J. OPERATOR THEORY
71:1(2014), 175–197
doi: 10.7900/jot.2011dec08.1932

ASYMPTOTIC COMMUTATIVITY

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Communicated by Şerban Strătilă

ABSTRACT. The standard commutants in a noncommutative algebra are derived from commutativity which in terms of Lie algebras means $(\mathrm{ad}T)(S) = 0$. Some "weaker commutativities" given by vanishing (asymptotic vanishing) properties of the powers of $\mathrm{ad}T$, for instance $(\mathrm{ad}T)^n(S) = 0$ or $\lim\limits_{n\to\infty} \|(\mathrm{ad}T)^n(S)\|^{1/n} = 0$ when T and S are bounded linear operators on some complex Banach space, describe in a similar way different type of "weaker commutants". This paper studies these "weaker commutants" and their corresponding compositions, in particular "weaker bicommutants", in connection with J. von Neumann's classical bicommutant theorem.

KEYWORDS: *Bounded operator, commutant, selfadjoint operator algebra.*

MSC (2010): Primary 46L10, 47B47; Secondary 47A11, 47B48.

1. INTRODUCTION

One of the main concepts derived from commutativity is "the commutant" and the well-known J. von Neumann's bicommutant theorem (see [14]) in the algebra of all bounded operators on a Hilbert space is one of the main results concerning it. On the other hand a lot of properties containing as particular case commutativity in a (normed) algebra, particularly for bounded operators on a Banach space, have been described and well studied in terms of Lie theory (see for instance [2], [4], [8], [13], [17]). So, if S, T are bounded operators on a complex Banach space, let $\mathrm{ad}T(S) = TS - ST$. T commutes with S means $\mathrm{ad}T(S) = 0$ and the commutant of a subset $\mathcal{S}$ of bounded operators is $\mathcal{S}' = \{T : (\mathrm{ad}T)(S) = 0, \forall S \in \mathcal{S}\}$. The two conditions

$$\lim_{n\to\infty} \|(\mathrm{ad}T)^n(S)\|^{1/n} = 0 \quad \text{and} \quad \lim_{n\to\infty} \|(\mathrm{ad}S)^n(T)\|^{1/n} = 0$$

have respectively as particular cases the following equalities,

$$(\mathrm{ad}T)^m(S) = 0 \quad \text{and} \quad (\mathrm{ad}S)^p(T) = 0$$

for some m, p natural numbers. For $m = 1$ or $p = 1$ these mean that T commutes with S. Using one of the above four mentioned equalities we can associate to a subset $\mathcal{S}$ of bounded operators different types of commutants $\mathcal{C}_\alpha(\mathcal{S})$ in the similar way as the equality $\mathrm{ad}T(S) = 0$ associates to $\mathcal{S}$ its commutant $\mathcal{C}_1(\mathcal{S}) = \mathcal{S}'$. This paper studies these "commutants" and the composition of such commutants $\mathcal{C}_\alpha(\mathcal{C}_\beta(\mathcal{S}))$, in particular $\mathcal{C}_\alpha\mathcal{C}_\alpha(\mathcal{S})$, in connection with the classical bicommutant $\mathcal{C}_1(\mathcal{C}_1(\mathcal{S})) = (\mathcal{S}')' = \mathcal{S}''$ and some closure operators on the set of all subsets of bounded operators on a Hilbert space.

2. THE SPACES $\mathcal{Y}^U(\{0\})$

Let $\mathcal{Y}$ be a complex Banach space and $\mathcal{B}(\mathcal{Y})$ the Banach algebra of all bounded linear operators on $\mathcal{Y}$. For an open subset D of $\mathbb{C}$ (the field of complex numbers), $\mathcal{O}(D, \mathcal{Y})$ denotes the Fréchet space of all analytic $\mathcal{Y}$-valued functions on D.

Recall that $S \in \mathcal{B}(\mathcal{Y})$ has the single valued extension property (s.v.e.p.) if for every open subset G of $\mathbb{C}$, the unique analytic solution $f \in \mathcal{O}(G, \mathcal{Y})$ of the equation $(\lambda - T)f(\lambda) = 0, \lambda \in G$, is identically 0. Thus, for every $y \in \mathcal{Y}$ and $S \in \mathcal{B}(\mathcal{Y})$ having (s.v.e.p.) there exists a maximal open subset $\rho_S(y) \subset \mathbb{C}$ and a unique analytic function $\check{y} \in \mathcal{O}(\rho_S(y), \mathcal{Y})$ such that $(\lambda - T)f(\lambda) = y$ for every $\lambda \in \rho_S(y)$. Usually $\sigma_S(y) = \mathbb{C} \setminus \rho_S(y)$ is the local spectrum of T at y and for every closed set $F \subset \mathbb{C}$, $\mathcal{Y}_S(F)$ denotes the linear manifold of all vectors satisfying $\sigma_S(y) \subset F$ (see [4], [6]).

In the following, we will consider an analogue of the above linear manifold $\mathcal{Y}_S(F)$ for an arbitrary $U \in \mathcal{B}(\mathcal{Y})$, U not necessarily having (s.v.e.p.). For an arbitrary $U \in \mathcal{B}(\mathcal{Y})$ we consider these spaces only for some closed subsets $F \subset \mathbb{C}$. More precisely, the following definition describes a class of closed subsets F of $\mathbb{C}$ attached to an arbitrary $U \in \mathcal{B}(\mathcal{Y})$. Every subset F in this class defines a subspace $\mathcal{Y}^U(F)$ of $\mathcal{Y}$ in a similar way as $\mathcal{Y}_S(F)$ are defined for S having (s.v.e.p.). In particular, for a complex Banach space $\mathcal{X}$, $\mathcal{Y} = \mathcal{B}(\mathcal{X}) \ni T$ and $U = \mathrm{ad}(T) \in \mathcal{B}(\mathcal{Y})$, the corresponding subspace $\mathcal{Y}^U(\{0\})$ will be employed in the following to describe "weaker commutantants" ($\{0\}$denotes the subset of $\mathbb{C}$ consisting of the null element 0 of $\mathbb{C}$).

DEFINITION 2.1. For a closed subset F of $\mathbb{C}$ we say that U has (s.v.e.p.) on $\mathbb{C} \setminus F$, or F is *analytic spectral compatible* with $U \in \mathcal{B}(\mathcal{Y})$, if the only solution $f \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$ of the equation

$$(\lambda - U)f(\lambda) = 0, \quad \lambda \in \mathbb{C} \setminus F$$

is the null function.

EXAMPLES 2.2. We mention the following examples:

(i) The spectrum of $U, \sigma(U)$ is analytic spectral compatible with U.

(ii) If F is a closed subset $F \subset \mathbb{C}$ and $\mathbb{C} \setminus F$ is a connex set having a nonempty intersection with the resolvent set $\rho(U)$ of U, then F is analytic spectral compatible with U.

(iii) Every compact subset K of $\mathbb{C}$ with $\mathbb{C} \setminus K$ a connex set is analytic spectral compatible with U.

(iv) Every finite subset of $\mathbb{C}$ is analytic spectral compatible with U.

(v) $S \in \mathcal{B}(\mathcal{Y})$ has the single valued extension property if and only if every closed subset F of $\mathbb{C}$ is analytic spectral compatible with S; every closed subset F of $\mathbb{C}$ is analytic spectral compatible with every $S \in \mathcal{B}(\mathcal{Y})$ having (s.v.e.p.).

Definition 2.1 can be rewritten as in the following lemma.

LEMMA 2.3. *For $U \in \mathcal{B}(\mathcal{Y})$ and a closed subset $F \subset \mathbb{C}$, the following assertions are equivalent:*

(i) *F is analytic spectral compatible with U;*

(ii) *there exists $y \in \mathcal{Y}$ such that the equation*

$$(\lambda - U)h(\lambda) = y, \quad \text{for all } \lambda \in \mathbb{C} \setminus F$$

has a unique solution $f \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$.

Proof. (i) $\Rightarrow$ (ii) because by (i) $y = 0$ satisfies (ii). It remains to prove (ii) $\Rightarrow$ (i). It is obvious if we have $y = 0$ in (ii). If in (ii) we have $y \neq 0$ and h is the unique solution given by (ii), then $(\lambda - U)(h(\lambda) + f(\lambda)) = y$, for every $\lambda \in \mathbb{C} \setminus F$ and $f \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$ verifying $(\lambda - U)f(\lambda) = 0$ for $\lambda \in \mathbb{C} \setminus F$. Then we deduce by (ii) $h(\lambda) + f(\lambda) = h(\lambda)$ for every $\lambda \in \mathbb{C} \setminus F$, which gives $f(\lambda) = 0$ for every $\lambda \in \mathbb{C} \setminus F$, hence (i) holds. ∎

Recall now a well known analogue of the above linear manifold $\mathcal{Y}_S(F)$ (associated to $S \in \mathcal{B}(\mathcal{Y})$ having (s.v.e.p.)) for an arbitrary $U \in \mathcal{B}(\mathcal{Y})$. The strong spectral manifold $M(F, U) = \overline{M_0(F, U)}$ was defined (see [3]) for an arbitrary $U \in \mathcal{B}(\mathcal{Y})$ and every closed subset $F \subset \mathbb{C}$. Sometimes called the "global spectral space" (see [9]), $M_0(F, T)$ consists of all $y \in \mathcal{Y}$ for which there exists an analytic function $f \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$ satisfying the equation $(\lambda - U)f(\lambda) = y$ for every $\lambda \in \mathbb{C} \setminus F$. Also used in describing property (δ) for an arbitrary bounded operator on $\mathcal{Y}$ (see [1]), the linear manifold $M_0(F, U)$ is $\mathcal{Y}_S(F)$ for every closed subset $F \subset \mathbb{C}$ if S has (s.v.e.p.). Therefore an analogue of $\mathcal{Y}_S(F)$ for an arbitrary $U \in \mathcal{B}(\mathcal{Y})$ can be the linear manifold $M_0(F, U)$. When $F \subset \mathbb{C}$ is a closed subset "analytic spectral compatible"with U, the following definition introduces a suggestive name and notation for $M_0(F, U)$ in order to simplify our exposition.

DEFINITION 2.4. If $U \in \mathcal{B}(\mathcal{Y})$ and $F \subset \mathbb{C}$ is a closed set that is analytic spectral compatible with U we denote

$$\mathcal{Y}^U(F) = \{y \in \mathcal{Y} : \exists f \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y}), (\lambda - U)f(\lambda) = y \text{ for all } \lambda \in \mathbb{C} \setminus F\}$$

and we call it the *resolvent space of U corresponding to F*.

REMARK 2.5. We note the following:

(i) Obviously $\mathcal{Y}^U(F) = M_0(F,U)$ is a linear manifold and by Definition 2.1 it is easy to deduce (see Lemma 2.3) that for every $y \in \mathcal{Y}^U(F)$ there exists a unique function $f \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$ satisfying the equation $(\lambda - U)f(\lambda) = y$ for every $\lambda \in \mathbb{C} \setminus F$. We denote this unique function by $\check{y}_F$.

(ii) The linear manifold $\mathcal{Y}^U(F)$ is defined for an arbitrary $U \in \mathcal{B}(\mathcal{Y})$ only for F analytic spectral compatible with U. If S has (s.v.e.p.) then $\mathcal{Y}^S(F) = \mathcal{Y}_S(F)$ for every closed subset $F \subset \mathbb{C}$.

The following proposition is a direct consequence of the definitions above.

