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SPECTRAL COMPARISONS BETWEEN NETWORKS WITH DIFFERENT CONDUCTANCE FUNCTIONS

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ABSTRACT. For an infinite network consisting of a graph with edge weights prescribed by a given conductance function c, we consider the effects of replacing these weights with a new function b that satisfies $b \leqslant c$ on each edge. In particular, we compare the corresponding energy spaces and the spectra of the Laplace operators acting on these spaces. We use these results to derive estimates for effective resistance on the two networks, and to compute a spectral invariant for the canonical embedding of one energy space into the other.

KEYWORDS: *Dirichlet form, graph energy, unbounded discrete Laplacian, weighted graph, spectral graph theory, effective resistance, harmonic analysis, Hilbert space, reproducing kernels.*

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1. INTRODUCTION

Motivated by recent work on the analysis of large networks, we consider the natural (discrete) Laplace operator on the network. If the weights on the edges of the network are given by a (conductance) function c, we denote this operator by Δ_c. In this paper, we study the dependence of the spectrum of Δ_c on c. In addition to operator theory and spectral theory, we use tools from metric geometry and variational calculus. Our main result (Theorem 3.20) deals with monotonicity of the spectrum of Δ_c. We illustrate this with explicit models on the binary tree, and on a 1-dimensional lattice.

We begin with a network structure defined by a set of vertices G and a conductance function $c : G \times G \to \mathbb{R}^+$ which specifies the both the adjacency relation and the edge weights; two vertices x and y are neighbours if and only if $c_{xy} > 0$. The case of primary interest is when G is infinite, in which case the Hilbert space $\mathcal{H}_{\mathcal{E}}$ (comprised of functions of finite Dirichlet energy) has a rich structure and the

Laplace operator Δ corresponding to the network may be an unbounded operator on $\mathcal{H}_{\mathcal{E}}$. (Precise definitions of these terms may be found in Definition 2.2, Definition 2.3, and Definition 2.4.)

The Hilbert space $\mathcal{H}_{\mathcal{E}}$ has a rather different geometry than the more familiar $\ell^2(G)$, and depends crucially on the choice of conductance function c. The same is true for the Laplacian Δ as a linear operator on $\mathcal{H}_{\mathcal{E}}$. In this paper, we use the framework developed in earlier projects (see [6], [7], [8], [9], [10], [11], [12], [13], [14], [15]) to compute certain spectral theoretic information; as well as resistance metrics on the underlying vertex set. In particular, we explore how certain quantities depend on the choice of c, in comparison to another conductance function, which we denote by b. It will be assumed that both b and c yield a connected weighted graph, although we allow for the case when $c_{xy} > 0$ and $b_{xy} = 0$ (so that x and y are neighbours in (G, c) but not in (G, b)). The data, defined from b and c, to be compared are as follows:

(i) the energy forms $\mathcal{E}^{(b)}$ and $\mathcal{E}^{(c)}$, and the respective energy Hilbert spaces $\mathcal{H}_{\mathcal{E}^{(b)}}$ and $\mathcal{H}_{\mathcal{E}^{(c)}}$ that they define;

(ii) the systems of dipole vectors that form reproducing kernels for the two Hilbert spaces; see Definition 2.6;

(iii) the respective Laplace operators $\Delta^{(b)}$ and $\Delta^{(c)}$, and their spectra;

(iv) the spaces of finite-energy harmonic functions on $\mathcal{H}_{\mathcal{E}^{(b)}}$ and $\mathcal{H}_{\mathcal{E}^{(c)}}$; and

(v) the effective resistance metrics on $\mathcal{H}_{\mathcal{E}^{(b)}}$ and $\mathcal{H}_{\mathcal{E}^{(c)}}$.

We focus our study on the case when one of the two energy-Hilbert spaces is contractively contained in the other, which corresponds to the inequality $b \leqslant c$. In this case, we believe that our results have applications to percolation theory and the study of random walks in random environments, as well as to dilation theory (see [1]) and the contractive inclusion of Hilbert spaces (see [20]).

Of special operator theoretic significance is the adjoint of the contractive inclusion mapping. The issues involved with the adjoint operator are subtle as the computation of the adjoint operator depends on the Hilbert-inner products used. It is the adjoint operator that allows one to compute the respective systems of dipole vectors that form reproducing kernels for the two Hilbert spaces; see Definition 2.6. We further derive an invariant (involving induced linear maps between the respective spaces of finite-energy harmonic functions) which distinguishes two networks when G is fixed and the conductance functions vary.

We also give a necessary and sufficient condition on a fixed conductance function c having its energy Hilbert space $\mathcal{E}^{(c)}$ boundedly contained in $\mathcal{H}_{\mathcal{E}^{(b)}}$ ($b = 1$); i.e., contractive containment in the "flat" energy Hilbert space corresponding to constant conductance b. The significance of this is that computations in $\mathcal{H}_{\mathcal{E}^{(b)}}$ are typically much easier, and that the conclusions obtained there may then be transferred to $\mathcal{H}_{\mathcal{E}^{(c)}}$.

The use of analysis on infinite discrete systems and spectral theory of associated Laplace operators is of relevance to operator algebras, and to mathematics of computation: sampling, approximation, learning theory, and more; see for example [2], [4], [5], [21], [22], [23], [26].

2. BASIC TERMS AND PREVIOUS RESULTS

In this section, we introduce the key notions used throughout this paper: resistance networks, the energy form $\mathcal{E}$, the Laplace operator Δ, and their elementary properties.

DEFINITION 2.1. A *(resistance) network* is a connected graph (G, c), where G is a graph with vertex set G^0, and c is the *conductance function* which defines adjacency by $x \sim y$ if and only if $c_{xy} > 0$, for $x, y \in G^0$. We assume $c_{xy} = c_{yx} \in [0, \infty)$, and write $c(x) := \sum_{y \sim x} c_{xy}$. We require that the graph is *locally finite*, i.e., that every vertex has only finitely many neighbors.

In this definition, connected means simply that for any $x, y \in G^0$, there is a finite sequence $\{x_i\}_{i=0}^n$ with $x = x_0$, $y = x_n$, and $c_{x_{i-1}x_i} > 0$, $i = 1, \ldots, n$. We may assume there is at most one edge from x to y, as two conductors c^1_{xy} and c^2_{xy} connected in parallel can be replaced by a single conductor with conductance $c_{xy} = c^1_{xy} + c^2_{xy}$. Also, we assume $c_{xx} = 0$ so that no vertex has a loop.

Since the edge data of (G, c) is carried by the conductance function, we will henceforth simplify notation and write $x \in G$ to indicate that x is a vertex. For any network, one can fix a reference vertex, which we shall denote by o (for "origin"). It will always be apparent that our calculations depend in no way on the choice of o.

