin \sigma_{\mathrm{ess}}(A)$ is treated analogously). Then for each $j \in \mathfrak{K}$ the value $\widetilde{\lambda}_j(1)$ is either a boundary point of $\sigma_{\mathrm{ess}}(B)$ or a discrete eigenvalue of B. Hence, the sequence (β_n) in (i) is an extended enumeration of discrete eigenvalues of B in (a,b). But it might happen that there exist indices $j \in \mathfrak{K}$ such that $\widetilde{\lambda}_j(0) = a$, which is not a boundary point of $\sigma_{\mathrm{ess}}(A)$ and not a discrete eigenvalue of A in (a,b). In the following we shall show that the number of such indices is finite. Then we just replace the corresponding values $\widetilde{\lambda}_j(0)$ in (α_n) by a point in $\partial\sigma_{\mathrm{ess}}(A) \cap (a,b]$ and obtain an extended enumeration (α_n) of discrete eigenvalues of A in (a,b) such that $(\beta_n - \alpha_n) \in \ell^p$.

Assume that $\widetilde{\lambda}_j(0) = a$ for all j from some infinite subset $\mathfrak{K}_a$ of $\mathfrak{K}$. Then $\widetilde{\lambda}_j(t) = a$ for all $t \in [0, t_j]$, where $t_j := \inf \widetilde{\Delta}_j$, $j \in \mathfrak{K}_a$. Observe that $a \in \sigma_{\mathrm{d}}(A(t_j))$ (cf. Lemma 3.1) and $\lambda_j(t_j) = a$, and as $a \notin \sigma_{\mathrm{ess}}(A(t))$ for all $t \in [0,1]$, the set $\{t_j : j \in \mathfrak{K}_a\}$ is an infinite subset of $[0,1]$. Hence we can assume that t_j converges to some t_0, $t_j \neq t_0$ for all $j \in \mathfrak{K}_a$, and that the functions λ_j are not constant. Choose $\varepsilon > 0$ such that $a - \varepsilon > 0$ and

$$([a-\varepsilon, a) \cup (a, a+\varepsilon]) \cap \sigma(A(t_0)) = \varnothing.$$

Either $t_0 \notin \Delta_j$ or $t_0 \in \Delta_j$, in which case $|\lambda_j(t_0) - a| > \varepsilon$ holds. As $\lambda_j(t_j) = a$ for each j there exists s_j between t_0 and t_j such that $|\lambda_j(s_j) - a| = \varepsilon$. Therefore, there exists ξ_j between s_j and t_j such that

$$\varepsilon = |\lambda_j(t_j) - \lambda_j(s_j)| = \lambda_j'(\xi_j)|t_j - s_j| \leqslant \lambda_j'(\xi_j)|t_j - t_0|.$$

Hence, $\lambda_j'(\xi_j) \to \infty$ as $j \to \infty$. On the other hand, by Lemma 3.2 there exists $\delta_0 > 0$ such that $[x,x] \geqslant \delta_0 \|x\|^2$ for all $x \in E_{A(t)}([a-\varepsilon, \infty))\mathcal{K}$ and $t \in [0,1]$. Together with (3.2) this implies

$$\lambda_j'(\xi_j) \leqslant \frac{\|C\|}{m_j} \sum_{l=1}^{m_j} \|x_j^l(\xi_j)\|^2 \leqslant \frac{\|C\|}{m_j \delta_0} \sum_{l=1}^{m_j} [x_j^l(\xi_j), x_j^l(\xi_j)] = \frac{\|C\|}{\delta_0},$$

a contradiction. Hence there exist at most finitely many $j \in \mathfrak{K}$ such that $\widetilde{\lambda}_j(0) = a$.

(iii) If $a, b \notin \sigma_{\mathrm{ess}}(A)$, we choose $c \in (a,b) \cap \sigma_{\mathrm{ess}}(A)$ and construct the extended enumerations (α_n) and (β_n) as the unions of the extended enumerations in (a,c) and (c,b), which exist by (ii). ∎

4. AN EXAMPLE

In this section we discuss an example where the unperturbed operator A is a multiplication operator and the additive perturbation C is a special integral operator from the Hilbert–Schmidt class.