PROPOSITION 2.6. *Let $U \in \mathcal{B}(\mathcal{Y})$ and $\mathcal{Y}^U(F)$ be a resolvent space for U as in Definition 2.4. Then we have:*

(i) *$X\mathcal{Y}^U(F) \subset \mathcal{Y}^U(F)$, for all $X \in \mathcal{B}(\mathcal{Y})$ commuting with U.*

(ii) *If $z = Xy$, then $\check{z}_F = X\check{y}_F$ for every $y \in \mathcal{Y}^U(F)$.*

PROPOSITION 2.7. *In the conditions of the Proposition 2.6, the following equality defines a linear operator $R_F(\lambda, U)$ on $\mathcal{Y}^U(F)$, if for each $y \in \mathcal{Y}^U(F)$ and $\lambda \in \mathbb{C} \setminus F$, we have:*

(i) *$R_F(\lambda, U) : \mathcal{Y}^U(F) \to \mathcal{Y}^U(F)$, $R_F(\lambda, U)y = \check{y}_F(\lambda)$.*

(ii) *For every $\lambda \in \mathbb{C} \setminus F$, $R_F(\lambda, U)$ is the inverse of the linear operator $(\lambda - U)|_{\mathcal{Y}^U(F)}$.*

Proof. First, we prove (i). It is easy to observe that the following equalities define for every $\lambda \in \mathbb{C} \setminus F$ a function $g_\lambda \in \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$,

$$g_\lambda(\mu) = \begin{cases} \frac{\check{y}_F(\mu) - \check{y}_F(\lambda)}{\lambda - \mu} & \mu \neq \lambda, \\ -\check{y}'_F(\lambda) & \mu = \lambda. \end{cases}$$

Now, a simple computation gives $(\mu - U)g_\lambda(\mu) = \check{y}_F(\lambda)$ for every $\lambda, \mu \in \mathbb{C} \setminus F$. This means that $\check{y}_F(\lambda) \in \mathcal{Y}^U(F)$ for every $\lambda \in \mathbb{C} \setminus F$ and (i) is proved. For proving (ii), let y be an arbitrary element $y \in \mathcal{Y}^U(F)$. By the definition of $R_F(\lambda, U)$ and $\check{y}_F(\lambda)$, we have

$$(\lambda - U)R_F(\lambda, U)y = (\lambda - U)\check{y}_F(\lambda) = y$$

for all $\lambda \in \mathbb{C} \setminus F$. We can also compute $R_F(\lambda, U)(\lambda - U)y = \check{z}_F(\lambda)$ where $z = (\lambda - U)y$. By Proposition 2.6, we have $\check{z}_F(\mu) = (\lambda - U)\check{y}_F(\mu)$ for every $\lambda, \mu \in \mathbb{C} \setminus F$ and finally we have for $\mu = \lambda$, $R_F(\lambda, U)(\lambda - U)y = (\lambda - U)\check{y}_F(\lambda) = y$ which concludes the proof. ∎

COROLLARY 2.8. *Let $U \in \mathcal{B}(\mathcal{X})$ be an arbitrary linear operator on $\mathcal{X}$ and $\rho(U)$ the resolvent set of U. Let $F \subset \mathbb{C}$ be a closed subset analytic spectral compatible with U and $R_F(\lambda, U)$ from the above Proposition 2.7. If $(\mathbb{C} \setminus F) \cap \rho(U) \neq \emptyset$, then we have:*

(i) *$R_F(\lambda, U)$ is an extension from $(\mathbb{C} \setminus F) \cap \rho(U)$ to $\mathbb{C} \setminus F$ of the resolvent $R(\lambda, U)$ of U.*

(ii) *For every* $y \in \mathcal{Y}^U(F)$, $R_F(\lambda, U)y = \check{y}_F(\lambda)$ *is the single analytic extension* g *of* $R(\lambda, U)y$ *from* $(\mathbb{C} \setminus F) \cap \rho(U)$ *to* $\mathbb{C} \setminus F$ *and satisfying the equation* $(\lambda - U)g(\lambda) = y$ *for* $\lambda \in \mathbb{C} \setminus F$.

Proof. First we observe that $R(\lambda, U)\mathcal{Y}^U(F) \subset \mathcal{Y}^U(F)$ for every $\lambda \in \rho(U)$, so $R(\lambda, U) = R_F(\lambda, U)$ on $(\mathbb{C} \setminus F) \cap \rho(U) \neq \emptyset$ as inverse of the same operator $(\lambda - U)|_{\mathcal{Y}^U(F)}$. This proves (i). For proving (ii), it suffices to recall that $R_F(\lambda, U)y = \check{y}_F(\lambda)$ is analytic on $\mathbb{C} \setminus F$ (see Proposition 2.7), $R(\lambda, U)$ is analytic on $(\mathbb{C} \setminus F) \cap \rho(U) \neq \emptyset$, and $\check{y}_F(\lambda)$ is the single analytic function on $\mathbb{C} \setminus F$ verifying $(\lambda - U)g(\lambda) = y$ because F is analytic spectral compatible with U. ∎

Although the linear manifold $\mathcal{Y}^U(F)$ is not generally closed, it is possible to describe some other topological properties of it. In [11] it is proved that for every closed subset F of $\mathbb{C}$ and $S \in \mathcal{B}(\mathcal{X})$ having (s.v.e.p.), the linear manifold $\mathcal{Y}_S(F)$ is the range of a continuous $\mathcal{Y}$-valued linear map defined on some Fréchet space. This property holds also for $\mathcal{Y}^U(F)$, the linear manifold corresponding to an arbitrary U and a closed subset $F \subset \mathbb{C}$ analytic spectral compatible with U, the above mentioned corresponding property when S has (s.v.e.p.) being a particular case (see Examples 2.2(v)).

PROPOSITION 2.9. *If* $U \in \mathcal{B}(\mathcal{Y})$ *and* $F \subset \mathbb{C}$ *is analytic spectral compatible with* U*, then there exist a Fréchet space* $\mathcal{Y}_{\mathrm{F}}$ *and a continuous linear injective map* $\phi_U : \mathcal{Y}_{\mathrm{F}} \to \mathcal{Y}$ *such that* Range $\phi_F = \mathcal{Y}^U(F)$.

Proof. The following assertions give the proof and will also be used in the follwing. Let us consider $\mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$ the standard Fréchet space of all $\mathcal{Y}$-valued analytic functions on $\mathbb{C} \setminus F$.

Step 1. We have

$$\begin{aligned}\mathcal{Y} &\simeq \{y : \mathbb{C} \setminus F \to \mathcal{Y}, y(\lambda) = y \text{ for all } \lambda \in \mathbb{C} \setminus F \text{ where } y \in \mathcal{Y}\} \\ &= \{f : \mathbb{C} \setminus F \to \mathcal{Y} \text{ where } f \text{ is constant}\} \subset \mathcal{O}(\mathbb{C} \setminus F).\end{aligned}$$

By this identification $\simeq$, the norm topology of $\mathcal{Y}$ is the topology induced by the Fréchet topology of $\mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$.

Step 2. $\mathcal{Y}$ can be considered a closed subspace of $\mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$.

Step 3. The following linear map is continuous,

$$\begin{aligned}&\phi_U : \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y}) \to \mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y}), \\ &[\phi_U(\varphi)](\lambda) = (\lambda - U)\varphi(\lambda) \quad \text{for all } \lambda \in \mathbb{C} \setminus F \quad \text{and} \\ &\phi_U(\check{y}_F) = y, \quad \text{for all } y \in \mathcal{Y}^U(F).\end{aligned}$$

Step 4. $\phi_U^{-1}(\mathcal{Y})$ is a closed subspace of $\mathcal{O}(\mathbb{C} \setminus F, \mathcal{Y})$, hence a Fréchet space.

Step 5. Because F is analytic spectral compatible with U we have

$$\phi_U^{-1}(\{y\}) = \begin{cases} \{\check{y}_F\} & \text{if } y \in \mathcal{Y}^U(F), \\ \emptyset & \text{if } y \notin \mathcal{Y}^U(F). \end{cases}$$

Obviously $\phi_U(\phi_U^{-1}(\mathcal{Y})) = \mathcal{Y}^U(F)$ and $\phi_U|_{\phi_U^{-1}(\mathcal{Y})}$ is an injective map.

So the conclusion of the Proposition 2.9 holds if we put $\mathcal{Y}_{\mathrm{F}} = \phi_U^{-1}(\mathcal{Y}) = \phi_U^{-1}(\mathcal{Y}^U(F))$, by observing that $\phi_U(\mathcal{Y}_{\mathrm{F}}) = \mathcal{Y}^U(F)$. ∎

The inverse of $\phi_U|_{\phi_U^{-1}(\mathcal{Y})}$ is defined on $\mathcal{Y}^U(F)$ as the following map

$$y \to \check{y}_F, \quad \text{for every } y \in \mathcal{Y}^U(F)$$

which can also be denoted by ϕ_U^{-1}. The inverse of this algebraic isomorphism is the continous map $\phi_U|_{\phi_U^{-1}(\mathcal{Y})}$.

Now we consider a particular case of a closed set analytic spectral compatible with $U \in \mathcal{B}(\mathcal{Y})$ and its corresponding resolvent space, namely the subset $\{0\}$ of $\mathbb{C}$ and the corresponding resolvent space $\mathcal{Y}^U(\{0\})$ (by Examples 2.2(iv), $\{0\}$ is spectral compatible with all $U \in \mathcal{B}(\mathcal{Y})$). Using Laurent's development of an analytic function on $\mathbb{C} \setminus \{0\}$, an appropriate description of the linear manifold $\mathcal{Y}^U(\{0\})$ can be given. The following proposition is devoted to it.

PROPOSITION 2.10. *For every $U \in \mathcal{B}(\mathcal{Y})$ the following equalities hold:*

$$\mathcal{Y}^U(\{0\}) = \left\{ y : \lim_{n\to\infty} \|(U)^n y\|^{1/n} = 0 \right\} \quad \textit{and}$$
$$\check{y}_{\{0\}}(\lambda) = \sum_{n\geqslant 0} \lambda^{-(n+1)} U^n y, \quad \textit{for every } \lambda \in \mathbb{C}\setminus\{0\}, y \in \mathcal{Y}^U(\{0\}).$$

The proof is an easy consequence of the following lemma.

LEMMA 2.11. *For an arbitrary fixed $y \in \mathcal{Y}$ and $U \in \mathcal{B}(\mathcal{Y})$ the following statements are equivalent:*

(i) *There exists $\varphi \in \mathcal{O}(\mathbb{C}\setminus\{0\}, \mathcal{Y})$ such that $(\lambda - U)\varphi(\lambda) = y$ for every $\lambda \in \mathbb{C}\setminus\{0\}$.*

(ii) $\lim_{n\to\infty} \|U^n y\|^{1/n} = 0$.

If (i) *holds, then*

$$\check{y}_{\{0\}}(\lambda) = \varphi(\lambda) = \sum_{n\geqslant 0} \lambda^{-(n+1)} U^n y, \quad \textit{for every } \lambda \in \mathbb{C}\setminus\{0\}.$$

Proof. (ii) $\Rightarrow$ (i) Indeed, by (ii) $\varphi(\lambda) = \sum_{n\geqslant 0} \lambda^{-(n+1)} U^n y, \lambda \in \mathbb{C}\setminus\{0\}$, is an analytic function on $\mathbb{C}\setminus\{0\}$ and obviously satisfies (i).