DEFINITION 2.2. The *Laplacian* on G is the linear difference operator which acts on a function $v : G \to \mathbb{C}$ by

$$(\Delta v)(x) := \sum_{y \sim x} c_{xy}(v(x) - v(y)). \tag{2.1}$$

The domain of Δ is discussed in detail in (3.3), below. A function $v : G \to \mathbb{C}$ is *harmonic* if and only if $\Delta v(x) = 0$ for each $x \in G$.

It is clear from (2.1) that Δ commutes with complex conjugation and therefore the subspace of real-valued functions is invariant under Δ.

We have adopted the physics convention (so that the spectrum is nonnegative) and thus our Laplacian is the negative of the one commonly found in the PDE literature. The network Laplacian (2.1) should not be confused with the stochastically renormalized Laplace operator $c^{-1}\Delta$ which appears in the probability literature, or with the spectrally renormalized Laplace operator $c^{-1/2}\Delta c^{-1/2}$ which appears in the literature on spectral graph theory (e.g., [3]).

DEFINITION 2.3. The *energy* of functions $u, v : G \to \mathbb{C}$ is given by the (closed, bilinear) Dirichlet form

$$\mathcal{E}(u,v) := \frac{1}{2} \sum_{x,y \in G} c_{xy}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y)), \tag{2.2}$$

with the energy of u given by $\mathcal{E}(u) := \mathcal{E}(u,u)$. The *domain* of the energy form is

$$\operatorname{dom} \mathcal{E} = \{u : G \to \mathbb{C} : \mathcal{E}(u) < \infty\}. \tag{2.3}$$

Note that (2.2) converges if and only if it converges absolutely, by the Schwarz inequality, so this summation is well defined. Since $c_{xy} = c_{yx}$ and $c_{xy} = 0$ for nonadjacent vertices, the initial factor of $1/2$ in (2.2) implies there is exactly one term in the sum for each edge in the network.

2.1. THE ENERGY SPACE $\mathcal{H}_\mathcal{E}$. The energy form $\mathcal{E}$ is sesquilinear and conjugate symmetric on $\operatorname{dom} \mathcal{E}$ and would be an inner product if it were positive definite.

DEFINITION 2.4. Let $\mathbf{1}$ denote the constant function with value 1 and recall that $\ker \mathcal{E} = \mathbb{C}\mathbf{1}$. Then $\mathcal{H}_\mathcal{E} := \operatorname{dom} \mathcal{E} / \mathbb{C}\mathbf{1}$ is a Hilbert space with inner product and corresponding norm given by

$$\langle u, v \rangle_\mathcal{E} := \mathcal{E}(u,v) \quad \text{and} \quad \|u\|_\mathcal{E} := \mathcal{E}(u,u)^{1/2}. \tag{2.4}$$

We call $\mathcal{H}_\mathcal{E}$ the *energy (Hilbert) space*.

REMARK 2.5. Since G is connected, it is possible to show (with the use of Fatou's lemma) that $\operatorname{dom} \mathcal{E} / \mathbb{C}\mathbf{1}$ is complete; see [10], [12] for further details regarding this point.

DEFINITION 2.6. Let v_x be defined to be the unique element of $\mathcal{H}_\mathcal{E}$ for which

$$\langle v_x, u \rangle_\mathcal{E} = u(x) - u(o), \quad \text{for every } u \in \mathcal{H}_\mathcal{E}. \tag{2.5}$$

The existence and uniqueness of v_x for each $x \in G$ is implied by the Riesz lemma. It follows from (2.5) that $\{v_x\}_{x \in G}$ forms a reproducing kernel for $\mathcal{H}_\mathcal{E}$ (called the *energy kernel*; see Corollary 2.7 of [10]) and that $\operatorname{span}\{v_x\}_{x \in G}$ is dense in $\mathcal{H}_\mathcal{E}$.

Note that v_o corresponds to a constant function, since $\langle v_o, u \rangle_\mathcal{E} = 0$ for every $u \in \mathcal{H}_\mathcal{E}$. Therefore, v_o may often be safely ignored or omitted during calculations.

DEFINITION 2.7. A *dipole* is any $v \in \mathcal{H}_\mathcal{E}$ satisfying the pointwise identity $\Delta v = \delta_x - \delta_y$ for some vertices $x, y \in G$. One can check that $\Delta v_x = \delta_x - \delta_o$; cf. Lemma 2.13 of [10].

Note that dipoles always exist for any pair of vertices $x, y \in G$, by Riesz's lemma, as in Definition 2.6.

DEFINITION 2.8. For $v \in \mathcal{H}_\mathcal{E}$, one says that v has *finite support* if and only if there is a finite set $F \subseteq G$ for which $v(x) = k \in \mathbb{C}$ for all $x \notin F$. The set of functions of finite support in $\mathcal{H}_\mathcal{E}$ is denoted $\operatorname{span}\{\delta_x\}$, where δ_x is the Dirac mass at x, i.e., the element of $\mathcal{H}_\mathcal{E}$ containing the characteristic function of the singleton

$\{x\}$. It is immediate from (2.2) that $\mathcal{E}(\delta_x) = c(x)$, whence $\delta_x \in \mathcal{H}_\mathcal{E}$. Define $\mathcal{F}in$ to be the closure of span$\{\delta_x\}$ with respect to $\mathcal{E}$.

DEFINITION 2.9. The *set of harmonic functions of finite energy* is denoted

$$\mathcal{H}arm := \{v \in \mathcal{H}_\mathcal{E} : \Delta v(x) = 0, \text{ for all } x \in G\}. \tag{2.6}$$

It may be the case that the only harmonic functions of finite energy are constant (and hence trivial in $\mathcal{H}_\mathcal{E}$). This is true, for example, on any finite network.

LEMMA 2.10 ([10], 2.11). *For any $x \in G$, one has $\langle \delta_x, u \rangle_\mathcal{E} = \Delta u(x)$.*

The next result follows easily from Lemma 2.10; cf. Theorem 2.15 of [10].

THEOREM 2.11 (Royden decomposition). *$\mathcal{H}_\mathcal{E} = \mathcal{F}in \bigoplus \mathcal{H}arm.$*

REMARK 2.12. The Royden decomposition illustrates one of the advantages of working with $\langle u, v \rangle_\mathcal{E}$, as opposed to the inner product on $\ell^2(G)$ or the grounded energy product $\langle u, v \rangle_o := \langle u, v \rangle_\mathcal{E} + u(o)v(o)$. Another advantage is the following: by combining (2.5) and the conclusion of Lemma 2.10, one can reconstruct the network (G, c) (or equivalently, the corresponding Laplacian) from the dual systems (i) $(\delta_x)_{x \in X}$ and (ii) $(v_x)_{x \in X}$. Indeed, from (ii), we obtain the (relative) reproducing kernel Hilbert space $\mathcal{H}_\mathcal{E}$ and from (ii), we get an associated operator $(\Delta u)(x) = \langle \delta_x, u \rangle_\mathcal{E}$ for $u \in \mathcal{H}_\mathcal{E}$.

