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ERGODIC QUANTUM MARKOV SEMIGROUPS AND DECOHERENCE

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ABSTRACT. We study the relationships between ergodicity and environment induced decoherence for quantum Markov semigroups on a von Neumann algebra. We show that these properties are equivalent when the set of fixed points is an algebra containing the maximal subalgebra on which the semigroup is authomorphic.

KEYWORDS: *Decoherence, ergodicity, quantum Markov semigroups.*

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INTRODUCTION

The term decoherence is used with many different meanings in literature, always in some sense related with the emergence of classical features in the evolution of a quantum open system. It is however quite difficult to find out a clear and well-recognized formal definition of decoherence. We study the mathematical definition proposed by Blanchard and Olkiewicz in 2003 ([2]) in the case of markovian systems. For the reader who has never encountered this topic, we suggest to see [2], [3], [6], [7], [19], [20], for an introduction to the technical details of the matter and for motivations for this research.

We want to investigate the relations between this kind of decoherence and the ergodic properties of the semigroup. First, we specify the context and the mathematical objects we are going to treat. We consider a von Neumann algebra $\mathcal{M}$ acting on a complex Hilbert space h and a quantum dynamical semigroup (QDS) $\mathcal{T} = (\mathcal{T}_t)_{t\geqslant 0}$ on $\mathcal{M}$, that is a weak* continuous semigroup of completely positive normal bounded maps $\mathcal{T}_t$ on $\mathcal{M}$. We call $\mathcal{T}$ a quantum Markov semigroup (QMS) if the maps $\mathcal{T}_t$ are also identity preserving.

The idea of decoherence introduced by Blanchard and Olkiewicz essentially purposes to decompose the space $\mathcal{M}$ into the direct sum of a decoherence-free subalgebra on which the evolution is unitary (i.e. resembles to the evolution of a closed system) and a remaining part of $\mathcal{M}$ on which the semigroup goes to 0 when time goes on. To be more precise, we say that there is *environment induced decoherence* (EID) on the system described by $\mathcal{T}$ if there exist a $\mathcal{T}_t$-invariant von Neumann subalgebra $\mathcal{M}_1$ of $\mathcal{M}$ and a $\mathcal{T}_t$-invariant and $*$-invariant weak* closed subspace $\mathcal{M}_2$ of $\mathcal{M}$ such that:

(EID1) $\mathcal{M} = \mathcal{M}_1 \oplus \mathcal{M}_2$ with $\mathcal{M}_2 \neq \{0\}$;

(EID2) $\mathcal{M}_1$ is a maximal von Neumann subalgebra of $\mathcal{M}$ on which the restriction of every $\mathcal{T}_t$ is a $*$-automorphism;

(EID3) $\text{w}^*\text{-}\lim_{t\to\infty} \mathcal{T}_t(x) = 0$ for all $x \in \mathcal{M}_2$.

$\mathcal{M}_1$ is called the algebra of *effective observables*, while $\mathcal{M}_2$ is the space of *non-detectable observables*.

In this paper, when not specified differently, by decoherence we will mean EID.

The above definition is a slight modification of the original one introduced by Blanchard and Olkiewicz, but it is equivalent at least when we have a QMS with a faithful normal invariant state (see [6] for the details about this).

In general it is very difficult to understand when such a decomposition occurs, also because it is not clear whether the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$ are univocally determined. A collection of questions arise about how to prove the existence of EID and about the determination of the corresponding decomposition of the algebra. A starting point can be finding some possible candidates for the spaces $\mathcal{M}_1$ and $\mathcal{M}_2$. The definition of EID, as we already emphasized in previous papers [6], [7], clearly suggests to consider the relations with the spaces

$$\mathcal{N}(\mathcal{T}) := \{a \in \mathcal{M} : \mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a), \mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*) \forall t \geqslant 0\},$$

the biggest von Neumann subalgebra of $\mathcal{M}$ on which the action of any $\mathcal{T}_t$ is a $*$-homomorphism (see Proposition 1.3), and

$$\mathcal{M}_0 := \left\{x \in \mathcal{M} : \text{w}^*\text{-}\lim_t \mathcal{T}_t(x) = 0\right\},$$

the space on which the semigroup weakly* converges to 0 as time goes on. When EID occurs, we clearly have $\mathcal{M}_1 \subset \mathcal{N}(\mathcal{T})$ and $\mathcal{M}_2 \subset \mathcal{M}_0$, and, in some cases, the previous inclusions are necessarily equalities: for example in the finite-dimensional case ([7]), or when $\mathcal{N}(\mathcal{T})$ is contained in the set $\mathcal{F}(\mathcal{T})$ of fixed points of the semigroup (see Proposition 1.5 and Theorem 1.6). Nevertheless, it can happen to have $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ (e.g. for uniformly continuous semigroups [13] or for QMSs possessing a faithful normal invariant state [22]), but its intersection with $\mathcal{M}_0$ can be not trivial, so that $\mathcal{M}_2 \neq \mathcal{M}_0$ (see [8]).

On the other hand, $\mathcal{N}(\mathcal{T})$ has been extensively studied (see [10], [15], [16]) in relation to asymptotic behavior of the semigroup: indeed, under the hypothesis of the existence of a faithful normal invariant state ω, if $\mathcal{N}(\mathcal{T})$ consists only of fixed points, then the semigroup is ergodic, i.e. the system converges toward a steady state (possibly different from ω) . Moreover, under additional conditions (the algebra $\mathcal{M}$ on which the semigroup acts is finite-dimensional, or the detailed balance property), the equality $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ is also necessary to have ergodicity (see [16]). Therefore, it is natural to wonder whether there are some relationships between decoherence and ergodicity, at least in the case $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$. This paper is aimed at answering to this question.

In Section 2 we show that in general, for an ergodic QMS, it is possible to define a normal norm one projection $\mathcal{E}$ onto $\mathcal{F}(\mathcal{T})$ which induces the decomposition $\mathcal{M} = \mathcal{F}(\mathcal{T}) \oplus \operatorname{Ker}\mathcal{E}$, but in general, we cannot state that it is an EID decomposition: indeed, $\mathcal{F}(\mathcal{T})$ could be not an algebra and, in any case, even if it was so, $\mathcal{F}(\mathcal{T})$ could be not an effective subalgebra. On the other hand, whenever $\mathcal{N}(\mathcal{T})$ consists only of fixed points, we show that any QMS displaying decoherence is necessarily ergodic and has $\mathcal{F}(\mathcal{T})$ which is an algebra (see Theorem 1.6). More precisely, when EID holds, we obtain that $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ coincides with the biggest subalgebra on which $\mathcal{T}$ acts in a unitary way, the decomposition induced by decoherence is unique (with $\mathcal{M}_1 = \mathcal{F}(\mathcal{T})$ and $\mathcal{M}_2 = \mathcal{M}_0$), and there exists a normal conditional expectation on $\mathcal{M}_1$. In addition, if $\mathcal{F}(\mathcal{T})$ is an algebra, ergodicity and decoherence are equivalent (see Corollary 1.8). Theorems 1.15 and 1.17 describe how these results can be improved in two special cases, respectively when $\mathcal{N}(\mathcal{T})$ is trivial (i.e. $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$) and when the semigroup has a faithful normal invariant state.

For applications, the value of our results also derives from the fact that the algebra $\mathcal{N}(\mathcal{T})$ can be determined explicitly in many cases: for example, for a wide class of QMSs acting on $\mathcal{B}(\mathsf{h})$, it has been completely characterized in terms of the generator (see [9], [13]). We shall come back to this point in Section 3. All the results of the second section hold for QMSs and, really, for a larger class of semigroups, since it is enough to consider the case when the maps $\mathcal{T}_t$ satisfy only Kadison–Schwarz property ($\mathcal{T}_t(x^*)\mathcal{T}_t(x) \leqslant \mathcal{T}_t(x^*x)$ for all $t \geqslant 0$ and $x \in \mathcal{M}$), instead of complete positivity. QMSs are obviously the most popular choice for applications, but weakening the positivity conditions can be however interesting if we think about new theories wondering about the reasonable positivity conditions for quantum dynamics (see for instance [18] and [23]).

Section 3 illustrates the use of our results for some QDSs with unbounded generator, which constitute in general the most difficult case, but include many interesting models. We recall the construction of minimal QDS and some relations between the canonical form of the generator and the space $\mathcal{N}(\mathcal{T})$; finally we explain how to prove decoherence for a couple of models: the two photon absorption and emission process and a generic semigroup.