(i) $\Rightarrow$ (ii) Let us consider φ as in (i). Then there exists a sequence $\{y_n\} \subset \mathcal{Y}$ so that

$$\varphi(\lambda) = \sum_{n=-\infty}^{n=+\infty} \lambda^n y_n$$

and $(\lambda - U)\varphi(\lambda) = y$, for every $\lambda \in \mathbb{C}\setminus\{0\}$. We then have,

$$(\lambda - U)\varphi(\lambda) = \sum_{n=-\infty}^{n=+\infty} \lambda^{n+1} y_n - \sum_{n=-\infty}^{n=+\infty} \lambda^n U y_n = \sum_{m\neq 0} \lambda^m (y_{m-1} - U y_m) + y_{-1} - U y_0$$

and by (i) it follows that

$$\sum_{m\neq 0} \lambda^m (y_{m-1} - Uy_m) + y_{-1} - Uy_0 = y$$

for every $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore, we have

$$y_{m-1} = Uy_m, \quad \text{for every } m \neq 0 \text{ and } y_{-1} - Uy_0 = y \tag{2.1}$$

and we can write for $n \geqslant 0$,

$$U^{n-k} y_n = y_k \quad \text{if } 0 \leqslant k \leqslant n.$$

Then the following inequalities hold:

$$\|y_k\| \leqslant \|U^{n-k}\| \|y_n\| \quad \text{if } 0 \leqslant k \leqslant n.$$

But $\varphi \in \mathcal{O}(\mathbb{C} \setminus \{0\}, \mathcal{Y})$ gives $\lim\limits_{n\to+\infty} \|y_n\|^{1/n} = 0$ and by the above inequalities we get

$$y_k = 0 \quad \text{for every } k \geqslant 0.$$

In particular $y_0 = 0$ and by (2.1) it is easy to obtain $y = y_{-1} - Uy_0 = y_{-1}, y_{-2} = Uy_{-1} = Uy, \ldots, y_{-(n+1)} = Uy_n = U^n y$ for every $n > 0$. Hence, we obtain a complete description of φ from (i) (in particular this gives also a direct proof of the uniqueness of φ in (i)) i.e.

$$\check{y}_{\{0\}}(\lambda) = \varphi(\lambda) = \sum_{n\geqslant 0} \lambda^{-(n+1)} U^n y, \quad \text{for every } \lambda \in \mathbb{C} \setminus \{0\}$$

and obviously we have $\lim\limits_{n\to\infty} \|U^n y\|^{1/n} = 0$. Hence (i) $\Rightarrow$ (ii) and the lemma has been proved. ∎

Now, we use the Proposition 2.10 in order to obtain $\mathcal{Y}^U(\{0\})$ as the range of an injective continuous linear map on the Fréchet space $\mathcal{Y}_{\{0\}}$ (see Proposition 2.9). More precisely, in the particular case $F = \{0\}$ we have to describe the following objects from Proposition 2.9: the Fréchet space $\mathcal{Y}_{\mathrm{F}}$ and the map $\phi_U : \mathcal{Y}_{\mathrm{F}} \to \mathcal{Y}$, a continuous linear injective map having $\operatorname{Range} \phi_U = \mathcal{Y}^U(\{0\})$. For $F = \{0\}$ we have

$$\phi_U : \mathcal{O}(\mathbb{C} \setminus \{0\}, \mathcal{Y}) \to \mathcal{O}(\mathbb{C} \setminus \{0\}, \mathcal{Y})$$

where

$$[\phi_U(\varphi)](\lambda) = (\lambda - U)\varphi(\lambda) = \sum_{n=-\infty}^{n=+\infty} \lambda^{n+1} y_n - \sum_{n=-\infty}^{n=+\infty} \lambda^n U y_n,$$

for every

$$\varphi \in O(\mathbb{C} \setminus \{0\}, \mathcal{Y}), \quad \varphi(\lambda) = \sum_{n=-\infty}^{n=+\infty} \lambda^n y_n \text{ and } \lambda \in \mathbb{C} \setminus \{0\}.$$

The Banach space $\mathcal{Y}$ is a closed subspace of $\mathcal{O}(\mathbb{C} \setminus \{0\}, \mathcal{Y})$ and

$$\mathcal{Y} = \Big\{ \sum_{n=-\infty}^{+\infty} \lambda^n y_n : y_n = 0 \text{ for every } n \neq 0 \text{ and } y_0 \in \mathcal{Y} \Big\}.$$

The structure of $\mathcal{Y}^U(\{0\})$ is described as in the Propositions 2.9, 2.10. So, we deduce $\mathcal{Y}_{\{0\}} = \phi_U^{-1}(\mathcal{Y}) = \phi_U^{-1}(\mathcal{Y}^U(\{0\}))$ and

$$\mathcal{Y}_{\{0\}} = \Big\{ \sum_{n\geqslant 0} \lambda^{-(n+1)} U^n y : \text{ every } \lambda \in \mathbb{C}\setminus\{0\}, y, \lim_{n\to\infty} \|U^n y\|^{1/n} = 0 \Big\}.$$

We thus have a linear isomorphism

$$\phi_U : \mathcal{Y}_{\{0\}} \to \mathcal{Y}^U(\{0\}) \subset \mathcal{Y}, \quad \sum_{n\geqslant 0} \lambda^{-(n+1)} U^n y \mapsto y$$

for all $y \in \mathcal{Y}$ verifying $\lim\limits_{n\to\infty} \|U^n y\|^{1/n} = 0$. So, $\phi_U(\mathcal{Y}_{\{0\}}) = \mathcal{Y}^U(\{0\})$ and ϕ_U is an injective linear continuous map from the Fréchet space $\mathcal{Y}_{\{0\}}$ into the Banach space $\mathcal{Y}$.

Finally we discuss the locally convex topologies on $\mathcal{Y}^U(\{0\})$. Given a family $\{\mathcal{Y}_\alpha\}_{\alpha\in A}$ of complex vector subspaces $\mathcal{Y}_\alpha$ of a complex vector space $\mathcal{Y}$, there exists a locally convex topology on $\mathcal{Y}$ having $\mathcal{Y}_\alpha$ as closed subspace for every $\alpha \in A$. Every linear functional $f \neq 0$ on $\mathcal{Y}$ is given by the equalities $f(e_\gamma) = 1, f(e_\beta) = 0$ if $\beta \neq \gamma$ for some linear base $\{e_\beta\}$ of $\mathcal{Y}$. As every linear independent system of vectors in $\mathcal{Y}$ can be completed to a base, there exists a family of linear functionals $\{f_{\alpha\beta}\}_{(\alpha,\,\beta)\in A\times B}$ on $\mathcal{Y}$ such that $\mathcal{Y}_\alpha = \bigcap\limits_{\beta\in B} \ker f_{\alpha\beta}$ for every $\alpha \in A$. The topology on $\mathcal{Y}$ given by the family of seminorms $p_{\alpha\beta}(\cdot) = |f_{\alpha\beta}(\cdot)|$, $(\alpha,\beta) \in A \times B$, has $\mathcal{Y}_\alpha$ as closed subspace for every $\alpha \in A$. In this way can be defined a locally convex topology on $\mathcal{Y}$ such that in this topology $\mathcal{Y}^U(\{0\})$ are closed subspaces for all $U \in \mathcal{B}(\mathcal{Y})$.

On the other hand, for an arbitrary fixed $U \in \mathcal{B}(\mathcal{Y})$ a Fréchet topology on $\mathcal{Y}^U(\{0\})$ can be derived from an equivalent assertion with the definition of this linear manifold given by the following lemma. As usual $\mathbb{N}$ denotes the set of all natural numbers.

LEMMA 2.12. *The following assertions are equivalent:*

(i) $y \in \mathcal{Y}^U(\{0\})$.

(ii) $\lim\limits_{n\to\infty} \|U^n y\|^{1/n} = 0$.

(iii) $\sup\limits_{n\in\mathbb{N}} \|(\alpha U)^n y\| < +\infty$ *for every* $\alpha \in \mathbb{N}, \alpha \geqslant 1$.

Proof. (i) $\Leftrightarrow$ (ii) is given by Proposition 2.10. It is easy to see that (iii) is equivalent with the following assertion:

$$\text{for every } \alpha > 0 \text{ there exist } k_\alpha \in \mathbb{N} \text{ such that } \sup_{n\geqslant k_\alpha} \|(\alpha U)^n x\| < +\infty.$$

For proving (ii) $\Rightarrow$ (iii) it is sufficient to observe that for every $\alpha > 0$ there exists $k_\alpha \in \mathbb{N}$ such that $\|U^n x\| < (1/\alpha)^n$ for every $n \geqslant k_\alpha$. For proving (iii) $\Rightarrow$ (ii) we deduce from (iii) that for every $\alpha > 0$ there exists M_α with $\|(\alpha U)^n x\| < M_\alpha$ for every $n \in \mathbb{N}$ i.e. $\|U^n x\| < M_\alpha (1/\alpha)^n$ for every $n \in \mathbb{N}$. This gives

$$\limsup_{n\to\infty} \|U^n x\|^{1/n} < \frac{1}{\alpha} \quad \text{for every } \alpha > 0 \quad \text{and} \quad \lim_{n\to\infty} \|U^n x\|^{1/n} = 0.$$

Hence (iii) $\Rightarrow$ (ii), (ii) $\Leftrightarrow$ (iii) and the lemma has been proved. ∎

NOTATION 2.13. For $U \in \mathcal{B}(\mathcal{Y})$ and $\alpha > 0$, we denote for every $y \in \mathcal{Y}$,

$$p_{U,\alpha}(y) = \sup_{n\in\mathbb{N}} \|(\alpha U)^n y\| = \sup\{\|y\|, \|\alpha U y\|, \dots, \|\alpha^n U^n y\|, \dots\}.$$

REMARK 2.14. We note the following:

(i) For every $\alpha > 0$, $p_{U,\alpha} : \mathcal{Y}^U(\{0\}) \to \mathbb{R}_+$ is a norm on $\mathcal{Y}^U(\{0\})$.

(ii) The equivalence given by Lemma 2.12 can be rewritten:

$$y \in \mathcal{Y}^U(\{0\}) \text{ if and only if } p_{U,\alpha}(y) < +\infty \text{ for every } \alpha \in \mathbb{N}, \alpha \geqslant 1.$$

PROPOSITION 2.15. *The increasing sequence of norms $\{p_{U,k}\}_{1\leqslant k\in\mathbb{N}}$ defines a Fréchet space topology on $\mathcal{Y}^U(\{0\})$. Every norm $p_{U,k}$, $1 \leqslant k \in \mathbb{N}$ on $\mathcal{Y}^U(\{0\})$ can be extended to a norm on $\mathcal{Y}$. The resulting sequence of norms on $\mathcal{Y}$ defines a metrizable locally convex topology on $\mathcal{Y}$, and in this topology $\mathcal{Y}^U(\{0\})$ is a closed subspace of $\mathcal{Y}$.*

Proof. First we verify that $\{p_{U,k}\}_{1\leqslant k\in\mathbb{N}}$ defines a Fréchet space topology on $\mathcal{Y}^U(\{0\})$. Consider $\{y_n\} \subset \mathcal{Y}^U(\{0\})$ a Cauchy sequence corresponding to the increasing sequence of norms $\{p_{U,k}\}_{1\leqslant k\in\mathbb{N}}$. We now prove that there exists $y \in \mathcal{Y}^U(\{0\})$ such that $\{p_{U,k}\}(y_n - y) \to 0$ for $n \to \infty$. By the Cauchy property of $\{y_n\}$, for every $1 \leqslant k \in \mathbb{N}$ and $\eta > 0$ there exists a natural number $m(k,\eta)$ such that

$$\|(kU)^n (y_m - y_p)\| < \eta \quad \text{for every } m, p > m(k,\eta) \text{ and every } n \in \mathbb{N}.$$

For $n = 0$ we deduce that $\{y_n\}$ is a Cauchy sequence in the Banach space $\mathcal{Y}$. So there exists $y \in \mathcal{Y}$ such that $\|y_n - y\| \to 0$ for $n \to \infty$ and taking the limit for $p \to \infty$ in the above inequalities, we obtain

$$\|(kU)^n (y_m - y)\| \leqslant \eta \quad \text{for every } m > m(k,\eta) \text{ and every } \; n \in \mathbb{N}.$$

Using Remark 2.14(ii) we deduce that $y_m - y \in \mathcal{Y}^U(\{0\})$, $y = y_m - (y_m - y) \in \mathcal{Y}^U(\{0\})$ and

$$\lim_{m\to\infty} p_{U,k}(y_m - y) = 0 \quad \text{for every } k, 1 \leqslant k \in \mathbb{N}$$

which means that $y_m \to y$ in the topology given by the increasing sequence of norms $\{p_{U,k}\}_{1\leqslant k\in\mathbb{N}}$. Therefore $\mathcal{Y}^U(\{0\})$ endowed with the increasing sequence of norm $\{p_{U,k}\}_{1\leqslant k\in\mathbb{N}}$ is a Fréchet space. Using Zorn's lemma as in the proof of the Hahn–Banach theorem it is possible to prove that there exists an extension of the seminorm $p_{U,k}$ and the inequality $\|y\| \leqslant p_{U,k}(y)$ from $\mathcal{Y}^U(\{0\})$ to a seminorm $\widetilde{p}_{U,k}$ on $\mathcal{Y}$ verifying the inequality $\|y\| \leqslant \widetilde{p}_{U,k}(y)$ for every $y \in \mathcal{Y}$. Indeed, we can prove the following lemma.

