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HÖRMANDER TYPE FUNCTIONAL CALCULUS AND SQUARE FUNCTION ESTIMATES

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ABSTRACT. We investigate Hörmander spectral multiplier theorems as they hold on $X = L^p(\Omega), 1 < p < \infty$, for many self-adjoint elliptic differential operators A including the standard Laplacian on $\mathbb{R}^d$. A strengthened matricial extension is considered, which coincides with a completely bounded map between operator spaces in the case that X is a Hilbert space. We show that the validity of the matricial Hörmander theorem can be characterized in terms of square function estimates for imaginary powers $A^{\mathrm{i}t}$, for resolvents $R(\lambda, A)$, and for the analytic semigroup $\exp(-zA)$. We deduce Hörmander spectral multiplier theorems for semigroups satisfying generalized Gaussian estimates.

KEYWORDS: *Functional calculus, square functions, Hörmander spectral multipliers, operator spaces.*

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1. INTRODUCTION

Let f be a bounded function on $(0, \infty)$ and $u(f)$ the operator on $L^p(\mathbb{R}^d)$ defined by $[u(f)g]\widehat{\ }(\xi) = f(|\xi|^2)\widehat{g}(\xi)$. Hörmander's theorem on Fourier multipliers ([20], Theorem 2.5) asserts that $u(f) : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for any $p \in (1, \infty)$ provided that for some integer N strictly larger than $d/2$

$$\sup_{R>0} \int_{R/2}^{2R} |t^k f^{(k)}(t)|^2 \frac{\mathrm{d}t}{t} < \infty \quad (k = 0, 1, \ldots, N). \tag{1.1}$$

This theorem has many refinements and generalisations to various similar contexts. For $\alpha > 1/2$, let $W_2^\alpha(\mathbb{R}) = \{f \in L^2(\mathbb{R}) : \|f\|_{W_2^\alpha(\mathbb{R})} = \|(1 + \xi^2)^{\alpha/2}\widehat{f}(\xi)\|_{L^2(\mathbb{R})} < \infty\}$ denote the usual Sobolev space, and $W^\alpha = \{f : (0, \infty) \to \mathbb{C} : f \circ \exp \in W_2^\alpha(\mathbb{R})\}$, which is a Banach algebra with respect to $\|f\|_{W^\alpha} = \|f \circ \exp\|_{W_2^\alpha(\mathbb{R})}$. Let $\phi_0 \in C_c^\infty(1/2, 2)$. For $n \in \mathbb{Z}$, let $\phi_n = \phi_0(2^{-n}\cdot)$ and assume that $\sum_{n\in\mathbb{Z}} \phi_n(t) = 1$ for any $t > 0$. Such a function exists ([2], Lemma 6.1.7) and we call $(\phi_n)_{n\in\mathbb{Z}}$ a dyadic partition of unity. We define the Banach algebra

$$\mathcal{H}^\alpha = \Big\{ f : (0,\infty) \to \mathbb{C} : \|f\|_{\mathcal{H}^\alpha} = \sup_{n\in\mathbb{Z}} \|\phi_n f\|_{W^\alpha} < \infty \Big\}.$$

Note that the restriction $\alpha > 1/2$ is natural since for $\alpha < 1/2$, the spaces W^α and $\mathcal{H}^\alpha$ contain unbounded functions and are not closed with respect to pointwise multiplication. This makes the spaces unsuitable for functional calculus which is the main subject of this article. The definition of $\mathcal{H}^\alpha$ is independent of the dyadic partition of unity, different choices resulting in equivalent norms. This latter fact can bee seen by an easy calculation, see also Section 4.2 of [25]. The space $\mathcal{H}^\alpha$ refines (1.1), more precisely, $f \in \mathcal{H}^\alpha$ implies that f satisfies (1.1) for $N \leqslant \alpha$, and the converse holds for $N \geqslant \alpha$. This fact is widely used in the literature and shown e.g. in Proposition 4.11 of [25].

Now if A is a self-adjoint positive operator on some $L^2(\Omega,\mu)$, then its functional calculus assigns to any bounded measurable function f on $(0,\infty)$ an operator $f(A)$ on $L^2(\Omega,\mu)$. In particular, if $A = -\Delta$ and $(\Omega,\mu) = (\mathbb{R}^d, \mathrm{d}x)$, then $f(A)$ equals the above $u(f)$. A theorem of Hörmander type holds true for many elliptic and sub-elliptic operators A, including sublaplacians on Lie groups of polynomial growth, Schrödinger operators and elliptic operators on Riemannian manifolds ([1], [3], [7], [12], [14]). By this, we mean that

$$u : \mathcal{H}^\alpha \to B(X), f \mapsto f(A) \text{ is a bounded homomorphism,} \tag{1.2}$$

where $X = L^p(\Omega)$, $p \in (1,\infty)$, α is the differentiation parameter typically larger than $d/2$, where d is the dimension of Ω, and $f(A)$ is given by (the unique bounded L^p-extension of) the self-adjoint functional calculus.

The aim of this article is to characterize the validity of the Hörmander multiplier theorem for A in terms of square function estimates.

The latter have been introduced in Stein's classical book [41] and have since then been used widely with applications to functional calculi and multiplier theorems. Note that $\|(\cdot)^{\mathrm{i}t}\|_{\mathcal{H}^\alpha} \lesssim (1+|t|^2)^{\alpha/2}$. Indeed, if $\alpha \in \mathbb{N}_0$, then $\|(\cdot)^{\mathrm{i}t}\|_{\mathcal{H}^\alpha} = \sup_{n\in\mathbb{Z}} \|\phi_n(\cdot)^{\mathrm{i}t}\|_{W^\alpha} \leqslant (1+t^2)^{\alpha/2}$ by an elementary calculation. Then the general case $\alpha > 1/2$ follows from complex interpolation. Thus for this particular function, (1.2) implies $\|A^{\mathrm{i}t}\| \leqslant C(1+|t|^2)^{\alpha/2}$. Then a natural square function estimate for our situation is

$$\|(1+t^2)^{-\alpha/2} A^{\mathrm{i}t} x\|_{\gamma(\mathbb{R},X)} \leqslant C\|x\|_X, \tag{1.3}$$

where $\gamma(\mathbb{R},X)$ is given by

$$\|x(t)\|_{\gamma(\mathbb{R},X)} \cong \Big\| \Big(\int_{\mathbb{R}} |x(t)|^2 \mathrm{d}t \Big)^{1/2} \Big\|_X$$

for $X = L^p(\Omega, \mu)$ and $p \in [1, \infty)$, which explains the name square function. The general definition of the space $\gamma(\mathbb{R}, X)$ involves Gaussian random sums in the Banach space X, see Section 2.

Our setting, developed in Section 2, is as follows: We let X be a Banach space having Pisier's property (α), which is a geometric property playing an important role for the theory of spectral multipliers. It is natural to assume the operator A to be 0-sectorial i.e. a negative generator of an analytic semigroup $(\exp(-zA))_{\operatorname{Re} z>0}$ which is uniformly bounded on the sector $\Sigma_\omega = \{z \in \mathbb{C}\backslash\{0\} : |\arg z| < \omega\}$ for each $\omega < \pi/2$. Indeed, $\exp(-z\,\cdot\,)$ belongs to $\mathcal{H}^\alpha$ with uniform norm bound on such sectors. Further, for simplicity we assume throughout that A has dense range.

We shall base the definition of u in (1.2) on the well-known H^∞ functional calculus ([8], [30]). This means that for f belonging to $H^\infty_0(\Sigma_\omega) = \{f \in H^\infty(\Sigma_\omega) : \exists \varepsilon, C > 0$ such that $|f(z)| \leqslant C \min(|z|^\varepsilon, |z|^{-\varepsilon})\}$ which is a subclass of $H^\infty(\Sigma_\omega) = \Big\{f : \Sigma_\omega \to \mathbb{C} : f$ is analytic, $\|f\|_{\infty,\omega} = \sup_{z\in\Sigma_\omega} |f(z)| < \infty\Big\}$, $f(A) \in B(X)$ is defined by a certain Cauchy integral formula, see (2.9). Secondly, under certain conditions, A has a bounded H^∞ calculus, which means that there is an extension to a bounded homomorphism $H^\infty(\Sigma_\omega) \to B(X), f \mapsto f(A)$. Note that $H^\infty(\Sigma_\omega)$ is a subclass of $\mathcal{H}^\alpha$. In Lemma 4.3 it will be shown in particular that an extension of the H^∞ calculus to a bounded homomorphism $u : \mathcal{H}^\alpha \to B(X)$ is unique.

For any such mapping u and $n \in \mathbb{N}$, we now consider the linear tensor extension

$$u_n : \begin{cases} M_n \otimes \mathcal{H}^\alpha & \to M_n \otimes B(X), \\ a \otimes f & \mapsto a \otimes u(f), \end{cases}$$

where M_n is the space of $n \times n$ scalar matrices. We will equip both $M_n \otimes \mathcal{H}^\alpha$ and $M_n \otimes B(X)$ with suitable norms. In fact, $\mathcal{H}^\alpha$ will become an operator space (see Section 4), $M_n \otimes B(X) \cong B(\ell^2_n \otimes_2 X)$ if X is a Hilbert space, and if X is a Banach space, $M_n \otimes B(X) \cong B(\mathrm{Gauss}_n(X))$ carries the norm induced by an action on X-valued Gaussian random sums. We call u matricially γ-bounded in this article if

$$\|u\|_{\text{mat-}\gamma} = \sup_{n\in\mathbb{N}} \|u_n\| < \infty. \tag{1.4}$$

This is in general strictly stronger than $\|u\| < \infty$ (see Proposition 5.9), and is related to the following two well-known boundedness notions, explained in Section 2. First, if X is a Hilbert space, then (1.4) is equivalent to the complete boundedness of u, and second, if X is a Banach space, then (1.4) entails that the set of spectral multipliers $\{u(f) : \|f\|_{\mathcal{H}^\alpha} \leqslant 1\}$ is γ-bounded.

The main result reads as follows.

THEOREM 1.1. *Let X be a space with property (α). Let A be a 0-sectorial operator on X with bounded H^∞ calculus. Let $\alpha > 1/2$. Then the following are equivalent:*

(i) *The square function estimate* (1.3) *holds.*

(ii) *The H^∞ calculus mapping $f \mapsto f(A)$ extends to a homomorphism $u : \mathcal{H}^\alpha \to B(X)$ which is matricially γ-bounded.*

Theorem 1.1 entails a spectral multiplier theorem in the following situations. The space $X = L^p(\Omega)$ for $p \in (1, \infty)$ has property (α). If (Ω, μ) is a d-dimensional space of homogeneous type, e.g. a sufficiently regular open subset of $\mathbb{R}^d$ with Lebesgue measure μ, and A is self-adjoint positive on $L^2(\Omega)$ such that the corresponding semigroup $\exp(-tA)$ has an integral kernel $k_t(x,y)$ that satisfies the Gaussian estimate for some $m \geqslant 2$

$$(1.5)\quad |k_t(x,y)| \leqslant C\mu(B(x,t^{1/m}))^{-1}\exp\Big(-c\Big(\frac{\mathrm{dist}(x,y)}{t^{1/m}}\Big)^{m/(m-1)}\Big) \quad (x,y\in\Omega, t>0),$$

then A has a bounded H^∞ calculus on X ([13], Theorem 3.4 and [4], Corollary 2.3). This is indeed the case for many operators listed before (1.2) ([3], Section 2). Moreover, the mappings u from (1.2) and Theorem 1.1(ii) are the same, so that we obtain as a corollary

COROLLARY 1.2. *Assume that A is a self-adjoint positive operator on $L^2(\Omega)$ satisfying* (1.5)*. Let $\alpha > 1/2$ and $p \in (1, \infty)$. If A satisfies the square function estimate*

$$(1.6)\qquad \Big\| \Big(\int_{\mathbb{R}} |(1+t^2)^{-\alpha/2} A^{it} x|^2 \mathrm{d}t \Big)^{1/2} \Big\|_p \leqslant C\|x\|_p,$$

then for any $f \in \mathcal{H}^\alpha$, the spectral multiplier $f(A)$ is bounded $L^p(\Omega) \to L^p(\Omega)$.

In Proposition 5.9, we will show a partial converse of Corollary 1.2. More precisely, (1.2) implies that a restriction to a smaller Hörmander space $\mathcal{H}^\beta$ is matricially γ-bounded.

Let us close the introduction with an overview of the rest of the article. In Section 2, we give the necessary background of the above mentioned notions of matricial norms, square functions, Gaussian random sums and functional calculus. Matricially γ-bounded mappings and the connection to square functions are explained in Section 3. Section 4 is devoted to homomorphisms $u : \mathcal{H}^\alpha \to B(X)$ and the connection to H^∞ functional calculus. Moreover Theorem 1.1 is proved. A main ingredient is to deduce a spectral decomposition of Paley–Littlewood type, see (4.6), under the hypotheses of Theorem 1.1. In Section 5, we discuss some extensions and applications. Firstly, the square function estimate in terms of imaginary powers A^{it} in Theorem 1.1 has several equivalent and almost equivalent rewritings in terms of other typical square functions, involving the analytic semigroup

$$(1.7)\quad \|A^{1/2}\exp(-te^{i\theta}A)x\|_{\gamma(\mathbb{R}_+,X)} \leqslant C\Big(\frac{\pi}{2}-|\theta|\Big)^{-\beta}\|x\| \qquad (\theta\in(-\pi/2,\pi/2)),$$

or resolvents

$$(1.8)\qquad \|A^{1/2}R(e^{i\theta}t, A)x\|_{\gamma(\mathbb{R}_+,X)} \leqslant C|\theta|^{-\beta}\|x\| \qquad (\theta \in (-\pi, \pi)\backslash\{0\}).$$

We have (1.3) $\Rightarrow$ (1.7) and (1.8) for $\alpha \leqslant \beta$, and conversely, (1.7) or (1.8) $\Rightarrow$ (1.3) for $\alpha > \beta$.

Secondly, we discuss Theorem 1.1 in the presence of generalized Gaussian estimates (see Assumption 5.5), which in particular covers semigroups satisfying (1.5). This is a well-studied property in connection with (Hörmander) functional calculus, see e.g. [12], [14]. In particular, we show the square function assumption of Corollary 1.2 in the form of (1.7) and improve the derivation order of the Hörmander theorem from $\alpha > d/2 + 1/2$ as proved in [3] to $\alpha > d|1/p_0 - 1/2| + 1/2$. We finally discuss the connections and differences between matricially γ-bounded Hörmander calculus and bounded Hörmander calculus. The last Section 6 contains some technical proofs of Section 4.

2. PRELIMINARIES ON OPERATOR SPACES, GAUSSIAN SUMS, SQUARE FUNCTIONS AND FUNCTIONAL CALCULUS

We will need in different contexts cross norms on a tensor product of two Banach spaces.

OPERATOR SPACES. A Banach space E is called operator space if it is isometrically embedded into $B(H)$, where H is a Hilbert space. Let M_n denote the space of scalar $n \times n$ matrices. What makes operator spaces different from mere Banach spaces is that there is a specific collection of norms on $M_n \otimes E$, the operator space structure of E. Namely for all $n \in \mathbb{N}$, it is equipped with the norm arising from the embedding $M_n \otimes E \hookrightarrow B(\ell^2_n(H))$, $[a_{ij}] \otimes x \mapsto ((h_i)_{i=1}^n \mapsto (\sum_{j=1}^n a_{ij} x(h_j))_{i=1}^n)$.

Let E and F be operator spaces and $u : E \to F$ a linear mapping. For any $n \in \mathbb{N}$, let u_n be the linear mapping $M_n \otimes E \to M_n \otimes F, a \otimes x \mapsto a \otimes u(x)$. Then u is called completely bounded (completely isometric) if $\|u\|_{\mathrm{cb}} = \sup_{n \in \mathbb{N}} \|u_n\| < \infty$ (for any $n \in \mathbb{N}, u_n$ is isometric).