1. SEMIGROUPS WITH TRIVIAL EVOLUTION ON $\mathcal{N}(\mathcal{T})$

In this section, we shall always assume that $\mathcal{T}$ is a QMS on the algebra $\mathcal{M}$, possibly with the weakened positivity condition that the maps $\mathcal{T}_t$ are only

Kadison–Schwarz instead of completely positive, as we mentioned above. Dropping complete positivity is actually a common practice that one can find in many article on similar topics ([10], [16], [22]). However, we underline that, even if, in most of the proofs here, only simple positivity of the maps $\mathcal{T}_t$ is used, the reader should not be tempted to think that simple positivity is sufficient for our results. Really, Kadison–Schwarz is a basic ingredient to demonstrate Proposition 1.3: this result is essential for our study and does not hold true in general when the maps $\mathcal{T}_t$ are only positivity preserving instead of Kadison–Schwarz (see [6] for a counterexample).

As we already explained, our interest consists in studying the link of EID decomposition with ergodicity. We shall say that the QMS $\mathcal{T}$ is *ergodic* if the net $(\mathcal{T}_t(x))_{t\geqslant 0}$ is weakly* convergent for all $x \in \mathcal{M}$ or, equivalently, if the limit $\text{w-}\lim_{t\to+\infty} \mathcal{T}_{*t}(\varphi)$ exists for all normal functional φ on $\mathcal{M}$, where $\mathcal{T}_{*t}$ denotes the pre-dual map of $\mathcal{T}_t$ and w-lim is the limit in the weak topology.

The following proposition illustrates how every ergodic semigroup naturally induces a Jacobs–DeLeeuw–Glicksberg splitting of the algebra $\mathcal{M}$ (see also Proposition 1.20), and we want to try to relate it to decoherence properties.

PROPOSITION 1.1. *The following conditions are equivalent:*

(i) *$\mathcal{T}$ is ergodic;*

(ii) *there exists a normal norm one projection $\mathcal{E}$ onto $\mathcal{F}(\mathcal{T})$ such that*

$$\operatorname{Ker}\mathcal{E} = \mathcal{M}_0 := \Big\{x \in \mathcal{M} : \text{w}^*\text{-}\lim_t \mathcal{T}_t(x) = 0\Big\};$$

(iii) $\mathcal{M} = \mathcal{F}(\mathcal{T}) \oplus \mathcal{M}_0$.

In particular, if one of the previous conditions holds, then $\operatorname{Ker}\mathcal{E}$ *is a $\mathcal{T}_t$-invariant and $*$-invariant weak* closed subspace of $\mathcal{M}$, $\mathcal{E}$ is the unique norm one projection onto $\mathcal{F}(\mathcal{T})$ such that* $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0$, *and it is given by*

$$\mathcal{E}(x) := \text{w}^*\text{-}\lim_t \mathcal{T}_t(x) \quad \forall\, x \in \mathcal{M}, \tag{1.1}$$

$$\operatorname{Ker}\mathcal{E} = \overline{\operatorname{span}}^{\,\text{w}^*}\{(I - \mathcal{T}_t)x : x \in \mathcal{M}, t \geqslant 0\},$$

where I denotes the identical operator.

Finally, $\mathcal{E}$ satisfies $\mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y)$ for all $x, y \in \mathcal{M}$ if and only if $\mathcal{F}(\mathcal{T})$ is an algebra. In particular, in this case, $\mathcal{E}$ is a conditional expectation onto $\mathcal{F}(\mathcal{T})$.

Proof. (i) $\Rightarrow$ (ii) Assume $\mathcal{T}$ ergodic and consider the linear operator $\mathcal{E}$: $\mathcal{M} \to \mathcal{M}$ given by (1.1). Clearly, the range of $\mathcal{E}$ coincides with $\mathcal{F}(\mathcal{T})$ and $\mathcal{E} \circ \mathcal{T}_t = \mathcal{T}_t \circ \mathcal{E} = \mathcal{E}$ for all $t \geqslant 0$, by the semigroup property and the normality of $\mathcal{T}_t$. As a consequence,

$$\mathcal{E}^2(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(\mathcal{E}(x)) = \text{w}^*\text{-}\lim_t \mathcal{E}(x) = \mathcal{E}(x) \quad \forall\, x \in \mathcal{M},$$

i.e. $\mathcal{E}$ is a projection. Since it is also a positive operator preserving the identity $\mathbb{1}$, $\mathcal{E}$ is a norm one projection onto $\mathcal{F}(\mathcal{T})$.

The normality of $\mathcal{E}$ follows by Theorem 2.1 in [16], since every ergodic semigroup is also mean ergodic. The equality $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0$ is clear by definition of $\mathcal{E}$.

(ii) $\Rightarrow$ (iii) It is trivial.

(iii) $\Rightarrow$ (i) If the decomposition $\mathcal{M} = \mathcal{F}(\mathcal{T}) \oplus \mathcal{M}_0$ holds, for every $x \in \mathcal{M}$, $x = x_1 + x_2$ with $x_1 \in \mathcal{F}(\mathcal{T})$ and $x_2 \in \mathcal{M}_0$, there exists $\text{w}^*\text{-}\lim_t \mathcal{T}_t(x) = x_1$, so that $\mathcal{T}$ is ergodic.

Assume now that one of conditions (i), (ii), (iii) is fulfilled, and take $\mathcal{E}$ defined by (1.1). So, $\operatorname{Ker}\mathcal{E}$ is weak* closed as a consequence of the normality of $\mathcal{E}$. If $\mathcal{E}'$ is another norm one projection onto $\mathcal{F}(\mathcal{T})$ such that $\operatorname{Ker}\mathcal{E}' = \mathcal{M}_0$, then, for $x = x_1 + x_2 \in \mathcal{M}$ with $x_1 \in \mathcal{F}(\mathcal{T})$ and $x_2 \in \mathcal{M}_0$, we necessarily have

$$\mathcal{E}'(x) = \mathcal{E}'(x_1 + x_2) = \mathcal{E}'(x_1) = x_1,$$

since x_1 and x_2 are respectively in the range and in the kernel of $\mathcal{E}'$. Therefore, $\mathcal{E}' = \mathcal{E}$.

To conclude, we prove that

$$\operatorname{Ker}\mathcal{E} = \overline{\operatorname{span}}^{\,\text{w}^*}\{(I - \mathcal{T}_t)x : x \in \mathcal{M}, t \geqslant 0\}. \tag{1.2}$$

If $\mathcal{E}(x) = 0$, then $x = (I - \mathcal{E})x = \text{w}^*\text{-}\lim_t (I - \mathcal{T}_t)x$, so one inclusion holds. Conversely, if $x = (I - \mathcal{T}_t)y$ for some $y \in \mathcal{M}$ and $t \geqslant 0$, then $\mathcal{E}(x) = (I - \mathcal{T}_t)\mathcal{E}(x) = 0$ by relation $\mathcal{E} \circ \mathcal{T}_t = \mathcal{T}_t \circ \mathcal{E} = \mathcal{E}$. This proves that

$$\operatorname{span}\{(I - \mathcal{T}_t)x : x \in \mathcal{M}, t \geqslant 0\} \subset \operatorname{Ker}\mathcal{E} \subset \overline{\operatorname{span}}^{\,\text{w}^*}\{(I - \mathcal{T}_t)x : x \in \mathcal{M}, t \geqslant 0\},$$

and so equality (1.2) follows by the weak* closure of $\operatorname{Ker}\mathcal{E}$. In particular, $\operatorname{Ker}\mathcal{E}$ is a $\mathcal{T}_t$-invariant and $*$-invariant subspace of $\mathcal{M}$.

Assume now that $\mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y)$ for all $x, y \in \mathcal{M}$. Then clearly the subspace $\mathcal{F}(\mathcal{T})$ becomes an algebra. Conversely, if $\mathcal{F}(\mathcal{T})$ is an algebra, then $\mathcal{E}$ satisfies the property $\mathcal{E}(x\mathcal{E}(y)) = \mathcal{E}(x)\mathcal{E}(y)$ for all $x, y \in \mathcal{M}$ by Tomiyama's theorem (see Theorem 1 in [24]), i.e. $\mathcal{E}$ is a conditional expectation onto $\mathcal{F}(\mathcal{T})$. ∎

REMARK 1.2. Note that, in general, $\mathcal{F}(\mathcal{T})$ is not automatically an algebra, even if the semigroup is ergodic. But, for instance, the existence of a faithful normal invariant state is a sufficient condition to have that $\mathcal{F}(\mathcal{T})$ is an algebra (see Lemma 3 in [14]).

The obtained decomposition $\mathcal{M} = \mathcal{F}(\mathcal{T}) \oplus \mathcal{M}_0$ shows some similarities with EID, but, unfortunately, it does not automatically imply decoherence. Indeed, $\mathcal{F}(\mathcal{T})$ is not necessarily an algebra and, anyway, also when it is an algebra, the semigroup acts in a unitary way on $\mathcal{F}(\mathcal{T})$ (the action is trivial), but $\mathcal{F}(\mathcal{T})$ could be not the maximal algebra with this property. We want then to understand when such a decomposition is an EID decomposition.