LEMMA 2.16. *Let $\mathcal{Y}$ be a complex vector space, s be a seminorm on $\mathcal{Y}$ and $\mathcal{Y}_1$ a subspace of $\mathcal{Y}$. The following assertions hold:*

(i) *Every seminorm p on $\mathcal{Y}_1$ can be extended to a seminorm $\widetilde{p}$ on $\mathcal{Y}$.*

(ii) *If $s(y_1) \leqslant p(y_1)$ for every $y_1 \in \mathcal{Y}_1$, then p can be extended to a seminorm $\widetilde{p}$ verifying $s(y) \leqslant \widetilde{p}(y)$ for every $y \in \mathcal{Y}$ and $\widetilde{p}$ is a norm on $\mathcal{Y}$ if the seminorm s is a norm.*

Proof. First we prove the lemma for $\mathcal{Y} = \mathcal{Y}_1 + \mathbb{C}x, x \notin \mathcal{Y}_1$. In this case an arbitrary $y \in \mathcal{Y}$ can be written uniquely $y = y_1 + \lambda x, y_1 \in \mathcal{Y}_1, \lambda \in \mathbb{C}$ and $\widetilde{p}(y) = p(y_1) + |\lambda|$ is well defined and $\widetilde{p}$ is a seminorm extension of p to $\mathcal{Y}$. If $s(y_1) \leqslant p(y_1)$ for every $y_1 \in \mathcal{Y}_1$, then

$$s(y_1 + \lambda x) \leqslant s(y_1) + |\lambda| \leqslant p(y_1) + |\lambda| = \widetilde{p}(y).$$

because s is a seminorm and we can chose $x \in \mathcal{Y} \setminus \mathcal{Y}_1$ with the property $s(x) = 1$ or $s(x) = 0$ for every $x \in \mathcal{Y} \setminus \mathcal{Y}_1$. Now it is easy to verify that for the following ordered set $\widetilde{\mathcal{S}}$ the hypothesis of Zorn's lemma hold. $\widetilde{\mathcal{S}}$ consists of all pairs $(\mathcal{Z}, q)$ where $\mathcal{Z}$ is a subspace of $\mathcal{Y}$, $\mathcal{Y}_1 \subset \mathcal{Z}$ and q is a seminorm extension of p to $\mathcal{Z}$ (verifying $s(z) \leqslant q(z)$ for every $z \in \mathcal{Z}$ when we use $\widetilde{\mathcal{S}}$ for the proof of the second part of the lemma). The order on $\widetilde{\mathcal{S}}$ is the following:

$$(\mathcal{Z}_1, q_1) \prec (\mathcal{Z}_2, q_2) \text{ if and only if } \mathcal{Z}_1 \subset \mathcal{Z}_2 \text{ and } q_2|_{\mathcal{Z}_1} = q_1.$$

Therefore $\widetilde{\mathcal{S}}$ has a maximal element $(\mathcal{Z}_0, q_o)$. If $\mathcal{Z}_0 \neq \mathcal{Y}$ there exists $x \in \mathcal{Y} \setminus \mathcal{Z}_0$ and we can consider $\mathcal{Z}_1 = \mathcal{Z}_0 + \mathbb{C}x$. From the first part of the proof there exists a seminorm extension of q_0 to $\mathcal{Z}_1$ verifying the conclusion of lemma which contradicts the maximality of $(\mathcal{Z}_0, q_o)$. Thus, $\mathcal{Z}_0 = \mathcal{Y}$ and q_0 will be $\widetilde{p}$ from the conclusion of the lemma which is now completely proved. ∎

We conclude by proving that $\mathcal{Y}^U(\{0\})$ is a closed subspace of $\mathcal{Y}$ in the locally convex topology given by the norms $\widetilde{p}_{U,k}$, $1 \leqslant k \in \mathbb{N}$. Let $\{y_n\} \subset \mathcal{Y}^U(\{0\})$ be a sequence in $\mathcal{Y}$ and $y_n \to y \in \mathcal{Y}$, $n \to \infty$, in the metrizable locally convex topology on $\mathcal{Y}$ given by the increasing sequence of norms $\{\widetilde{p}_{U,k}\}_{1 \leqslant k \in \mathbb{N}}$ on $\mathcal{Y}$. Then $\{y_n\}$ is a Cauchy sequence in the Fréchet topology of $\mathcal{Y}^U(\{0\})$ given by the norms $p_{U,k}$ because the norms $\widetilde{p}_{U,k}$ are extensions of the norms $p_{U,k}$. So there exists $y_0 \in \mathcal{Y}^U(\{0\})$ such that $y_n \to y_0$ in the Fréchet topology of $\mathcal{Y}^U(\{0\})$ given by the norms $p_{U,k}$ therefore $y = y_0$ and Proposition 2.15 is proved. ∎

REMARK 2.17. We note the following:

(i) The locally convex metrizable topology on $\mathcal{Y}$ is stronger than the topology given by the initial norm $\|\cdot\|$ of $\mathcal{Y}$ ($\|y\| \leqslant \widetilde{p}_{U,k}(y)$ for every $y \in \mathcal{Y}$ and $1 \leqslant k \in \mathbb{N}$).

(ii) The set $\{y \in \mathcal{Y} : p_{U,k}(y) \leqslant m\}$ is for every $1 \leqslant k \in \mathbb{N}$ a closed subset of the normed space $\mathcal{Y}$ for every $1 \leqslant m \in \mathbb{N}$.

PROPOSITION 2.18. *$\mathcal{Y}^U(\{0\})$ is a subset of type F_σ of the Banach space $\mathcal{Y}$.*

Proof. The following decomposition holds

$$\mathcal{Y}^U(\{0\}) = \bigcup_{m\geqslant 1} \{p_{U,k}(y) \leqslant m\}, \quad \text{for every } 1 \leqslant k \in \mathbb{N},$$

and the above Remark 2.17 concludes the proof. ∎

All the above results concerning the space $\mathcal{Y}^U(\{0\})$ can be rewritten for $\mathcal{Y} = \mathcal{A}$ a Banach algebra and $U = \mathrm{ad}T \in \mathcal{B}(\mathcal{A}), T \in \mathcal{A}, \mathrm{ad}T(S) = TS - ST$ for $S \in \mathcal{A}$ (in particular for $\mathcal{A} = \mathcal{B}(\mathcal{X})$ the Banach algebra of all bounded linear operator on a complex Banach space $\mathcal{X}$).

NOTATION 2.19. For $\mathcal{Y} = \mathcal{A}$, including the particular case when $\mathcal{A} = \mathcal{B}(\mathcal{X})$, we denote

$$\mathcal{Y}^U(\{0\}) = \mathcal{A}^{\mathrm{ad}T}(\{0\})$$

when $U = \mathrm{ad}T, T \in \mathcal{A}, \mathrm{ad}T(S) = TS - ST$ for $S \in \mathcal{A}$.

We have a specific description for $\mathcal{A}^{\mathrm{ad}T}(\{0\})$ when $\mathcal{A} = \mathcal{B}(\mathcal{X})$ and $T \in \mathcal{B}(\mathcal{X})$,

$$\mathcal{A}^{\mathrm{ad}T}(\{0\}) = \Big\{S \in \mathcal{B}(\mathcal{X}) : \lim_{n\to\infty} \|(\mathrm{ad}T)^n(S)\| = 0\Big\}.$$

First we can attach to $x \in \mathcal{X}, 1 \leqslant k \in \mathbb{N}$ and $T \in \mathcal{B}(\mathcal{X})$ the following function on $\mathcal{B}(\mathcal{X})$:

$$p_{\mathrm{ad}T,k,x}(S) = \sup_{n\in\mathbb{N}}\{\|(k \cdot \mathrm{ad}T)^n(S)x\|\} \quad \text{for every } S \in \mathcal{B}(\mathcal{X}).$$

The following inequalities hold:

$$\|Sx\| \leqslant p_{\mathrm{ad}T,k,x}(S) \leqslant p_{\mathrm{ad}T,k}(S)\|x\|. \tag{2.2}$$

If we denote

$$\widetilde{\mathcal{A}}^{\mathrm{ad}T}(\{0\}) = \{S : p_{\mathrm{ad}T,k,x}(S) < +\infty \text{ for every } x \in \mathcal{X} \text{ and } 1 \leqslant k \in \mathbb{N}\},$$

it is easy to observe by the above inequalities (2.2) that $\mathcal{A}^{\mathrm{ad}T}(\{0\}) \subset \widetilde{\mathcal{A}}^{\mathrm{ad}T}(\{0\})$. By the Banach–Steinhaus theorem we obtain the equality $\mathcal{A}^{\mathrm{ad}T}(\{0\}) = \widetilde{\mathcal{A}}^{\mathrm{ad}T}(\{0\})$ and the following description of $\mathcal{A}^{\mathrm{ad}T}(\{0\})$.

PROPOSITION 2.20. *If $\mathcal{X}$ is a complex Banach space and $T \in \mathcal{B}(\mathcal{X})$ is a bounded linear operator on $\mathcal{X}$ then the following equalities hold:*

$$\begin{aligned}\mathcal{A}^{\mathrm{ad}T}(\{0\}) &= \{S : p_{\mathrm{ad}T,k,x}(S) < +\infty \textit{ for every } x \in \mathcal{X} \textit{ and } 1 \leqslant k \in \mathbb{N}\}\\ &= \{S : p_{\mathrm{ad}T,k}(S) < +\infty \textit{ for every } 1 \leqslant k \in \mathbb{N}\}\\ &= \Big\{S \in \mathcal{B}(\mathcal{X}) : \lim_{n\to\infty} \|(\mathrm{ad}T)^n(S)\| = 0\Big\}.\end{aligned}$$

By applying Lemma 2.16 in two steps using the two inequalities from (2.2) we deduce now that the families of seminorms on $\mathcal{A}^{\mathrm{ad}T}(\{0\})$ $\{p_{\mathrm{ad}T,k,x}\}_{k\geqslant 1, x\in\mathcal{X}}$ and $\{p_{\mathrm{ad}T,k}\}_{k\geqslant 1}$ can be extended to a family of seminorms $\{\widetilde{p}_{\mathrm{ad}T,k,x}\}_{k\geqslant 1,x\in\mathcal{X}}$ and a family of norms $\{\widetilde{p}_{\mathrm{ad}T,k}\}_{k\geqslant 1}$ on $\mathcal{B}(\mathcal{X})$. These satisfay the inequalities,

$$\|Sx\| \leqslant \widetilde{p}_{\mathrm{ad}T,k,x}(S) \leqslant \widetilde{p}_{\mathrm{ad}T,k}(S)\|x\|. \tag{2.3}$$

So we can associate to each $T \in \mathcal{B}(\mathcal{X})$ two locally convex topologies on $\mathcal{B}(\mathcal{X})$ stronger than τ_{so}, the so-topology on $\mathcal{B}(\mathcal{X})$. If we denote $\tau_{\text{sa},T}$ (*the strong asymptotic commutativity topology associated to* T) the locally convex topology given by the family of seminorms $\{\widetilde{p}_{\text{ad}T,k,x}\}_{k\geqslant 1,x\in\mathcal{X}}$ and if we denote $\tau_{\text{a},T}$ (*the asymptotic commutativity topology associated to* T) the local convex topology given by the family of norms $\{p_{\text{ad}T,k}\}_{k\geqslant 1}$, then we have

$$\tau_{\text{so}} \prec \tau_{\text{sa},T} \prec \tau_{\text{a},T}. \tag{2.4}$$

Finally we recall that for $\mathcal{A} = \mathcal{B}(\mathcal{X})$, $\mathcal{A}^{\text{ad}T}(\{0\})$ is a closed subspace of $\mathcal{B}(\mathcal{X})$ in the Fréchet topology $\tau_{\text{a},T}$.