Clearly, any space $B(H)$ itself is an operator space, so in particular $M_m = B(\ell^2_m)$ is. Further we will consider the Hilbert row space $\ell^2_{\mathrm{r}} = \{h \mapsto \langle h, x \rangle e : x \in \ell^2\} \subset B(\ell^2)$ where $e \in \ell^2$ is a fixed element of norm 1 and $\langle h, x \rangle$ is the scalar product. Different choices of e give isometric norms of $M_n \otimes \ell^2_{\mathrm{r}}$ and ℓ^2_{r} is isometric to ℓ^2 as a Banach space. We shall also consider the m-dimensional subspaces $\ell^2_{m,\mathrm{r}} \subset \ell^2_{\mathrm{r}}$. These are completely isometrically determined by the following embedding, which also explains the name of row space:

$$i_m : \ell^2_m \hookrightarrow M_m, (a_1, \ldots, a_m) \mapsto \begin{pmatrix} a_1 & \ldots & a_m \\ 0 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & 0 \end{pmatrix}. \tag{2.1}$$

We refer to the books [15], [39] for further information on operator spaces.

γ-BOUNDED SETS, PROPERTY (α) AND SQUARE FUNCTIONS. We let Ω be a probability space and $(\gamma_k)_{k\in\mathbb{Z}}$ a sequence of independent standard Gaussian random variables on Ω. For a Banach space X, we let $\mathrm{Gauss}(X)$ be the closure of $\mathrm{span}\{\gamma_k \otimes x_k : k \in \mathbb{Z}\}$ in $L^2(\Omega; X)$ with respect to the norm

$$\Big\| \sum_k \gamma_k \otimes x_k \Big\|_{\mathrm{Gauss}(X)} = \Big(\int_\Omega \Big\| \sum_{k=1}^n \gamma_k(\omega) x_k \Big\|^2 \mathrm{d}\omega \Big)^{1/2}. \tag{2.2}$$

It will be convenient to denote $\mathrm{Gauss}_n(X)$ the subspace of $\mathrm{Gauss}(X)$ of elements of the form $\sum_{k=1}^n \gamma_k \otimes x_k$.

Note that if X is a Hilbert space, then

$$\Big\| \sum_k \gamma_k \otimes x_k \Big\|^2_{\mathrm{Gauss}_n(X)} = \sum_{k=1}^n \|x_k\|^2. \tag{2.3}$$

A collection $\tau \subset B(X)$ is called γ-bounded if there exists $C > 0$ such that

$$\Big\| \sum_k \gamma_k \otimes T_k x_k \Big\|_{\mathrm{Gauss}(X)} \leqslant C \Big\| \sum_k \gamma_k \otimes x_k \Big\|_{\mathrm{Gauss}(X)}$$

for any finite families $T_1, \ldots, T_n \in \tau$ and $x_1, \ldots, x_n \in X$. The least admissible constant is denoted by $\gamma(\tau)$ (and $\gamma(\tau) := \infty$ if such a C does not exist). Note that a γ-bounded set is automatically uniformly norm bounded, since one has $\gamma(\tau) \geqslant \sup_{T\in\tau} \|T\|$. For $\sigma, \tau \subset B(X)$ and $\sigma \circ \tau = \{S \circ T : S \in \sigma, T \in \tau\}$, one has $\gamma(\sigma \circ \tau) \leqslant \gamma(\sigma)\gamma(\tau)$. The set $\tau = \{a\,\mathrm{id}_X : a \in \mathbb{C}, |a| \leqslant 1\}$ is γ-bounded with constant 1.

We say that X has property (α) if there is a constant $C \geqslant 1$ such that for any finite family (x_{ij}) in X, we have

$$\begin{aligned} \frac{1}{C} \Big\| \sum_{i,j} \gamma_{ij} \otimes x_{ij} \Big\|_{\mathrm{Gauss}(X)} &\leqslant \Big\| \sum_{i,j} \gamma_i \otimes \gamma_j \otimes x_{ij} \Big\|_{\mathrm{Gauss}(\mathrm{Gauss}(X))} \\ &\leqslant C \Big\| \sum_{i,j} \gamma_{ij} \otimes x_{ij} \Big\|_{\mathrm{Gauss}(X)}, \end{aligned} \tag{2.4}$$

where γ_{ij} is a doubly indexed family of independent standard Gaussian variables. Property (α) is inherited by closed subspaces and isomorphic spaces. The L^p spaces have property (α) for $1 \leqslant p < \infty$ and moreover, if X has property (α), then also $L^p(\Omega; X)$ has. Property (α) is usually defined in terms of independent Rademacher variables ε_i, i.e. $\mathrm{Prob}(\varepsilon_i = \pm 1) = 1/2$ instead of Gaussian variables [38]. In analogy with (2.2), we define $\mathrm{Rad}(X) \subset L^2(\Omega; X)$ by

$$\Big\| \sum_k \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)} = \Big(\int_\Omega \Big\| \sum_{k=1}^n \varepsilon_k(\omega) x_k \Big\|^2 \mathrm{d}\omega \Big)^{1/2}.$$

It turns out that the two definitions are the same:

LEMMA 2.1. *The property* (2.4) *is equivalent to the following equivalence uniform in finite families* (x_{ij}) *in* X.

$$\Big\| \sum_{i,j} \varepsilon_i \otimes \varepsilon_j \otimes x_{ij} \Big\|_{\mathrm{Rad}(\mathrm{Rad}(X))} \cong \Big\| \sum_{i,j} \varepsilon_{ij} x_{ij} \Big\|_{\mathrm{Rad}(X)}. \tag{2.5}$$

Proof. First observe that the Schatten classes S^p for $p \in (1,\infty)\backslash\{2\}$ are spaces which do not satisfy (2.4) nor (2.5). This is shown in [38] for the Rademachers. On the other hand, S^p has finite cotype, which implies that on this space, Rademacher sums and Gaussian sums are equivalent ([10], Theorem 12.27), i.e.

$$\Big\| \sum_{k\in F} \gamma_k \otimes x_k \Big\|_{\mathrm{Gauss}(X)} \cong \Big\| \sum_{k\in F} \varepsilon_k \otimes x_k \Big\|_{\mathrm{Rad}(X)}, \tag{2.6}$$

uniformly in $F \subset \mathbb{Z}$. From this one easily deduces that (2.4) does not hold.

Next observe that by the Banach–Mazur theorem and Theorem 3.2 of [10], S^p has the property that all finite dimensional subspaces are isomorphic to a subspace of some ℓ^∞_n, with one fixed isomorphism constant. This implies that ℓ^∞ does not satisfy (2.4) nor (2.5).

Therefore, by the characterization of finite cotype in Theorem 14.1 of [10], a space X satisfying (2.4) or (2.5) has finite cotype. As cited above, Rademacher and Gaussian sums are then equivalent, so the corresponding expressions in (2.4) and (2.5) are, which shows the lemma. ∎

We recall the construction of Gaussian function spaces from [24], see also Section 1.3 of [22].

Let H be a separable Hilbert space. We consider the tensor product $H \otimes X$ as a subspace of $B(H,X)$ in the usual way, i.e. by identifying $\sum_{k=1}^n h_k \otimes x_k \in H \otimes X$ with the mapping $u : h \mapsto \sum_{k=1}^n \langle h, h_k\rangle x_k$ for any finite families $h_1, \ldots, h_n \in H$ and $x_1, \ldots, x_n \in X$. Choose such families with corresponding u, where the h_k shall be orthonormal. Let $\gamma_1, \ldots, \gamma_n$ be independent standard Gaussian random variables over some probability space. We equip $H \otimes X$ with the norm

$$\|u\|_{\gamma(H,X)} = \Big\| \sum_k \gamma_k \otimes x_k \Big\|_{\mathrm{Gauss}(X)}.$$

By Corollary 12.17 of [10], this expression is independent of the choice of the h_k representing u. We let $\gamma(H,X)$ be the completion of $H \otimes X$ in $B(H,X)$ with respect to that norm. Then for $u \in \gamma(H,X)$, $\|u\|_{\gamma(H,X)} = \Big\| \sum_k \gamma_k \otimes u(e_k) \Big\|_{\mathrm{Gauss}(X)}$, where the e_k form an orthonormal basis of H ([43], Definition 3.7).

A particular subclass of $\gamma(H,X)$ will be important, which is obtained by the following procedure. Assume that (Ω,μ) is a σ-finite measure space and $H = L^2(\Omega)$. Denote $P_2(\Omega,X)$ the space of Bochner-measurable functions $f : \Omega \to X$ such that $x' \circ f \in L^2(\Omega)$ for all $x' \in X'$. We identify $P_2(\Omega,X)$ with a subspace of

$B(L^2(\Omega), X'')$ by assigning to f the operator u_f defined by

$$\langle u_f h, x' \rangle = \int_\Omega \langle f(t), x' \rangle h(t) \mathrm{d}\mu(t). \tag{2.7}$$

An application of the uniform boundedness principle shows that, in fact, u_f belongs to $B(L^2(\Omega), X)$ ([17], Section 5.5). Then we let

$$\gamma(\Omega, X) = \{f \in P_2(\Omega, X) : u_f \in \gamma(L^2(\Omega), X)\}$$

and set

$$\|f\|_{\gamma(\Omega,X)} = \|u_f\|_{\gamma(L^2(\Omega),X)}.$$

The space $\{u_f : f \in \gamma(\Omega, X)\}$ is a dense and in general proper subspace of $\gamma(L^2(\Omega), X)$. Resuming the above, we have the following embeddings of spaces, cf. also Section 3 of [32].

$$L^2(\Omega) \otimes X \to \gamma(\Omega, X) \to \gamma(L^2(\Omega), X) \to B(L^2(\Omega), X).$$

In some cases, $\gamma(L^2(\Omega), X)$ and $\gamma(\Omega, X)$ can be identified with more classical spaces. If X is a Banach function space with finite cotype, e.g. an L^p space for some $p \in [1, \infty)$, then for any step function $f = \sum_{k=1}^n x_k \chi_{A_k} : \Omega \to X$, where $x_k \in X$ and the A_k are measurable and disjoint with $\mu(A_k) \in (0, \infty)$, one easily checks (see also Subsection 2.1 of [26])

$$\begin{aligned} \|f\|_{\gamma(\Omega,X)} &= \Big\| \sum_k \gamma_k \otimes \mu(A_k)^{1/2} x_k \Big\|_{\mathrm{Gauss}(X)} \cong \Big\| \Big(\sum_k \mu(A_k) |x_k|^2 \Big)^{1/2} \Big\|_X \\ &= \Big\| \Big(\int_\Omega |f(t)(\cdot)|^2 \mathrm{d}\mu(t) \Big)^{1/2} \Big\|_X. \end{aligned} \tag{2.8}$$

The second equivalence follows from Theorem 16.18 of [10]. The last expression above is a classical square function (see e.g. Section 6 of [8]), whence for an arbitrary space X, $\|u\|_{\gamma(H,X)}$ is called (generalized) square function. In particular, if X is a Hilbert space, then $\gamma(\Omega, X) = L^2(\Omega, X)$ with equal norms.

We have the following well-known properties of square functions.

LEMMA 2.2. *Let (Ω, μ) be a σ-finite measure space and X a Banach space with property (α).*

(i) *Suppose that $f_n, f \in P_2(\Omega, X)$ and $f_n(t) \to f(t)$ for almost all $t \in \Omega$. Then $\|f\|_{\gamma(\Omega,X)} \leqslant \liminf_n \|f_n\|_{\gamma(\Omega,X)}$.*

(ii) *Let $K \in B(H_2, H_1)$, where H_1, H_2 are separable Hilbert spaces. Then for $u \in \gamma(H_1, X)$ we have $u \circ K \in \gamma(H_2, X)$ and $\|u \circ K\|_{\gamma(H_2,X)} \leqslant \|u\|_{\gamma(H_1,X)} \|K\|$.*

(iii) *If $\Omega \to B(X), t \mapsto N(t)$ is a strongly continuous map such that $\tau = \{N(t) : t \in \Omega\}$ is γ-bounded, and $f \in \gamma(\Omega, X)$, then $\|N \cdot f\|_{\gamma(\Omega,X)} \leqslant \gamma(\tau) \|f\|_{\gamma(\Omega,X)}$.*

Proof. As X has property (α), it does not contain c_0 isomorphically. Using this fact, a proof of (i) can be found in Lemma 4.10 of [24], or in Proposition 3.18 of [43]. For (ii), we refer to Proposition 4.3 of [24] or Corollary 6.3 of [43]. Finally, (iii) is proved in Proposition 4.11 of [24], see also Theorem 5.2 of [43]. ∎

SECTORIAL OPERATORS AND H^∞ FUNCTIONAL CALCULUS. Let $\theta \in (0, \pi)$ and $A : D(A) \subset X \to X$ a densely defined linear mapping on some Banach space X. A is called θ-sectorial, if

(i) The spectrum $\sigma(A)$ is contained in $\overline{\Sigma}_\theta$.

(ii) For all $\omega > \theta$ there is a $C_\omega > 0$ such that $\|\lambda(\lambda - A)^{-1}\| \leqslant C_\omega$ for all $\lambda \in \overline{\Sigma}_\omega^c$.

(iii) $R(A)$ is dense in X.

We call A 0-sectorial if it is θ-sectorial for all $\theta > 0$. In the literature, property (iii) is sometimes omitted. It entails that A is injective ([30], Proposition 15.2). For such an operator A and $f \in H_0^\infty(\Sigma_\omega), \omega \in (\theta, \pi)$, one defines the operator

$$f(A) = \frac{1}{2\pi \mathrm{i}} \int_{\partial \Sigma_{(\theta+\omega)/2}} f(\lambda)(\lambda - A)^{-1} \mathrm{d}\lambda, \tag{2.9}$$

where $\partial \Sigma_{(\theta+\omega)/2}$ is the sector boundary which is parametrized as usual counterclockwise. It is easy to check that $f(A)$ is bounded and that $u : H_0^\infty(\Sigma_\omega) \to B(X)$ is a linear and multiplicative mapping. Suppose that there exists $C > 0$ such that

$$\|f(A)\| \leqslant C \|f\|_{\infty,\omega} \quad (f \in H_0^\infty(\Sigma_\omega)). \tag{2.10}$$

Then there exists an extension of u to a bounded mapping $H^\infty(\Sigma_\omega) \to B(X), f \mapsto f(A)$, satisfying the so-called convergence lemma ([8], Lemma 2.1).

LEMMA 2.3. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $H^\infty(\Sigma_\omega)$ such that $\sup_{n\in\mathbb{N}} \|f_n\|_{\infty,\omega} < \infty$ and $f_n(\lambda) \to f(\lambda)$ for all $\lambda \in \Sigma_\omega$ and some f (which then necessarily belongs to $H^\infty(\Sigma_\omega)$). Then $f(A)x = \lim_{n\to\infty} f_n(A)x$ for any $x \in X$.*

Note that the extension is uniquely determined by Lemma 2.3 since for any $f \in H^\infty(\Sigma_\omega)$,

$$f_n(\lambda) = f(\lambda)(\lambda/(1+\lambda)^2)^{1/n} \tag{2.11}$$

is a sequence in $H_0^\infty(\Sigma_\omega)$ approximating f in the sense of that lemma. As a consequence, if (2.10) is satisfied, then it also holds for any $f \in H^\infty(\Sigma_\omega)$. In this case, we say that A has a bounded $H^\infty(\Sigma_\omega)$ calculus, or without precising the angle $\omega \in (\theta, \pi)$, a bounded H^∞ calculus.

3. SQUARE FUNCTION ESTIMATE AND MATRICIALLY BOUNDED HOMOMORPHISM

Throughout the section, we let X be a Banach space. For any $n \in \mathbb{N}$, we identify $M_n \otimes B(X)$ with $B(\mathrm{Gauss}_n(X))$ by associating $[a_{ij}] \otimes T \in M_n \otimes B(X)$ with the operator

$$\sum_{k=1}^{n} \gamma_k \otimes x_k \mapsto \sum_{k,j=1}^{n} \gamma_k \otimes a_{kj} T(x_j). \tag{3.1}$$

Via this identification, we get a norm on the tensor product space, which we note by $M_n \otimes_\gamma B(X)$.