DEFINITION 2.13. Denote the *(free) effective resistance* from x to y by

$$R(x, y) := R^{\mathrm{F}}(x, y) = \mathcal{E}(v_x - v_y) = \|v_x - v_y\|_\mathcal{E}^2. \tag{2.7}$$

This quantity represents the voltage drop measured when one unit of current is passed into the network at x and removed at y, and the central equality in (2.7) is proved in [6] and elsewhere in the literature; see [18], [19] for different formulations.

The following results will be useful in the sequel; for further details, please see [6], [7], [8], [10] and [12].

LEMMA 2.14 ([10], Lemma 2.23). *Every v_x is $\mathbb{R}$-valued, with $v_x(y) - v_x(o) > 0$ for all $y \neq o$.*

LEMMA 2.15 ([8], Lemma 6.9). *Every v_x is bounded. In particular, if we define*

$$\|u\|_\infty := \sup_{x,y \in G} |u(x) - u(y)|. \tag{2.8}$$

then we always have $\|v_x\|_\infty \leqslant R(x, o)$.

LEMMA 2.16 ([8], Lemma 6.8). *If $v \in \mathcal{H}_\mathcal{E}$ is bounded, then $P_{\mathcal{F}in}\, v$ is also bounded.*

DEFINITION 2.17. Let $p(x, y) := c_{xy}/c(x)$ so that $p(x, y)$ defines a random walk on the network, with transition probabilities weighted by the conductances.

Then let

$$\mathbb{P}[x \to y] := \mathbb{P}_x(\tau_y < \tau_x^+) \tag{2.9}$$

be the probability that the random walk started at x reaches y before returning to x. In (2.9), τ_z is the hitting time of the vertex z and $\tau_z^+ := \min\{\tau_z, 1\}$.

COROLLARY 2.18 ([6], Corollary 3.13 and Corollary 3.15). *For any $x \neq o$, one has*

$$\mathbb{P}[x \to o] = \frac{1}{c(x)R(x,o)}. \tag{2.10}$$

3. COMPARING DIFFERENT CONDUCTANCE FUNCTIONS

In our proofs, we will make use of tools from the theory of unbounded operators, and readers may find the reference [16] helpful. Similarly, we refer to [24] for graphs and networks.

Given a network (G,c), we will be interested in comparing its energy space $\mathcal{H}_\mathcal{E} = \mathcal{H}_{\mathcal{E}^{(c)}}$ and Laplace operator $\Delta = \Delta^{(c)}$ with those corresponding to a different conductance function b. To clarify dependence on the conductance functions, we use scripts to distinguish between objects corresponding to different underlying conductance functions. For example, $\Delta^{(c)} = \Delta$ in (2.1) and $\mathcal{E}^{(c)} = \mathcal{E}$ in (2.2), as opposed to

$$(\Delta^{(b)}v)(x) := \sum_{y\sim x} b_{xy}(v(x) - v(y)), \tag{3.1}$$

and

$$\mathcal{E}_b(u,v) = \langle u,v\rangle_{\mathcal{E}^{(b)}} = \frac{1}{2}\sum_{x,y\in G} b_{xy}(\overline{u}(x) - \overline{u}(y))(v(x) - v(y)), \tag{3.2}$$

with domain $\operatorname{dom}\mathcal{E}^{(b)} = \{u : G \to \mathbb{C} : \mathcal{E}^{(b)}(u) < \infty\}$. It is clear that $\mathcal{H}_{\mathcal{E}^{(b)}}$ also depends on b, and so too does the energy kernel $\{v_x^{(b)}\}_{x\in G}$. We will take the domains to be

$$\operatorname{dom}\Delta^{(b)} = \operatorname{span}\{v_x^{(b)}\}_{x\in G} \quad \text{and} \quad \operatorname{dom}\Delta^{(c)} = \operatorname{span}\{v_x^{(c)}\}_{x\in G}. \tag{3.3}$$

REMARK 3.1. Given a network (G,c) and a new conductance function $b \leqslant c$, it may be that $b_{xy} = 0$ even though $c_{xy} > 0$, and consequently the edge structure of (G,b) may be very different from (G,c). *However*, we will *always* make the assumption that (G,b) is connected, so that Lemma 3.5 may be applied.

DEFINITION 3.2. Let $b : G^0 \times G^0 \to [0,\infty)$ be a symmetric function satisfying

$$b_{xy} \leqslant c_{xy}, \quad \text{for all } x,y \in G^0.$$

In this case, we write $b \leqslant c$. Note that we will always assume (G,b) is connected; see Remark 3.1.

LEMMA 3.3. *Inclusion gives natural contractive embedding* $\mathcal{I} : \mathcal{H}_{\mathcal{E}^{(c)}} \hookrightarrow \mathcal{H}_{\mathcal{E}^{(b)}}$.

Proof. Since $b \leqslant c$, one has

$$\mathcal{E}^{(b)}(u) = \frac{1}{2} \sum_{x,y \in G} b_{xy} |u(x) - u(y)|^2 \leqslant \frac{1}{2} \sum_{x,y \in G} c_{xy} |u(x) - u(y)|^2 = \mathcal{E}^{(c)}(u) \tag{3.4}$$

for any function $u : G \to \mathbb{R}$, and hence $\|\mathcal{I}u\|_{\mathcal{E}^{(b)}} \leqslant \|u\|_{\mathcal{E}^{(c)}}$. ∎

LEMMA 3.4. $\mathcal{I}(\mathcal{F}in^{(c)}) \hookrightarrow \mathcal{F}in^{(b)}$, *and* $\mathcal{I}^\star(\mathcal{H}arm^{(b)}) \hookrightarrow \mathcal{H}arm^{(c)}$.

Proof. The first follows from the fact that $\mathcal{I}(\delta_x) = \delta_x$, whence the second follows because adjoints preserve the orthocomplements (see Theorem 2.11), i.e.,

$$\mathcal{I}^\star(\mathcal{H}arm^{(b)}) = \mathcal{I}^\star((\mathcal{F}in^{(b)})^\perp) \subseteq (\mathcal{F}in^{(c)})^\perp = \mathcal{H}arm^{(c)}. \quad ∎$$

Lemma 3.5 clarifies the nature of the blanket assumption that (G, b) is connected; see Remark 3.1.

LEMMA 3.5. *If* (G, c) *is a network and* $b \leqslant c$*, then the following are equivalent:*

(i) (G, b) *is connected.*

(ii) $\ker \mathcal{E}^{(b)} = \ker \mathcal{E}^{(c)} = \mathbb{C}\mathbf{1}$.