3. ASYMPTOTIC COMMUTATIVITY, NIL-COMMUTATIVITY, ASYMPTOTIC COMMUTANTS, AND "OTHER" COMMUTANTS

In this section we discuss some extensions of commutativity. Let $\mathcal{A}$ be an associative algebra. Recall the following notation for commutativity which will be used in the following (see [12]). For $S, T \in \mathcal{A}$, $S \smile T$ means that S commutes with T i.e. $ST = TS$ or, in terms of Lie algebras, $\text{ad}S(T) = 0$. This relation is symmetric i.e. $S \smile T \Rightarrow T \smile S$ and for $\mathcal{L} \subset \mathcal{A}$ we denote

$$\mathcal{L}' = \{X \in \mathcal{A} : X \smile S \text{ for every } S \in \mathcal{L}\} \text{ the commutant of } \mathcal{L},$$
$$\mathcal{L}'' \text{ the bicommutant of } \mathcal{L}, \text{etc.}$$

The usual commutativity is embedded as a particular case in the following straightforward implications:

$$S \smile T \Rightarrow (\text{ad}S)^n(T) = 0 \quad \text{for } 1 \leqslant n \in \mathbb{N},$$

and if $\mathcal{A}$ is a normed algebra we have,

$$(\text{ad}S)^k(T) = 0 \quad \text{for some } k \in \mathbb{N} \Rightarrow \lim_{n\to\infty} \|(\text{ad}S)^n(T)\|^{1/n} = 0.$$

We now define extensions of the usual commutativity concept, and the associated "commutants", by using the (weaker) properties of commutativity which are contained in the two implications above. We begin by defining n-commutativity in an associative algebra $\mathcal{A}$, which corresponds to the first implication above.

DEFINITION 3.1. Let $1 \leqslant n \in \mathbb{N}$ be a fixed natural number. If A and B are two elements of $\mathcal{A}$, we say that A *n-commutes* with B if we have

$$(\text{ad}A)^n(B) = 0$$

and we denote this property by $A \overset{n}{\smile} B$.

REMARK 3.2. (i) For $n = 1$ we have the commutativity,

$$A \overset{1}{\smile} B \text{ if and only if } A \smile B.$$

(ii) For $n \neq 1$, in general $A \overset{n}{\smile} B$ does not imply $B \overset{n}{\smile} A$ (the relation $A \overset{n}{\smile} B$ is not symmetric).

(iii) $A \overset{n}{\smile} B \Rightarrow A \overset{m}{\smile} B$ for every $m \geqslant n$, in particular

$$A \smile B \Rightarrow A \overset{m}{\smile} B \quad \text{for every } m \geqslant 1.$$

The n-commutants are now defined in a natural way.

DEFINITION 3.3. Let $\mathcal{L} \subset \mathcal{A}$ be a subset of $\mathcal{A}$ and m a natural number. The *right n-commutant* of $\mathcal{L}$ is

$$C_{\mathrm{r}n}(\mathcal{L}) = \{T \in \mathcal{A} : S \overset{n}{\smile} T \text{ for every } S \in \mathcal{L}\} = \bigcap_{S \in \mathcal{L}} \ker(\mathrm{ad}S)^n.$$

The *left n-commutant* of $\mathcal{L}$ is

$$C_{\mathrm{l}n}(\mathcal{L}) = \{T \in \mathcal{A} : T \overset{n}{\smile} S \text{ for every } S \in \mathcal{L}\} = \{T \in \mathcal{A} : \mathcal{L} \subset \ker(\mathrm{ad}T)^n\}.$$

The *n-commutant* of $\mathcal{L}$ is

$$C_n(\mathcal{L}) = C_{\mathrm{r}n}(\mathcal{L}) \cap C_{\mathrm{l}n}(\mathcal{L}) = \{T \in \mathcal{A} : (\mathrm{ad}S)^n(T) = (\mathrm{ad}T)^n(S) = 0\}.$$

The *regular right (respectively left) nil-commutants* of $\mathcal{L}$ are

$$C_{\underline{\mathrm{r}}}(\mathcal{L}) = \bigcup_{n \geqslant 1} C_{\mathrm{r}n}(\mathcal{L}) \quad \text{and} \quad C_{\underline{\mathrm{l}}}(\mathcal{L}) = \bigcup_{n \geqslant 1} C_{\mathrm{l}n}(\mathcal{L}).$$

The *regular nil-commutant* of $\mathcal{L}$ is

$$C_{\smile}(\mathcal{L}) = C_{\underline{\mathrm{r}}}(\mathcal{L}) \cap C_{\underline{\mathrm{l}}}(\mathcal{L}) = \bigcup_{n \geqslant 1} C_n(\mathcal{L}).$$

REMARK 3.4. We note the following:

(i) $C_{\mathrm{r}n}(\mathcal{L}) \subset C_{\mathrm{r}m}(\mathcal{L})$, $C_{\mathrm{l}n}(\mathcal{L}) \subset C_{\mathrm{l}m}(\mathcal{L})$, $\forall\, n \leqslant m$.

(ii) $C_{\mathrm{r}n}(\mathcal{L}) \cap C_{\mathrm{l}k}(\mathcal{L}) \subset C_{n \vee k}(\mathcal{L})$.

(iii) $C_{\mathrm{r}1}(\mathcal{L}) = C_{\mathrm{l}1}(\mathcal{L}) = C_1(\mathcal{L}) = \mathcal{L}' \subset C_n(\mathcal{L}) \subset C_{\smile}(\mathcal{L}), \forall\, n \geqslant 1$.

(iv) $C_{\mathrm{r}n}(\{S\}) = \ker(\mathrm{ad}S)^n$, $C_{\mathrm{l}n}(\{S\}) = \{T \in \mathcal{A} : S \in \ker(\mathrm{ad}T)^n\}$, $C_n(\{S\}) = \{T \in \mathcal{A} : T \overset{n}{\smile} S, S \overset{n}{\smile} T\}, \forall\, S \in \mathcal{A}$.

The second type of commutativity will be called *nil-commutativity* and means n-commutativity with variable n.

DEFINITION 3.5. If $A, B \in \mathcal{A}$, *A nil-commutes or ν-commutes with B* if there exists a natural number $k \geqslant 1$ such that $A \overset{k}{\smile} B$. We will denote this property by $A \overset{\nu}{\smile} B$.

REMARK 3.6. We also note the following:

(i) $A \overset{n}{\smile} B$ for $n \in \mathbb{N}, n \geqslant 1 \Rightarrow A \overset{\nu}{\smile} B$.

(ii) If $A \overset{\nu}{\smile} B$, then there exists $\min\{k : A \overset{\nu}{\smile} B\} = k(A,B)$ which is called the *index of nil-commutativity (or ν-commutativity)* when $A \overset{\nu}{\smile} B$.

(iii) If $A \overset{\nu}{\smile} B$ then $A \overset{n}{\smile} B$ for every $n \geqslant k(A,B), n \in \mathbb{N}$.

(iv) The relation given by $A \overset{\nu}{\smile} B$ is not symmetric. We can also define "commutants"corresponding to ν-commutativity.

DEFINITION 3.7. Let $\mathcal{L} \subset \mathcal{A}$ be a subset of $\mathcal{A}$.
The *right nil-commutant (rν-commutant)* of $\mathcal{L}$ is

$$C_{\mathrm{r}\nu}(\mathcal{L}) = \{T \in \mathcal{A} : S \overset{\nu}{\smile} T \text{ for every } S \in \mathcal{L}\} = \bigcap_{S\in\mathcal{L}} \bigcup_{n\in\mathbb{N}} \ker(\mathrm{ad}S)^n.$$

The *left nil-commutant (lν-commutant)* of $\mathcal{L}$ is

$$C_{\mathrm{l}\nu}(\mathcal{L}) = \{T \in \mathcal{A} : T \overset{\nu}{\smile} S \text{ for every } S \in \mathcal{L}\} = \Big\{T \in \mathcal{A} : \mathcal{L} \subset \bigcup_{n\in\mathbb{N}} \ker(\mathrm{ad}T)^n\Big\}.$$

The *nil-commutant (ν-commutant)* of $\mathcal{L}$ is

$$C_\nu(\mathcal{L}) = C_{\mathrm{r}\nu}(\mathcal{L}) \cap C_{\mathrm{l}\nu}(\mathcal{L}) = \{T \in \mathcal{A} : (\forall S \in \mathcal{L})(\exists n \in \mathbb{N})(\mathrm{ad}S)^n(T) = (\mathrm{ad}T)^n(S) = 0\}.$$

REMARK 3.8. We also note that:
(i) $C_{\underline{\mathrm{r}}}(\mathcal{L}) \subset C_{\mathrm{r}\nu}(\mathcal{L})$, and $C_{\underline{\mathrm{l}}}(\mathcal{L}) \subset C_{\mathrm{l}\nu}(\mathcal{L}), C_{-}(\mathcal{L}) \subset C_\nu(\mathcal{L}), \mathcal{L}' \subset C_\nu(\mathcal{L})$;
(ii) $C_{\mathrm{r}\nu}(\{S\}) = \bigcup\limits_{n\in\mathbb{N}} \ker(\mathrm{ad}S)^n$;
(iii) $C_{\mathrm{l}\nu}(\{S\}) = \Big\{T \in \mathcal{A} : S \in \bigcup\limits_{n\in\mathbb{N}} \ker(\mathrm{ad}T)^n\Big\}$, $C_\nu(\{S\}) = \{T \in \mathcal{A} : \exists n \in \mathbb{N}, (\mathrm{ad}S)^n(T) = (\mathrm{ad}T)^n(S) = 0\}$.

In closing, let us consider the case when $(\mathcal{A}, \|\cdot\|)$ is a Banach algebra. In this case, we introduce asymptotic commutativity, which corresponds to the second implication mentioned in the begining of this section.

DEFINITION 3.9. We say that $S \in \mathcal{A}$ *asymptotically commutes with* $T \in \mathcal{A}$ if

$$\lim_{n\to\infty} \|(\mathrm{ad}S)^n(T)\|^{1/n} = 0,$$

that is $T \in \mathcal{A}^{\mathrm{ad}S}(\{0\})$ and we denote it $S \overset{a}{\smile} T$.

REMARK 3.10. It is worth mentioning the following:
(i) $T \overset{a}{\smile} S$ means

$$\lim_{n\to\infty} \|(\mathrm{ad}T)^n(S)\|^{1/n} = 0, \quad \text{i.e. } S \in \mathcal{A}^{\mathrm{ad}T}(\{0\})$$

and the relation $S \overset{a}{\smile} T$ is not symmetric.
(ii) $S \smile T \Rightarrow S \overset{n}{\smile} T \Rightarrow S \overset{\nu}{\smile} T \Rightarrow S \overset{a}{\smile} T$.
(iii) $S \smile T \Rightarrow T \smile S \Rightarrow T \overset{n}{\smile} S \Rightarrow T \overset{\nu}{\smile} S \Rightarrow T \overset{a}{\smile} S$.