In view of this, we analyze the relations between $\mathcal{F}(\mathcal{T})$ and the space

$$\mathcal{N}(\mathcal{T}) = \{a \in \mathcal{M} : \mathcal{T}_t(a^*a) = \mathcal{T}_t(a^*)\mathcal{T}_t(a), \mathcal{T}_t(aa^*) = \mathcal{T}_t(a)\mathcal{T}_t(a^*)\ \forall\, t \geqslant 0\}, \tag{1.3}$$

which has been introduced by Evans in [10] and then extensively studied precisely in relation to ergodic properties of dissipative semigroups (see also [15], [22]).

The following statement demonstrates some very interesting properties of $\mathcal{N}(\mathcal{T})$ (see Theorem 4 in [10] or Proposition 1 in [6] for the proof).

PROPOSITION 1.3. *$\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{M}$ on which the action of any $\mathcal{T}_t$ is a $*$-homomorphism. In particular $\mathcal{N}(\mathcal{T})$ is $\mathcal{T}_t$-invariant and we have*

$$\mathcal{T}_t(x^*y) = \mathcal{T}_t(x^*)\mathcal{T}_t(y) \quad \forall\, t \geqslant 0, \quad \textit{if either } x \textit{ or } y \textit{ belongs to } \mathcal{N}(\mathcal{T}). \tag{1.4}$$

By definition, it is clear that every decoherence-free space is contained in $\mathcal{N}(\mathcal{T})$, and then the following result follows.

COROLLARY 1.4. *If $\mathcal{F}(\mathcal{T})$ is an algebra, then it is contained in $\mathcal{N}(\mathcal{T})$.*

As we already recalled, when the semigroup possesses a faithful normal invariant state, the equality $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ is a sufficient condition to have ergodicity. Therefore, it is natural to wonder what happens about decoherence when the action of the semigroup on $\mathcal{N}(\mathcal{T})$ is trivial, i.e. $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$. In this case, the analysis of decoherence is considerably simplified, and we can give fairly satisfactory answers to the following general questions about decoherence:

(1) Is it possible to find necessary and sufficient conditions to have EID?

And, if EID holds,

(2) $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$? $\mathcal{M}_2 = \mathcal{M}_0$?
(3) There exists a conditional expectation onto $\mathcal{M}_1$?
(4) Is the decoherence decomposition unique?

In particular, in Corollary 1.8, we shall show that, under the condition $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$, EID is equivalent to the ergodicity of $\mathcal{T}$.

In the following *we will assume that $\mathcal{F}(\mathcal{T})$ does not coincide with all the algebra* $\mathcal{M}$ (this case is obviously trivial). As a consequence, since we suppose $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$, the condition $\mathcal{M}_2 \neq \{0\}$ required by EID definition is always satisfied.

PROPOSITION 1.5. *If $\mathcal{N}(\mathcal{T})$ is contained in $\mathcal{F}(\mathcal{T})$, then $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{M}$ on which the restriction of every $\mathcal{T}_t$ is a $*$-automorphism. In particular, the action of $\mathcal{T}_t$ is trivial on $\mathcal{N}(\mathcal{T})$.*

Proof. We know that $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann algebra on which every $\mathcal{T}_t$ acts as a $*$-homomorphism. When $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$, the restriction of $\mathcal{T}_t$ to $\mathcal{N}(\mathcal{T})$ is the identity operator, so the action is surely automorphic. ∎

THEOREM 1.6. *Assume $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$. Then the following facts are equivalent:*
(i) EID *holds;*
(ii) *there exists a normal conditional expectation $\mathcal{E}$ onto $\mathcal{N}(\mathcal{T})$ such that* Ker $\mathcal{E}=\mathcal{M}_0$.
In particular, in this case:

(a) *the* EID *decomposition is uniquely determined by* $\mathcal{M}_1 = \mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ *and* $\mathcal{M}_2 = \mathcal{M}_0$*;*

(b) $\mathcal{T}$ *is ergodic;*

(c) $\mathcal{E}$ *is the unique normal conditional expectation onto* $\mathcal{M}_1$ *such that* $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0 = \mathcal{M}_2$*, and it is given by* $\mathcal{E}(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x)$.

Proof. (ii) $\Rightarrow$ (i) If $\mathcal{E}$ is a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ such that $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0$, then we have $\mathcal{M} = \mathcal{N}(\mathcal{T}) \oplus \mathcal{M}_0$ with $\mathcal{M}_0$ a $\mathcal{T}_t$ and $*$-invariant weak* closed subspace of $\mathcal{M}$. Therefore, since $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ by Proposition 1.5, EID holds.

(i) $\Rightarrow$ (ii) If EID holds, we necessarily have $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ by Proposition 1.5 and by EID definition. Therefore, since $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$, for all $x = x_1 + x_2 \in \mathcal{M}$ with $x_i \in \mathcal{M}_i$, $i = 1, 2$, there exists

$$\text{w}^*\text{-}\lim_t \mathcal{T}_t(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x_1) = x_1,$$

i.e. the semigroup is ergodic. This proves that (i) implies condition (b). Moreover, for x in $\mathcal{F}(\mathcal{T})$, the same decomposition written above will give us $x = \mathcal{T}_t(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x) = x_1 \in \mathcal{N}(\mathcal{T})$, so we have the equality $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$. This means in particular that $\mathcal{F}(\mathcal{T})$ is an algebra and so, by Proposition 1.1, there is a conditional expectation $\mathcal{E}$ onto $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ such that $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0$.

Moreover, Propositions 1.1 and 1.5 assure the uniqueness of the EID decomposition, with $\mathcal{M}_1 = \mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ and $\mathcal{M}_2 = \mathcal{M}_0 = \operatorname{Ker}\mathcal{E}$, where $\mathcal{E}$ is the unique normal conditional expectation $\mathcal{E}$ onto $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ with $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0$, and it is given by $\mathcal{E}(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x)$. ∎

REMARK 1.7. Note that it is possible that $\mathcal{N}(\mathcal{T})$ is strictly contained in $\mathcal{F}(\mathcal{T})$ (see Example 2, item (4b) in [7]).

On the other hand, in general, the condition $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ alone does not imply EID (see Example 2.6).

Theorem 1.6 immediately gives that, if $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$, the condition "$\mathcal{F}(\mathcal{T})$ is an algebra" is necessary in order to have EID. By exploiting the proof of the same theorem, we obtain that it is also sufficient when the semigroup is ergodic.

COROLLARY 1.8. *Assume* $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$*. The following facts are equivalent:*

(i) $\mathcal{T}$ *is ergodic and* $\mathcal{F}(\mathcal{T})$ *is an algebra;*

(ii) EID *holds.*

We now want to find other equivalent conditions to EID. In order to search for these, in the following proposition we restrict to the case of the existence of a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$, since, by Theorem 1.6, this is a necessary condition to have EID when the inclusion $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$ holds.

PROPOSITION 1.9. *Assume* $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$ *and suppose there exists a normal conditional expectation* $\mathcal{E}$ *onto* $\mathcal{N}(\mathcal{T})$*. Then the following facts are equivalent:*

(i) $(\mathcal{T}_t(x))_{t\geqslant 0}$ *weakly* converges to* 0 *for all* $x \in \operatorname{Ker}\mathcal{E}$*;*
(ii) $(\mathcal{T}_{*t}(\varphi))_{t\geqslant 0}$ *weakly converges to* $\varphi \circ \mathcal{E}$ *for all normal states* φ*;*
(iii) $\mathcal{E}(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x)$ *for all* $x \in \mathcal{M}$*;*
(iv) $\operatorname{Ker}\mathcal{E} = \mathcal{M}_0$.
Moreover, if one of the previous conditions holds, then $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$*.*

Proof. The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i) are clear.

Now, note that $\mathcal{T}_t \circ \mathcal{E} = \mathcal{E}$ for all $t \geqslant 0$, being $\mathcal{T}_t$ the identical operator on $\mathcal{N}(\mathcal{T})$. Moreover, since for any x in $\mathcal{M}$ we have $\mathcal{E}(x) \in \mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$ and $(I-\mathcal{E})(x) \in \operatorname{Ker}\mathcal{E}$ (where I always denotes the identical operator on $\mathcal{M}$), we get

$$\mathcal{T}_t(x) = \mathcal{T}_t(\mathcal{E}(x)) + \mathcal{T}_t((I-\mathcal{E})(x)) = \mathcal{E}(x) + \mathcal{T}_t((I-\mathcal{E})(x)). \tag{1.5}$$

Therefore, equivalence between (i) and (iii) immediately follows, recalling that $(I-\mathcal{E})(\mathcal{M}) = \operatorname{Ker}\mathcal{E}$.

Now, take a normal state φ on $\mathcal{M}$, always using (1.5), we deduce

$$\mathcal{T}_{*t}(\varphi) = \varphi \circ \mathcal{T}_t = \varphi \circ \mathcal{E} + \varphi \circ \mathcal{T}_t \circ (1-\mathcal{E})$$

and the equivalence of (i) with (ii) easily follows.