(iii) $\ker \mathcal{I} = 0$.

Proof. To see (i) ⇔ (ii), observe that $\mathcal{E}^{(b)}(u)$ is given by a sum of nonnegative terms and hence vanishes if and only if each summand does. Thus $\mathcal{E}^{(b)}(u) = 0$ if and only if u is locally constant. For (ii) ⇒ (iii), note that $\mathcal{I}(u) = 0$ implies $\|u\|_{\mathcal{E}^{(b)}} = 0$ and hence that u is a constant function, whence $u = 0$ in $\mathcal{H}_{\mathcal{E}^{(b)}}$. For (iii) ⇒ (ii), suppose (G, b) is not connected, and define $u = 1$ on one component and $u = 0$ on the complement. Then $\|\mathcal{I}(u)\|_{\mathcal{E}^{(b)}} = 0$ but $u \neq 0$ in $\mathcal{H}_{\mathcal{E}^{(c)}}$. ∎

LEMMA 3.6. $\mathcal{I}^\star v_x^{(b)} = v_x^{(c)}$*, and for general* $u \in \mathcal{H}_{\mathcal{E}^{(b)}}$*, one can compute* $\mathcal{I}^\star$ *via*

$$(\mathcal{I}^\star u)(x) - (\mathcal{I}^\star u)(y) = \frac{b_{xy}}{c_{xy}}(u(x) - u(y)). \tag{3.5}$$

Proof. For $u \in \mathcal{H}_{\mathcal{E}^{(c)}} \subseteq \mathcal{H}_{\mathcal{E}^{(b)}}$,

$$\langle \mathcal{I}^\star v_x^{(b)}, u \rangle_{\mathcal{E}^{(c)}} = \langle v_x^{(b)}, \mathcal{I}u \rangle_{\mathcal{E}^{(b)}} = u(x) - u(o) = \langle v_x^{(c)}, u \rangle_{\mathcal{E}^{(c)}}.$$

Now for $u \in \mathcal{H}_{\mathcal{E}^{(b)}}$ and $v \in \mathcal{H}_{\mathcal{E}^{(c)}}$, the latter claim follows from the fact that

$$\langle u, \mathcal{I}v \rangle_{\mathcal{E}^{(b)}} = \frac{1}{2} \sum_{x,y \in G} b_{xy}(u(x) - u(y))(v(x) - v(y))$$

is equal to

$$\langle \mathcal{I}^\star u, v \rangle_{\mathcal{E}^{(c)}} = \frac{1}{2} \sum_{x,y \in G} c_{xy}((\mathcal{I}^\star u)(x) - (\mathcal{I}^\star u)(y))(v(x) - v(y)). \quad ∎$$

COROLLARY 3.7. $\mathcal{I}$ *is injective.*

Proof. $\ker \mathcal{I} = \{0\}$ because $\operatorname{span}\{v_x^{(c)}\} = \operatorname{ran} \mathcal{I}^\star$ is dense in $\mathcal{H}_{\mathcal{E}(c)}$. ∎

LEMMA 3.8. *If δ_{xy} is the Kronecker delta, then*

$$\langle v_x^{(b)}, \Delta^{(b)} v_y^{(b)} \rangle_{\mathcal{E}^{(b)}} = \delta_{xy} + 1 = \langle v_x^{(c)}, \Delta^{(c)} v_y^{(c)} \rangle_{\mathcal{E}^{(c)}}, \quad \forall x, y \in G \setminus \{o\}. \tag{3.6}$$

Proof. Note that

$$\langle v_x^{(b)}, \Delta^{(b)} v_y^{(b)} \rangle_{\mathcal{E}^{(b)}} = (\Delta^{(b)} v_y^{(b)})(x) - (\Delta^{(b)} v_y^{(b)})(o) = \langle \delta_x, v_y^{(b)} \rangle_{\mathcal{E}^{(b)}} - \langle \delta_o, v_y^{(b)} \rangle_{\mathcal{E}^{(b)}},$$

because $\delta_x \in \mathcal{H}_{\mathcal{E}^{(b)}}$ and $\langle \delta_x, u \rangle_{\mathcal{E}^{(b)}} = \Delta^{(b)} u(x)$. Now the result follows via

$$\langle \delta_x, v_y^{(b)} \rangle_{\mathcal{E}^{(b)}} - \langle \delta_o, v_y^{(b)} \rangle_{\mathcal{E}^{(b)}} = (\delta_x(y) - \delta_x(o)) - (\delta_o(y) - \delta_o(o)) = \delta_{xy} + 1,$$

since $x, y \neq o$. ∎

LEMMA 3.9. *For $1 < b \leqslant c$, one has $\Delta^{(b)} = \mathcal{I} \Delta^{(c)} \mathcal{I}^\star$.*

Proof. Applying Lemma 3.8 and Lemma 3.6,

$$\begin{aligned} \langle v_x^{(b)}, \Delta^{(b)} v_y^{(b)} \rangle_{\mathcal{E}^{(b)}} &= \langle v_x^{(c)}, \Delta^{(c)} v_y^{(c)} \rangle_{\mathcal{E}^{(c)}} = \langle \mathcal{I}^\star v_x^{(b)}, \Delta^{(c)} \mathcal{I}^\star v_y^{(b)} \rangle_{\mathcal{E}^{(c)}} \\ &= \langle v_x^{(b)}, \mathcal{I} \Delta^{(c)} \mathcal{I}^\star v_y^{(b)} \rangle_{\mathcal{E}^{(c)}}. \end{aligned}$$ ∎

Thus we have a commuting square

$$\begin{tikzcd} \mathcal{H}_{\mathcal{E}^{(c)}} \arrow[d, "\Delta^{(c)}"'] & \mathcal{H}_{\mathcal{E}^{(b)}} \arrow[l, "\mathcal{I}^\star"'] \arrow[d, "\Delta^{(b)} = \mathcal{I}\Delta^{(c)}\mathcal{I}^\star"] \\ \mathcal{H}_{\mathcal{E}^{(c)}} \arrow[r, "\mathcal{I}"'] & \mathcal{H}_{\mathcal{E}^{(b)}} \end{tikzcd} \tag{3.7}$$

Note that one can recover the dipole property of $v_x^{(b)}$ from Lemma 3.6 and Lemma 3.9: $\Delta^{(b)} v_x^{(b)} = \mathcal{I} \Delta^{(c)} \mathcal{I}^\star v_x^{(b)} = \mathcal{I} \Delta^{(c)} v_x^{(c)} = \mathcal{I}(\delta_x - \delta_o) = \delta_x - \delta_o$.