As in the case of n-commutativity or ν-commutativity we can define commutants for the case of asymptotic-commutativity.

DEFINITION 3.11. Let $\mathcal{L} \subset \mathcal{A}$ be a subset of $\mathcal{A}$.

The *right asymptotic commutant* of $\mathcal{L}$ is

$$C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}) = \{T \in \mathcal{A} : S \overset{a}{\smile} T \text{ for every } S \in \mathcal{L}\} = \bigcap_{S \in \mathcal{L}} \mathcal{A}^{\mathrm{ad}S}(\{0\}).$$

The *left asymptotic commutant* of $\mathcal{L}$ is

$$C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}) = \{T \in \mathcal{A} : T \overset{a}{\smile} S \text{ for every } S \in \mathcal{L}\} = \{T \in \mathcal{A} : \mathcal{L} \subset \mathcal{A}^{\mathrm{ad}T}(\{0\})\}.$$

The *asymptotic commutant* of $\mathcal{L}$ is

$$C_{\sim}(\mathcal{L}) = C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}) \cap C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}).$$

REMARK 3.12. We note the following facts:

(i) Using the seminorms introduced in Section 2, see (2.2) and (2.3), we can describe $C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}), C_{\underset{\sim}{\mathrm{r}}}((L), C_{\sim}(\mathcal{L})$ for an arbitrary $\mathcal{L} \subset \mathcal{A}$,

$$C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}) = \{T \in \mathcal{A} : p_{k,\mathrm{ad}T}(S) < +\infty, \text{ for every } 1 \leqslant k \in \mathbb{N} \text{ and } S \in \mathcal{L}\},$$
$$C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}) = \{T \in \mathcal{A} : p_{k,\mathrm{ad}S}(T) < +\infty, \text{ for every } 1 \leqslant k \in \mathbb{N} \text{ and } S \in \mathcal{L}\}.$$

(ii) $C_{\underset{\sim}{\mathrm{r}}}(S) = \mathcal{A}^{\mathrm{ad}S}(\{0\}), C_{\underset{\sim}{\mathrm{l}}}(S) = \{T \in \mathcal{A} : S \in \mathcal{A}^{\mathrm{ad}T}(\{0\})\}$.

(iii) $C_{\underline{\mathrm{r}}}(\mathcal{L}) \subset C_{\mathrm{r}\nu}(\mathcal{L}) \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L})$, $C_{\underline{\mathrm{l}}}(\mathcal{L}) \subset C_{\mathrm{l}\nu}(\mathcal{L}) \subset C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L})$ and $\mathcal{L}' \subset C_n(\mathcal{L}) \subset C_{-}(\mathcal{L}) \subset C_{\nu}(\mathcal{L}) \subset C_{\sim}(\mathcal{L})$, for $1 \leqslant k \in \mathbb{N}$ and $\mathcal{L} \subset \mathcal{A}$.

The above introduced "commutants"(Definitions 3.3, 3.7, 3.11) define maps on $\mathcal{P}(\mathcal{A}) = \{\mathcal{L} : \mathcal{L} \subset \mathcal{A}\}$ for any Banach algebra $\mathcal{A}$ (some of them for a general associative algebra $\mathcal{A}$ as in Definitions 3.3, 3.7). If we use the generic notation $C(\mathcal{L})$ for one of these commutants of $\mathcal{L} \subset \mathcal{A}$, then

$$\pi : \mathcal{P}(\mathcal{A}) \to \mathcal{P}(\mathcal{A}), \quad \pi(\mathcal{L}) = C(\mathcal{L}) \quad \text{for } \mathcal{L} \subset \mathcal{A},$$

denotes one of these maps. The following proposition summarizes some basic properties concerning these maps.

PROPOSITION 3.13. *The map π has the following properties:*

(i) *π is a decreasing map, i.e. $\mathcal{L} \subset \mathcal{M} \Rightarrow \pi(\mathcal{M}) \subset \pi(\mathcal{L})$.*

(ii) *If $\mathcal{A}$ is a $*$-complex Banach algebra and $\mathcal{L} \subset \mathcal{A}$ is selfadjoint then $\pi(\mathcal{L})$ is selfadjoint.*

If π is one of the maps attached to $C_{\mathrm{l}}, C_n, C_\nu, C_\sim$, we have for every $\mathcal{L} \subset \mathcal{A}$:

(a) *$\mathcal{L} \subset \pi^2(\mathcal{L}) = \pi^{2k}(\mathcal{L})$ and $\pi^{2k+1} = \pi$.*

(b) *$\mathcal{L} \subset C_{\mathrm{l}n}(C_{\mathrm{r}n}(\mathcal{L})) \cap C_{\mathrm{r}n}(C_{\mathrm{l}n}(\mathcal{L}))$, $\mathcal{L} \subset C_{\mathrm{l}\nu}(C_{\mathrm{r}\nu}(\mathcal{L})) \cap C_{\mathrm{r}\nu}(C_{\mathrm{l}\nu}(\mathcal{L}))$, and $\mathcal{L} \subset C_{\underset{\sim}{\mathrm{l}}}(C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L})) \cap C_{\underset{\sim}{\mathrm{r}}}(C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}))$.*

Proof. Properties (i), (ii) and (b) follow directly from the definitions. For proving (a) it is easy to verify using definitions that $\mathcal{L} \subset \pi^2(\mathcal{L})$ and then we proceed as in the well known case when $\pi(\mathcal{L}) = \mathcal{L}'$. ∎

PROPOSITION 3.14. *For every $\mathcal{L} \subset \mathcal{A}$ the following assertions hold:*

(i) $\mathcal{L}' \subset C_{\underline{\mathrm{r}}}(\mathcal{L}) \subset C_{\mathrm{r}\nu}(\mathcal{L}) \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L})$, $\mathcal{L}' \subset C_{\underline{\mathrm{l}}}(\mathcal{L}) \subset C_{\mathrm{l}\nu}(\mathcal{L}) \subset C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L})$, *and* $\mathcal{L}' \subset C_n(\mathcal{L}) \subset C_{-}(\mathcal{L}) \subset C_{\nu}(\mathcal{L}) \subset C_{\sim}(\mathcal{L})$, *for* $1 \leqslant k \in \mathbb{N}$ *and* $\mathcal{L} \subset \mathcal{A}$.

(ii) $\mathcal{L} \subset \mathcal{L}'' \subset C_{\sim}(\mathcal{L}')$.

(iii) $C_{\underset{\sim}{\mathrm{l}}}C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}) \cup C_{\underset{\sim}{\mathrm{l}}}C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}) \subset C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}')$ *and* $C_{\underset{\sim}{\mathrm{r}}}C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}) \cup C_{\underset{\sim}{\mathrm{r}}}C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}) \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}')$.

(iv) $C_{\sim}C_{\sim}(\mathcal{L}) \subset C_{\sim}(\mathcal{L}')$.

Proof. One applies the definitions and (i) of the above Proposition 3.13.
For (i) see also (iii) from the Remark 3.12.
For (ii) replace $\mathcal{L}$ by $\mathcal{L}'$ in $\mathcal{L}' \subset C_{\sim}(\mathcal{L})$ from (i).
For (iii) one applies (i) for $C_{\underset{\sim}{\mathrm{l}}}, C_{\underset{\sim}{\mathrm{l}}}$ in $\mathcal{L}' \subset C_{\underline{\mathrm{r}}}(\mathcal{L})$ and $\mathcal{L}' \subset C_{\underline{\mathrm{l}}}(\mathcal{L})$.
For (iv), (i) holds also for $C_{\sim}$ and from (i) we have $\mathcal{L}' \subset C_{\sim}(\mathcal{L})$. ∎

4. COMMUTATORS COMPOSITION FOR SELFADJOINT SUBSETS OF BOUNDED OPERATORS ON A COMPLEX HILBERT SPACE

Let $\mathcal{L}$ be a set of linear bounded operators on a complex Banach space $\mathcal{X}$. Because $C_1C_1(\mathcal{L}) = (\mathcal{L}')' = \mathcal{L}''$, if C_α, C_β are given by Definitions 3.3, 3.7, 3.11 we call $C_\alpha C_\beta(\mathcal{L})$ *a bicommutant type set*. This section is devoted to initiating a study of these bicommutants type sets, including the cases when $\mathcal{L}$ is a selfadjoint set of bounded operators on a complex Hilbert space $\mathcal{H}$, or a selfadjoint unital subalgebra $\mathcal{L}$ of $\mathcal{B}(\mathcal{H})$.

We start by associating to $\mathcal{L}$ some well known objects suggested by the proof of J. von Neuman's bicommutant theorem. If $\mathcal{B}(\mathcal{X})$ is the Banach algebra of all bounded linear operators on a Banach complex space $\mathcal{X}$ and $\mathcal{L} \subset \mathcal{B}(\mathcal{X})$ we denote:

$$\mathrm{Lat}\mathcal{L} = \{\mathcal{Y} \subset \mathcal{X} : \mathcal{Y} \text{ closed subspace}, S\mathcal{Y} \subset \mathcal{Y} \text{ for every } S \in \mathcal{L}\},$$

$$\overline{\mathcal{L}}^{\mathrm{si}} = \{T \in \mathcal{B}(\mathcal{X}) : T\mathcal{Y} \subset \mathcal{Y} \text{ for every } \mathcal{Y} \in \mathrm{Lat}\mathcal{L}\},$$

$$\mathcal{X}_{\mathcal{L}}(x) = \overline{\mathrm{sp}}\{Sx : S \in \mathcal{L}\} \subset \mathcal{X} \text{ for every } x \in \mathcal{X},$$

$$\overline{\mathcal{L}}^{\mathrm{i}} = \{T \in \mathcal{B}(\mathcal{X}) : Tx \in \mathcal{X}_{\mathcal{L}}(x) \text{ for every } x \in \mathcal{X}\}.$$

The following basic properties are easy consequences of the definitions.

PROPERTIES 4.1. In what follows, $\mathcal{L}, \mathcal{L}_1, \mathcal{L}_2$ are subsets of $\mathcal{B}(\mathcal{X})$ and the following properties hold:

(i) $\mathcal{L} \subset \overline{\mathcal{L}}^{\mathrm{si}}$.

(ii) $\mathcal{L}_1 \subset \mathcal{L}_2 \Rightarrow \mathrm{Lat}\mathcal{L}_1 \subset \mathrm{Lat}\mathcal{L}_2 \Rightarrow \overline{\mathcal{L}}_1^{\mathrm{si}} \subset \overline{\mathcal{L}}_2^{\mathrm{si}}$.

(iii) $\mathcal{L}_1 \subset \mathcal{L}_2 \Rightarrow \mathcal{X}_{\mathcal{L}_1}(x) \subset \mathcal{X}_{\mathcal{L}_2}(x)$ for every $x \in \mathcal{X} \Rightarrow \overline{\mathcal{L}}_1^{\mathrm{i}} \subset \overline{\mathcal{L}}_2^{\mathrm{i}}$.

(iv) $\mathcal{L} \subset \overline{\mathcal{L}}^{\mathrm{i}}$.

(v) $\mathcal{X}_{\mathcal{L}}(x) = \mathcal{X}_{\overline{\mathcal{L}}^{\mathrm{i}}}(x)$ for every $x \in \mathcal{X}$.

(vi) $\overline{(\overline{\mathcal{L}}^{\mathrm{i}})}^{\mathrm{i}} = \overline{\mathcal{L}}^{\mathrm{i}}$ and by the above properties (iii) and (iv), " $\overline{(\cdot)}^{\mathrm{i}}$ "is a closure operator on the subsets of $\mathcal{B}(\mathcal{X})$.