Finally, if (iii) holds, then any $x \in \mathcal{F}(\mathcal{T})$ satisfies $x = \mathcal{E}(x)$, so it is in the range of $\mathcal{E}$, which coincides with $\mathcal{N}(\mathcal{T})$. This proves $\mathcal{F}(\mathcal{T}) \subset \mathcal{N}(\mathcal{T})$ and, since the opposite inclusion holds by hypothesis, equality follows. ∎

COROLLARY 1.10. *Assume* $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$*. Then* EID *holds if and only if there exists a normal conditional expectation onto* $\mathcal{N}(\mathcal{T})$ *satisfying one of the equivalent conditions* (i), (ii), (iii) *in Proposition* 1.9.

Proposition 1.9 has an important consequence relating decoherence and invariant states.

COROLLARY 1.11. *Assume* $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$*, then the semigroup can display decoherence only if it has at least an invariant state.*

Proof. By Corollary 1.10, we know that, if decoherence takes place, then there exists a conditional expectation $\mathcal{E}$ on $\mathcal{N}(\mathcal{T})$ such that $(\mathcal{T}_{*t}(\varphi))_t$ weakly converges to $\varphi \circ \mathcal{E}$ for all normal states φ. This assures that all the states of the form $\varphi \circ \mathcal{E}$ are invariant. ∎

We conclude this section by a remark about the relationships between the vector space $\mathcal{M}_0$ and the "orthogonal"of $\mathcal{N}(\mathcal{T})$ with respect to an invariant state.

REMARK 1.12. Assume that $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$ and ω is an invariant state of the semigroup. Then

$$\mathcal{M}_0 \subset \mathcal{N}(\mathcal{T})^{\perp,\omega} := \{x \in \mathcal{M} : \omega(y^*x) = 0 \text{ for all } y \in \mathcal{N}(\mathcal{T})\}$$

Indeed, if we consider $y \in \mathcal{N}(\mathcal{T})$, $x \in \mathcal{M}_0$, then

$$\omega(y^*x) = \omega(\mathcal{T}_t(y^*x)) = \omega(\mathcal{T}_t(y^*)\mathcal{T}_t(x)) = \omega(y^*\mathcal{T}_t(x)) \to_t 0.$$

In the following subsections, we consider the cases when $\mathcal{N}(\mathcal{T})$ is trivial and/or there exists an invariant faithful normal state for the semigroup.

1.1. SEMIGROUPS WITH $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$. By assuming that $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$, we have an easy way to write a conditional expectation onto the trivial algebra $\mathbb{C}\mathbb{1}$, by which we can improve the results of Theorem 1.6 and Proposition 1.9 and give sufficient and necessary conditions for EID.

PROPOSITION 1.13. *Consider the algebra $\mathcal{A} = \mathbb{C}\mathbb{1}$. A linear map $\mathcal{E} : \mathcal{M} \to \mathcal{A}$ is a conditional expectation onto $\mathcal{A}$ if and only if it is of the form*

$$\mathcal{E}(x) = \omega(x)\mathbb{1} \quad \forall\, x \in \mathcal{M} \tag{1.6}$$

for some state ω on $\mathcal{M}$. Moreover, the following facts hold:

(i) *$\mathcal{E}$ is normal if and only if ω is;*

(ii) *ω is the only compatible state with $\mathcal{E}$, i.e. $\omega = \omega \circ \mathcal{E}$;*

(iii) $\operatorname{Ker}\omega = \operatorname{Ker}\mathcal{E}$;

(iv) $\operatorname{Ker}\mathcal{E}$ *is $\mathcal{T}_t$-invariant for all $t \geqslant 0$ if and only if ω is an invariant state for the semigroup $\mathcal{T}$.*

Proof. Let $\mathcal{E}$ be a conditional expectation onto $\mathcal{A}$; for all $x \in \mathcal{M}$, let $\omega(x) \in \mathbb{C}$ be such that $\mathcal{E}(x) = \omega(x)\mathbb{1}$. By construction, the map $\omega : \mathcal{M} \to \mathbb{C}$ is a linear and positive functional satisfying $\omega(\mathbb{1}) = \mathbb{1}$, i.e. ω is a state on $\mathcal{M}$.

On the other hand, if we consider a state ω and introduce a map $\mathcal{E}$ defined as in (1.6), $\mathcal{E}$ is clearly a positive and linear operator satisfying $\mathcal{E}^2 = \mathcal{E}$ and $\mathcal{E}(\mathbb{1}) = \mathbb{1}$. Therefore, $\|\mathcal{E}\| = \|\mathcal{E}(\mathbb{1})\| = 1$, i.e. $\mathcal{E}$ is a conditional expectation onto $\mathbb{C}\mathbb{1}$.

Statements (i)–(iii) trivially follow by definition of $\mathcal{E}$.

For the proof of point (iv), we can remember that $\operatorname{Ker}\mathcal{E} = (I - \mathcal{E})(\mathcal{M})$, so $\operatorname{Ker}\mathcal{E}$ is preserved by $\mathcal{T}_t$ if and only if $\mathcal{E}(\mathcal{T}_t((I - \mathcal{E})(x))) = 0$ for all x, so if and only if $\mathcal{E}\mathcal{T}_t(x) = \mathcal{E}\mathcal{T}_t\mathcal{E}(x)$ for all x. Since

$$\mathcal{E}\mathcal{T}_t(x) = \omega(\mathcal{T}_t(x))\mathbb{1} \quad \text{and} \quad \mathcal{E}\mathcal{T}_t\mathcal{E}(x) = \omega(x)\mathbb{1} \quad \forall\, x \in \mathcal{M}, \tag{1.7}$$

the preservation of the kernel of $\mathcal{E}$ by $\mathcal{T}_t$ is equivalent to the invariance of ω under $\mathcal{T}_t$, so that the statement is proved. ∎

REMARK 1.14. Note that, if $\mathcal{N}(\mathcal{T})$ is the image of a normal conditional expectation compatible with a normal invariant state ω but $\mathcal{N}(\mathcal{T}) \neq \mathbb{C}\mathbb{1}$, the equality $\operatorname{Ker}\mathcal{E} = \operatorname{Ker}\omega$ does not hold true in general. An example is given by the two-photon absorption and emission process; see next section, and in particular Remark 2.4.

THEOREM 1.15. *Assume $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$. Then the following facts are equivalent:*

(i) EID *holds;*

(ii) *there exists a normal state ω such that $(\mathcal{T}_t(x))_{t\geqslant 0}$ weakly* converges to 0 for all $x \in \operatorname{Ker}\omega$;*

(iii) *there exists a normal state ω which is the weak limit of $(\mathcal{T}_{*t}(\varphi))_{t\geqslant 0}$ for all normal states φ;*

(iv) *there exists a normal state ω such that* $\text{w}^*\text{-}\lim_t \mathcal{T}_t(x) = \omega(x)\mathbb{1}$ *for all $x \in \mathcal{M}$.*

Moreover, if one of these conditions is satisfied, then ω is the unique normal invariant state, $\mathcal{M}_1 = \mathbb{C}\mathbb{1}$ and $\mathcal{M}_2 = \operatorname{Ker}\omega$.

Proof. (ii), (iii) and (iv) are equivalent thanks to Proposition 1.9, since every normal conditional expectation $\mathcal{E}$ onto $\mathbb{C}\mathbb{1}$ is given by equation (1.6) for some normal state ω (see Proposition 1.13) and $\varphi(\mathcal{E}(x)) = \omega(x)\varphi(\mathbb{1}) = \omega(x)$ for all states φ.

Moreover, since $\operatorname{Ker}\omega = \operatorname{Ker}\mathcal{E} = \mathcal{M}_0$ by Propositions 1.9 and 1.13, statements (i) and (ii) are equivalent by Theorem 1.6 and we have $\mathcal{M}_1 = \mathbb{C}\mathbb{1}$ and $\mathcal{M}_2 = \operatorname{Ker}\omega$.

If conditions (i)–(iv) are satisfied, then we have $\varphi = \text{w-}\lim_t \mathcal{T}_{*t}(\varphi) = \omega$ for every normal invariant state φ, i.e. ω is the unique normal invariant state. ∎

REMARK 1.16. In general, it is not true that if the semigroup has a unique invariant state, then it displays decoherence (see Example 2.6).

Moreover, we will see in the next subsection that the existence of a faithful normal invariant state implies EID (see Theorem 1.17). However, in general, this condition is not necessary to have decoherence, even if $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$. Indeed, we describe a generic semigroup displaying decoherence even if its unique invariant state is not faithful (see Proposition 2.5).