DEFINITION 3.1. Let E be an operator space. Let further $u : E \to B(X)$ be a linear mapping. We call u *matricially γ-bounded*, if $\mathrm{id}_{M_n} \otimes u : M_n \otimes E \to M_n \otimes_\gamma B(X)$ is bounded uniformly in $n \in \mathbb{N}$, i.e. if there is a constant $C > 0$ such that for any $n \in \mathbb{N}$,

$$\Big\| \sum_{i,j=1}^{n} \gamma_i \otimes u(f_{ij}) x_j \Big\|_{\mathrm{Gauss}_n(X)} \leqslant C \|[f_{i,j}]\|_{M_n \otimes E} \Big\| \sum_{i=1}^{n} \gamma_i \otimes x_i \Big\|_{\mathrm{Gauss}_n(X)}.$$

We denote the least admissible constant C by $\|u\|_{\text{mat-}\gamma}$.

REMARK 3.2. (i) If $u : E \to B(X)$ is matricially γ-bounded then

$$\{u(x) : \|x\|_E \leqslant 1\} \tag{3.2}$$

is γ-bounded. Indeed, from the definition of γ-boundedness in Section 2, we immediately deduce that (3.2) is satisfied if and only if $\mathrm{id}_{M_n} \otimes u|_{D_n \otimes E}$ is bounded uniformly in $n \in \mathbb{N}$, where $D_n \subset M_n$ denotes the subspace of diagonal matrices. We call a linear mapping u γ-bounded if (3.2) holds.

(ii) Assume that X is a Hilbert space. Then by (2.3), u is matricially γ-bounded if and only if u is completely bounded, and in this case, $\|u\|_{\mathrm{cb}} = \|u\|_{\text{mat-}\gamma}$.

A first example for Definition 3.1 is given by

PROPOSITION 3.3. *For a given space X and $m \in \mathbb{N}$, consider*

$$\sigma_{m,X} : M_m \to M_m \otimes_\gamma B(X), [a_{ij}] \mapsto [a_{ij}\, \mathrm{id}_X].$$

Then $\sigma_{m,X}$ is matricially γ-bounded with $\sup_{m \in \mathbb{N}} \|\sigma_{m,X}\|_{\text{mat-}\gamma} < \infty$ if and only if X has property (α).

It is shown in Lemma 4.3 of [27] that the $\sigma_{m,X}$ are γ-bounded uniformly in $m \in \mathbb{N}$ if and only if X has property (α) (with Rademachers in place of Gaussians). Actually the same proof applies to Proposition 3.3.

Mappings which are γ-bounded or matricially γ-bounded have been studied so far in connection with functional calculi and unconditional decompositions [11], [27] where E is a $C(K)$-space and representations of amenable groups [32], where E is a nuclear C^*-algebra. We shall focus in this section on the row Hilbert space $E = \ell^2_{\mathrm{r}}$.

THEOREM 3.4. *Let $u : \ell^2 \to B(X)$ be a bounded linear mapping. Assume that X has Pisier's property (α). For $n \in \mathbb{N}$, denote by $C_n \subset M_n$ the subspace of matrices vanishing outside the first column. Then the following conditions are equivalent:*

(i) $\|u(\cdot)x\|_{\gamma(\ell^2,X)} \leqslant C\|x\|$ $(x \in X)$.

(ii) $u : \ell^2_{\mathrm{r}} \to B(X)$ *is matricially γ-bounded.*

(iii) *The restriction* $\mathrm{id} \otimes u : C_n \otimes \ell^2_{\mathrm{r}} \to M_n \otimes_\gamma B(X)$ *is bounded uniformly in $n \in \mathbb{N}$.*

Proof. We fix an orthonormal basis $(e_k)_k$ of ℓ^2. Write $T_k = u(e_k)$. Then condition (i) of the statement rewrites

$$\|u(\cdot)x\|_\gamma = \sup_{n\in\mathbb{N}} \Big\| \sum_{k=1}^n \gamma_k \otimes T_k x \Big\|_{\mathrm{Gauss}_n(X)} \leqslant C\|x\|. \tag{3.3}$$

On the other hand, for $[f_{ij}] \in M_n \otimes \ell^2$, we have with $f_{ij}^{(k)} = \langle f_{ij}, e_k\rangle$,

$$\|(\mathrm{id}_{M_n} \otimes u)[f_{ij}]\|_{M_n\otimes_\gamma B(X)} = \lim_m \Big\| \Big[\sum_{k=1}^m f_{ij}^{(k)} T_k \Big] \Big\|_{M_n\otimes_\gamma B(X)}.$$

Thus, condition (ii) is equivalent to

$$\Big\| \sum_{j=1}^n \sum_{i=1}^n \sum_{k=1}^m \gamma_i \otimes f_{ij}^{(k)} T_k x_j \Big\|_{\mathrm{Gauss}_n(X)} \leqslant C\|[f_{ij}]\|_{M_n\otimes \ell^2_{m,\mathrm{r}}} \Big\| \sum_{i=1}^n \gamma_i \otimes x_i \Big\|_{\mathrm{Gauss}_n(X)}, \tag{3.4}$$

where $f_{ij} = (f_{ij}^{(k)})_k \in \ell^2_m$, and C is independent of n and m. Denote the linear bounded mapping $\mathrm{Gauss}_n(X) \to \mathrm{Gauss}_n(X)$ arising from (3.4) by $u_{n,m}([f_{ij}])$. Finally, condition (iii) is equivalent to (3.4) with $f_{ij} = 0$ for $j \geqslant 2$.

(i) $\Rightarrow$ (ii) For $m \in \mathbb{N}$ fixed, let $Y = \mathrm{Gauss}_m(X)$ and define the operators

$$V : X \to Y,\ x \mapsto \sum_{k=1}^m \gamma_k \otimes T_k x, \quad W : Y \to X, \sum_{k=1}^m \gamma_k \otimes x_k \mapsto x_1.$$

By assumption (3.3), V is bounded with constant C independent of m. Further, W is bounded ([10], Corollary 12.17). For $n \in \mathbb{N}$, denote

$$V_n = \mathrm{id}_{\ell^2_n} \otimes V : \mathrm{Gauss}_n(X) \to \mathrm{Gauss}_n(Y), \quad \sum_{k=1}^n \gamma_k \otimes x_k \mapsto \sum_{k=1}^n \gamma_k \otimes V(x_k).$$

It is easy to check that $\|V_n\| = \|V\|$. Similarly, defining $W_n = \mathrm{id}_{\ell^2_n} \otimes W : \mathrm{Gauss}_n(Y) \to \mathrm{Gauss}_n(X)$, one has $\|W_n\| = \|W\|$. Let $i_m : \ell^2_{m,\mathrm{r}} \to M_m$ be the first row identification as in (2.1) which is completely bounded of cb-norm 1. Then by Remark 3.2 and Proposition 3.3, along with property (α), $\pi_m = \sigma_{m,X} \circ i_m : \ell^2_{m,\mathrm{r}} \to B(M_m \otimes X)$ is a matricially γ-bounded mapping and $\sup_{m\in\mathbb{N}} \|\pi_m\|_{\text{mat-}\gamma} < \infty$. For $f = [f_{ij}]_{ij} \in$

$M_n \otimes \ell^2_{m,\mathrm{r}}$, we have the identity

$$u_{n,m}(f) = \Big[\sum_{k=1}^m f_{ij}^{(k)} T_k \Big] = [W \pi_m(f_{ij}) V] = W_n [\pi_m(f_{ij})] V_n.$$

Therefore,

$$\|u_{n,m}(f)\| \leqslant \|W_n\| \, \|V_n\| \, \|[\pi_m(f_{ij})]\| \leqslant \|W\| \, \|V\| \, \|\pi_m\|_{\text{mat-}\gamma} \|[f_{ij}]\|_{M_n \otimes \ell^2_{m,\mathrm{r}}},$$

so (3.4) follows.

(ii) $\Rightarrow$ (iii) This is clear, since (iii) is an obvious restriction of (ii).

(iii) $\Rightarrow$ (i) Choose $n = m \in \mathbb{N}$ and $f = [f_{ij}] \in M_n \otimes \ell^2_{n,r}$ with $f_{ij} = \delta_{j1} e_i$, where (e_i) is the standard basis of ℓ^2_n. By definition of the row norm, we have

$$\|f\|_{M_n \otimes \ell^2_{n,r}} = \Big\| \Big[\sum_k \langle f_{ik}, f_{jk} \rangle \Big] \Big\|_{M_n}^{1/2} = \Big\| \Big[\sum_k \delta_{k1} \langle e_i, e_j \rangle \Big] \Big\|_{M_n}^{1/2} = \|[\delta_{ij}]\|_{M_n}^{1/2} = 1.$$

As f is supported by the first column, by assumption (3.4) there is some $C < \infty$ such that

$$\begin{aligned} C \geqslant \|u_{n,n}(f)\|_{M_n \otimes_\gamma B(X)} &= \left\| \begin{pmatrix} T_1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ T_n & 0 & \dots & 0 \end{pmatrix} \right\|_{M_n \otimes_\gamma B(X)} \\ &= \sup \Big\{ \Big\| \sum_k \gamma_k \otimes T_k x_1 \Big\|_{\mathrm{Gauss}_n(X)} : \Big\| \sum_k \gamma_k \otimes x_k \Big\|_{\mathrm{Gauss}_n(X)} \leqslant 1 \Big\} \\ &= \sup \Big\{ \Big\| \sum_k \gamma_k \otimes T_k x \Big\|_{\mathrm{Gauss}_n(X)} : \|x\| \leqslant 1 \Big\}. \end{aligned}$$

Letting $n \to \infty$ shows that (3.3) holds. ∎

REMARK 3.5. Theorem 3.4 is a generalization of Proposition 3.3 of [31], where X is an L^p-space, and Corollary 3.19 of [18], where more generally X has property (α). There it is shown that condition (i) of the theorem implies that $u : \ell^2 \to B(X)$ is γ-bounded, which is weaker than matricial γ-boundedness by Remark 3.2. Indeed the last line in Proposition 3.3 of [31] says that

$$\{u(a) : a \in L^2(I), \|a\|_{L^2(I)} \leqslant 1\} \tag{3.5}$$

is R-bounded, under the condition that $\Big\| \Big(\sum_{n \in \mathbb{N}} |u(e_n)x|^2 \Big)^{1/2} \Big\| \leqslant C\|x\|$, where $(e_n)_n$ is an orthonormal basis of $L^2(I)$. However, as the space $X = L^q(\Sigma)$ has finite cotype, R-boundedness and γ-boundedness are equivalent, which means that (3.2) holds. Also $\Big\| \Big(\sum_{n \in \mathbb{N}} |u(e_n)x|^2 \Big)^{1/2} \Big\| \cong \|u(\cdot)x\|_{\gamma(L^2(I),X)}$. (In [18], [31], u maps to $B(Y,X)$ instead of $B(X)$. A corresponding version of Theorem 3.4 with $B(Y,X)$ in place of $B(X)$ holds with the same proof).

4. THE HÖRMANDER FUNCTIONAL CALCULUS

Recall the spaces W^α and $\mathcal{H}^\alpha$ and the dyadic partition of unity $(\phi_n)_{n\in\mathbb{Z}}$ from the introduction. Clearly the space W^α is a Hilbert space. We equip $\mathcal{H}^\alpha$ with an operator space structure by putting

$$\|[f_{ij}]\|_{M_n\otimes\mathcal{H}^\alpha} = \sup_{k\in\mathbb{Z}} \|[\phi_k f_{ij}]\|_{M_n\otimes W_r^\alpha}, \tag{4.1}$$

where the index r refers to the row space structure. It is easy to check that (4.1) indeed defines an operator space, arising from the embedding

$$\mathcal{H}^\alpha \hookrightarrow B(\bigoplus\nolimits^2_{k\in\mathbb{Z}} W^\alpha), f \mapsto ((g_k)_{k\in\mathbb{Z}} \mapsto (\langle g_k, \phi_k f\rangle e)_{k\in\mathbb{Z}}),$$

where $\bigoplus\limits^2_{k\in\mathbb{Z}} W^\alpha$ is the Hilbert sum, and e is some fixed element in W^α of norm 1.

In this section we focus on (unital) homomorphisms

$$u : \mathcal{H}^\alpha \to B(X). \tag{4.2}$$

We give a characterization when such mappings are matricially γ-bounded. An example of a bounded homomorphism of this type is given by Hörmander's classical theorem mentioned in the introduction, which states that for $\alpha > d/2, X = L^p(\mathbb{R}^d)$ and $p \in (1,\infty)$, the radial Fourier multiplier representation $u_{-\Delta} : \mathcal{H}^\alpha \to B(X)$ given by

$$u_{-\Delta}(f)g = [f(|\cdot|^2)\hat{g}]^\vee = f(-\Delta)g \tag{4.3}$$

is bounded. In fact, by means of our characterization, we will show in Section 5 that $u_{-\Delta}$ is even matricially γ-bounded provided that $\alpha > (d+1)/2$.

For $n \in \mathbb{N}$, let M^n be the space consisting of n-times continuously differentiable functions f defined on $(0,\infty)$ such that $\|f\|_{\mathrm{M}^n} = \sum\limits_{k=0}^n \sup\limits_{t>0} |t^k f^{(k)}(t)|$ is finite. Let us record how $W^\alpha, \mathcal{H}^\alpha, H^\infty(\Sigma_\omega)$ and the auxiliary space M^n compare.

LEMMA 4.1. *Let $\omega \in (0,\pi)$ and $\alpha > 1/2$.*

(i) $H^\infty(\Sigma_\omega) \hookrightarrow \mathcal{H}^\alpha$.

(ii) $W^\alpha \hookrightarrow \mathcal{H}^\alpha$, *where the embedding is completely bounded.*

(iii) *For any $\omega \in (0,\pi)$, $H^\infty(\Sigma_\omega) \cap W^\alpha$ is a dense subset of W^α.*

(iv) $\mathrm{M}^n \hookrightarrow \mathcal{H}^\alpha$ *for $n > \alpha$.*

(v) $H^\infty(\Sigma_\omega)$ *is a dense subset of* M^n.

Moreover, any $f \in W^\alpha \cap \mathrm{M}^n$ can be simultaneously approximated by a sequence $(f_k)_{k\in\mathbb{N}} \subset H^\infty(\Sigma_\omega) \cap W^\alpha \cap \mathrm{M}^n$.

Proof. (i) If f belongs to $H^\infty(\Sigma_\omega)$, then by the Cauchy integral formula for derivatives to any order n, $f^{(n)}(t) = n!/2\pi\mathrm{i} \int_{\Gamma_t} f(z)/(z-t)^n \mathrm{d}z$, where we choose the contour depending on t to be the biggest circle with center t which is still contained in $\overline{\Sigma}_\omega$, so with a radius proportional to t. A simple estimate of the above integral then shows that $\sup_{t>0} |t^n f^{(n)}(t)|$ is finite, with constant depending on $\|f\|_{\infty,\omega}$, so that $H^\infty(\Sigma_\omega) \hookrightarrow \mathrm{M}^n$.

(iv) The embedding is elementary. Then the above gives $H^\infty(\Sigma_\omega) \hookrightarrow \mathrm{M}^n \hookrightarrow \mathcal{H}^\alpha$, so (i) is shown.

(ii) This follows directly from the definitions of the operator spaces W^α and $\mathcal{H}^\alpha$.

(iii) + (v) This is proved as in Proposition 3.2 of [26], see also Lemma 4.15, proof of Proposition 4.22 and Proposition 4.9 of [25]. ∎

The main interest of M^n is the following convergence lemma, which is proved in Section 4.2.4 of [25], see also [28].