Fix some $\varphi \in L^\infty((-1,1))$ such that $\varphi \leqslant 0$ on $(-1,0)$ and $\varphi \geqslant 0$ on $(0,1)$, and let A be the corresponding multiplication operator in $L^2 := L^2((-1,1))$,

$$(Ah)(x) := \varphi(x)h(x), \quad x \in (-1,1),\ h \in L^2.$$

Moreover, let $q \in L^1((-1,1))$, $q \geqslant 0$, and let u and v be the solutions of the differential equation $\psi'' = q\psi$ satisfying

$$u(-1) = 0, \quad u'(-1) = 1, \quad \text{and} \quad v(1) = 0, \quad v'(1) = 1.$$

Next, define the integral operator C in L^2 by

$$(Ch)(x) := \int_{-1}^{1} k(x,y)h(y)\mathrm{d}y, \quad x \in (-1,1),\ h \in L^2, \tag{4.1}$$

where the kernel k has the form

$$k(x,y) = \frac{1}{vu' - uv'} \begin{cases} v(x)u(y)\,\mathrm{sgn}(y) & -1 < y < x, \\ u(x)v(y)\,\mathrm{sgn}(y) & x < y < 1. \end{cases}$$

In this situation our main result Theorem 1.1 yields the following corollary.

COROLLARY 4.1. *Let A and C be as above and let $B = A + C$. Then for each finite union of open intervals Δ with $0 \notin \overline{\Delta}$ there exist an extended enumeration (β_n) of the discrete eigenvalues of B in Δ and a sequence (α_n) of boundary points of $\sigma_{\mathrm{ess}}(A)$ in $\mathbb{R}$, such that*

$$(\beta_n - \alpha_n) \in \ell^2.$$

Proof. Define an indefinite inner product $[\cdot\,,\cdot]$ on L^2 by

$$[f,g] := \int_{-1}^{1} f(x)\overline{g(x)}\,\mathrm{sgn}(x)\mathrm{d}x, \quad f,g \in L^2.$$

It is easy to see that A is selfadjoint and non-negative in $(L^2, [\cdot\,,\cdot])$, and that $\sigma(A) = \sigma_{\mathrm{ess}}(A) = \mathrm{essran}\varphi$ holds. Moreover, as in Satz 13.16 of [29] it follows that $C^{-1}f = \mathrm{sgn}\cdot(-f'' + qf)$ is the (unbounded) Sturm–Liouville differential operator with Dirichlet boundary conditions at -1 and 1, which is selfadjoint in

$(L^2, [\cdot\,,\cdot])$ and non-negative since q is assumed to be non-negative. Furthermore, by Theorem 3.6(iii) of [13] the point ∞ is a regular critical point of C^{-1}, and hence 0 is a regular critical point of C. Clearly, $\ker C = \ker C^2 = \{0\}$, and as k is an L^2-kernel we have $C \in \mathfrak{S}_2(L^2)$.

Hence, the operators A and $B = A + C$ satisfy the assumptions of Theorem 1.1. Therefore, for each finite union of open intervals Δ with $0 \notin \overline{\Delta}$ there exist extended enumerations (α_n) and (β_n) of the discrete eigenvalues of A and B in Δ, respectively, such that $(\beta_n - \alpha_n) \in \ell^2$. But A does not have any discrete eigenvalues, and hence each α_n is a boundary point of $\sigma_{\text{ess}}(A)$ in $\mathbb{R}$. ∎

We remark that Corollary 4.1 does not claim the existence of a finite or infinite set of discrete eigenvalues of $B = A + C$, e.g. the extended enumeration (β_n) may consist only of boundary points of $\sigma_{\text{ess}}(B)$. In the next example we consider the case that φ is constant on $(-1, 0)$ and $(0, 1)$. In this situation it turns out that every integral operator C of the form (4.1) in fact leads to a sequence of discrete eigenvalues of $A + C$ accumulating to $\sigma_{\text{ess}}(A)$.