(vii) $\mathcal{L} \subset \mathcal{L}_1 \subset \overline{\mathcal{L}}^{\mathrm{i}} \Rightarrow \overline{\mathcal{L}}^{\mathrm{i}} = \overline{\mathcal{L}}_1^{\mathrm{i}}$.

(viii) If $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$, $\mathcal{A}$ multiplicatively closed, then

$$\mathcal{X}_{\mathcal{A}}(x) = \overline{\mathrm{sp}}\{Ax : A \in \mathcal{A}\} \in \mathrm{Lat}\mathcal{A} \quad \text{for every } x \in \mathcal{X}.$$

(ix) If $\mathcal{A} \subset \mathcal{B}(\mathcal{X})$, $\mathcal{A}$ multiplicatively closed and $I \in \mathcal{A}$, then

$$x \in \mathcal{X}_{\mathcal{A}}(x) \quad \text{for every } x \in \mathcal{X} \text{ and } \overline{\mathcal{L}}^{\mathrm{si}} \subset \overline{\mathcal{L}}^{\mathrm{i}}.$$

Proof. We only prove (v), (vi) and (vii); the proofs of the other properties are simple verifications. By (iv) and (iii) we have $\mathcal{X}_{\mathcal{L}}(x) \subset \mathcal{X}_{\overline{\mathcal{L}}^{\mathrm{i}}}(x)$ for every $x \in \mathcal{X}$. On the other hand, $T \in \overline{\mathcal{L}}^{\mathrm{i}}$ implies $Tx \in \mathcal{X}_{\mathcal{L}}(x)$ for every $x \in \mathcal{X}$, so we deduce $\mathcal{X}_{\mathcal{L}}(x) \supset \mathcal{X}_{\overline{\mathcal{L}}^{\mathrm{i}}}(x)$ for every $x \in \mathcal{X}$ and the property (v). Then property (vi) follows from (v) and definitions, because the following implication holds:

$$\mathcal{X}_{\mathcal{L}_1}(x) = \mathcal{X}_{\mathcal{L}_2}(x) \quad \text{for every } x \in \mathcal{X} \Rightarrow \overline{\mathcal{L}}_1^{\mathrm{i}} = \overline{\mathcal{L}}_2^{\mathrm{i}}.$$

Property (vii) is a well known property of the closure operators and it follows from properties (iii) and (vi). ∎

Now we recall that a projector $P \in \mathcal{B}(\mathcal{X})$, i.e. an idempotent bounded linear operator on $\mathcal{X}$, has the spectrum $\sigma(P) = \{0, 1\}$, is decomposable, its maximal spectral spaces being $P(\mathcal{X})$ and $(I - P)(\mathcal{X})$. Moreover $P = e_{\{1\}}(P)$ is the value of analytic functional calculus of P in $e_{\{1\}}$ (an analytic function equals 1, respectively 0 in a neighborhood of 1, respectively 0). The following lemma describes the right asymptotic commutant and asymptotic commutant (see (3.11)) of a projector $P \in \mathcal{B}(\mathcal{X})$.

LEMMA 4.2. *If $X, P \in \mathcal{B}(\mathcal{X})$ and $P^2 = P$, then the following assertions are equivalent:*

($\mathrm{c_a}$) $P \overset{a}{\smile} X$, *i.e.* $\lim_{n\to\infty} \|(\mathrm{ad}P)^n X\|^{1/n} = 0$;

(c) $PX = XP$;

($\widetilde{\mathrm{c}}$) $P \overset{a}{\smile} X$ *and* $X \overset{a}{\smile} P$.

Proof. It is enough to prove ($\mathrm{c_a}$) $\Rightarrow$ (c), the other implications being obvious. The implication is an easy consequence of an asymptotic formula for the commutator (see [2]) observing that all the derivatives of $e_{\{1\}} \in \mathcal{O}(\sigma(P))$ are identically zero and by the asymptotic formula for the commutator we deduce $[e_{\{1\}}(P), X] = 0$. ∎

By Definition 3.11 the above lemma can be rewritten.

LEMMA 4.3. *For every projector $P \in \mathcal{B}(\mathcal{X})$ we have*

$$C_{\underset{\sim}{\mathrm{r}}}(\{P\}) = C_{\sim}(\{P\}) = \{P\}'.$$

Let now $\mathcal{H}$ be a complex Hilbert space. In the following all the results refer to the case when $\mathcal{X} = \mathcal{H}$. We denote by $\mathcal{P}$ the set of all selfadjoint projectors on $\mathcal{H}$, $\mathcal{P} = \{P \in \mathcal{B}(\mathcal{H}) : P^2 = P = P^*\}$. As usual $* : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is the adjoint operation, T^* is the adjoint of $T \in \mathcal{B}(\mathcal{H})$ and a subset $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is selfadjoint if $T^* \in \mathcal{L}$ for every $T \in \mathcal{L}$. We denote $P_{\mathcal{Y}} \in \mathcal{P}$ the ortogonal projection on $\mathcal{Y}$ for every closed subspace $\mathcal{Y}$ of $\mathcal{H}$. We recall also the following well known result, one of the keys of the bicommutant theorem.

LEMMA 4.4. *If $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is a selfadjoint set, then the following assertions are equivalent:*

(i) $\mathcal{Y} \in \mathrm{Lat}\mathcal{L}$.

(ii) $P_{\mathcal{Y}} \in \mathcal{L}'$.

Proof. (ii) $\Rightarrow$ (i) is obvious. If (i) holds then $P_{\mathcal{Y}} S P_{\mathcal{Y}} = S P_{\mathcal{Y}}$ for every $S \in \mathcal{L}$. So $P_{\mathcal{Y}} S^* P_{\mathcal{Y}} = P_{\mathcal{Y}} S^*$ for every $S \in \mathcal{L}$, which gives $P_{\mathcal{Y}} S P_{\mathcal{Y}} = P_{\mathcal{Y}} S$ because $\mathcal{L}$ is a selfadjoint set. ∎

PROPOSITION 4.5. *If $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is a selfadjoint set, then $\overline{\mathcal{L}}^{\mathrm{si}} = (\mathcal{P} \cap \mathcal{L}')'$, i.e. for every $X \in \mathcal{B}(\mathcal{H})$ the following assertions are equivalent:*

(i) $X \in \overline{\mathcal{L}}^{\mathrm{si}}$.

(ii) $X \in (\mathcal{P} \cap \mathcal{L}')'$ *(i.e. $XP = PX$ for every $P \in \mathcal{P} \cap \mathcal{L}'$).*

Proof. (ii) $\Rightarrow$ (i) is obvious because $\mathcal{L}$ is selfadjoint and by the above Lemma 4.4 we have $\mathcal{Y} \in \mathrm{Lat}\mathcal{L}$ if and only if $P_{\mathcal{Y}} \in \mathcal{L}'$. For proving (i) $\Rightarrow$ (ii) we observe that for $P \in \mathcal{P} \cap \mathcal{L}'$ we have also $(I - P) \in \mathcal{P} \cap \mathcal{L}'$ and by the above lemma $P\mathcal{H}$ and $(I - P)\mathcal{H}$ are both in $\mathrm{Lat}\mathcal{L}$. But $P\mathcal{H}$ and $(I - P)\mathcal{H}$ are all maximal spectral spaces of P and by [4] we get that $\lim\limits_{n\to\infty} \|(\mathrm{ad}P)^n X\|^{1/n} = 0$, i.e. $P \overset{a}{\smile} X$ and by Lemma 4.2 we have $PX = XP$, hence (ii). ∎

PROPOSITION 4.6. *If $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is a selfadjoint set, then $\overline{\mathcal{L}}^{\mathrm{si}} = \mathcal{L}''$.*

Proof. Having $\mathcal{P} \cap \mathcal{L}' \subset \mathcal{L}'$ we deduce by Proposition 4.5 the inclusion

$$\mathcal{L}'' \subset (\mathcal{P} \cap \mathcal{L}')' = \overline{\mathcal{L}}^{\mathrm{si}}.$$

To prove the equality we have to prove the other inclusion and it suffices to prove that $X \in \overline{\mathcal{L}}^{\mathrm{si}}$ commutes with every selfadjoint element $A \in \mathcal{L}'$. Indeed, $\mathcal{L}$ hence $\mathcal{L}'$ being selfadjoint sets, every element $T \in \mathcal{L}'$ is a linear combination of selfadjoint elements of $\mathcal{L}'$. So, let $X \in \overline{\mathcal{L}}^{\mathrm{si}}$ and $A = A^* \in \mathcal{L}'$. If $\{P_\lambda\}$ is the family of spectral projectors of A, $\{P_\lambda\}$ commutes with every operator commuting with A, in particular with $\mathcal{L}$ because A is supposed to be in $\mathcal{L}'$. Then, by the above Proposition 4.5 X commutes with $\{P_\lambda\}$, hence with A, which concludes the proof. ∎

PROPOSITION 4.7. *If $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ is a selfadjoint set, then*

$$C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') \subset \overline{\mathcal{L}}^{\mathrm{si}},$$

i.e. the following implication holds:

$$X \in C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') \Rightarrow X\mathcal{Y} \subset \mathcal{Y} \quad \textit{for every } \mathcal{Y} \in \mathrm{Lat}\mathcal{L}.$$

Proof. $X \in C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}')$ means $\lim\limits_{n\to\infty} \|(\mathrm{ad}T)^n X\|^{1/n} = 0$ for every $T \in \mathcal{L}'$. For $\mathcal{Y} \in \mathrm{Lat}\mathcal{L}$ we have $P_{\mathcal{Y}} \in \mathcal{L}'$ (see Lemma 4.4). Thus,

$$\lim_{n\to\infty} \|(\mathrm{ad}P_{\mathcal{Y}})^n X\|^{1/n} = 0$$

and by Lemma 4.3 we deduce $P_{\mathcal{Y}}X = XP_{\mathcal{Y}}$. Hence $X\mathcal{Y} \subset \mathcal{Y}$ for every $X \in C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L})$ and $\mathcal{Y} \in \mathrm{Lat}\mathcal{L}$. ∎

With the following corollary we start to describe the sets of bicommutant type anounced in the beginning of this section.

COROLLARY 4.8. *For every selfadjoint set $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$ we have:*

$$C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \overline{\mathcal{L}}^{\mathrm{si}} = \mathcal{L}''.$$

Proof. By first inclusion of (i) from Proposition 3.14, Proposition 4.6 and Proposition 4.7, we have $\mathcal{L}'' \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}')$ and $C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') \subset \overline{\mathcal{L}}^{\mathrm{si}} = \mathcal{L}''$. ∎

Another result concerning the sets of bicommutant type, in particular $C_{\sim}C_{\sim}(\mathcal{L})$, can be obtained using this corollary and properties (iii) and (vi), from Properties 4.1.