LEMMA 4.2. *Let $u : \mathrm{M}^n \to B(X)$ be bounded such that $u(f) = f(A)$ for some 0-sectorial operator A and any $f \in \bigcup_{\theta \in (0,\pi)} H^\infty(\Sigma_\theta)$. Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity and $(a_n)_{n\in\mathbb{Z}}$ a bounded sequence. Then $\sum_{n\in\mathbb{Z}} a_n\phi_n$ belongs to M^n and*

$$\sum_{n\in\mathbb{Z}} a_n u(\phi_n)x = u\Big(\sum_{n\in\mathbb{Z}} a_n\phi_n\Big)x \quad (x \in X). \tag{4.4}$$

Many spectral multiplier theorems for Laplace type operators A consist in the boundedness of u in (4.2), which in turn is the functional calculus u_A of some 0-sectorial operator. For example, in the case of (4.3) one has $A = -\Delta$. In the sequel we will only consider homomorphisms of the form $u = u_A$. The next lemma gives a criterion when this is the case.

LEMMA 4.3. *Let $\omega \in (0,\pi)$.*

(i) *Let u be a bounded homomorphism $u : \mathcal{H}^\alpha \to B(X)$. There exists a 0-sectorial operator A such that*

$$u(f) = f(A) \quad (f \in H^\infty(\Sigma_\omega)), \tag{4.5}$$

if and only if the restriction of u to $H^\infty(\Sigma_\omega)$ satisfies the convergence Lemma 2.3, i.e. for any $f_n, f \in H^\infty(\Sigma_\omega)$ with $\sup_n \|f_n\|_{\infty,\omega} < \infty$ and $f_n(\lambda) \to f(\lambda)$ pointwise, we have $u(f)x = \lim_n u(f_n)x$ for any $x \in X$. In this case, we write $f(A)$ in place of $u(f)$ for any $f \in \mathcal{H}^\alpha$.

(ii) *Let A be a 0-sectorial operator. Then there exists a bounded homomorphism $u : \mathcal{H}^\alpha \to B(X)$ satisfying* (4.5) *if and only if $\|f(A)\| \leqslant C\|f\|_{\mathcal{H}^\alpha}$ $(f \in H^\infty(\Sigma_\omega))$. Moreover, u is uniquely determined by* (4.5).

Proof. (i) By Lemma 4.1(i), only the "if" part has to be shown. Suppose that $u : \mathcal{H}^\alpha \to B(X)$ satisfies the convergence Lemma 2.3. Since u is a homomorphism, one has $u((\lambda - \cdot)^{-1}) - u((\mu - \cdot)^{-1}) = (\mu - \lambda)u((\lambda - \cdot)^{-1})u((\mu - \cdot)^{-1})$ for any $\lambda, \mu \in \mathbb{C}\backslash[0,\infty)$, i.e. $u((\lambda - \cdot)^{-1})$ is a pseudo resolvent ([37], Definition 9.1). By Lemma 2.3, $u(\lambda(\lambda - \cdot)^{-1})x \to x$ for any $x \in X$ and $|\lambda| \to \infty$. Thus, by Corollary 9.5 of [37], there exists a densely defined operator A such that $u((\lambda - \cdot)^{-1}) =$

$(\lambda - A)^{-1}$ for $\lambda \in \mathbb{C}\backslash[0,\infty)$. Again by Lemma 2.3, $(-1/n)(1/n + A)^{-1}x \to 0$ for any $x \in X$ and $n \to \infty$. Thus, A has dense range ([30], Proposition 15.2). As u is a bounded homomorphism, it now follows that A is 0-sectorial and for any rational function $r \in \mathcal{H}^\alpha$, $\|r(A)\| \lesssim \|r\|_{\mathcal{H}^\alpha} \lesssim \|r\|_{\infty,\omega}$.

We claim that A has an H^∞ calculus coinciding with u. Indeed, a given $f \in H^\infty_0(\Sigma_\omega)$, we write

$$f = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Sigma_{\omega/2}} f(\lambda)(\lambda - \cdot)^{-1} \mathrm{d}\lambda.$$

As $f(\lambda)(\lambda - \cdot)^{-1} : \partial\Sigma_{\omega/2} \to H^\infty(\Sigma_{\omega/4})$ is continuous, we find a sequence $r_n = \sum_{k=1}^{K} c_k f(\lambda_k)(\lambda_k - \cdot)^{-1}$ such that $r_n \to f$ in $H^\infty(\Sigma_{\omega/4})$, so in particular in $\mathcal{H}^\alpha$. Clearly, r_n are rational functions. Inserting formally $(\cdot) = A$ in the Cauchy integral, the same arguments apply, and $r_n(A) \to f(A)$. We conclude $u(f) = \lim_n u(r_n) = \lim_n r_n(A) = f(A)$.

We have shown that $u(f) = f(A)$ for any $f \in H^\infty_0(\Sigma_\omega)$. For a general $f \in H^\infty(\Sigma_\omega)$ we use the approximation (2.11).

(ii) The "only if" part is clear and the "if" part follows from the density in Lemma 4.1 together with the convergence Lemma 4.2. Using density and Lemma 4.1, u is uniquely determined on W^α and M^n for any $n > \alpha$. Thus u satisfies the decomposition (4.4). Then for $f \in \mathcal{H}^\alpha$, we have $u(f)x = u(f) \sum_{k\in\mathbb{Z}} u(\phi_k)x = \sum_{k\in\mathbb{Z}} u(f\phi_k)x$. As $f\phi_k \in W^\alpha$, we conclude the uniqueness of u. ∎

The strategy to prove matricial γ-boundedness of a mapping from $\mathcal{H}^\alpha$ to $B(X)$ will be to show the matricial γ-boundedness from W^α to $B(X)$, and then to pass to $\mathcal{H}^\alpha$ by means of a spectral decomposition, given by (4.6) in the following theorem. The restriction of the H^∞ calculus angle ω to $(0, \pi/4)$ is only for technical reasons.

THEOREM 4.4. *Let X be a Banach space with property (α). Let $\alpha > 1/2, \omega \in (0, \pi/4)$ and A be a 0-sectorial operator on X having a bounded $H^\infty(\Sigma_\omega)$ calculus. Assume that*

$$\|f(A)\| \leqslant C\|f\|_{W^\alpha} \quad (f \in H^\infty(\Sigma_\omega) \cap W^\alpha)$$

and that the extension resulting from density $u : W^\alpha \to B(X), f \mapsto f(A)$ is matricially γ-bounded. Let $(\phi_n)_{n\in\mathbb{Z}}$ be a dyadic partition of unity. Then

$$\|x\| \cong \Big\| \sum_{n\in\mathbb{Z}} \gamma_n \otimes \phi_n(A)x \Big\|_{\mathrm{Gauss}(X)}, \tag{4.6}$$

the sum on the right hand side being convergent in Gauss(X).

As the proof is rather long we separate four preliminary lemmas, whose proofs are annexed in Section 6.

LEMMA 4.5. *Let X have property (α), let $\alpha > 1/2$. Let A be as in Theorem 4.4. Then for $\beta \in \mathbb{N}, \beta > \alpha$,*

$$\{\exp(-2^k zA) : k \in \mathbb{Z}\} \text{ is } \gamma\text{-bounded with constant } \lesssim \left|\frac{z}{\operatorname{Re} z}\right|^{\beta} \quad (\operatorname{Re} z > 0). \tag{4.7}$$

LEMMA 4.6. *Let A be a 0-sectorial operator on some Banach space X such that for some $\beta > 0$, (4.7) holds. Then for $\gamma = \beta + 1$, we have*

$$\{\lambda^{1/2}(2^k A)^{1/2}(\lambda - 2^k A)^{-1} : k \in \mathbb{Z}\} \text{ is } \gamma\text{-bounded with constant } \lesssim |\arg \lambda|^{-\gamma} \quad (\operatorname{Re} \lambda > 0). \tag{4.8}$$

LEMMA 4.7. *Let A be a 0-sectorial operator on some space X with property (α) having a bounded $H^\infty(\Sigma_\omega)$ calculus. Assume that A satisfies (4.8) for some $\gamma > 0$. Then for any $n > \gamma$,*

$$\|f(A)\| \leqslant C\|f\|_{\mathrm{M}^n} \quad (f \in H^\infty(\Sigma_\omega)).$$

LEMMA 4.8. *Let $n \in \mathbb{N}$. Let $(g_k)_{k\in\mathbb{Z}}$ satisfy $\sup_{k\in\mathbb{Z}} \|g_k\|_{\mathrm{M}^n} < \infty$. Suppose that the supports of g_k satisfy the following overlapping condition*

$$\sup_{x>0} \#\{k \in \mathbb{Z} : \operatorname{supp} g_k \cap [1/2x, 2x] \neq \emptyset\} < \infty.$$

Then $\sum_{k\in\mathbb{Z}} g_k$, which is consequently pointwise a finite sum belongs to M^n, and

$$\Big\| \sum_{k\in\mathbb{Z}} g_k \Big\|_{\mathrm{M}^n} \lesssim \sup_{k\in\mathbb{Z}} \|g_k\|_{\mathrm{M}^n} < \infty.$$

Proof of Theorem 4.4. Using Lemmas 4.5, 4.6 and 4.7 one after another shows that $\|f(A)\| \lesssim \|f\|_{\mathrm{M}^n}$ for n sufficiently large ($n > \lfloor\alpha\rfloor + 2$). For any $k \in \mathbb{Z}$, let $a_k \in \{1, -1\}$. Apply Lemma 4.8 with $g_k = a_k\phi_k$. It is easy to check that $\|g_k\|_{\mathrm{M}^n}$ is independent of $k \in \mathbb{Z}$. Further, the overlapping condition is clearly satisfied with constant 2. Thus we have, for any finite $F \subset \mathbb{Z}$,

$$\Big\| \sum_{k\in F} a_k\phi_k(A)x \Big\| \lesssim \Big\| \sum_{k\in F} a_k\phi_k \Big\|_{\mathrm{M}^n} \|x\| \lesssim \|x\|.$$

Replacing a_k by independent Rademacher variables ε_k and taking expectation gives

$$\Big\| \sum_{k\in F} \varepsilon_k \otimes \phi_k(A)x \Big\|_{\operatorname{Rad}(X)} \lesssim \|x\|.$$

Since X has property (α), X has finite cotype (see the reasoning in Lemma 2.1), the equivalence of Gaussian and Rademacher sums (2.6) holds. By (4.4), $\sum_{k\in\mathbb{Z}} a_k\phi_k(A)x$ converges in X. By dominated convergence, convergence holds also in $\operatorname{Rad}(X)$ (respectively $\operatorname{Gauss}(X)$), when a_k is replaced by ε_k (respectively γ_k).

We have shown

$$\Big\| \sum_{k\in\mathbb{Z}} \gamma_k \otimes \phi_k(A)x \Big\|_{\operatorname{Gauss}(X)} \lesssim \|x\|.$$

For the reverse inequality, we argue by duality. Let $x' \in X'$, write $\mathbb{E}$ for expectation and $\widetilde{\phi}_l = \sum_{k=l-1}^{l+1} \phi_k$. By the support condition on the ϕ_k, $\widetilde{\phi}_l \phi_l = \phi_l$. Then using the independence of the γ_k, we have

$$\begin{aligned}|\langle x, x'\rangle| &= \Big|\mathbb{E}\Big\langle \sum_{k\in\mathbb{Z}} \gamma_k \phi_k(A)x, \sum_{l\in\mathbb{Z}} \gamma_l \widetilde{\phi}_l(A)'x'\Big\rangle\Big| \\ &\leqslant \Big\| \sum_{k\in\mathbb{Z}} \gamma_k \otimes \phi_k(A)x\Big\|_{\mathrm{Gauss}(X)} \Big\| \sum_{l\in\mathbb{Z}} \gamma_l \otimes \widetilde{\phi}_l(A)'x'\Big\|_{\mathrm{Gauss}(X')}.\end{aligned}$$

We conclude the proof by the same argument as above which shows that

$$\Big\| \sum_{l\in\mathbb{Z}} \gamma_l \otimes \widetilde{\phi}_l(A)'x'\Big\|_{\mathrm{Gauss}(X')} \lesssim \|x'\|. \quad \blacksquare$$

The main result of this section reads as follows.

THEOREM 4.9. *Let A be a 0-sectorial operator with bounded $H^\infty(\Sigma_\theta)$ calculus for some $\theta \in (0, \pi)$ on a space X with property (α). Let $\alpha > 1/2$. Then the following are equivalent:*

(i) *For any $x \in X$, $(1+t^2)^{-\alpha/2}A^{\mathrm{i}t}x$ belongs to $\gamma(\mathbb{R}, X)$ and*

$$\|(1+t^2)^{-\alpha/2}A^{\mathrm{i}t}x\|_{\gamma(\mathbb{R},X)} \leqslant C\|x\|.$$

(ii) *The H^∞ calculus of A extends to a matricially γ-bounded mapping $u : \mathcal{H}^\alpha \to B(X)$.*

Proof. Assume first that the theorem is shown under the additional assumption that $\theta < \pi/4$. For a general $\theta \in (0, \pi)$, we can reduce to this case by considering $B = A^{1/4}$. Namely, by Theorem 2.4.2 of [19], B has a bounded $H^\infty(\Sigma_\omega)$ calculus for some $\omega \leqslant \theta/4 < \pi/4$. Moreover,

$$(1+t^2)^{-\alpha/2}B^{\mathrm{i}t}x = \Big(\frac{1+(t/4)^2}{1+t^2}\Big)^{\alpha/2} \cdot \Big(1+\Big(\frac{t}{4}\Big)^2\Big)^{-\alpha/2} A^{\mathrm{i}t/4}x.$$

The first factor is bounded, so its multiplication with an $L^2(\mathbb{R})$ function is a bounded operation on $L^2(\mathbb{R})$. The same holds for its inverse, and also for the change of variables $f \mapsto f(\cdot/4)$, and its inverse. Thus, by Lemma 2.2, if A satisfies (i) then so does B, so B satisfies (ii). As $H^\infty(\Sigma_\omega) \hookrightarrow \mathcal{H}^\alpha$, B has an $H^\infty(\Sigma_\omega)$ calculus actually for any $\omega > 0$, so by Theorem 2.4.2 of [19], A has an $H^\infty(\Sigma_\theta)$ calculus for some $\theta < \pi/4$. The same is true, provided that A satisfies (ii). Thus (i) or (ii) imply that the assumption of the theorem actually holds with $\theta < \pi/4$. We suppose from now on that $\theta < \pi/4$.