1.2. SEMIGROUPS WITH AN INVARIANT FAITHFUL STATE.

In this case, $\mathcal{N}(\mathcal{T})$ always includes $\mathcal{F}(\mathcal{T})$, since $\mathcal{F}(\mathcal{T})$ is an algebra and the semigroup is surely automorphic on it. So, the condition $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$ implies $\mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$, and decoherence and ergodicity are equivalent (see Corollary 1.8).

THEOREM 1.17. *Let ω be a faithful normal invariant state for $\mathcal{T}$ and assume $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$. Then:*

(i) $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$.

(ii) *(Frigerio–Verri) There exists a unique normal conditional expectation $\mathcal{E}$ onto $\mathcal{N}(\mathcal{T})$ which is compatible with ω. In particular, $\mathcal{E}$ is defined by*

$$\mathcal{E}(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x), \quad x \in \mathcal{M}.$$

(iii) EID *holds and the decomposition is unique with $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ and $\mathcal{M}_2 = \mathcal{M}_0 = \mathcal{N}(\mathcal{T})^{\perp,\omega}$.*

Proof. Since $\mathcal{T}$ possesses a faithful normal invariant state, Frigerio–Verri theorem [16] assures the existence of a normal conditional expectation $\mathcal{E}$ onto $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$, which is also $\mathcal{T}_t$-invariant and given by

$$\mathcal{E}(x) = \text{w}^*\text{-}\lim_t \mathcal{T}_t(x), \quad x \in \mathcal{M}. \tag{1.8}$$

In particular, we have $\varphi \circ \mathcal{E} = \varphi$ for every normal invariant state φ. Note that if $\mathcal{E}'$ is a normal conditional expectation onto $\mathcal{N}(\mathcal{T})$ which is compatible with ω, then $\mathrm{Ker}\,\mathcal{E}' = \mathcal{N}(\mathcal{T})^{\perp,\omega}$, and this proves the uniqueness of $\mathcal{E}$. Indeed, if $x \in \mathrm{Ker}\,\mathcal{E}'$ and $y \in \mathcal{N}(\mathcal{T})$, then

$$\omega(y^*x) = \omega(\mathcal{E}'(y^*x)) = \omega(y^*\mathcal{E}'(x)) = 0.$$

On the other hand, if $x \in \mathcal{N}(\mathcal{T})^{\perp,\omega}$, then $\omega(y^*x) = 0$ for each $y \in \mathcal{N}(\mathcal{T})$. In particular, since $\mathcal{E}'$ is compatible with ω,

$$\omega(\mathcal{E}'(x)^*x) = \omega(\mathcal{E}'(x)^*\mathcal{E}'(x)) = 0,$$

so $\mathcal{E}'(x) = 0$, i.e. $x \in \mathrm{Ker}\,\mathcal{E}'$.

Moreover, Theorem 1.6 implies that EID holds with $\mathcal{M}_1 = \mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T})$ and $\mathcal{M}_2 = \mathcal{M}_0 = \mathrm{Ker}\,\mathcal{E} = \mathcal{N}(\mathcal{T})^{\perp,\omega}$. ∎

If we assume $\mathcal{M} = \mathcal{B}(\mathsf{h})$ and $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$, we can say something more about the behaviour of off-diagonal terms of the dynamics of a density operator. By a density operator ρ, we mean a nonnegative operator with trace 1. Any normal state ω on $\mathcal{B}(\mathsf{h})$ can therefore be written in a unique way as $\omega(x) = \mathrm{tr}(\rho x)$. Thus, in the following, we shall identify normal states and their densities.

COROLLARY 1.18. *Assume $\mathcal{M} = \mathcal{B}(\mathsf{h})$, $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$, and let ρ be a faithful invariant state for $\mathcal{T}$. If $(e_n)_n$ is an* onb *of* h *given by eigenvectors of ρ, then, for every normal state φ, we have*

$$\langle e_n, \mathcal{T}_{*t}(\varphi)e_m\rangle \to_t 0 \quad \forall\, n \neq m. \tag{1.9}$$

Proof. Since there exists a faithful invariant state, we have $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$ by item (i) of Theorem 1.17. Therefore, there exists $\mathcal{E} : \mathcal{B}(\mathsf{h}) \to \mathcal{N}(\mathcal{T})$ normal conditional expectation given by formula (1.8) and satisfying $\mathrm{tr}(\rho\mathcal{E}(x)) = \mathrm{tr}(\rho x)$. The triviality of $\mathcal{N}(\mathcal{T})$ clearly implies $\mathcal{E}(x) = \mathrm{tr}(\rho x)\mathbb{1}$ for all $x \in \mathcal{B}(\mathsf{h})$, so that, for all normal states φ, we have

$$\begin{aligned}\lim_t \langle e_n, \mathcal{T}_{*t}(\varphi)e_m\rangle &= \lim_t \mathrm{tr}((\mathcal{T}_{*t}(\varphi))(|e_m\rangle\langle e_n|)) = \lim_t \mathrm{tr}(\varphi\mathcal{T}_t(|e_m\rangle\langle e_n|)) \\ &= \mathrm{tr}(\varphi\mathcal{E}(|e_m\rangle\langle e_n|)) = \mathrm{tr}(\rho|e_m\rangle\langle e_n|) = 0.\end{aligned}$$

for all $n \neq m$. ∎

REMARK 1.19. Our analysis always starts by the definition of decoherence given by Blanchard and Olkiewicz, but in fact the term decoherence can be used with different meanings; the original interpretation surely indicates the loss of coherences. Indeed, given a density matrix ρ, its dynamics (Schrödinger picture) is given by the predual semigroup $\rho_t = \mathcal{T}_{*t}\rho$. If we select an orthonormal basis $(e_n)_n$ of h, each density matrix ρ is characterized by its components $\rho(m,n) = \langle e_m, \rho e_n\rangle$. The off-diagonal terms $\rho_t(m,n)$, with $n \neq m$, are called the coherences of ρ. The intuitive idea is that decoherence consists of the disappearance of these terms as time increases, that is $\rho_t(m,n)$ goes to 0 as t tends to ∞, for $m \neq n$.

So that, for large times, the evolution of states becomes essentially described by diagonal matrices with respect to a privileged bases. Rebolledo proposed in [21] an alternative mathematical definition of decoherence which tries to catch this aspect.

The previous result assures that, if $\mathcal{M} = \mathcal{B}(\mathsf{h})$, $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$ and there exists a faithful normal invariant state, then EID implies the loss of coherences, in the sense explained above, and the privileged basis is determined by the invariant state.

Finally, when there exists a faithful normal invariant state, different authors (see [3] and [17]) have already underlined that it is natural to study the relation of EID with the so called Jacobs–DeLeeuw–Glicksberg splitting (see in particular Corollary 3.3 and Propositions 3.3 and 3.6 in [17]).

THEOREM 1.20 (Jacobs–DeLeeuw–Glicksberg splitting). *If there exists a faithful normal invariant state ω, then*

$$V_0 := \overline{\operatorname{span}}^{w^*}\{x \in \mathcal{M} : \exists\, \lambda \in \mathbb{R} \textit{ such that } \mathcal{T}_t(x) = e^{i\lambda t}\ \forall\, t \geqslant 0\}$$

is a von Neumann subalgebra of $\mathcal{M}$ and we have $\mathcal{M} = V_0 \oplus \mathfrak{M}_s$ with

$$\mathfrak{M}_s := \{x \in \mathcal{M} : 0 \in \overline{\{\mathcal{T}_t(x)\}_{t\geqslant 0}}^{w^*}\}$$

a weak Banach subspace which is $*$-invariant and $\mathcal{T}_t$ invariant.*

Moreover, the action of every $\mathcal{T}_t$ on V_0 is a $*$-automorphism, and V_0 is the image of a normal conditional expectation Q compatible with ω.

Finally, when $V_0 = \mathcal{N}(\mathcal{T})$ we have $\mathfrak{M}_s = \mathcal{M}_0$.

In the case we have studied, also this aspect is now clear.

COROLLARY 1.21. *If $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$ and the semigroup has a faithful normal invariant state, then $\mathcal{N}(\mathcal{T}) = V_0$ and $\mathfrak{M}_s = \mathcal{M}_0$, so* EID *and Jacobs–DeLeeuw–Glicksberg decompositions coincide. In particular, the conditional expectation Q appearing in Theorem* 1.20 *is the orthogonal projection onto $\mathcal{N}(\mathcal{T})$ with respect to the scalar product induced by the faithful invariant state.*

Proof. Since there is a faithful normal invariant state, we have $\mathcal{F}(\mathcal{T}) \subset V_0 \subset \mathcal{N}(\mathcal{T})$, and these three spaces have to coincide by the assumption $\mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T})$. Moreover, Theorem 1.17 assures EID with $\mathcal{M}_2 = \mathcal{M}_0$. Finally, the equality $\mathfrak{M}_s = \mathcal{M}_0 = \mathcal{M}_2$ follows by Theorem 1.20, since in general we have $\mathcal{M}_0 \subset \mathfrak{M}_s$. ∎

2. MINIMAL QUANTUM DYNAMICAL SEMIGROUPS

In this section, we describe some cases when our results can be easily applied. The class of uniformly continuous semigroups is already interesting, but it is sometimes too small for the applications in quantum probability and mathematical physics: in general, the generator $\mathcal{L}$ can be given formally in a "generalized" Lindblad form with unbounded operators G and $(L_k)_k$ (see formulas before). In this case, under some assumptions, it is possible to define a quantum dynamical semigroup associated with $\mathcal{L}$ and called *minimal QDS*. For these QDSs, with possibly unbounded generator, we recall the essential definitions and some results, in particular about the characterization of the space $\mathcal{N}(\mathcal{T})$ starting from the canonical form of the generator. This is obviously a central point here, because the results of the previous section are based on certain assumptions on $\mathcal{N}(\mathcal{T})$, so it is crucial that we can have a way to determine this space.