COROLLARY 3.10. $\mathcal{I}^\star \in \operatorname{Hom}(\mathcal{H}arm^{(b)}, \mathcal{H}arm^{(c)})$ *is a spectral invariant.*

The proof is basically a restatement of Lemma 3.4.

This spectral invariant is also apparent from the formula $\Delta^{(b)} = \mathcal{I} \Delta^{(c)} \mathcal{I}^\star$ of Lemma 3.9. While $\mathcal{I}$ is not a norm-preserving map, it is standard from spectral theory that one can write $\mathcal{I}$ in terms of its polar decomposition as $\mathcal{I} = UP$ and then $\Delta^{(b)} = \mathcal{I} \Delta^{(c)} \mathcal{I}^\star$ implies that a unitary equivalence is given by $\Delta^{(b)} = U \Delta^{(c)} U^\star$.

In the case when $\dim \mathcal{H}arm^{(b)} = \dim \mathcal{H}arm^{(c)} = 1$, the spectral invariant of Corollary 3.10 is just a number. This is computed explicitly for the geometric integers in Example 5.1.

REMARK 3.11 (Open Question). For a fixed conductance function $b : G^0 \times G^0 \to [0, \infty)$, what are the closed subspaces $\mathcal{K} \subseteq \mathcal{H}_{\mathcal{E}^{(b)}}$ such that $\mathcal{K} \cong \mathcal{H}_{\mathcal{E}^{(c)}}$ for some conductance functions c with $b \leqslant c$?

COROLLARY 3.12. *If $b \leqslant c$ and $\Delta^{(c)}$ is bounded on $\mathcal{H}_{\mathcal{E}(c)}$, then $\Delta^{(b)}$ is bounded on $\mathcal{H}_{\mathcal{E}(b)}$.*

Proof. Lemma 3.9 immediately implies

$$\|\Delta^{(b)}\|_{\mathcal{H}_{\mathcal{E}(b)} \to \mathcal{H}_{\mathcal{E}(b)}} \leqslant \|\Delta^{(c)}\|_{\mathcal{H}_{\mathcal{E}(c)} \to \mathcal{H}_{\mathcal{E}(c)}}. \qquad \blacksquare$$

COROLLARY 3.13. *If $c \equiv 1$ and $\Delta^{(c)}$ is bounded on $\mathcal{H}_{\mathcal{E}(c)}$, then $\Delta^{(b)}$ is bounded on $\mathcal{H}_{\mathcal{E}(b)}$ for any bounded conductance function b.*

Proof. Writing $\|b\|_\infty$ for the supremum of b, we have

$$b_{xy} \leqslant \|b\|_\infty c_{xy} = \|b\|_\infty,$$

so Corollary 3.12 applies to the network with conductances all equal to $\|b\|_\infty$. $\blacksquare$

THEOREM 3.14. *Let c be an arbitrary conductance function, and let $\mathbf{1}$ be the conductance function which assigns a conductance of 1 to every edge. Then $\mathcal{H}_{\mathcal{E}(c)}$ is contained in $\mathcal{H}_{\mathcal{E}(\mathbf{1})}$ if and only if there is an $\varepsilon > 0$ such that $c_{xy} \geqslant \varepsilon$ for all $x, y \in G$ with $c_{xy} > 0$.*

Proof. For the forward direction, suppose $K < \infty$ satisfies $\|u\|^2_{\mathcal{E}(\mathbf{1})} \leqslant K\|u\|^2_{\mathcal{E}(c)}$, for all $u \in \mathcal{H}_{\mathcal{E}(c)}$. Note that $\mathcal{E}^{(c)}(\delta_x) = c(x)$ follows directly from (2.2), so

$$c(x) = \|\delta_x\|^2_{\mathcal{E}(c)} \geqslant \frac{1}{K}\|\delta_x\|^2_{\mathcal{E}(\mathbf{1})} \geqslant \frac{1}{K}$$

since $\|\delta_x\|_{\mathcal{E}(\mathbf{1})} \geqslant 1$ by the connectedness of the network.

For the converse,

$$\|u\|^2_{\mathcal{E}(\mathbf{1})} = \frac{1}{2}\sum_{x,y \in G}(u(x) - u(y))^2 \leqslant \frac{1}{2}\sum_{x,y \in G}\frac{c_{xy}}{\varepsilon}(u(x) - u(y))^2 = \frac{1}{\varepsilon}\|u\|^2_{\mathcal{E}(c)},$$

so $\mathcal{I} : \mathcal{H}_{\mathcal{E}(c)} \to \mathcal{H}_{\mathcal{E}(\mathbf{1})}$ is a bounded operator with $\|\mathcal{I}\|_{\mathcal{H}_{\mathcal{E}(c)} \to \mathcal{H}_{\mathcal{E}(\mathbf{1})}} \leqslant 1/\sqrt{\varepsilon}$. $\blacksquare$

EXAMPLE 3.15 (Horizontally connected binary tree). This example shows that the boundedness of the conductance function is not sufficient to imply boundedness of the Laplacian, and illustrates the interplay between spectral reciprocity and effective resistance (see also [9]). To begin, let (G, b) be the binary tree where every edge has conductance $c_{xy} = 1$. Now let (G, c) be the network obtained by connecting all vertices at level k with an edge of conductance c_k as in Figure 1. The resulting network is no longer a tree, but we call it the *horizontally connected binary tree* for lack of a better name. Note that $b \leqslant c$.

Suppose that $c_k = 1$ for each k, so c_{xy} is globally constant on G^1. However, $c(x) = 2^k + 2$ for x in level k, so $c(x)$ is clearly unbounded on G^0. (As usual, level k consists of all vertices in (G, b) for which the shortest path to o contains exactly k edges.) Let K_n be the complete graph on n vertices. Using Schur complements (for example, as in [6], [12] or [17], [18]), one can compute $R_{K_n}(x, y) = 2^{1-n}$ for

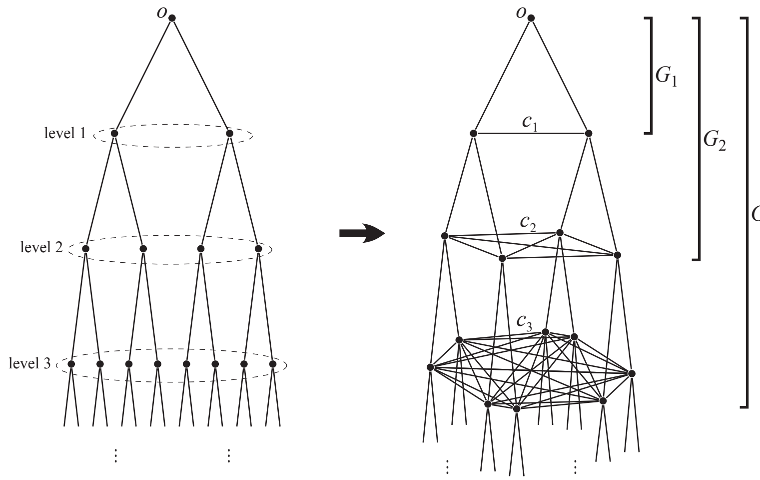


FIGURE 1. Construction of the "horizontally connected binary tree" of Example 3.15.

any $x, y \in K_n$. Consequently, it is easy to see that $R^{\mathrm{F}}_{(G,c)}(x,y)$ can be made arbitrarily small by choosing x, y in level k, for sufficiently large k. By spectral reciprocity (see [9]), this implies that $\Delta^{(c)}$ is unbounded on $\mathcal{H}_{\mathcal{E}(c)}$. Thus, this network provides an example of how boundedness of c_{xy} does not imply boundedness of $\Delta^{(c)}$. For an example of how boundedness of c_{xy} does not imply boundedness of Δ on other spaces, see [25].