(i) $\Rightarrow$ (ii) Fix an arbitrary orthonormal basis $(f_k)_k$ of $L^2(\mathbb{R})$. Let $T_k \in B(X)$ be defined by

$$\langle T_k x, x'\rangle = \frac{1}{2\pi} \int_{\mathbb{R}} f_k(t)(1+t^2)^{-\alpha/2}\langle A^{\mathrm{i}t}x, x'\rangle \mathrm{d}t$$

and $v : \ell^2 \to B(X)$ the linear mapping given by $e_k \mapsto T_k$. Then (i) implies

$$\|v(\cdot)x\|_{\gamma(\ell^2,X)} = \Big\| \sum_k \gamma_k \otimes v(e_k)x \Big\|_{\mathrm{Gauss}(X)} = \Big\| \sum_k \gamma_k \otimes T_k x \Big\|_{\mathrm{Gauss}(X)}$$
$$= \|(1+t^2)^{-\alpha/2} A^{\mathrm{i}t} x\|_\gamma \lesssim \|x\|,$$

so that by Theorem 3.4, $v : \ell^2_{\mathrm{r}} \to B(X)$ is matricially γ-bounded. Consider the mapping $w : W^\alpha \to \ell^2, f \mapsto \left(\frac{1}{2\pi} \int_{\mathbb{R}} (f \circ \exp)\hat{}\,(t)(1+t^2)^{\alpha/2} \overline{f}_k(t) \mathrm{d}t \right)_k$. It is easy to check that w is unitary and consequently, $\widetilde{u} = v \circ w : W^\alpha_r \to B(X)$ is also matricially γ-bounded. On the other hand, $\widetilde{u}(f) = f(A)$ for any $f \in H^\infty(\Sigma_\theta) \cap W^\alpha$. Indeed, by a representation formula (see e.g. in Proposition 4.6 or proof of Theorem 4.9 of [26]),

$$\begin{aligned} 2\pi\langle f(A)x, x'\rangle &= \int_{\mathbb{R}} (f \circ \exp)\hat{}\,(t) \langle A^{\mathrm{i}t} x, x'\rangle \mathrm{d}t \\ &= \int_{\mathbb{R}} (f \circ \exp)\hat{}\,(t)(1+t^2)^{\alpha/2}(1+t^2)^{-\alpha/2} \langle A^{\mathrm{i}t} x, x'\rangle \mathrm{d}t \\ &= \sum_k \int_{\mathbb{R}} (f \circ \exp)\hat{}\,(t)(1+t^2)^{\alpha/2} \overline{f}_k(t) \mathrm{d}t \langle T_k x, x'\rangle = 2\pi\langle \widetilde{u}(f)x, x'\rangle. \end{aligned}$$

Let $n \in \mathbb{N}$ and $F = [f_{ij}] \in M_n \otimes H^\infty(\Sigma_\theta)$. We show that $\|[f_{ij}(A)]\|_{M_n \otimes_\gamma B(X)} \lesssim \|[f_{ij}]\|_{M_n \otimes \mathcal{H}^\alpha}$. For $N \in \mathbb{N}$ consider

$$F_N = \begin{bmatrix} [\phi_{-N} f_{ij}] & 0 & \dots & 0 \\ 0 & [\phi_{-N+1} f_{ij}] & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & [\phi_N f_{ij}] \end{bmatrix} \in M_{(2N+1)n} \otimes W^\alpha.$$

By (4.1), we have $\sup_N \|F_N\|_{M_{(2N+1)n} \otimes W^\alpha} = \|F\|_{M_n \otimes \mathcal{H}^\alpha}$. Observe first that for any scalars $g_1, \dots, g_n$, by Theorem 4.4, with $\widetilde{\phi}_k = \sum_{l=k-1}^{k+1} \phi_l$,

$$\Big\| \sum_{i,j=1}^n g_i f_{ij}(A) x_j \Big\| \cong \Big\| \sum_{i,j=1}^n \sum_{k \in \mathbb{Z}} \gamma_k \otimes g_i \widetilde{u}(f_{ij}\phi_k) x_j \Big\| \cong \Big\| \sum_{i,j=1}^n \sum_{k \in \mathbb{Z}} \gamma_k \otimes g_i \widetilde{u}(f_{ij}\phi_k) \widetilde{u}(\widetilde{\phi}_k) x_j \Big\|.$$

Replacing g_i by Gaussian variables and taking expectations shows that

$$\begin{aligned} &\Big\| \sum_{i,j=1}^n \sum_{k \in \mathbb{Z}} \gamma_i \otimes \widetilde{u}(f_{ij}) x_j \Big\|_{\mathrm{Gauss}_n(X)} \\ &\quad \cong \Big\| \sum_{i,j=1}^n \sum_{k \in \mathbb{Z}} \gamma_i \otimes \gamma_k \otimes \widetilde{u}(f_{ij}\phi_k) \widetilde{u}(\widetilde{\phi}_k) x_j \Big\|_{\mathrm{Gauss}_n(\mathrm{Gauss}(X))}. \end{aligned} \tag{4.9}$$

Further we have

$$\begin{aligned}
&\Big\| \sum_{i,j=1}^{n} \sum_{k=-N}^{N} \gamma_i \otimes \gamma_k \otimes \widetilde{u}(f_{ij}\phi_k)\widetilde{u}(\widetilde{\phi}_k)x_j \Big\|_{\mathrm{Gauss}_n(\mathrm{Gauss}(X))} \\
&\qquad \cong \Big\| \sum_{i,j,k} \gamma_{ik} \otimes \widetilde{u}(f_{ij}\phi_k)\widetilde{u}(\widetilde{\phi}_k)x_j \Big\|_{\mathrm{Gauss}(X)} \\
&\qquad \lesssim \|F_N\|_{M_{(2N+1)n}\otimes W^\alpha} \Big\| \sum_{i,k} \gamma_{i,k} \otimes \widetilde{u}(\widetilde{\phi}_k)x_i \Big\|_{\mathrm{Gauss}(X)} \\
&\qquad \lesssim \|F\|_{M_n\otimes \mathcal{H}^\alpha} \Big\| \sum_{i} \gamma_i \otimes x_i \Big\|_{\mathrm{Gauss}_n(X)}. \qquad (4.10)
\end{aligned}$$

Finally taking the supremum over $N \in \mathbb{N}$, (4.9) and (4.10) give

$$\|[f_{ij}(A)]\|_{M_n\otimes_\gamma B(X)} \lesssim \|[f_{ij}]\|_{M_n\otimes\mathcal{H}^\alpha}.$$

In particular, $\|f(A)\| \lesssim \|f\|_{\mathcal{H}^\alpha}$, so that by Lemma 4.3, there exists a bounded mapping $u : \mathcal{H}^\alpha \to B(X)$ extending the H^∞ calculus in the sense of (4.5). Now repeat the above argument with an arbitrary $F = [f_{ij}] \in M_n \otimes \mathcal{H}^\alpha$ and $u(f_{ij})$ in place of $f_{ij}(A)$, to deduce that u is matricially γ-bounded.

(ii) $\Rightarrow$ (i) Denote $\widetilde{u}$ the restriction of u to W^α which by Lemma 4.1 is again matricially γ-bounded. Thus also the mapping $v = \widetilde{u} \circ w^{-1}$ from the first part of the proof is matricially γ-bounded and by Theorem 3.4, $\|(1+t^2)^{-\alpha/2}A^{it}x\|_{\gamma(\mathbb{R},X)} \leqslant C\|x\|$. ∎

5. EXTENSIONS AND APPLICATIONS

We have characterized in Theorem 4.9 the matricially γ-bounded Hörmander calculus in terms of square functions of A. In fact, the imaginary powers A^{is} appearing in these square functions can be replaced by several other typical operator families associated with A, such as resolvents $R(\lambda, A)$ for $\lambda \in \mathbb{C}\backslash[0,\infty)$ and the semigroup generated by $-A$, $T(z) = \exp(-zA)$ for $\operatorname{Re} z > 0$. This gives (almost) equivalent conditions, see Proposition 5.2 below. Subsequently, we use the semigroup condition of this proposition to apply Theorem 4.9 to some examples. The starting point for us will be semigroups that satisfy (generalized) Gaussian estimates (see (GE)).

The following lemma serves as a preparation for Proposition 5.2.

LEMMA 5.1. *For $i = 1,2$, let (Ω_i, μ_i) be σ-finite measure spaces and $K \in B(L^2(\Omega_1), L^2(\Omega_2))$.*

(i) *Assume that $f \in \gamma(\Omega_1, X)$ and that there exists a Bochner-measurable $g : \Omega_2 \to X$ such that*

$$\langle g(\cdot), x'\rangle = K(\langle f(\cdot), x'\rangle) \quad (x' \in X').$$

Then $g \in \gamma(\Omega_2, X)$ *and*

$$\|g\|_{\gamma(\Omega_2,X)} \leqslant \|K\| \, \|f\|_{\gamma(\Omega_1,X)}.$$

(ii) *Let* $\Omega_1 \to B(X), t \mapsto N(t)$ *and* $\Omega_2 \to B(X), t \mapsto M(t)$ *be weakly measurable. Assume that* $\|N(\cdot)x\|_\gamma \leqslant C\|x\|$ *and that there is* $K \in B(L^2(\Omega_1), L^2(\Omega_2))$ *such that* $K[\langle N(\cdot)x, x'\rangle] = \langle M(\cdot)x, x'\rangle$ *for* $x \in D$, *where* D *is some dense subset of* X. *Then* $M(\cdot)x \in \gamma(\Omega_2, X)$ *for any* $x \in X$ *and* $\|M(\cdot)x\|_\gamma \lesssim \|N(\cdot)x\|_\gamma$.

(iii) *Let* (Ω, μ) *be a measure space and* $g : \Omega \to X$ *measurable. For* $n \in \mathbb{N}$, *let* $\varphi_n : \Omega \to [0,1]$ *measurable with* $\sum_{n=1}^{\infty} \varphi_n(t) = 1$ *for all* $t \in \Omega$. *Then*

$$\|g\|_{\gamma(\Omega,X)} \leqslant \sum_{n=1}^{\infty} \|\varphi_n g\|_{\gamma(\Omega,X)}.$$

Proof. (i) By assumption, $g \in P_2(\Omega_2, X)$. Consider the associated operator $u_g : L^2(\Omega_2) \to X$ as in (2.7). We have $u_g = u_f \circ K'$. Thus, by Lemma 2.2, $\|u_g\|_{\gamma(L^2(\Omega_2),X)} \leqslant \|K'\| \, \|u_f\|_{\gamma(L^2(\Omega_1),X)}$, which proves (i).

(ii) We show first that $M(\cdot)x$ belongs to $P_2(\Omega_2, X)$ for any $x \in X$. For $x \in D$, this follows immediately from the assumption. For $x \in X$, we let $x_n \in D$ such that $x_n \to x$ $(n \to \infty)$. Then $\langle M(t)x, x'\rangle = \lim_n \langle M(t)x_n, x'\rangle$ for almost every $t \in \Omega_2$. On the other hand, $\langle M(\cdot)x_n, x'\rangle$ is convergent in $L^2(\Omega_2)$. Indeed,

$$\begin{aligned} \|\langle M(\cdot)(x_n - x_m), x'\rangle\|_{L^2(\Omega_2)} &= \|K[\langle N(\cdot)(x_n - x_m), x'\rangle]\|_{L^2(\Omega_1)} \\ &\lesssim \|N(\cdot)(x_n - x_m)\|_\gamma \, \|x'\| \lesssim \|x_n - x_m\| \, \|x'\|, \end{aligned}$$

which converges to 0 $(n, m \to \infty)$. Thus, $\langle M(\cdot)x, x'\rangle$ the pointwise limit, so necessarily equal to the L^2 limit, belongs to $L^2(\Omega_2)$. Consequently, by (i) and Lemma 2.2, $\|M(\cdot)x\|_\gamma = \lim_n \|M(\cdot)x_n\|_\gamma \leqslant \|K\| \lim_n \|N(\cdot)x_n\|_\gamma = \|K\| \, \|N(\cdot)x\|_\gamma$, which shows (ii).

(iii) For $n \in \mathbb{N}$, put $\phi_n = \sum_{k=1}^{n} \varphi_k$. Then $\phi_n : \Omega \to [0,1]$ and $\phi_n(t) \to 1$ monotonically for all $t \in \Omega$. Then $\sup_n \|\phi_n g\|_\gamma \leqslant \sup_n \sum_{k=1}^{n} \|\varphi_k g\|_\gamma = \sum_{k=1}^{\infty} \|\varphi_k g\|_\gamma$. It remains to show $\|g\|_\gamma \leqslant \sup_n \|\phi_n g\|_\gamma$. Let us show first that $g \in P_2(\Omega, X)$, i.e. for any $x' \in X', \langle g(\cdot), x'\rangle \in L^2(\Omega)$. By assumption, we have $|\langle g(t), x'\rangle| = \lim_n \phi_n(t)|\langle g(t), x'\rangle|$ for any $t \in \Omega$, and this convergence is monotone. Then by Beppo Levi's theorem,

$$\|\langle g(\cdot), x'\rangle\|_{L^2(\Omega)} = \lim_n \|\langle \phi_n(\cdot)g(\cdot), x'\rangle\|_{L^2(\Omega)} \leqslant \limsup_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)} \|x'\|,$$

where we have used that $\|\langle f(\cdot), x'\rangle\|_{L^2(\Omega)} \leqslant \|f\|_{\gamma(\Omega,X)} \, \|x'\|$ for any $f \in \gamma(\Omega, X)$. Thus we have shown that $g \in P_2(\Omega, X)$. Then by Lemma 2.2,

$$\|g\|_{\gamma(\Omega,X)} \leqslant \liminf_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)} \leqslant \sup_n \|\phi_n \cdot g\|_{\gamma(\Omega,X)}. \quad \blacksquare$$

PROPOSITION 5.2. *Let A be a 0-sectorial operator having a bounded H^∞ calculus on some space X with property (α). Let $\alpha > 1/2$. Consider the following conditions:*

(1) Hörmander functional calculus

(i) *The H^∞ calculus of A extends to a matricially γ-bounded mapping $\mathcal{H}^\alpha \to B(X)$.*

(2) Imaginary powers

(ii) $\|(1+t^2)^{-\alpha/2}A^{it}x\|_{\gamma(\mathbb{R},X)} \leqslant C\|x\|$.

(3) Resolvents

(iii) *For some $\beta \in (0,1)$ and $\theta \in (-\pi,\pi)\backslash\{0\}$:*

$$\|t^\beta A^{1-\beta}R(e^{i\theta}t,A)x\|_{\gamma(\mathbb{R}_+,dt/t,X)} \lesssim |\theta|^{-\alpha}\|x\|.$$

(iv) *For some $\beta \in (0,1), \theta_0 \in (0,\pi]$:*

$$\||\theta|^{\alpha-1/2}t^\beta A^{1-\beta}R(e^{i\theta}t,A)x\|_{\gamma(\mathbb{R}_+\times[-\theta_0,\theta_0],dt/td\theta,X)} \lesssim \|x\|.$$

(4) Analytic semigroup

(v) *For $\theta \in (-\pi/2,\pi/2)$, $\|A^{1/2}T(e^{i\theta}t)x\|_{\gamma(\mathbb{R}_+,dt,X)} \lesssim (\pi/2-|\theta|)^{-\alpha}\|x\|$.*

(vi) $\|(1+|b/a|^2)^{-\alpha/2+1/4}(|a|+|b|)^{-1/2}A^{1/2}T(a+ib)x\|_{\gamma(\mathbb{R}_+\times\mathbb{R},dadb,X)} \lesssim \|x\|$.

Then the conditions (i), (ii), (iv), (vi) *are equivalent. Further these conditions imply the remaining ones* (iii), (v), *which conversely imply that the H^∞ calculus of A extends to a matricially γ-bounded homomorphism $\mathcal{H}^{\alpha+\varepsilon} \to B(X)$ for any $\varepsilon > 0$.*

Proof. (i) $\Leftrightarrow$ (ii) This is Theorem 4.9.