THEOREM 4.9. *The following relations hold when $\mathcal{L}$ is a selfadjoint subset of $\mathcal{B}(\mathcal{H})$:*

$$\mathcal{L} \subset C_{\sim}C_{\sim}(\mathcal{L}) \subset C_{\sim}(\mathcal{L}') = C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'' = \overline{\mathcal{L}}^{\mathrm{si}}.$$

Proof. The last two equalities are given by Corollary 4.8. The first inclusion is an easy consequence of the definition of $C_{\sim}$ and the second is (iv) from Proposition 3.14. By (ii) of Proposition 3.14 we have $\mathcal{L}'' \subset C_{\sim}(\mathcal{L}')$. On the other hand, the inclusion $C_{\sim}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}')$ results directly by definition. So we deduce $\mathcal{L}'' \subset C_{\sim}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}''$, hence the equalities $\mathcal{L}'' = C_{\sim}(\mathcal{L}') = C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}''$ and the theorem is proved. ∎

COROLLARY 4.10. *If $\mathcal{L}$ is a selfadjoint subset of $\mathcal{B}(\mathcal{H})$, then*

$$\overline{(\overline{\mathcal{L}}^{\mathrm{si}})}^{\mathrm{si}} = \overline{\mathcal{L}}^{\mathrm{si}} \quad \textit{and} \quad \overline{C_{\sim}C_{\sim}(\mathcal{L})}^{\mathrm{si}} = \overline{\mathcal{L}}^{\mathrm{si}}.$$

Proof. Like $\mathcal{L}$, $\mathcal{L}'$ is also selfadjoint set, hence $\mathcal{L}''$ is selfadjoint. The last equality from the above theorem gives $\overline{(\overline{\mathcal{L}}^{\mathrm{si}})}^{\mathrm{si}} = \overline{\mathcal{L}}^{\mathrm{si}}$. We can also replace in the above theorem $\mathcal{L}$ by $\mathcal{L}''$ and using (a) of Proposition 3.13 we obtain $\overline{\mathcal{L}''}^{\mathrm{si}} = (\mathcal{L}'')'' = \mathcal{L}''$. We thus proved

$$\overline{(\overline{\mathcal{L}}^{\mathrm{si}})}^{\mathrm{si}} = \overline{\mathcal{L}''}^{\mathrm{si}} = \mathcal{L}'' = \overline{\mathcal{L}}^{\mathrm{si}}.$$

By the above theorem we have

$$\mathcal{L} \subset C_{\sim}C_{\sim}(\mathcal{L}) \subset \overline{\mathcal{L}}^{\mathrm{si}},$$

and using (ii) from Properties 4.1 we obtain

$$\overline{\mathcal{L}}^{\mathrm{si}} \subset \overline{C_{\sim}C_{\sim}(\mathcal{L})}^{\mathrm{si}} \subset \overline{(\overline{\mathcal{L}}^{\mathrm{si}})}^{\mathrm{si}} = \overline{\mathcal{L}}^{\mathrm{si}}$$

which conclude the proof. ∎

For a selfadjoint multiplicative closed set $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ which contains the identity operator I on $\mathcal{H}$ we can compare the asymptotic bicommutants of $\mathcal{A}$ with $\overline{\mathcal{A}}^{\mathrm{i}}$. We begin with a preliminary result concerning the asymptotic bicommutant type set given by $C_{\underset{\sim}{\mathrm{r}}}C_1(\mathcal{A})$.

PROPOSITION 4.11. *Let $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ be a multiplicatively closed, selfadjoint set containing the identity $I \in \mathcal{B}(\mathcal{H})$. The following relations hold:*

$$C_{\underset{\sim}{\mathrm{r}}}C_1(\mathcal{A}) = C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}') \subset \overline{\mathcal{A}}^{\mathrm{i}} \quad \textit{and} \quad \overline{C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}')}^{\mathrm{i}} = \overline{\mathcal{A}}^{\mathrm{i}}.$$

Proof. From (viii) of Properties 4.1 we get that $\overline{\mathcal{A}}^{\mathrm{si}} \subset \overline{\mathcal{A}}^{\mathrm{i}}$ because $\mathcal{A}$ is multiplicatively closed and $I \in \mathcal{A}$. Because $\mathcal{A}$ is selfadjoint, by Proposition 4.7 we have that

$$C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}') \subset \overline{\mathcal{A}}^{\mathrm{si}} \subset \overline{\mathcal{A}}^{\mathrm{i}}.$$

Since $\mathcal{A} \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}') \subset \overline{\mathcal{A}}^{\mathrm{i}}$, (iii) and (vi) of Properties 4.1 imply the equality $\overline{C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}')}^{\mathrm{i}}$ $= \overline{\mathcal{A}}^{\mathrm{i}}$ and the proof is complete. ∎

The following theorem sumarizes some results concerning the composition of the following maps given by the "special" commutants $C_{\underset{\sim}{\mathrm{r}}}, C_{\underset{\sim}{1}}, C_{\sim}, C_1$ (C_1 denotes the classical comutant $\mathcal{L}'$ and the others are defined in Section 2). We focus on the composition of commutants mentioned above on the subset of $\mathcal{B}(\mathcal{H})$ for which the bicommutant theorem holds. In other words we talk about "bicommutant" type sets resulting from the composition of the "special" commutants introduced earlier.

THEOREM 4.12. *Let $\mathcal{A}$ be a selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$ such that $I \in \mathcal{A}$. Then the following relations hold:*

(i) $\mathcal{A} \subset C_{\sim}C_{\sim}(\mathcal{A}) \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}') = \mathcal{A}'' = \overline{\mathcal{A}}^{\mathrm{so}} = \overline{\mathcal{A}}^{\mathrm{si}} \subset \overline{\mathcal{A}}^{\mathrm{i}}$.

(ii) $C_{\underset{\sim}{\mathrm{r}}}C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}) \cup C_{\underset{\sim}{\mathrm{r}}}C_{\underset{\sim}{1}}(\mathcal{A}) \cup C_{\underset{\sim}{\mathrm{r}}}C_{\sim}(\mathcal{A}) \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}') \subset \overline{\mathcal{A}}^{\mathrm{i}}$.

(iii) $\overline{\mathcal{A}}^{\mathrm{i}} = \overline{C_{\sim}C_{\sim}(\mathcal{A})}^{\mathrm{i}} = \overline{C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}')}^{\mathrm{i}} = \overline{C_{\sim}(\mathcal{A}')}^{\mathrm{i}} = \overline{\mathcal{A}''}^{\mathrm{i}} = \overline{\overline{\mathcal{A}}^{\mathrm{so}}}^{\mathrm{i}} = \overline{\overline{\mathcal{A}}^{\mathrm{si}}}^{\mathrm{i}}$.

Proof. For (i) we apply Theorem 4.9, J. von Neumann's bicommutant theorem and (ix) of Properties 4.1 for $\overline{\mathcal{A}}^{\mathrm{si}} \subset \overline{\mathcal{A}}^{\mathrm{i}}$. For proving (iii), we apply the operator $\overline{(\cdot)}^{\mathrm{i}}$ in (i) taking into account Properties 4.1. For proving (ii), we deduce by definitions that $\mathcal{A}'$ is contained in the asymptotic commutant $C_{\sim}(\mathcal{A}) = C_{\underset{\sim}{\mathrm{r}}}(\mathcal{A}) \cap C_{\underset{\sim}{1}}(\mathcal{A})$ and the first inclusion of (ii) results from the decreasing property of $C_{\underset{\sim}{\mathrm{r}}}$ (see property (i) of Proposition 3.13) and then we can use (i). ∎

To conclude we describe how we order by inclusion other bicommutant type sets attached to a selfadjoint subset $\mathcal{L} \subset \mathcal{B}(\mathcal{H})$. All these bicommutant type sets attached to $\mathcal{L}$ are contained in the bicommutant $C_1 C_1(\mathcal{L}) = \mathcal{L}''$. The following relations follow from $\mathcal{L}' \subset C_n(\mathcal{L}) \subset C_{\mathrm{l}n}(\mathcal{L}) \cap C_{\mathrm{r}n}(\mathcal{L})$ using the decreasing property (i) of $C_{\mathrm{r}n}$, C_n and properties (a) and (b) from Proposition 3.13.

(I) We have

$$C_{\mathrm{r}n}C_{\mathrm{r}n}(\mathcal{L}) \subset C_{\mathrm{r}n}C_n(\mathcal{L}) \subset C_{\mathrm{r}n}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'',$$
$$C_n C_{\mathrm{r}n}(\mathcal{L}) \subset C_n C_n(\mathcal{L}) \subset C_n(\mathcal{L}') \subset C_{\mathrm{r}n}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'',$$
$$\mathcal{L} \subset C_n C_n(\mathcal{L}) \subset C_n(\mathcal{L}') \subset C_{\mathrm{r}n}(\mathcal{L}') \subset C_{\mathrm{r}\nu}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'',$$
$$\mathcal{L} \subset C_{\mathrm{r}n}C_{\mathrm{l}n}(\mathcal{L}) \subset C_{\mathrm{r}n}C_n(\mathcal{L}) \subset C_{\mathrm{r}n}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}''.$$

(II) In the same way as above we obtain,

$$C_{\mathrm{r}\nu}C_{\mathrm{r}\nu}(\mathcal{L}) \subset C_{\mathrm{r}\nu}C_\nu(\mathcal{L}) \subset C_{\mathrm{r}\nu}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'',$$
$$C_\nu C_{\mathrm{r}\nu}(\mathcal{L}) \subset C_\nu C_\nu(\mathcal{L}) \subset C_\nu(\mathcal{L}') \subset C_{\sim}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'',$$
$$\mathcal{L} \subset C_\nu C_\nu(\mathcal{L}) \subset C_\nu(\mathcal{L}') \subset C_{\sim}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}'',$$
$$\mathcal{L} \subset C_{\mathrm{r}\nu}C_{\mathrm{l}\nu}(\mathcal{L}) \subset C_{\mathrm{r}\nu}C_\nu(\mathcal{L}) \subset C_{\mathrm{r}\nu}(\mathcal{L}') \subset C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}') = \mathcal{L}''.$$

It is easy to verify that $C_{\mathrm{r}n}(\mathcal{M}), C_{\mathrm{l}n}(\mathcal{M}), C_n(\mathcal{M})$ are so-closed for every $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$. If we pass to so-closure (usually denoted by $\overline{(\cdot)}^{\mathrm{so}}$) in the last two chains of inclusions from (I) we obtain by the bicommutant theorem the following equalities.

PROPOSITION 4.13. *If $\mathcal{L}$ is a unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then the following equalities hold:*

$$\overline{\mathcal{L}}^{\mathrm{so}} = \mathcal{L}'' = C_{\mathrm{r}n}C_{\mathrm{l}n}(\mathcal{L}) = C_{\mathrm{r}n}C_n(\mathcal{L}) = C_n C_n(\mathcal{L}) = C_n(\mathcal{L}') = C_{\mathrm{r}n}(\mathcal{L}') = C_{\underset{\sim}{\mathrm{r}}}(\mathcal{L}').$$

When $\mathcal{L}$ is a so-closed, unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, the two last chains from inclusions (I) and (II), the inclusion $\mathcal{L} \subset C_{\underset{\sim}{\mathrm{r}}}C_{\underset{\sim}{1}}(\mathcal{L}) \cap C_{\underset{\sim}{\mathrm{r}}}C_{\sim}(\mathcal{L})$, (i) and (ii) from Theorem 4.12 and the J. von Neumann's bicommutant theorem give besides the equalities from the conclusion of the above proposition the following equalities.

PROPOSITION 4.14. *If $\mathcal{L}$ is a* so*-closed, unital $*$-subalgebra of $\mathcal{B}(\mathcal{H})$, then the following equalities hold:*

$$\begin{aligned}\mathcal{L} = \mathcal{L}'' &= C_{\sim}C_{\sim}(\mathcal{L}) = C_{\sim}(\mathcal{L}') = C_{\underset{\sim}{\mathrm{r}}}C_{\underset{\sim}{\mathrm{l}}}(\mathcal{L}) = C_{\underset{\sim}{\mathrm{r}}}C_{\sim}(\mathcal{L})\\ &= C_{\nu}C_{\nu}\mathcal{L} = C_{\mathrm{r}\nu}(\mathcal{L}') = C_{\nu}(\mathcal{L}') = C_{\mathrm{r}\nu}C_{\mathrm{l}\nu}(\mathcal{L}) = C_{\mathrm{r}\nu}C_{\nu}(\mathcal{L}).\end{aligned}$$

Acknowledgements. The author thanks Daniel Beltiţă and Florin Şabac for their useful remarks which improved the first versions of the present paper.

This work was supported by CNCSIS-UEFISCSU project number 538/2009 PNII-IDEI code 1905/2008.

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Received December 8, 2011; posted on February 17, 2014.