EXAMPLE 4.2. Assume that the function φ is equal to a constant $\varphi_+ > 0$ on $(0, 1)$ and $\varphi_- < 0$ on $(-1, 0)$, let $q \in L^1((-1, 1))$, $q \geqslant 0$, and let C be the corresponding integral operator in (4.1). Then the discrete eigenvalues of $B = A + C$ accumulate to φ_+ and φ_-, and every sequence (β_n) of eigenvalues of B, converging to φ_+ (φ_-) satisfies

$$(\beta_n - \varphi_+) \in \ell^2 \quad ((\beta_n - \varphi_-) \in \ell^2, \text{respectively}).$$

In fact, since $\sigma_{\text{ess}}(B) = \sigma_{\text{ess}}(A) = \sigma(A) = \sigma_p(A) = \{\varphi_-, \varphi_+\}$ and every isolated spectral point of a non-negative operator is an eigenvalue, it is sufficient to show that φ_+ and φ_- are no eigenvalues of $B = A + C$. We verify that the operator $A + C - \varphi_-$ is injective; a similar argument shows that $A + C - \varphi_+$ is injective. Let $f \in L^2$ such that $(A + C - \varphi_-)f = 0$. Then we have

$$g(x) := (Cf)(x) = (\varphi_- - A)f(x) = \begin{cases} (\varphi_- - \varphi_+)f(x) & x \in (0,1), \\ 0 & x \in (-1,0), \end{cases} \tag{4.2}$$

and since C^{-1} is the Sturm–Liouville operator corresponding to the expression $\operatorname{sgn}(-\mathrm{d}^2/\mathrm{d}x^2 + q)$ with Dirichlet boundary conditions at ± 1 (cf. Satz 13.16 of [29]) we conclude that g and g' are absolutely continuous on $(-1, 1)$ and

$$f(x) = (C^{-1}g)(x) = \operatorname{sgn}(x)(-g''(x) + q(x)g(x)), \quad x \in (-1,1). \tag{4.3}$$

Since $g = 0$ on $(-1, 0)$ we have $f = 0$ on $(-1, 0)$ from (4.3). Moreover, from (4.3) we obtain $f = -g'' + qg$ on the interval $(0, 1)$. Now, (4.2) and the continuity of g and g' yield

$$-g''(x) + \Big(q(x) + \frac{1}{\varphi_+ - \varphi_-}\Big)g(x) = 0, \quad g(0) = g'(0) = 0,$$

for almost all $x \in (0, 1)$. Therefore, $g = 0$ on $(0, 1)$ and hence also $f = 0$ on $(0, 1)$.

REFERENCES

[1] S. ALBEVERIO, A.K. MOTOVILOV, A.A. SHKALIKOV, Bounds on variation of spectral subspaces under J-self-adjoint perturbations, *Integral Equations Operator Theory* **64**(2009), 455–486.

[2] S. ALBEVERIO, A.K. MOTOVILOV, C. TRETTER, Bounds on the spectrum and reducing subspaces of a J-self-adjoint operator, *Indiana Univ. Math. J.* **59**(2010), 1737–1776.

[3] T.YA. AZIZOV, I.S. IOKHVIDOV, *Linear Operators in Spaces with an Indefinite Metric*, John Wiley & Sons Ltd., Chichester 1989.

[4] T.YA. AZIZOV, J. BEHRNDT, P. JONAS, C. TRUNK, Spectral points of definite type and type π for linear operators and relations in Krein spaces, *J. London Math. Soc.* **83**(2011), 768–788.

[5] T.YA. AZIZOV, J. BEHRNDT, F. PHILIPP, C. TRUNK, On domains of powers of linear operators and finite rank perturbations, *Oper. Theory Adv. Appl.* **188**(2008), 31–37.

[6] T.YA. AZIZOV, J. BEHRNDT, C. TRUNK, On finite rank perturbations of definitizable operators, *J. Math. Anal. Appl.* **339**(2008), 1161–1168.

[7] T.YA. AZIZOV, P. JONAS, C. TRUNK, Spectral points of type π_+ and π_- of selfadjoint operators in Krein spaces, *J. Funct. Anal.* **226**(2005), 114–137.

[8] T.YA. AZIZOV, P. JONAS, C. TRUNK, Small perturbations of selfadjoint and unitary operators in Krein spaces, *J. Operator Theory* **64**(2010), 401–416.