(ii) $\Leftrightarrow$ (iv) Consider

$$\begin{aligned} K &: L^2(\mathbb{R},ds) \to L^2(\mathbb{R}\times(-\pi,\pi),dsd\theta), \\ f(s) &\mapsto (\pi-|\theta|)^{\alpha-1/2}\frac{1}{\sin\pi(\beta+is)}e^{\theta s}\langle s\rangle^\alpha f(s), \end{aligned} \tag{5.1}$$

where we write in short

$$\langle s\rangle = (1+|s|^2)^{1/2}.$$

Note that $|\sin\pi(\beta+is)| \cong \cosh(\pi s)$ for $\beta \in (0,1)$ fixed. K is an isomorphic embedding. Indeed,

$$\|Kf\|_2^2 = \int_{\mathbb{R}}\int_{-\pi}^{\pi}((\pi-|\theta|)^{\alpha-1/2}e^{\theta s})^2 d\theta \frac{1}{|\sin^2(\pi(\beta+is))|}\langle s\rangle^{2\alpha}|f(s)|^2 ds \quad \text{and}$$

$$\int_{-\pi}^{\pi}(\pi-|\theta|)^{2\alpha-1}e^{2\theta s}d\theta \cong \int_0^\pi \theta^{2\alpha-1}e^{2(\pi-\theta)|s|}d\theta \cong \cosh^2(\pi s)\int_0^\pi \theta^{2\alpha-1}e^{-2\theta|s|}d\theta.$$

For $|s| \geqslant 1$,

$$\int_0^\pi \theta^{2\alpha-1}e^{-2\theta|s|}d\theta = (2|s|)^{-2\alpha}\int_0^{2|s|\pi}\theta^{2\alpha-1}e^{-\theta}d\theta \cong |s|^{-2\alpha}.$$

This clearly implies that $\|Kf\|_2 \cong \|f\|_2$. Applying Lemma 5.1, we get

$$\|\langle s\rangle^{-\alpha} A^{\mathrm{i}s} x\|_{\gamma(\mathbb{R},\mathrm{d}s,X)} \cong \Big\|(\pi - |\theta|)^{\alpha-1/2} \frac{1}{\cosh(\pi s)} \mathrm{e}^{\theta s} A^{\mathrm{i}s} x\Big\|_{\gamma(\mathbb{R}\times(-\pi,\pi),\mathrm{d}s\mathrm{d}\theta,X)}.$$

In p. 228 and Theorem 15.18 of [30], the following formula is derived for $x \in A(D(A^2))$ and $|\theta| < \pi$:

$$\frac{\pi}{\sin \pi(\beta + \mathrm{i}s)} \mathrm{e}^{\theta s} A^{\mathrm{i}s} x = \int_0^\infty t^{\mathrm{i}s} [t^\beta \mathrm{e}^{\mathrm{i}\theta\beta} A^{1-\beta} (\mathrm{e}^{\mathrm{i}\theta} t + A)^{-1} x] \frac{\mathrm{d}t}{t}. \tag{5.2}$$

Note that $A(D(A^2))$ is a dense subset of X. As the Mellin transform $f(s) \mapsto \int_0^\infty t^{\mathrm{i}s} f(s)(\mathrm{d}s/s)$ is an isometry $L^2(\mathbb{R}_+, \mathrm{d}s/s) \to L^2(\mathbb{R}, \mathrm{d}t)$, we get by Lemma 5.1(ii)

$$\begin{aligned}\|\langle s\rangle^{-\alpha} A^{\mathrm{i}s} x\|_{\gamma(\mathbb{R},X)} &\cong \|(\pi - |\theta|)^{\alpha-1/2} t^\beta A^{1-\beta} (\mathrm{e}^{\mathrm{i}\theta} t + A)^{-1} x\|_{\gamma(\mathbb{R}_+\times(-\pi,\pi),\mathrm{d}t/t\mathrm{d}\theta,X)} \\ &\cong \||\theta|^{\alpha-1/2} t^\beta A^{1-\beta} R(\mathrm{e}^{\mathrm{i}\theta} t, A) x\|_{\gamma(\mathbb{R}_+\times(0,2\pi),\mathrm{d}t/t\mathrm{d}\theta,X)}.\end{aligned}$$

so that (ii) $\Leftrightarrow$ (iv) for $\theta_0 = \pi$.

For a general $\theta_0 \in (0, \pi]$, consider K from (5.1) with restricted image, i.e.

$$K : L^2(\mathbb{R}, \mathrm{d}s) \to L^2(\mathbb{R} \times (-\pi, -(\pi - \theta_0)] \cup [\pi - \theta_0, \pi), \mathrm{d}s\mathrm{d}\theta).$$

Then argue as in the case $\theta_0 = \pi$.

(iv) $\Leftrightarrow$ (vi). The proof of (ii) $\Leftrightarrow$ (iv) above shows that condition (iv) is independent of $\theta_0 \in (0, \pi]$ and $\beta \in (0, 1)$. Put $\theta_0 = \pi$ and $\beta = 1/2$. Apply Lemma 5.1 with

$$(\mathrm{e}^{\mathrm{i}\theta} \mu + \mathrm{i}t)^{-1} = K[\exp(-(\cdot)\mathrm{e}^{\mathrm{i}\theta} \mu) \chi_{(0,\infty)}(\cdot)](t), \tag{5.3}$$

where $K : L^2(\mathbb{R}, \mathrm{d}s) \to L^2(\mathbb{R}, \mathrm{d}t)$ is the Fourier transform. This yields that (iv) is equivalent to

$$\||\theta|^{\alpha-1/2} A^{1/2} T(\exp(\pm(\pi/2 - \theta))t) x\|_{\gamma((0,\pi/2)\times\mathbb{R}_+,\mathrm{d}\theta\mathrm{d}t)} \lesssim \|x\|.$$

Applying the change of variables $\theta \rightsquigarrow \pi/2 \pm \theta$ and $\mathrm{d}t \rightsquigarrow t\mathrm{d}t$ shows that this is equivalent to

$$\Big\|\Big(\frac{\pi}{2} - |\theta|\Big)^{\alpha-1/2} t^{-1/2} A^{1/2} T(\mathrm{e}^{\mathrm{i}\theta} t) x\Big\|_{\gamma((-\pi/2,\pi/2)\times\mathbb{R}_+,\mathrm{d}\theta t\mathrm{d}t)} \lesssim \|x\|.$$

Now the equivalence to (vi) follows from the change of variables $a = t\cos\theta, b = t\sin\theta, t = |a + \mathrm{i}b|, \mathrm{d}\theta\, t\mathrm{d}t = \mathrm{d}a\,\mathrm{d}b$.

(iii) $\Leftrightarrow$ (v) for $\beta = 1/2$. Use K and the first argument from (iv) $\Leftrightarrow$ (vi).

(ii) $\Rightarrow$ (iii). We use a similar K_θ as in the proof of (ii) $\Leftrightarrow$ (iv), fixing $\theta \in (-\pi, \pi)$:

$$K_\theta : L^2(\mathbb{R}, \mathrm{d}s) \to L^2(\mathbb{R}, \mathrm{d}s), \quad f(s) \mapsto (\pi - |\theta|)^\alpha \frac{1}{\sin \pi(\beta + \mathrm{i}s)} \mathrm{e}^{\theta s} \langle s\rangle^\alpha f(s).$$

We have

$$\sup_{|\theta|<\pi}\|K_\theta\| = \sup_{|\theta|<\pi, s\in\mathbb{R}} \langle s\rangle^\alpha(\pi-|\theta|)^\alpha \frac{e^{\theta s}}{|\sin\pi(\beta+is)|} \lesssim \sup_{\theta,s}\langle s(\pi-|\theta|)\rangle^\alpha e^{-|s|(\pi-|\theta|)} < \infty.$$

Thus, by (5.2),

$$\begin{aligned}
(5.4)\qquad & \sup_{0<|\theta|\leqslant\pi} |\theta|^\alpha \|t^\beta A^{1-\beta} R(te^{i\theta}, A)x\|_{\gamma(\mathbb{R}_+, dt/t, X)} \\
&= \sup_{|\theta|<\pi} (\pi-|\theta|)^\alpha \|t^\beta A^{1-\beta}(e^{i\theta}t + A)^{-1}x\|_{\gamma(\mathbb{R}_+, dt/t, X)} \\
&= \sup_{|\theta|<\pi} (\pi-|\theta|)^\alpha \Big\|\frac{\pi}{\sin\pi(\beta+is)} e^{\theta s} A^{is}x\Big\|_{\gamma(\mathbb{R}, ds, X)} \\
&\lesssim \|\langle s\rangle^{-\alpha} A^{is}x\|_{\gamma(\mathbb{R}, ds, X)}.
\end{aligned}$$

Next we claim that for any $\varepsilon > 0$, (iii) implies (ii), where in (ii), α is replaced by $\alpha+\varepsilon$. First we consider $\langle s\rangle^{-(\alpha+\varepsilon)}A^{is}x$ for $s \geqslant 1$.

$$(5.5)\qquad \|\langle s\rangle^{-(\alpha+\varepsilon)}A^{is}x\|_{\gamma([1,\infty), X)} \leqslant \sum_{n=0}^\infty 2^{-n\varepsilon}\|\langle s\rangle^{-\alpha}A^{is}x\|_{\gamma([2^n, 2^{n+1}], X)}.$$

For $s \in [2^n, 2^{n+1}]$, we have

$$\langle s\rangle^{-\alpha} \lesssim 2^{-n\alpha} \lesssim 2^{-n\alpha}e^{-2^{-n}s} \lesssim (\pi-\theta_n)^\alpha \frac{e^{\theta_n s}}{\sin\pi(\beta+is)},$$

where $\theta_n = \pi - 2^{-n}$. Therefore

$$\begin{aligned}
\|\langle s\rangle^{-\alpha}A^{is}x\|_{\gamma([2^n,2^{n+1}],X)} &\lesssim (\pi-\theta_n)^\alpha \Big\|\frac{\pi}{\sin\pi(\beta+is)}e^{\theta_n s}A^{is}x\Big\|_{\gamma(\mathbb{R},X)} \\
&\overset{(5.4)}{\lesssim} \sup_{0<|\theta|\leqslant\pi} |\theta|^\alpha \|t^\beta A^{1-\beta}R(te^{i\theta}, A)x\|_{\gamma(\mathbb{R}_+, dt/t, X)} < \infty.
\end{aligned}$$

Thus, the sum in (5.5) is finite.

The part $\langle s\rangle^{-(\alpha+\varepsilon)}A^{is}x$ for $s \leqslant -1$ is treated similarly, whereas $\|\langle s\rangle^{-\alpha}A^{is}x\|_{\gamma((-1,1),X)} \cong \|A^{is}x\|_{\gamma((-1,1),X)}$. It remains to show that the last expression is finite. We have assumed that X has property (α). Then the fact that A has an H^∞ calculus implies that $\{A^{is} : |s| < 1\}$ is γ-bounded ([30], Theorem 12.8). Then by Lemma 2.2(iii), we have $\|A^{is}x\|_{\gamma((-1,1),X)} \leqslant \gamma(\{A^{is} : |s| < 1\})\|1\|_{L^2(-1,1)}\|x\|$. ∎

Condition (v) of the preceding proposition can be checked in the following way.

LEMMA 5.3. *Let A be a 0-sectorial operator on a space X with property (α) having an H^∞ calculus. If for some $\beta > 0$*

$$(5.6)\qquad \{T(te^{\pm i(\pi/2-\theta)}) : t > 0\} \text{ is } \gamma\text{-bounded with constant } \lesssim \theta^{-\beta},$$

then $\|A^{1/2}T(e^{\pm i(\pi/2-\theta)}t)x\|_{\gamma(\mathbb{R}_+,X)} \lesssim \theta^{-\alpha}\|x\|$ with $\alpha = \beta + 1/2$.

Proof. Decompose

$$t\exp(\pm \mathrm{i}(\pi/2-\theta)) = s(t,\theta) + r(t,\theta)\exp(\pm \mathrm{i}(\pi/2-\theta/2))$$

where the reals s and r are uniquely determined by t and θ. We have $s(t,\theta) = \kappa(\theta)t$ with $\kappa(\theta) \cong \theta$. Then by Lemma 2.2,

$$\begin{aligned}\|A^{1/2}T(t\mathrm{e}^{\pm \mathrm{i}(\pi/2-\theta)})x\|_{\gamma(\mathbb{R}_+,X)} \\ &= \|T(r\mathrm{e}^{\pm \mathrm{i}(\pi/2-\theta/2)})A^{1/2}T(s)x\|_{\gamma(\mathbb{R}_+,X)} \\ &\leqslant \gamma(\{T(r\mathrm{e}^{\pm \mathrm{i}(\pi/2-\theta/2)}) : r>0\})\|A^{1/2}T(s(t,\theta))x\|_{\gamma(\mathbb{R}_+,X)} \\ &\lesssim (\theta/2)^{-\beta}\theta^{-1/2}\|A^{1/2}T(t)x\|_{\gamma(\mathbb{R}_+,X)}.\end{aligned}$$

By (5.3) and Theorem 7.2 of [24], (see Theorem 11.9 of [30] for the Hilbert space case which is proved in a very similar way) $\|A^{1/2}T(t)x\|_{\gamma(\mathbb{R}_+,X)} = \|A^{1/2}(\mathrm{i}t-A)^{-1}x\|_{\gamma(\mathbb{R},X)} \leqslant C\|x\|$, which finishes the proof. ∎

Let us now turn to some examples.

DEFINITION 5.4. Let Ω be a topological space which is equipped with a distance ρ and a Borel measure μ. Let $d \geqslant 1$ be an integer. Ω is called a homogeneous space of dimension d if there exists $C>0$ such that for any $x \in \Omega, r>0$ and $\lambda \geqslant 1$:

$$\mu(B(x,\lambda r)) \leqslant C\lambda^d \mu(B(x,r)).$$

Typical cases of homogeneous spaces are open subsets of $\mathbb{R}^d$ with Lipschitz boundary and Lie groups with polynomial volume growth, in particular stratified nilpotent Lie groups (see e.g. [16]).

We will consider operators satisfying the following assumption.

ASSUMPTION 5.5. A is a self-adjoint positive (injective) operator on $L^2(\Omega)$, where Ω is a homogeneous space of a certain dimension d. Further, there exists some $p_0 \in [1,2)$ such that the semigroup generated by $-A$ satisfies the so-called generalized Gaussian estimate (see e.g. (GGE) of [3]):

$$\text{(GGE)}\quad \begin{aligned}&\|\chi_{B(x,r_t)}\mathrm{e}^{-tA}\chi_{B(y,r_t)}\|_{p_0\to p_0'} \\ &\leqslant C\mu(B(x,r_t))^{1/p_0'-1/p_0}\exp(-c(\rho(x,y)/r_t)^{m/(m-1)}) \quad (x,y\in\Omega, t>0).\end{aligned}$$

Here, p_0' is the conjugated exponent to $p_0, C, c > 0, m \geqslant 2$ and $r_t = t^{1/m}$, χ_B denotes the characteristic function of B, $B(x,r)$ is the ball $\{y \in \Omega : \rho(y,x) < r\}$ and $\|\chi_{B_1}T\chi_{B_2}\|_{p_0\to p_0'} = \sup_{\|f\|_{p_0}\leqslant 1}\|\chi_{B_1}\cdot T(\chi_{B_2}f)\|_{p_0'}$.

If $p_0 = 1$, then it is proved in [5] that (GGE) is equivalent to the usual Gaussian estimate, i.e. e^{-tA} has an integral kernel $k_t(x,y)$ satisfying the pointwise estimate (cf. e.g. Assumption 2.2 of [14])

$$\text{(GE)}\quad |k_t(x,y)| \lesssim \mu(B(x,t^{1/m}))^{-1}\exp(-c(\rho(x,y)/t^{1/m})^{m/(m-1)}) \quad (x,y\in\Omega, t>0).$$

This is satisfied in particular by sublaplacian operators on Lie groups of polynomial growth [45] as considered e.g. in [1], [7], [12], [33], [34], or by more general elliptic and sub-elliptic operators [9], [35], and Schrödinger operators [36]. It is also satisfied by all the operators in Section 2 of [14].

Examples of operators satisfying a generalized Gaussian estimate for $p_0 > 1$ are higher order operators with bounded coefficients and Dirichlet boundary conditions on domains of $\mathbb{R}^d$, Schrödinger operators with singular potentials on $\mathbb{R}^d$ and elliptic operators on Riemannian manifolds as listed in Section 2 of [3] and the references therein.