Following the construction given by Fagnola in [11], let G and $(L_k)_{k\geqslant 1}$ be operators in h satisfying the following hypothesis (by $D(G)$ we denote the domain of G and similarly for other operators):

(H-1) G is the infinitesimal generator of a strongly continuous contraction semigroup in h, $\mathrm{D}(G) \subset \mathrm{D}(L_k)$ for all $k \geqslant 1$, and for all $v, u \in \mathrm{D}(G)$,

$$\langle v, Gu\rangle + \sum_{k\geqslant 1} \langle L_k v, L_k u\rangle + \langle Gv, u\rangle = 0.$$

For each $x \in \mathcal{B}(\mathsf{h})$ we define the sesquilinear form $\mathfrak{L}(x)$ in h with domain $\mathrm{D}(G) \times \mathrm{D}(G)$ by setting

$$\mathfrak{L}(x)(v,u) = \langle v, xGu\rangle + \sum_{k\geqslant 1} \langle L_k v, xL_k u\rangle + \langle Gv, xu\rangle. \tag{2.1}$$

If (H-1) holds, then there exists a minimal QDS $(\mathcal{T}_t^{(\min)})_{t\geqslant 0}$ on $\mathcal{B}(\mathsf{h})$ that is sub-Markov (i.e. $\mathcal{T}_t^{(\min)}(\mathbb{1}) \leqslant \mathbb{1}$) and such that:

$$\langle v, \mathcal{T}_t^{(\min)}(x)u\rangle = \langle v, xu\rangle + \int_0^t \mathfrak{L}(\mathcal{T}_s^{(\min)}(x))(v,u)\mathrm{d}s \quad \forall\, v, u \in \mathrm{D}(G). \tag{2.2}$$

In addition, if $\mathcal{T}^{(\min)}$ is Markov, then the following facts hold:

(i) $\mathcal{T}^{(\min)}$ is the unique QDS on $\mathcal{B}(\mathsf{h})$ satisfying (2.2).

(ii) The domain of the infinitesimal generator $\mathcal{L}^{(\min)}$ of $\mathcal{T}^{(\min)}$ is given by all elements $x \in \mathcal{B}(\mathsf{h})$ such that the sesquilinear form $(v,u) \mapsto \mathfrak{L}(x)(v,u)$ is norm continuous on $\mathrm{D}(G) \times \mathrm{D}(G)$. In this case, we have $\langle v, \mathcal{L}(x)u\rangle = \mathfrak{L}(x)(v,u)$ for all $x \in \mathrm{D}(\mathcal{L})$ and $(v,u) \in \mathrm{D}(G) \times \mathrm{D}(G)$.

For a QMS on $\mathcal{B}(\mathsf{h})$ with an unbounded generator represented in the generalized canonical form (2.1), it is possible to obtain a characterization of $\mathcal{N}(\mathcal{T})$ based on the generalized commutator of the set

$$\mathcal{D}(\mathcal{T}) := \{\mathrm{e}^{-\mathrm{i}tH} L_k \mathrm{e}^{\mathrm{i}tH}, \mathrm{e}^{-\mathrm{i}tH} L_k^* \mathrm{e}^{\mathrm{i}tH} : k \geqslant 1, t \geqslant 0\}, \tag{2.3}$$

where H is the Hamiltonian of the system, i.e. the operator defined by $Hu := (\mathrm{i}/2)(Gu - G^*u)$ for $u \in \mathrm{D}(G) \cap \mathrm{D}(G^*)$.

But the sole hypothesis (H-1) is not sufficient to make sense of the operators $e^{-itH}L_k e^{itH}$, so, following [9], we shall also assume that

(H-2) there exists a linear manifold D dense in h, $D \subset \mathrm{D}(G) \cap \mathrm{D}(G^*)$, which is a core for G and

(i) L_k is closed and D is a core for every L_k,

(ii) H is essentially self-adjoint on D and the unitary group $(e^{itH})_{t\in\mathbb{R}}$ generated by iH satisfies $e^{itH}(D) \subset \mathrm{D}(G)$ for all $t \in \mathbb{R}$,

(iii) the operator G_0 given by $G_0 u = (Gu + G^* u)/2$ for $u \in D$, is essentially self-adjoint and $\mathrm{D}(G) \subset \mathrm{D}(G_0) \subset \mathrm{D}(L_k)$ for every $k \geqslant 1$.

THEOREM 2.1 ([9], Theorem 3.2). *Let $\mathcal{T}$ be the minimal* QDS *associated with the operators G and L_k satisfying* (H-1) *and* (H-2). *If $\mathcal{T}$ is Markov, then we have*

$$\mathcal{T}_t(x) = e^{itH} x e^{-itH} \quad \forall\, x \in \mathcal{N}(\mathcal{T}).$$

In particular, $\mathcal{N}(\mathcal{T})$ is contained in the generalized commutator of $\mathcal{D}(\mathcal{T})$.

From this theorem and Proposition 1.3, we immediately deduce

COROLLARY 2.2. *Suppose all the assumptions of previous theorem hold, then $\mathcal{N}(\mathcal{T})$ is the biggest von Neumann subalgebra of $\mathcal{B}(\mathsf{h})$ on which $\mathcal{T}$ acts as a $*$-automorphism.*

Always in [9], it is proved that really equality $\mathcal{N}(\mathcal{T}) = \mathcal{D}(\mathcal{T})'$ holds under some additional assumptions. We do not write the technical details here since we shall not need this result.

2.1. TWO-PHOTON ABSORPTION AND EMISSION PROCESS.

We choose the Hilbert system space $\mathsf{h} = \ell^2(\mathbb{N})$, the space of complex-valued square summable sequences, with canonical orthonormal basis $(e_n)_{n\geqslant 0}$. The usual annihilation and creation operators a, a^+ can then be defined by $ae_0 = 0, ae_n = \sqrt{n}e_{n-1}$ for $n \geqslant 1$, $a^+ e_n = \sqrt{n+1}e_{n+1}$.

Here we consider the quantum Markov semigroup generated by the unbounded operator (which should be read as a bilinear form, as explained before)

$$\mathfrak{L}(x) = i\kappa[a^{+2}a^2, x] - \frac{\mu^2}{2}(a^{+2}a^2 x - 2a^{+2}xa^2 + xa^{+2}a^2) - \frac{\lambda^2}{2}(a^2 a^{+2} x - 2a^2 x a^{+2} + x a^2 a^{+2}) \tag{2.4}$$

where κ is a real constant and μ^2 (respectively λ^2) is the absorption (respectively emission) rate, $0 < \lambda^2 < \mu^2$.

The generator (2.4) can be written in canonical form (2.1) with $L_1 = \mu a^2$, $L_2 = \lambda a^{+2}$ and $L_k = 0$ for $k \geqslant 3$. The associated operators H and G will then be given by $H = \kappa a^{+2}a^2$, $G = -(1/2)\sum L_k^* L_k - iH$.

The associated process is called the *two-photon absorption and emission process* when $\lambda > 0$, while, for $\lambda = 0$, we would obtain the two-photon absorption process, that we shall not consider here. We refer to [12] and references therein for the construction of the quantum Markov semigroup $\mathcal{T}$ associated with the formal generator (2.4) by the minimal semigroup method and the characterization of invariant states. At positive temperature, i.e. $\lambda > 0$ (see [4]), there exist infinitely many commuting invariant states ρ_α, $\alpha \in [0,1]$, which are convex combinations $\rho_\alpha = \alpha\rho_{\rm e} + (1-\alpha)\rho_{\rm o}$ of the even and odd extremal invariant states

$$\rho_{\rm e} = (1-\nu^2)\sum_{k\geqslant 0}\nu^{2k}|e_{2k}\rangle\langle e_{2k}|, \quad \rho_{\rm o} = (1-\nu^2)\sum_{k\geqslant 0}\nu^{2k}|e_{2k+1}\rangle\langle e_{2k+1}|, \quad \nu = \frac{\lambda}{\mu}.$$

All these states are equilibrium (detailed balance) states and any initial state σ_0 converges to an invariant state as time goes to infinity (essentially by an application of the well-known Frigerio–Verri result but see always [12] for details).