[9] H. BAUMGÄRTEL, Zur Störungstheorie beschränkter linearer Operatoren eines Banachschen Raumes, *Math. Nachr.* **26**(1964), 361–379.

[10] J. BEHRNDT, Finite rank perturbations of locally definitizable selfadjoint operators in Krein spaces, *J. Operator Theory* **58**(2007), 415–440.

[11] J. BEHRNDT, P. JONAS, On compact perturbations of locally definitizable selfadjoint relations in Krein spaces, *Integral Equations Operator Theory* **52**(2005), 17–44.

[12] J. BOGNÁR, *Indefinite Inner Product Spaces*, Springer-Verlag, Berlin-Heidelberg-New York 1974.

[13] B. ĆURGUS, H. LANGER, A Krein space approach to symmetric ordinary differential operators with an indefinite weight function, *J. Differential Equations* **79**(1989), 31–61.

[14] I.C. GOHBERG M.G. KREĬN, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Transl. Math. Monographs, vol. 18, Amer. Math. Soc., Providence, RI 1969.

[15] P. JONAS, Compact perturbations of definitizable operators. II, *J. Operator Theory* **8**(1982), 3–18.

[16] P. JONAS, On a class of selfadjoint operators in Krein space and their compact perturbations, *Integral Equations Operator Theory* **11**(1988), 351–384.

[17] P. JONAS, A note on perturbations of selfadjoint operators in Krein spaces, *Oper. Theory Adv. Appl.* **43**(1990), 229–235.

[18] P. JONAS, On a problem of the perturbation theory of selfadjoint operators in Krein spaces, *J. Operator Theory* **25**(1991), 183–211.

[19] P. JONAS, Riggings and relatively form bounded perturbations of nonnegative operators in Krein spaces, *Oper. Theory Adv. Appl.* **106**(1998), 259–273.

[20] P. JONAS, H. LANGER, Compact perturbations of definitizable operators, *J. Operator Theory* **2**(1979), 63–77.

[21] P. JONAS, H. LANGER, Some questions in the perturbation theory of J-nonnegative operators in Krein spaces, *Math. Nachr.* **114**(1983), 205–226.

[22] T. KATO, Variation of discrete spectra, *Comm. Math. Phys.* **111**(1987), 501–504.

[23] T. KATO, *Perturbation Theory for Linear Operators*, Springer, Berlin-Heidelberg 1995.

[24] V. KOSTRYKIN, K.A. MAKAROV, A.K. MOTOVILOV, Perturbation of spectra and spectral subspaces, *Trans. Amer. Math. Soc.* **359**(2007), 77–89.

[25] H. LANGER, Spectral functions of definitizable operators in Krein spaces, in *Functional Analysis (Dubrovnik, 1981)*, Lect. Notes in Math., vol. 948, Springer, Berlin-New York 1982, pp. 1–46.

[26] H. LANGER, A. MARKUS, V. MATSAEV, Locally definite operators in indefinite inner product spaces, *Math. Ann.* **308**(1997), 405–424.

[27] H. LANGER, B. NAJMAN, Perturbation theory for definitizable operators in Krein spaces, *J. Operator Theory* **9**(1983), 297–317.

[28] C. TRETTER, Spectral inclusion for unbounded block operator matrices, *J. Funct. Anal.* **256**(2009), 3806–3829.

[29] J. WEIDMANN, *Lineare Operatoren in Hilberträumen, Teil* II. *Anwendungen*, Teubner, Stuttgart 2003.

JUSSI BEHRNDT, INSTITUT FÜR NUMERISCHE MATHEMATIK, TECHNISCHE UNIVERSITÄT GRAZ, 8010 GRAZ, AUSTRIA
E-mail address: behrndt@tugraz.at

LESLIE LEBEN, INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT ILMENAU, 98684 ILMENAU, GERMANY
E-mail address: leslie.leben@tu-ilmenau.de

FRIEDRICH PHILIPP, INSTITUT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN, 10623 BERLIN, GERMANY
E-mail address: philipp@math.tu-berlin.de

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