THEOREM 5.6. *Let Assumption* 5.5 *hold. Then for any* $p \in (p_0, p_0')$, *the* H^∞ *calculus of* A *extends to a matricially* γ*-bounded homomorphism* $\mathcal{H}^\alpha \to B(L^p(\Omega))$ *with*

$$\alpha > d\Big|\frac{1}{p_0} - \frac{1}{2}\Big| + \frac{1}{2}.$$

Proof. We show that (5.6) holds with $\beta = d(1/p_0 - 1/2)$. By Proposition 2.1 of [6], the assumption (GGE) implies that

$$\|\chi_{B(x,r_t)} e^{-tA} \chi_{B(y,r_t)}\|_{p_0 \to 2} \leqslant C_1 \mu(B(x,r_t))^{1/2-1/p_0} \exp(-c_1(\rho(x,y)/r_t)^{m/(m-1)}) \qquad (x,y \in \Omega, t > 0)$$

for some $C_1, c_1 > 0$. By Theorem 2.1 of [4], this can be extended from real t to complex $z = te^{i\theta}$ with $\theta \in (-\pi/2, \pi/2)$:

$$\begin{aligned} &\|\chi_{B(x,r_z)} e^{-zA} \chi_{B(y,r_z)}\|_{p_0 \to 2} \\ &\quad \leqslant C_2 \mu(B(x,r_z))^{1/2-1/p_0} (\cos\theta)^{-d(1/p_0-1/2)} \exp(-c_2(\rho(x,y)/r_z)^{m/(m-1)}), \end{aligned}$$

for $r_z = (\cos\theta)^{-(m-1)/m} t^{1/m}$, and some $C_2, c_2 > 0$. By Proposition 2.1(i), (i) $\Rightarrow$ (iii) with $R = e^{-zA}, \gamma = \alpha = 1/p_0 - 1/2, \beta = 0, r = r_z, u = p_0$ and $v = 2$ of [6], this gives for any $x \in \Omega, \operatorname{Re} z > 0$ and $k \in \mathbb{N}_0$

$$\begin{aligned} &\|\chi_{B(x,r_z)} e^{-zA} \chi_{A(x,r_z,k)}\|_{p_0 \to 2} \\ &\qquad \leqslant C_3 \mu(B(x,r_z))^{1/2-1/p_0} (\cos\theta)^{-d(1/p_0-1/2)} \exp(-c_3 k^{m/(m-1)}), \end{aligned}$$

where $A(x, r_z, k)$ denotes the annular set $B(x,(k+1)r_z) \backslash B(x, kr_z)$. By Theorem 2.2 with $q_0 = p_0, q_1 = s = 2, \rho(z) = r_z$ and $S(z) = (\cos\theta)^{d(1/p_0-1/2)} e^{-zA}$ of [29] and property (α), we deduce that

$$\{(\cos\theta)^{d(1/p_0-1/2)} e^{-zA} : \operatorname{Re} z > 0\}$$

is γ-bounded. Now apply Lemma 5.3 and Proposition 5.2, noting that A has an H^∞ calculus on $L^p(\Omega)$ ([4], Corollary 2.3). ∎

REMARK 5.7. (i) Theorem 5.6 improves on Theorem 1.1 of [3] in that it includes the matricial γ-boundedness of the Hörmander calculus. Note that [3] obtains also a weak-type result for $p = p_0$. If p_0 is strictly larger than 1, then Theorem 5.6 improves the order of derivation α of the calculus from $d/2 + 1/2 + \varepsilon$

in [3] to $d|1/p_0 - 1/2| + 1/2 + \varepsilon$. In Theorem 6.4 a) of [44], under the assumptions of Theorem 5.6, a $\mathcal{H}^{\beta,r}$ calculus with with $\beta > (d+1)|1/p_0 - 1/2|$ and $r > |1/2 - 1/p|^{-1}$ is derived. Here $\mathcal{H}^{\beta,r}$ is defined similarly to $\mathcal{H}^\alpha$ by

$$\mathcal{H}^{\beta,r} = \Big\{ f : (0,\infty) \to \mathbb{C} : \sup_{k\in\mathbb{Z}} \|(f \circ \exp)\phi_k\|_{W^{\beta,r}} < \infty \Big\},$$

where $W^{\beta,r}$ is the usual Sobolev space with norm $\|g\|_{W^{\beta,r}}{=}\|[(1{+}\xi^2)^{\beta/2}\widehat{g}(\xi)]^{\check{}}\|_{L^r(\mathbb{R})}$. Note that $\mathcal{H}^{\beta,r}$ is larger than $\mathcal{H}^\alpha$. In the classical case of Gaussian estimates, i.e. $p_0 = 1$, [14] yields a $\mathcal{H}^{\alpha_2,\infty}$ calculus under Assumption 5.5 and even a $\mathcal{H}^{\alpha_2}$ calculus for many examples, e.g. homogeneous operators, with the better derivation order $\alpha_2 > d/2$.

(ii) The theorem also holds for the weaker assumption that Ω is an open subset of a homogeneous space $\widetilde{\Omega}$. In that case, the ball $B(x, r_t)$ on the right hand side in (GGE) is the one in $\widetilde{\Omega}$. This variant can be applied to elliptic operators on irregular domains $\Omega \subset \mathbb{R}^d$ as discussed in Section 2 of [3].

In Theorem 5.6, the operator A was assumed to be self-adjoint, and thus admits a functional calculus $L^\infty \to B(L^2(\Omega))$. The space $L^\infty = L^\infty((0,\infty); d\mu_A)$ is larger than $\mathcal{H}^\alpha$, and one can use this fact to ameliorate the functional calculus of A on $L^q(\Omega)$ by complex interpolation.

PROPOSITION 5.8. *Let A satisfy Assumption 5.5. Then for $q \in (p_0, p_0')$, $\alpha > d|1/p_0 - 1/2| + 1/2$ and $\theta \in (0,1)$ with $\theta > |1/q - 1/p_0|/|1/2 - 1/p_0|$, the functional calculus of A on L^q*

$$u_{L^q} : (L^\infty, \mathcal{H}^\alpha)_\theta \to B(L^q(\Omega)) \text{ is matricially } \gamma\text{-bounded.} \tag{5.7}$$

Here $(L^\infty, \mathcal{H}^\alpha)_\theta$ is the complex interpolation space which is given an operator space structure ([39], *p.* 56) *by*

$$M_n \otimes (L^\infty, \mathcal{H}^\alpha)_\theta = (M_n \otimes L^\infty, M_n \otimes \mathcal{H}^\alpha)_\theta.$$

Proof. The self-adjoint calculus $u_{L^2} : L^\infty \to B(L^2(\Omega))$ is completely bounded since it is a $*$-representation ([39], Proposition 1.5), so by Remark 3.2, u_{L^2} is matricially γ-bounded. Moreover, we have $(\mathrm{Gauss}(L^p), \mathrm{Gauss}(L^2))_\theta{=}\mathrm{Gauss}((L^p, L^2)_\theta)$ ([21], Proposition 3.7). Then by bilinear interpolation between

$$M_n{\otimes}L^\infty{\times}\mathrm{Gauss}_n(L^2){\to}\mathrm{Gauss}_n(L^2), \quad ([a_{ij}]{\otimes}f, \textstyle\sum_k \gamma_k{\otimes}x_k) {\mapsto} \displaystyle\sum_{k,j} \gamma_k a_{kj} u_{L^2}(f) x_j$$

and, with the mapping u_{L^p} resulting from Theorem 5.6, p given by $\theta = |1/q - 1/p|/|1/2 - 1/p|$,

$$M_n{\otimes}\mathcal{H}^\alpha{\times}\mathrm{Gauss}_n(L^p){\to}\mathrm{Gauss}_n(L^p), \quad ([a_{ij}]{\otimes}f, \textstyle\sum_k \gamma_k{\otimes}x_k) {\mapsto} \displaystyle\sum_{k,j} \gamma_k a_{kj} u_{L^p}(f) x_j,$$

one deduces (5.7). ∎

Note that the space $(L^\infty, \mathcal{H}^\alpha)_\theta$ contains $\mathcal{H}_0^{\beta,r}$, where $1/r > \theta/2$, $\beta > \alpha\theta + (1/r - \theta/2)$. Here $\mathcal{H}_0^{\beta,r} = \{f \in \mathcal{H}^{\beta,r} : \|(f \circ \exp)\phi_k\|_{W^{\beta,r}} \to 0 \text{ for } |k| \to \infty\}$. Then (5.7) implies that in particular, $\mathcal{H}_0^{\beta,r} \to B(L^q), f \mapsto f(A)$ is (norm) bounded and by Section 4.6.1 of [25], this extends moreover boundedly to $\mathcal{H}^{\beta,r} \to B(L^q)$.

In Section 7 of [14], for many examples of operators A satisfying (GE), it is shown that the functional calculus

$$u : \mathcal{H}^\alpha \to B(L^p(\Omega)), f \mapsto f(A) \text{ is bounded for } 1 < p < \infty \text{ and } \alpha > \frac{d}{2}. \tag{5.8}$$

Moreover, for the fundamental example $A = -\Delta$ on $L^p(\mathbb{R}^d)$, the critical order $d/2$ in (5.8) is optimal ([40], IV.7.4 and [25], Proposition 4.12 (2)). Note that the derivation order for the matricially γ-bounded calculus obtained in Theorem 5.6 under the assumption (GE) (i.e. $p_0 = 1$) is only $(d+1)/2$, and therefore gives a weaker result in the derivation order compared to (5.8).

Thus the question arises if an arbitrary A that has a norm-bounded Hörmander calculus also has a matricially γ-bounded Hörmander calculus. In contrast to the self-adjoint L^∞ calculus on Hilbert space, which is always matricially γ-bounded (see the proof above), we have the following result.

PROPOSITION 5.9. *Let A be a 0-sectorial operator on a space X with property (α). Let $\alpha > 1/2$ and $\beta > \alpha + 1$. Suppose that its functional calculus $f \mapsto f(A)$ is bounded $u_\alpha : \mathcal{H}^\alpha \to B(X)$, and denote u_β the restriction of u_α to $\mathcal{H}^\beta$. Then $u_\beta : \mathcal{H}^\beta \to B(X)$ is matricially γ-bounded.*

On the other hand, for any $\alpha > 0$, there exists some A on a Hilbert space X such that $u_\alpha : \mathcal{H}^\alpha \to B(X)$ is bounded (even γ-bounded because of (2.3)*), but its restriction $u_{\alpha+1/2} : \mathcal{H}^{\alpha+1/2} \to B(X)$ is not matricially γ-bounded.*

Proof. For $t \in \mathbb{R}$, let $f_t(\lambda) = \lambda^{it}$. It is easy to check that $\|f_t\|_{\mathcal{H}^\alpha} \lesssim \langle t\rangle^\alpha$ (see the introduction). By Lemma 4.1(i), A has an H^∞ calculus. By Corollary 6.3 of [26], the set $\{T(te^{\pm i(\pi/2-\theta)}) : t > 0\}$ is γ-bounded with constant $\leqslant C\theta^{-\alpha-1/2}$ $\quad (\theta \in (0, \pi/2))$. By Lemma 5.3, condition (v) of Proposition 5.2 is satisfied with $\alpha + 1$ in place of α and therefore, $u_\beta : \mathcal{H}^\beta \to B(X)$ is matricially γ-bounded.

For the second statement, let $\alpha > 1/2$. Consider $X = W^\alpha$ and the group $U(t)g = (\cdot)^{it}g$ on X. Note that

$$\|(\cdot)^{it}g\|_X = \|(g \circ \exp)\widehat{\ }(\cdot - t)\langle\cdot\rangle^\alpha\|_2 = \|(g \circ \exp)\widehat{\ }(\cdot)\langle(\cdot) + t\rangle^\alpha\|_2 \cong \langle t\rangle^\alpha \|g\|_X.$$

In particular, $\|U(t)\| \cong \langle t\rangle^\alpha$. It is easy to check that $U(t) = A^{it}$ are the imaginary powers of a 0-sectorial operator A and that $f(A)g = fg$ for any $g \in X$ and $f \in \bigcup_{\omega>0} H^\infty(\Sigma_\omega)$. By [42], one has $\|fg\|_{W^\alpha} \lesssim \|f\|_{\mathcal{H}^\alpha}\|g\|_{W^\alpha}$. Thus, A has a bounded $\mathcal{H}^\alpha$ calculus.

On the other hand, since X is a Hilbert space, the square function condition of Theorem 4.10 reads

$$\|\langle t\rangle^{-\beta}A^{\mathrm{i}t}x\|_{\gamma(\mathbb{R},X)} = \|\langle t\rangle^{-\beta}A^{\mathrm{i}t}x\|_{L^2(\mathbb{R},X)} \cong \Big(\int_{\mathbb{R}} \langle t\rangle^{-2\beta+2\alpha}\mathrm{d}t\Big)^{1/2}\|x\|,$$

which is finite if and only if $\beta > \alpha + 1/2$. ∎

6. PROOFS OF LEMMAS 4.5–4.8

Proof of Lemma 4.5. Since X has property (α), the fact that A has a bounded $H^\infty(\Sigma_\omega)$ calculus implies ([30], Theorem 12.8) that for any $\theta > \omega$,

$$\{g(A) : \|g\|_{\infty,\theta} \leqslant 1\} \text{ is } \gamma\text{-bounded.}$$

We fix some $\theta \in (\omega, \pi/4)$. As the mapping $u : W^\alpha \to B(X)$ is matricially γ-bounded, by Remark 3.2,

$$\{h(A) : \|h\|_{W^\alpha} \leqslant 1\} \text{ is } \gamma\text{-bounded.}$$

The lemma stated that $\gamma(\{f_{2^kz}(A) : k \in \mathbb{Z}\}) \lesssim |z/\operatorname{Re} z|^\beta$, where $f_{2^kz}(\lambda) = \exp(-2^kz\lambda)$. Thus it suffices to decompose $f_{2^kz} = g + h$, where $\|g\|_{\infty,\theta}, \|h\|_{W^\alpha} \lesssim |z/\operatorname{Re} z|^\beta$.

As $\Psi : f \mapsto f(r\cdot)$ is an isomorphism $H^\infty(\Sigma_\theta) \to H^\infty(\Sigma_\theta)$ and $W^\alpha \to W^\alpha$, with $\|\Psi\| \cdot \|\Psi^{-1}\| \leqslant C$, C independent of $r > 0$, it suffices to have the above decomposition for $|z| = 1$ and $k = 0$. We choose $g(\lambda) = \exp(-(z+1)\lambda)$ and $h(\lambda) = \exp(-z\lambda)(1-\mathrm{e}^{-\lambda})$. As $|\arg(z+1)| + \theta \leqslant \pi/4 + \pi/4 = \pi/2$, we actually have $\|g\|_{\infty,\theta} \leqslant 1 \lesssim |\operatorname{Re} z|^{-\beta}$. Further it is a simple matter to check that $\|h\|_{W^\alpha} \lesssim |\operatorname{Re} z|^{-\beta}$ for any $\beta > \alpha$. For example, if $\alpha = 1$, then

$$\begin{aligned}
\|h\|_{W^\alpha}^2 &\cong \int_0^\infty |h(t)|^2\frac{\mathrm{d}t}{t} + \int_0^\infty |th'(t)|^2\frac{\mathrm{d}t}{t}\\
&\lesssim \int_0^\infty \mathrm{e}^{-2\operatorname{Re} zt}\min(t^2,1) + |tz(1-\mathrm{e}^{-t})|^2\mathrm{e}^{-2\operatorname{Re} zt} + t^2\mathrm{e}^{-2t}\mathrm{e}^{-2\operatorname{Re} zt}\frac{\mathrm{d}t}{t}\\
&\lesssim \int_0^1 \mathrm{e}^{-2\operatorname{Re} zt}t^2\frac{\mathrm{d}t}{t} + \int_1^\infty \mathrm{e}^{-2\operatorname{Re} zt}\frac{\mathrm{d}t}{t}\\
&\qquad + 1 + \int_0^1 t^2(1-\mathrm{e}^{-t})^2\mathrm{e}^{-2\operatorname{Re} zt}\frac{\mathrm{d}t}{t} + \int_1^\infty t^2(1-\mathrm{e}^{-t})^2\mathrm{e}^{-2\operatorname{Re} zt}\frac{\mathrm{d}t}{t}\\
&\lesssim (\operatorname{Re} z)^{-2} + (1 + |\log(\operatorname{Re} z)|) + 1 + 1 + \int_{\operatorname{Re} z}^\infty \frac{t^2}{(\operatorname{Re} z)^2}\mathrm{e}^{-2t}\frac{\mathrm{d}t}{t}
\end{aligned}$$

$$\lesssim 1 + 1 + \frac{1}{(\operatorname{Re} z)^2}. \qquad \blacksquare$$

Proof of Lemma 4.6. The assumption of the lemma was

$$\gamma(\{\exp(-2^k zA) : k \in \mathbb{Z}\}) \lesssim \left| \frac{z}{\operatorname{Re} z} \right|^\beta \quad (\operatorname{Re} z > 0). \tag{6.1}$$

We first show that

$$\gamma(\{(2^k tA)^{1/2} \exp(-2^k t e^{\pm i(\pi/2-\omega)} A) : k \in \mathbb{Z}\}) \lesssim \omega^{-(\beta+1/2)} \quad (\omega \in (0, \pi/2)). \tag{6.2}$$