Suppose that we choose c_k so as to make $c(x)$ bounded on G^0. Then we must have $c_k = O(2^{-k})$ as $k \to \infty$, so define $c_k = 2^{-k}$. Using this, one can compute that $R_{G_k}(x,y) = 1$ for x, y in level k of G_k, for every k.

LEMMA 3.16. *Suppose $b \leqslant c$. If $\Delta^{(c)}$ is self-adjoint, then $\Delta^{(b)}$ is self-adjoint also.*

Proof. Take adjoints on both sides of $\Delta^{(b)} = \mathcal{I}\Delta^{(c)}\mathcal{I}^{\star}$ (see Lemma 3.9). Note that the domains are as in (3.3). ∎

EXAMPLE 3.17 (Geometric integers). For a fixed constant $c > 1$, let $(\mathbb{Z}, c^n)$ denote the network with integers for vertices, and with geometrically increasing conductances defined by $c_{n-1,n} = c^{\max\{|n|,|n-1|\}}$ so that the network under consideration is

$$\cdots \overset{c^3}{\text{———}} -2 \overset{c^2}{\text{———}} -1 \overset{c}{\text{———}} 0 \overset{c}{\text{———}} 1 \overset{c^2}{\text{———}} 2 \overset{c^3}{\text{———}} 3 \overset{c^4}{\text{———}} \cdots$$

as in Example 6.2 of [10], and fix $o = 0$. It is shown in Section 4.2 of [9] that $\Delta^{(c)}$ is not self-adjoint, and a defect vector $\varphi \in \mathcal{H}_{\mathcal{E}(c)}$ is constructed which satisfies

$$\Delta^{(c)}\varphi = -\varphi. \tag{3.8}$$

However, for $b \equiv 1$, $\Delta^{(b)}$ is bounded and Hermitian, and thus clearly self-adjoint. This example shows that the converse of Lemma 3.16 does not hold. Using Fourier theory, one can show that $\mathcal{H}_{\mathcal{E}(b)} \cong L^2((-\pi,\pi), \sin^2(t/2))$; see Section 6.3 of [13], for example.

So Lemma 3.9 gives $\Delta^{(b)} = \mathcal{I}\Delta^{(c)}\mathcal{I}^\star$, where $\Delta^{(b)}$ is bounded and $\Delta^{(c)}$ is unbounded and not self-adjoint. The inclusion $\mathcal{I} : \mathcal{H}_{\mathcal{E}(c)} \to \mathcal{H}_{\mathcal{E}(b)}$ indicates that

$$\mathcal{H}_{\mathcal{E}(b)} = \mathcal{H}_{\mathcal{E}(c)} \oplus \mathcal{H}_{\mathcal{E}(c)}^{\perp},$$

where $\mathcal{H}_{\mathcal{E}(c)}^{\perp} = \mathcal{H}_{\mathcal{E}(b)} \ominus \mathcal{H}_{\mathcal{E}(c)}$, and that $\Delta^{(c)}$ is a matrix corner of $\Delta^{(b)}$:

$$\Delta^{(b)} = \begin{bmatrix} \Delta^{(c)} & A \\ A^\star & B \end{bmatrix}. \tag{3.9}$$

This gives another way to relate the operators $\Delta^{(b)}$ and $\Delta^{(c)}$.

3.1. THE ADJOINT OF $\Delta^{(b)}$ WITH RESPECT TO $\mathcal{E}^{(c)}$. For the results in this section we consider the adjoint of $\Delta^{(b)}$ with respect to $\mathcal{E}^{(c)}$and denote it by ${\Delta^{(b)}}^{\bigstar[2]c}$, in other words, we are interested in

$$\langle {\Delta^{(b)}}^{\bigstar[2]c} u, v\rangle_{\mathcal{E}(c)} = \langle u, \Delta^{(b)} v\rangle_{\mathcal{E}(c)}.$$

It will be helpful to know the action of $\mathcal{I}^\star$ on $\mathcal{F}in$, as given in Lemma 3.18; this result also generalizes the dipole property $\Delta v = \delta_x - \delta_y$ of Definition 2.7.

LEMMA 3.18. *For* $1 < b \leqslant c$, *one has* $\operatorname{span}\{v_x^{(c)}\} \subseteq \operatorname{dom} {\Delta^{(b)}}^{\bigstar[2]c}$ *and*

$${\Delta^{(b)}}^{\bigstar[2]c} v_x^{(c)} = \mathcal{I}^\star(\delta_x - \delta_o). \tag{3.10}$$

Proof. For any fixed $x \in G$ and $u \in \mathcal{H}_{\mathcal{E}(c)}$, we have the estimate

$$\langle v_x^{(c)}, \Delta^{(b)} u\rangle_{\mathcal{E}(c)} = \Delta^{(b)} u(x) - \Delta^{(b)} u(o) = \langle \delta_x - \delta_o, u\rangle_{\mathcal{E}(b)} \leqslant \|\delta_x - \delta_o\|_{\mathcal{E}(b)} \cdot \|u\|_{\mathcal{E}(b)},$$

by Lemma 2.10 followed by (2.5). This shows $\operatorname{span}\{v_x^{(c)}\} \subseteq \operatorname{dom} {\Delta^{(b)}}^{\bigstar[2]c}$ and $\langle v_x^{(c)}, \Delta^{(b)} u\rangle_{\mathcal{E}(c)} = \langle \delta_x - \delta_o, u\rangle_{\mathcal{E}(b)}$, which gives (3.10). ∎

For Theorem 3.20, we need to define ${\Delta^{(c)}}^{-1}$ via the spectral theorem. To this end, we introduce the following blanket assumption (which remains in place for the remainder of this paper).