We shall denote by $p_{\rm e}$ and $p_{\rm o}$ the support projections of the invariant states $\rho_{\rm e}$ and $\rho_{\rm o}$, respectively, i.e.,

$$p_{\rm e} = \sum_{k\geqslant 0}|e_{2k}\rangle\langle e_{2k}|, \quad p_{\rm o} = \sum_{k\geqslant 0}|e_{2k+1}\rangle\langle e_{2k+1}|.$$

PROPOSITION 2.3. *The two-photon absorption and emission semigroup displays decoherence with*

$$\mathcal{M}_1 = \mathcal{N}(\mathcal{T}) = \mathcal{F}(\mathcal{T}) = \operatorname{span}\{p_{\rm e}, p_{\rm o}\},$$
$$\mathcal{M}_2 = \mathcal{M}_0 = \mathcal{N}(\mathcal{T})^{\perp,\rho} = \{x \in \mathcal{M} : \operatorname{tr}(\rho_{\rm e}x) = \operatorname{tr}(\rho_{\rm o}x) = 0\},$$

where ρ is any invariant faithful state of the semigroup.

Proof. By [9], Subsection 5.1 and Corollary 3.1, for this model we have

$$\operatorname{span}\{p_{\rm e}, p_{\rm o}\} = \mathcal{D}(\mathcal{T})' = \mathcal{N}(\mathcal{T}) \subset \mathcal{F}(\mathcal{T}) = \{H, L_1, L_2\}',$$

so $\mathcal{F}(\mathcal{T}) = \{L_1, L_2\}' \subset \mathcal{D}(\mathcal{T})'$. Then the equality $\mathcal{F}(\mathcal{T}) = \mathcal{N}(\mathcal{T})$ easily follows.

Since the semigroup has an invariant faithful density ρ, it satisfies the conditions of Theorem 1.17 and this implies EID with $\mathcal{M}_1 = \mathcal{N}(\mathcal{T})$ and $\mathcal{M}_2 = \mathcal{M}_0 = \mathcal{N}(\mathcal{T})^{\perp,\rho}$. Moreover, Remark 1.12 tells us that $\mathcal{M}_0 \subset \mathcal{N}(\mathcal{T})^{\perp,\omega}$ for every invariant state ω and so, here, in particular $\mathcal{M}_0 = \mathcal{N}(\mathcal{T})^{\perp,\rho} \subset \mathcal{N}(\mathcal{T})^{\perp,\rho_{\rm e}} \cap \mathcal{N}(\mathcal{T})^{\perp,\rho_{\rm o}}$. But the last inclusion easily turns out to be an equality since we know that $\rho = \rho_\alpha = \alpha\rho_{\rm e} + (1-\alpha)\rho_{\rm o}$ for some $\alpha \in (0,1)$ and

$$\begin{aligned} x \in \mathcal{N}(\mathcal{T})^{\perp,\rho_{\rm e}} \cap \mathcal{N}(\mathcal{T})^{\perp,\rho_{\rm o}} &\Leftrightarrow \operatorname{tr}(\rho_{\rm e}p_{\rm e}x) = \operatorname{tr}(\rho_{\rm e}p_{\rm o}x) = \operatorname{tr}(\rho_{\rm o}p_{\rm e}x) = \operatorname{tr}(\rho_{\rm o}p_{\rm o}x) = 0 \\ &\Leftrightarrow \operatorname{tr}(\rho_{\rm e}x) = \operatorname{tr}(\rho_{\rm o}x) = 0, \end{aligned}$$

where the last equivalence follows by the fact that $\rho_k p_k = \rho_k$ and $\rho_k p_j = 0$ for $k \neq j$, $k, j \in \{{\rm o}, {\rm e}\}$. Then, for $x \in \mathcal{N}(\mathcal{T})^{\perp,\rho_{\rm e}} \cap \mathcal{N}(\mathcal{T})^{\perp,\rho_{\rm o}}$ and $k \in \{{\rm o}, {\rm e}\}$, $\operatorname{tr}(\rho p_k x) = \alpha\operatorname{tr}(\rho_{\rm e}p_k x) + (1-\alpha)\operatorname{tr}(\rho_{\rm o}p_k x) = 0$, so $x \in \mathcal{N}(\mathcal{T})^{\perp,\rho}$. We deduce

$$\mathcal{M}_2 = \mathcal{M}_0 = \mathcal{N}(\mathcal{T})^{\perp,\rho} = \{x \in \mathcal{M} : \operatorname{tr}(\rho_{\rm e}x) = \operatorname{tr}(\rho_{\rm o}x) = 0\}. \quad ∎$$

Moreover, in this case, we can state further that decoherence takes place with exponential speed since the action of the semigroup on the set $\mathcal{M}_2$ of non-detectable observables converges to 0 uniformly exponentially fast in a suitable L^2 norm induced by an invariant faithful state of the system, i.e. we have a spectral gap inequality (see [4]).

REMARK 2.4. In relation with Remark 1.14, we can underline that, in this two-photon absorption and emission process, the normal conditional expectation $\mathcal{E}$ onto $\mathcal{N}(\mathcal{T})$ compatible with ρ_α (given by Theorem 1.17) is such that $\mathrm{Ker}\,\mathcal{E} \neq \mathrm{Ker}\,\rho_\alpha$. Indeed, if we consider

$$x = |e_0\rangle\langle e_0| - \alpha(1-\alpha)^{-1}|e_1\rangle\langle e_1|,$$

then we have $\mathrm{tr}(\rho_\alpha x) = 0$ (i.e. x belongs to the kernel of the normal state defined by ρ_α) but $\mathrm{tr}(\rho_\alpha p_e x) = \alpha(1-\nu^2) \neq 0$, i.e. $x \notin \mathcal{N}(\mathcal{T})^{\perp,\rho_\alpha} = \mathrm{Ker}\,\mathcal{E}$.

2.2. GENERIC SEMIGROUPS. Take again $\mathsf{h} = l^2(\mathbb{N})$ and $\mathcal{M} = \mathcal{B}(\mathsf{h})$, with $(e_n)_n$ the canonical orthonormal basis of h as in the previous example. We consider a class of form generators which can be represented in the generalized canonical form

$$\mathfrak{L}(x) = G^* x + \sum_{\{j,m\in\mathbb{N}|j\neq m\}} L^*_{mj} x L_{mj} + xG$$

where

$$G = -\sum_{m\in\mathbb{N}} \Big(-\frac{\gamma_{mm}}{2} + \mathrm{i}\kappa_m \Big) |e_m\rangle\langle e_m|, \quad L_{mj} = \sqrt{\gamma_{mj}}|e_j\rangle\langle e_m| \quad \text{for } j \neq m,$$

with $\kappa_m \in \mathbb{R}$ and $\gamma_{mj} \geqslant 0$ for every $m \neq j$ such that $\gamma_{mm} := -\sum_{j\neq m} \gamma_{mj}$ is finite for any m.

Notice that the operators L_{mj} are bounded, so their domain coincide with h, while G is not necessarily bounded, its domain is the set

$$D(G) = \Big\{ u \in \mathsf{h} : \sum_k (\gamma^2_{mm} + \kappa^2_m)|u_m|^2 < +\infty \Big\},$$

which is obviously dense in h since it contains all the sequences with a finite number of non-zero elements. Anyway hypothesis (H-1) is always verified for these forms. So, by using the standard techniques described at the beginning of this section, we can introduce an operator $\mathcal{L}$ that generates a minimal QDS $\mathcal{T} = \mathcal{T}^{(\min)}$ associated with the form $\mathfrak{L}$. Moreover, if we call $\mathcal{D}$ the algebra of diagonal bounded operators, we have that $\mathcal{T}_t(\mathcal{D}) \subset \mathcal{D}$ and the restriction of the generator $\mathcal{L}$ to $\mathcal{D}$ is completely described by the classical minimal sub-Markov generator Γ defined on $l^\infty(\mathbb{N})$ by (see [5])

$$\Gamma f = \sum_{j,k\in\mathbb{N}} \gamma_{kj} f(j) e_k.$$

In particular, $\mathcal{T}$ is Markov if and only if the classical process generated by Γ is conservative.

Notice that also hypothesis (H-2) is verified, for instance taking $D = D(G)$, which is a core for H and G_0 in this case.

We address the semigroups of this form as generic semigroups after the definition introduced by Accardi and Kozyrev in [1] (even if the semigroups described here are a generalization of their original family). In this class, we determine a subset of semigroups without normal faithful invariant states and displaying decoherence.