Decompose

$$e^{\pm i(\pi/2-\omega)} t = s + e^{\pm i(\pi/2-\omega/2)} r,$$

where $s, r > 0$ are uniquely determined by t and ω. Then

$$\begin{aligned} &(2^k tA)^{1/2} \exp(-e^{\pm i(\pi/2-\omega)} 2^k tA) \\ &\qquad = \left(\frac{t}{s}\right)^{1/2} (2^k sA)^{1/2} \exp(-2^k sA) \exp(-2^k r e^{\pm i(\pi/2-\omega/2)} A), \end{aligned}$$

and consequently,

$$\begin{aligned} &\gamma(\{(2^k tA)^{1/2} \exp(-e^{\pm i(\pi/2-\omega)} 2^k tA) : k \in \mathbb{Z}\}) \\ &\qquad \leqslant \sup_t \left(\frac{t}{s}\right)^{1/2} \times \gamma(\{(2^k sA)^{1/2} \exp(-2^k sA) : k \in \mathbb{Z}\}) \\ &\qquad\quad \times \gamma(\{\exp(-2^k r e^{\pm i(\pi/2-\omega/2)} A) : k \in \mathbb{Z}\}). \end{aligned} \tag{6.3}$$

We will show that the right hand side of (6.3) can be estimated by $\lesssim \omega^{-1/2} \times 1 \times \omega^{-\beta}$. The estimate for the first factor follows from the law of sines

$$\frac{t}{s} = \frac{\sin(\pi/2 + \omega/2)}{\sin(\omega/2)} \cong \omega^{-1}.$$

For the second estimate, note that by Example 2.16 of [30], (6.1) implies that $\{\exp(-zA) : z \in \Sigma_\delta\}$ is γ-bounded for any $\delta < \pi/2$ and consequently, by Theorem 2.20 of [30], (iii) $\Rightarrow$ (i), $\{\lambda(\lambda - A)^{-1} : -\lambda \in \Sigma_\theta\}$ is γ-bounded for any $\theta \in (\pi/2, \pi)$. Then with $f(\lambda) = \lambda^{1/2} e^{-\lambda}$, the Cauchy integral formula (2.9) gives

$$\begin{aligned} (2^k tA)^{1/2} \exp(-2^k tA) &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\pi-\theta}} f(\lambda)(\lambda - 2^k tA)^{-1} d\lambda \\ &= \frac{1}{2\pi i} \int_{\partial\Sigma_{\pi-\theta}} \frac{f(\lambda)}{\lambda} \times \frac{\lambda}{2^k t} \left(\frac{\lambda}{2^k t} - A\right)^{-1} d\lambda. \end{aligned}$$

The first factor in the last integral belongs to $L^1(\partial\Sigma_{\pi-\theta}, |d\lambda|)$ and the second factor is γ-bounded by the above for any $\theta < \pi$. Thus by the well-known integral lemma for γ-bounds ([30], Corollary 2.14), the second factor in (6.3) is finite.

The estimate for the third factor in (6.3) follows from the assumption (6.1), so that we have shown (6.2).

Now we will write the expression in (4.8) as an integral of the expression in (6.2). Let $\theta \in (0, \pi/2), \lambda = t\mathrm{e}^{\mathrm{i}\theta}$ and set $\phi = \pi/2 - \theta/2$, so that $\mathrm{Re}(\mathrm{e}^{\mathrm{i}\phi}\lambda) < 0$. Then

$$\begin{aligned}\lambda^{1/2}(2^kA)^{1/2}(\lambda-2^kA)^{-1} &= \lambda^{1/2}(2^kA)^{1/2}\mathrm{e}^{\mathrm{i}\phi}(\mathrm{e}^{\mathrm{i}\phi}\lambda - \mathrm{e}^{\mathrm{i}\phi}2^kA)^{-1}\\ &= \int_0^\infty -\mathrm{e}^{\mathrm{i}\phi}s^{-1/2}\lambda^{1/2}\exp(\mathrm{e}^{\mathrm{i}\phi}\lambda s)\times(2^ksA)^{1/2}\exp(-2^k\mathrm{e}^{\mathrm{i}\phi}sA)\mathrm{d}s.\end{aligned}$$

The second factor of the integral is γ-bounded by (6.2) and the first factor is integrable, as the following lines show:

$$\begin{aligned}\int_0^\infty s^{-1/2}|\lambda^{1/2}\exp(\mathrm{e}^{\mathrm{i}\phi}\lambda s)|\mathrm{d}s &= \int_0^\infty s^{-1/2}|\exp(\mathrm{e}^{\mathrm{i}\phi}\mathrm{e}^{\mathrm{i}\theta}s)|\mathrm{d}s\\ &= \int_0^\infty s^{-1/2}\exp(\cos(\pi/2+\theta/2)s)\mathrm{d}s\\ &= \int_0^\infty s^{-1/2}\exp(-s)\mathrm{d}s\,|\cos(\pi/2+\theta/2)|^{-1/2} \lesssim \theta^{-1/2}.\end{aligned}$$

Then $\tau = \{\lambda^{1/2}(2^kA)^{1/2}(\lambda - 2^kA)^{-1} : k \in \mathbb{Z}\}$ is γ-bounded since by Kahane's contraction principle, we have

$$\begin{aligned}\gamma(\tau) &\leqslant \int_0^\infty s^{-1/2}|\lambda^{1/2}\exp(\mathrm{e}^{\mathrm{i}\phi}\lambda s)|\mathrm{d}s \times \sup_{t>0}\gamma(\{(2^ktA)^{1/2}\exp(-2^kt\mathrm{e}^{\pm\mathrm{i}(\pi/2-\omega)}A) : k \in \mathbb{Z}\})\\ &\lesssim |\arg\lambda|^{-1/2} \times |\arg\lambda|^{-\beta-1/2}.\end{aligned}$$

The same reasoning applies for $\lambda = t\mathrm{e}^{\mathrm{i}\theta}$ and $\theta \in (-\pi/2, 0)$. ∎

Proof of Lemma 4.7. By p. 73 and Theorem 4.10 of [8], it suffices to show that for some $\delta \in (\gamma, n)$,

$$\|f(A)\| \lesssim \theta^{-\delta}\|f\|_{\infty,\theta} \quad \text{for any } f \in \bigcup_{\theta>0} H_0^\infty(\Sigma_\theta). \tag{6.4}$$

To show (6.4), we use the Kalton–Weis characterization of the bounded $H^\infty(\Sigma_\theta)$ calculus in terms of γ-bounded operator families ([23], see also Theorem 12.7 of [30]). More precisely, we follow that characterization in the form of the proof of Theorem 12.7 of [30] and keep track of the dependence of appearing constants on the angle θ. It is shown there that for $f \in H_0^\infty(\Sigma_{2\theta}), x \in X$ and $x' \in X'$,

$$|\langle f(A)x, x'\rangle| = \Big|\frac{1}{2\pi\mathrm{i}}\int_{\partial\Sigma_\theta}\langle\lambda^{-1/2}f(\lambda)A^{1/2}(\lambda - A)^{-1}x, x'\rangle\mathrm{d}\lambda\Big|$$

$$\leqslant \frac{1}{2\pi} \sum_{j=\pm 1} \int_0^\infty |\langle f(t^{-1}e^{ij\theta})(tA)^{1/2}(e^{ij\theta} - tA)^{-1}x, x'\rangle| \frac{dt}{t}$$
$$= (*).$$

We put

$$\phi_{j\theta}(\lambda) = \frac{\lambda^{1/4}(1+\lambda)^{1/2}}{e^{ij\theta} - \lambda} \quad \text{and} \quad \psi(\lambda) = \Big(\frac{\lambda}{(1+\lambda)^2}\Big)^{1/8},$$

so that $(tA)^{1/2}(e^{ij\theta} - tA)^{-1} = \phi_{j\theta}(tA)\psi(tA)\psi(tA)$. By Lemma 12.6 of [30], the integral $(*)$ can be controlled by Gauss-norms. More precisely, we have

$$\begin{aligned}(*) \lesssim \sup_{j=\pm1}\sup_{t>0}\sup_N \Big\| \sum_{k=-N}^N \gamma_k \otimes f(2^k t^{-1} e^{ij\theta})\phi_{j\theta}(2^k tA)\psi(2^k tA)x\Big\|_{\mathrm{Gauss}(X)} \\ \cdot \Big\| \sum_{k=-N}^N \gamma_k \otimes \psi(2^k tA)' x'\Big\|_{\mathrm{Gauss}(X')} \end{aligned} \tag{6.5}$$

$$\lesssim \|f\|_{\infty,\theta} \sup_{j,t} \gamma(\{\phi_{j\theta}(2^k tA) : k \in \mathbb{Z}\}) \sup_{N,t} \Big\| \sum_{k=-N}^N \gamma_k \otimes \psi(2^k tA)x\Big\|_{\mathrm{Gauss}(X)}$$
$$\cdot \sup_{N,t} \Big\| \sum_{k=-N}^N \gamma_k \otimes \psi(2^k tA)' x'\Big\|_{\mathrm{Gauss}(X')}.$$

By Theorem 12.2 of [30], the fact that A has a bounded H^∞ calculus implies that $\sup_{N,t} \Big\| \sum_{k=-N}^N \gamma_k \otimes \psi(2^k tA)x\Big\|_{\mathrm{Gauss}(X)} \lesssim \|x\|$ and $\sup_{N,t} \Big\| \sum_{k=-N}^N \gamma_k \otimes \psi(2^k tA)' x'\Big\|_{\mathrm{Gauss}(X')} \lesssim \|x'\|$. Note that there is no dependence on θ in these two inequalities. It remains to show that

$$\sup_{j=\pm1, t>0} \gamma(\{\phi_{j\theta}(2^k tA) : k \in \mathbb{Z}\}) \lesssim \theta^{-\delta}. \tag{6.6}$$

We have

$$\phi_{j\theta}(2^k tA) = \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta/2}} \phi_{j\theta}(\lambda)\lambda^{1/2}(2^k tA)^{1/2}(\lambda - 2^k tA)^{-1}\frac{d\lambda}{\lambda}$$
$$= \frac{1}{2\pi i} \int_{\partial\Sigma_{\theta/2}} \phi_{j\theta}(t\lambda)\lambda^{1/2}(2^k A)^{1/2}(\lambda - 2^k A)^{-1}\frac{d\lambda}{\lambda}.$$

By Kahane's contraction principle,

$$\sup_{j=\pm1,t>0} \gamma(\{\phi_{j\theta}(2^k tA) : k \in \mathbb{Z}\})$$
$$\lesssim \sup_{j=\pm1,t>0} \|\phi_{j\theta}(t\lambda)\|_{L^1(\partial\Sigma_{\theta/2}, |d\lambda/\lambda|)} \times \sup_{\lambda\in\partial\Sigma_{\theta/2}\setminus\{0\}} \gamma(\{\lambda^{1/2}(2^k A)^{1/2}(\lambda - 2^k A)^{-1} : k \in \mathbb{Z}\}).$$

By assumption, it suffices to show that for any $\varepsilon > 0$

$$(6.7)\quad \sup_{t>0} \|\phi_{j\theta}(t\lambda)\|_{L^1(\partial\Sigma_{\theta/2},|\mathrm{d}\lambda/\lambda|)} \leqslant C_\varepsilon \theta^{-\varepsilon},$$

$$\int_{\partial\Sigma_{\theta/2}} |\phi_{j\theta}(t\lambda)|\Big|\frac{\mathrm{d}\lambda}{\lambda}\Big| = \int_{\partial\Sigma_{\theta/2}} |\phi_{j\theta}(\lambda)|\Big|\frac{\mathrm{d}\lambda}{\lambda}\Big| = \sum_{l=\pm 1}\int_0^\infty \Big|\frac{s^{1/4}(1+\mathrm{e}^{\mathrm{i}l\theta/2}s)^{1/2}}{\mathrm{e}^{\mathrm{i}j\theta}-\mathrm{e}^{\mathrm{i}l\theta/2}s}\Big|\frac{\mathrm{d}s}{s}.$$

The denominator is estimated from below by

$$\begin{aligned}|\mathrm{e}^{\mathrm{i}j\theta}-\mathrm{e}^{\mathrm{i}l\theta/2}s| = |\mathrm{e}^{\mathrm{i}\theta(j-l/2)}-s| &\gtrsim |\cos(\theta(j-l/2))-s| + |\sin(\theta(j-l/2))| \\ &\gtrsim |1-s| - |\cos(\theta(j-l/2))-1| + \theta \\ &\gtrsim |1-s| - \theta^2 + \theta \gtrsim |1-s| + \theta\end{aligned}$$

for the crucial case of small θ. Thus

$$\int_{\partial\Sigma_{\theta/2}} |\phi_{j\theta}(\lambda)|\Big|\frac{\mathrm{d}\lambda}{\lambda}\Big| \lesssim \int_0^\infty \frac{s^{1/4}(1+s)^{1/2}}{\theta+|1-s|}\,\frac{\mathrm{d}s}{s}.$$

We split the integral into the parts $\int_0^\infty = \int_0^{1/2} + \int_{1/2}^{1-\theta} + \int_{1-\theta}^{1+\theta} + \int_{1+\theta}^{2} + \int_2^\infty$.

$$\int_0^{1/2} \frac{s^{1/4}(1+s)^{1/2}}{\theta+|1-s|}\,\frac{\mathrm{d}s}{s} \leqslant \int_0^{1/2} \frac{s^{1/4}(1+s)^{1/2}}{|1-s|}\,\frac{\mathrm{d}s}{s} < \infty$$

is independent of θ. The same estimate applies to $\int_2^\infty$.

$$\int_{1/2}^{1-\theta} \frac{s^{1/4}(1+s)^{1/2}}{\theta+|1-s|}\,\frac{\mathrm{d}s}{s} \lesssim \int_{1/2}^{1-\theta} \frac{1}{\theta+|1-s|}\mathrm{d}s \leqslant \int_{1/2}^{1-\theta} \frac{1}{1-s}\mathrm{d}s \lesssim |\log\theta|.$$

Similarly,

$$\int_{1+\theta}^{2} \frac{s^{1/4}(1+s)^{1/2}}{\theta+|1-s|}\,\frac{\mathrm{d}s}{s} \lesssim \int_{1+\theta}^{2} \frac{1}{s-1}\mathrm{d}s \lesssim |\log\theta|.$$

Finally,

$$\int_{1-\theta}^{1+\theta} \frac{s^{1/4}(1+s)^{1/2}}{\theta+|1-s|}\,\frac{\mathrm{d}s}{s} \lesssim \int_{1-\theta}^{1+\theta} \frac{1}{\theta}\mathrm{d}s \lesssim 1.$$

Since $1+|\log\theta| \leqslant C_\varepsilon\theta^{-\varepsilon}$, the lemma is shown. ∎

Proof of Lemma 4.8. Denote $N = \sup_{x>0} \#\{k \in \mathbb{Z} : \operatorname{supp} g_k \cap [1/2x, 2x] \neq \emptyset\} < \infty$. Fix $x > 0$ and $j \in \{0, 1, \ldots, n\}$. Then almost all g_k vanish in a neighborhood of x, and thus

$$\Big| x^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} \Big(\sum_{k\in\mathbb{Z}} g_k \Big)(x) \Big| = \Big| \sum_{k\in\mathbb{Z}} x^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} g_k(x) \Big| \leqslant N \sup_{k\in\mathbb{Z}} \Big| x^j \frac{\mathrm{d}^j}{\mathrm{d}x^j} g_k(x) \Big| \leqslant N \sup_{k\in\mathbb{Z}} \|g_k\|_{\mathrm{M}^n}.$$

Taking the supremum over x and j gives $\Big\| \sum_{k\in\mathbb{Z}} g_k \Big\|_{\mathrm{M}^n} \leqslant N \sup_{k\in\mathbb{Z}} \|g_k\|_{\mathrm{M}^n}$. ∎

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