ASSUMPTION 3.19. Suppose a conductance function c has been fixed. If the corresponding Laplace operator $\Delta^{(c)}$ is not self-adjoint, then we replace it by the Friedrichs extension as described in [13].

With Assumption 3.19 in place, we can work with $\Delta^{(c)}$ as a self-adjoint operator. Then by the spectral theorem: for any $u \in \mathcal{H}_{\mathcal{E}(c)}$, there is a Borel measure

$\mu_u^{(c)}$ on $[0,\infty)$ such that

$$\langle u, \psi(\Delta^{(c)})u\rangle_{\mathcal{E}^{(c)}} = \int_0^\infty \psi(\lambda)\,\mathrm{d}\mu_u^{(c)}(\lambda) = \int_0^\infty \psi(u)\|P(\mathrm{d}\lambda)u\|^2_{\mathcal{E}^{(c)}}, \tag{3.11}$$

where P is the projection-valued measure in the spectral resolution of $\Delta^{(c)}$. This will be useful for Theorem 4.2. Furthermore, we also have that

$$\Delta^{(c)^{-1}} := \int_0^\infty \mathrm{e}^{-\lambda\Delta^{(c)}}\,\mathrm{d}\lambda. \tag{3.12}$$

This definition of the inverse is a standard application of the spectral theorem, and is based on the fact that $\int_0^\infty \mathrm{e}^{-\lambda t}\,\mathrm{d}\lambda = 1/t$.

THEOREM 3.20. *For $1 < b \leqslant c$, one has $\Delta^{(b)^{\bigstar[2]c}} = \Delta^{(c)^{-1}}\Delta^{(b)}\Delta^{(c)}$, where $\Delta^{(c)^{-1}}$ is the inverse of the Friedrichs extension, defined as in* (3.12).

Proof. We will first show $\Delta^{(c)}\Delta^{(b)^{\bigstar[2]c}} = \Delta^{(b)}\Delta^{(c)}$, which is equivalent to $\mathcal{I}(\Delta^{(c)}\Delta^{(b)^{\bigstar[2]c}} - \Delta^{(b)}\Delta^{(c)}) = 0$ by Corollary 3.7. Applying Lemma 3.18 and Lemma 3.9, one has

$$\Delta^{(c)}\Delta^{(b)^{\bigstar[2]c}} v_x^{(c)} = \mathcal{I}\Delta^{(c)}\Delta^{(b)^{\bigstar[2]c}} v_x^{(c)} = \mathcal{I}\Delta^{(c)}\mathcal{I}^\star(\delta_x - \delta_o) = \Delta^{(b)}(\delta_x - \delta_o).$$

Then using the dipole property $\Delta^{(c)}v_x^{(c)} = \delta_x - \delta_o$ yields

$$\Delta^{(b)}(\delta_x - \delta_o) = \Delta^{(b)}(\Delta^{(c)}v_x^{(c)}) = \Delta^{(b)}(\Delta^{(c)}v_x^{(c)}) = \Delta^{(b)}\Delta^{(c)}(v_x^{(c)}).$$

Now we have $\Delta^{(c)}\Delta^{(b)^{\bigstar[2]c}}(v_x^{(c)}) = \Delta^{(b)}\Delta^{(c)}(v_x^{(c)})$ for any x, whence $\Delta^{(c)}\Delta^{(b)^{\bigstar[2]c}} = \Delta^{(b)}\Delta^{(c)}$ follows by the density of $\operatorname{span}\{v_x^{(c)}\}$ in $\mathcal{H}_{\mathcal{E}^{(c)}}$. It follows from the preceding argument that $\Delta^{(b)}\Delta^{(c)}(\operatorname{span}\{v_x^{(c)}\}) \subseteq \operatorname{dom}\Delta^{(c)^{-1}}$, and so the proof is complete. ∎

4. MOMENTS OF $\Delta^{(c)}$ AND MONOTONICITY OF SPECTRAL MEASURES

Note that we continue to assume $\Delta^{(c)}$ is a self-adjoint operator as discussed in Assumption 3.19.

LEMMA 4.1. *For $u = v_x^{(c)} - v_y^{(c)}$ and $\psi(\lambda) = \lambda^k$, $k = 0,1,2$, we have*

$$k=0: \qquad \langle u, u\rangle_{\mathcal{E}^{(c)}} = R^{\mathrm{F}}(x,y),$$

$$k=1: \qquad \langle u, \Delta^{(c)}u\rangle_{\mathcal{E}^{(c)}} = 2 - 2\delta_{xy},$$

$$k=2: \qquad \langle v_x^{(c)}, \Delta^{(c)^2}v_x^{(c)}\rangle_{\mathcal{E}^{(c)}} = c(x) + 2c_{xy} + c(y).$$

Proof. The case $k = 0$ follows immediately from (2.7). For $k = 1$, (3.6) gives

$$\begin{aligned}\langle v_x^{(c)}, \Delta^{(c)} v_x^{(c)} \rangle_{\mathcal{E}^{(c)}} - \langle v_x^{(c)}, \Delta^{(c)} v_y^{(c)} \rangle_{\mathcal{E}^{(c)}} - \langle v_y^{(c)}, \Delta^{(c)} v_x^{(c)} \rangle_{\mathcal{E}^{(c)}} + \langle v_y^{(c)}, \Delta^{(c)} v_y^{(c)} \rangle_{\mathcal{E}^{(c)}} \\ = 2 - (\delta_{xy} + 1) - (\delta_{xy} + 1) + 2.\end{aligned}$$

For $k = 2$, we use the fact that the Friedrichs extension is self-adjoint and the dipole property (2.5) to compute

$$\begin{aligned}\langle v_x^{(c)}, {\Delta^{(c)}}^2 v_x^{(c)} \rangle_{\mathcal{E}^{(c)}} &= \langle \Delta^{(c)} v_x^{(c)}, \Delta^{(c)} v_x^{(c)} \rangle_{\mathcal{E}^{(c)}} = \langle \delta_x - \delta_y, \delta_x - \delta_y \rangle_{\mathcal{E}^{(c)}} \\ &= c(x) + 2c_{xy} + c(y).\end{aligned}$$

For the last step, we used $\mathcal{E}(\delta_x) = c(x)$, which is immediate from (2.2). ∎

In Theorem 4.2, we use $\mu_u^{(c)}$ as given by (3.11). Also, let

$$m_k^{(c)}(u) := \int_0^\infty \lambda^k \, \mathrm{d}\mu_u^{(c)} \tag{4.1}$$

be the k^{th} moment of $\mu_u^{(c)}$, and similarly for $\mu_u^{(b)}$. We now consider the moments of Δ via spectral theory.