PROPOSITION 2.5. *Suppose that the classical process associated with Γ is conservative and has the set $\{0\}$ as an absorbing class and an attractor; moreover, assume $\inf_{k\geqslant 1}|\gamma_{kk}| > 0$. Then the associated* QMS $\mathcal{T}$ *has the unique normal invariant state $\rho = |e_0\rangle\langle e_0|$ and displays decoherence with*

$$\mathcal{M}_1 = \mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}, \quad \mathcal{M}_2 = \mathcal{M}_0 = \{x \in \mathcal{M} : \mathrm{tr}(\rho x) = 0\}.$$

By $\{0\}$ *absorbing state* we mean that $\mathbb{P}\{X_t = 0 | X_0 = 0\} = 1$ for all $t \geqslant 0$; by $\{0\}$ *attractor* we mean that the set $\{0\}$ is visited infinitely often starting from any initial measure, i.e., in this case, $\lim_t \mathbb{P}\{X_t = 0 | X_0 \sim \pi\} = 1$ for any probability measure π on $\mathbb{N}$. We can obtain a generator Γ satisfying the conditions of the theorem for instance by choosing

$$\gamma_{00} = 0, \quad \gamma_{j,j+1} = a \text{ and } \gamma_{j,j-1} = b \ \forall j \geqslant 1, \quad \gamma_{jk} = 0 \text{ for } |j-k| \geqslant 2$$

with $0 < a < b$ constants.

Proof. First we want to determine the normal invariant states of the semigroup. Notice that, for $l \neq n$, $|e_l\rangle\langle e_n|$ is in the domain of the generator, since $\mathcal{T}$ is Markov and so we can use the characterization of the domain of $\mathcal{L}$ recalled at the beginning of this section. We can compute

$$\mathcal{L}(|e_l\rangle\langle e_n|) = \Big(\frac{\gamma_{ll} + \gamma_{nn}}{2} + \mathrm{i}(\kappa_l - \kappa_n)\Big)|e_l\rangle\langle e_n|.$$

Now, if ρ is an invariant state, then

$$0 = \mathrm{tr}(\rho\mathcal{L}(|e_l\rangle\langle e_n|)) = \Big(\frac{\gamma_{ll} + \gamma_{nn}}{2} + \mathrm{i}(\kappa_l - \kappa_n)\Big)\rho_{nl} \quad \text{for } l \neq n.$$

This implies ρ is diagonal, because, if, by contradiction, $\rho_{nl} \neq 0$ for $l \neq n$, we obtain $\gamma_{ll} = \gamma_{nn} = 0$ and this is not possible since $\inf_{k\geqslant 1}|\gamma_{kk}| > 0$.

Consequently any invariant state ρ is diagonal, $\rho = \sum \rho_k |e_k\rangle\langle e_k|$, and the invariance property is fulfilled if and only if the vector $(\rho_k)_k$ is an invariant density for the classical process, i.e. if and only if $\sum_k \rho_k \gamma_{kj} = 0$ for any j. But, since the class $\{0\}$ is an attractor, the unique invariant measure for the classical process is the Dirac measure concentrated in $\{0\}$. So $\rho = |e_0\rangle\langle e_0|$ is the unique invariant state for the quantum semigroup.

The second step is proving that $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$. Since we have already noticed that conditions (H-1) and (H-2) are verified, by Theorem 2.1 we get $\mathcal{N}(\mathcal{T}) \subset$

$\mathcal{D}(\mathcal{T})'$. We prove that $\mathcal{D}(\mathcal{T})' = \mathbb{C}\mathbb{1}$ and so $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$. $\mathcal{D}(\mathcal{T})$ contains all elements $|e_j\rangle\langle e_m|$ with $j \neq m$ and $(\gamma_{jm} \vee \gamma_{mj}) > 0$, so, take $x \in \mathcal{B}(\mathsf{h})$, $n \neq m$ and $\gamma_{nm} \neq 0$, we have

$$
\begin{aligned}
[L_{nm}, x] = 0 &\Leftrightarrow \sum_{jk} x_{jk}(\delta_{mj}|e_n\rangle\langle e_k| - \delta_{kn}|e_j\rangle\langle e_m|) = 0 \\
&\Leftrightarrow \sum_{k} x_{mk}|e_n\rangle\langle e_k| - \sum_{j} x_{jn}|e_j\rangle\langle e_m| = 0 \\
&\Leftrightarrow \begin{cases} x_{mk} = 0 & \text{for } k \neq m, \\ x_{jn} = 0 & \text{for } j \neq n, \\ x_{mm} = x_{nn}. \end{cases}
\end{aligned}
$$

Now we look at the classical process: since $\{0\}$ is an attractor, we have that, for any state k, there exist an integer $n \geqslant 1$ and some states $l_1, \ldots, l_n$ such that $\gamma_{kl_1}\gamma_{l_1 l_2}\cdots\gamma_{l_n 0} > 0$. So, for $x \in \mathcal{B}(\mathsf{h})$, $x_{kk} = x_{l_1 l_1} = \cdots = x_{00}$ and $x_{jk} = 0$ for $j \neq k$. Therefore any x in $\mathcal{D}(\mathcal{T})'$ has to be a multiple of the identity.

We still have to prove EID. Thanks to Theorem 1.15, it is enough to show that $(\mathcal{T}_t(x))_{t\geqslant 0}$ weakly* converges to 0 for all x in the kernel of the invariant state, i.e. for all $x \in \mathcal{B}(\mathsf{h})$ such that $\mathrm{tr}(|e_0\rangle\langle e_0|x) = x_{00} = 0$. So, take such an element x and write it as $x = y + z$ with

$$
y = \sum_{j\geqslant 1} x_{jj}|e_j\rangle\langle e_j|, \quad z = \sum_{\{j,m\in\mathbb{N}: j\neq m\}} x_{jm}|e_j\rangle\langle e_m|.
$$

We prove that $\mathrm{w}^*\text{-}\lim_t \mathcal{T}_t(y) = \mathrm{w}^*\text{-}\lim_t \mathcal{T}_t(z) = 0$.

Given the trace class operator $\sigma = |e_m\rangle\langle e_j|$, then

$$
\begin{aligned}
|\mathrm{tr}(\sigma\mathcal{T}_t(z))| &= x_{jm}\exp(t(\gamma_{mm} + \gamma_{jj})/2 + \mathrm{i}t(\kappa_j - \kappa_m))| \\
&\leqslant \mathrm{e}^{-\inf_{j\geqslant 1}|\gamma_{jj}|(t/2)}|x_{jm}| \to 0 \quad \text{as } t \to \infty.
\end{aligned}
$$

For the diagonal part y, we want to exploit the ergodic properties of the classical Markov associated process, say $(X_t)_t$, with generator Γ. Call $f : \mathbb{N} \to \mathbb{C}$ the bounded function defined by $f(n) = x_{nn}$, and restrict to consider initially only positive σ with normalized trace; then

$$
\mathrm{tr}(\sigma\mathcal{T}_t(y)) = \mathbb{E}[f(X_t)|X_0 \sim \pi],
$$

where π is the law on $\mathbb{N}$ defined by $\pi(\{k\}) = \langle e_k, \sigma e_k\rangle$. Since $\{0\}$ is an attractor and $x_{00} = 0$ implies that $f(0) = 0$, we have

$$
|\mathrm{tr}(\sigma\mathcal{T}_t(y))| = |\mathbb{E}[f(X_t)|X_0 \sim \pi]| \leqslant \|f\|_\infty \mathbb{P}\{X_t \neq 0|X_0 \sim \pi\} \to 0.
$$

We can then conclude that EID holds with $\mathcal{M}_1 = \mathbb{C}\mathbb{1}$ and $\mathcal{M}_2 = \mathcal{M}_0 = \{x \in \mathcal{B}(\mathsf{h}) : x_{00} = 0\}$. ∎

However, we underline that the uniqueness of the invariant state and triviality of $\mathcal{N}(\mathcal{T})$ do not assure EID, as the following example shows.

EXAMPLE 2.6. Consider a generic QMS associated with a Γ which is the generator of a classical birth and death process with rates

$$\gamma_{00} = 0, \quad \gamma_{j,j+1} = a \text{ and } \gamma_{j,j-1} = b \ \forall j \geqslant 1, \quad \gamma_{jk} = 0 \text{ for } |j-k| \geqslant 2,$$

now with $0 < b < a$ constants. Here $\{0\}$ is an absorbing class and is accessible from any other state, but it is no more an attractor, while the set $\{k, k \geqslant 1\}$ is a transient class as before. Similarly as in previous proposition, we can prove that $\mathcal{N}(\mathcal{T}) = \mathbb{C}\mathbb{1}$ and that $\rho = |e_0\rangle\langle e_0|$ is the unique normal invariant state of the semigroup. But EID does not hold: we can proceed as in the previous proof, but the conclusion will be different for the action of the semigroup on diagonal elements, since the classical Markov process, starting from a transient state, will display a propensity to escape to infinity.

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