THEOREM 4.2 (Monotonicity of spectral measures). *Let (G, c) be a given network, and let $b \leqslant c$. Then*

$$m_1^{(b)}(u) = m_1^{(c)}(\mathcal{I}^\star u) \quad \textit{and} \quad m_2^{(b)}(u) \leqslant m_2^{(c)}(\mathcal{I}^\star u). \tag{4.2}$$

Proof. First, note that Lemma 3.9 gives

$$m_1^{(b)} = \langle u, \Delta^{(b)} u \rangle_{\mathcal{E}^{(b)}} = \langle u, \mathcal{I}\Delta^{(c)}\mathcal{I}^\star u \rangle_{\mathcal{E}^{(b)}} = \langle \mathcal{I}^\star u, \Delta^{(c)}\mathcal{I}^\star u \rangle_{\mathcal{E}^{(c)}} = m_1^{(c)}.$$

For the second moments, using Lemma 3.9 again gives

$$\begin{aligned}m_2^{(b)} &= \langle u, (\Delta^{(b)})^2 u \rangle_{\mathcal{E}^{(b)}} = \langle u, \mathcal{I}\Delta^{(c)}\mathcal{I}^\star\mathcal{I}\Delta^{(c)}\mathcal{I}^\star u \rangle_{\mathcal{E}^{(b)}} \\ &= \langle {\Delta^{(c)}}^\star \mathcal{I}^\star u, \mathcal{I}^\star\mathcal{I}\Delta^{(c)}\mathcal{I}^\star u \rangle_{\mathcal{E}^{(c)}}.\end{aligned}$$

Since $\mathcal{I}^\star\mathcal{I}$ is contractive by Lemma 3.3,

$$\begin{aligned}\langle {\Delta^{(c)}}^\star \mathcal{I}^\star u, \mathcal{I}^\star\mathcal{I}\Delta^{(c)}\mathcal{I}^\star u \rangle_{\mathcal{E}^{(c)}} &\leqslant \|\mathcal{I}^\star\mathcal{I}\| \cdot \langle {\Delta^{(c)}}^\star \mathcal{I}^\star u, \Delta^{(c)}\mathcal{I}^\star u \rangle_{\mathcal{E}^{(c)}} \\ &\leqslant \langle u, \mathcal{I}(\Delta^{(c)})^2 \mathcal{I}^\star u \rangle_{\mathcal{E}^{(c)}},\end{aligned}$$

whence $m_2^{(b)} \leqslant m_2^{(c)}$. ∎

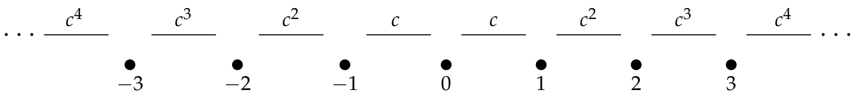
REMARK 4.3. If $b_{xy} < c_{xy}$ for some edge (xy), then $m_2^{(b)}(v_x^{(b)}) < m_2^{(c)}(\mathcal{I}^\star v_x^{(b)})$.

5. EXAMPLE

EXAMPLE 5.1 (Geometric integers). Let $(\mathbb{Z}, c^n)$ be the network whose vertices are the integers with conductances given by

$$c_{m,n} = \begin{cases} c^{\max\{|m|,|n|\}} & |m-n| = 1, \\ 0 & \text{else}, \end{cases}$$

as in the following diagram:



It is known that $\mathcal{H}arm$ is 1-dimensional for this network; see [10]. It was also shown in [9] that Δ is not essentially self-adjoint (as an operator on $\mathcal{H}_\mathcal{E}$) for this network.

We compare $(\mathbb{Z}, b^n)$ and $(\mathbb{Z}, c^n)$, where $1 < b \leqslant c$. In this case, $\dim \mathcal{H}arm^{(b)} = \dim \mathcal{H}arm^{(c)} = 1$ and we can compute the (numerical) spectral invariant of Corollary 3.10. Choose unit vectors $h_b \in \mathcal{H}arm^{(b)}$ and $h_c \in \mathcal{H}arm^{(c)}$:

$$h_b(n) = \frac{\operatorname{sgn}(n)}{2\sqrt{b-1}}\Big(1 - \frac{1}{b^{|n|}}\Big), \quad h_c(n) = \frac{\operatorname{sgn}(n)}{2\sqrt{c-1}}\Big(1 - \frac{1}{c^{|n|}}\Big). \tag{5.1}$$

Now since $\langle \mathcal{I}^\star h_b, u\rangle_{\mathcal{E}^{(c)}} = \langle h_b, u\rangle_{\mathcal{E}^{(b)}}$ for all $u \in \mathcal{H}_{\mathcal{E}^{(c)}}$, we have

$$\langle h_b, v_n^{(c)}\rangle_{\mathcal{E}^{(b)}} = \langle \mathcal{I}^\star h_b, v_n^{(c)}\rangle_{\mathcal{E}^{(c)}} = \langle K h_c, v_n^{(c)}\rangle_{\mathcal{E}^{(c)}} = K\langle h_c, v_n^{(c)}\rangle_{\mathcal{E}^{(c)}}, \tag{5.2}$$

following the ansatz that $\mathcal{I}^\star$ should be just a numerical constant (scaling factor). Suppose for simplicity that $n > 0$, as the other computation is similar. On the left side of (5.2), we can compute directly from (2.2):

$$\begin{aligned} \langle h_b, v_n^{(c)}\rangle_{\mathcal{E}^{(b)}} &= 2\sum_{j=1}^{\infty} b^j \Big(\frac{1-b^{-j}}{2\sqrt{b-1}} - \frac{1-b^{1-j}}{2\sqrt{b-1}}\Big)(v_n^{(c)}(j) - v_n^{(b)}(j-1)) \\ &= \sqrt{b-1}\,v_n^{(c)}(n) = \sqrt{b-1}\sum_{j=1}^{n}\frac{1}{c^n} = \sqrt{b-1}\frac{1-c^{-n}}{c-1}. \end{aligned} \tag{5.3}$$

Meanwhile, on the right side of (5.2), the reproducing property gives

$$\langle h_c, v_n^{(c)}\rangle_{\mathcal{E}^{(c)}} = h_c(n) - [2]h_c(o) = \frac{1}{2\sqrt{c-1}}\Big(1 - \frac{1}{c^n}\Big). \tag{5.4}$$

Substituting (5.3) and (5.4) into (5.2) gives

$$\sqrt{b-1}\frac{1-c^{-n}}{c-1} = K\frac{1}{2\sqrt{c-1}}\Big(1 - \frac{1}{c^n}\Big),$$

and so the corresponding spectral invariant is

$$K = \|\mathcal{I}^\star|_{\mathcal{H}arm^{(b)}}\| = \sqrt{\frac{1-b}{1-c}},$$

and this is the factor by which $\mathcal{I}^\star$ scales the basis vector h_b; see Corollary 3.10.

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