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HAAGERUP APPROXIMATION PROPERTY AND POSITIVE CONES ASSOCIATED WITH A VON NEUMANN ALGEBRA

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ABSTRACT. We introduce the notion of the α-Haagerup approximation property (α-HAP) for $\alpha \in [0, 1/2]$ using a one-parameter family of positive cones studied by Araki and show that the α-HAP actually does not depend on the choice of α. This enables us to prove the fact that the Haagerup approximation properties introduced in two ways are actually equivalent, one in terms of the standard form and the other in terms of completely positive maps. We also discuss the L^p-Haagerup approximation property (L^p-HAP) for a non-commutative L^p-space associated with a von Neumann algebra for $p \in (1, \infty)$ and show the independence of the L^p-HAP on the choice of p.

KEYWORDS: *von Neumann algebra, Haagerup approximation property, non-commutative L^p-space.*

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1. INTRODUCTION

This is a continuation of our previous work [38] on the Haagerup approximation property (HAP) for a von Neumann algebra. The origin of the HAP is the remarkable paper [21], where U. Haagerup proved that the reduced group C^*-algebra of the non-amenable free group has Grothendieck's metric approximation property. After his work, M. Choda [10] showed that a discrete group has the HAP if and only if its group von Neumann algebra has a certain von Neumann algebraic approximation property with respect to the natural faithful normal tracial state. Furthermore, P. Jolissaint [25] studied the HAP in the framework of finite von Neumann algebras. In particular, it was proved that it does not depend on the choice of a faithful normal tracial state.

In the last few years, the Haagerup type approximation property for the quantum groups with respect to the Haar states was actively investigated by many researchers (e.g. [5], [6], [14], [15], [34], [35]). The point here is that the

Haar state on a quantum group is not necessarily tracial, and so to fully understand the HAP for quantum groups, we need to characterize this property in the framework of arbitrary von Neumann algebras.

In the former work [38], we introduce the notion of the HAP for arbitrary von Neumann algebras in terms of the standard form. Namely, the HAP means the existence of contractive completely positive compact operators on the standard Hilbert space which are approximating the identity. In [9], M. Caspers and A. Skalski independently introduce the notion of the HAP based on the existence of completely positive maps approximating the identity with respect to a given faithful normal semifinite weight such that the associated implementing operators on the GNS Hilbert space are compact.

Now one may wonder whether these two approaches are different or not. Actually, by combining several results in [9] and [38], it is possible to show that these two formulations are equivalent. (See [8], and Remark 5.8 of [38] for details.) This proof, however, relies on the permanence results of the HAP for a core von Neumann algebra. One of our purposes in the present paper is to give a simple and direct proof for the above mentioned question.

Our strategy is to use the positive cones due to H. Araki. He introduced in [2] a one-parameter family of positive cones P^α with a parameter α in the interval $[0,1/2]$ that is associated with a von Neumann algebra admitting a cyclic and separating vector. This family is "interpolating" the three distinguished cones P^0, $P^{1/4}$ and $P^{1/2}$, which are also denoted by $P^\sharp$, $P^\natural$ and $P^\flat$ in the literature [42]. Among them, the positive cone $P^\natural$ at the middle point plays remarkable roles in the theory of the standard representation [2], [11], [19]. See [2], [29], [30] for comprehensive studies of that family.

In view of the positive cones P^α, on the one hand, our definition of the HAP is, of course, related with $P^\natural$. On the other hand, the associated L^2-GNS implementing operators in the definition due to Caspers and Skalski are, in fact, "completely positive" with respect to $P^\sharp$. Motivated by these facts, we will introduce the notion of the "interpolated" HAP called α-HAP and prove the following result (Theorem 3.11):

THEOREM A. *A von Neumann algebra M has the α-*HAP *for some $\alpha \in [0,1/2]$ if and only if M has the α-*HAP *for all $\alpha \in [0,1/2]$.*

As a consequence, it gives a direct proof that two definitions of the HAP introduced in [9], [38] are equivalent.

In the second part of the present paper, we discuss the Haagerup approximation property for non-commutative L^p-spaces ($1 < p < \infty$) [3], [20], [23], [24], [31], [43], [44]. One can introduce the natural notion of the complete positivity of operators on $L^p(M)$, and hence we will define the HAP called the L^p-HAP when there exists a net of completely positive compact operators approximating to the identity on $L^p(M)$. Since $L^2(M)$ is the standard form of M, it follows from the definition that a von Neumann algebra M has the HAP if and only if M

has the L^2-HAP. Furthermore, by using the complex interpolation method due to A.P. Calderón [7], we can show the following result (Theorem 4.12):

THEOREM B. *Let M be a von Neumann algebra. Then the following statements are equivalent:*
(i) *M has the* HAP*;*
(ii) *M has the L^p-*HAP *for all* $1 < p < \infty$*;*
(iii) *M has the L^p-*HAP *for some* $1 < p < \infty$*.*

We remark that a von Neumann algebra M has the completely positive approximation property (CPAP) if and only if $L^p(M)$ has the CPAP for some/all $1 \leqslant p < \infty$. In the case where $p = 1$, this is proved by E.G. Effros and E.C. Lance in [16]. In general, this is due to M. Junge, Z.-J. Ruan and Q. Xu in [27]. Therefore Theorem B is the HAP version of this result.

2. PRELIMINARIES

We first fix the notation and recall several facts studied in [38]. Let M be a von Neumann algebra. We denote by M_{sa} and M^+, the set of all self-adjoint elements and all positive elements in M, respectively. We also denote by M_* and M_*^+, the space of all normal linear functionals and all positive normal linear functionals on M, respectively. The set of faithful normal semifinite (f.n.s.) weights is denoted by $W(M)$. Recall the definition of a standard form of a von Neumann algebra.

DEFINITION 2.1 ([19], Definition 2.1). Let (M, H, J, P) be a quadruple, where M denotes a von Neumann algebra, H a Hilbert space on which M acts, J a conjugate-linear isometry on H with $J^2 = 1_H$, and $P \subset H$ a closed convex cone which is self-dual, i.e., $P = P^\circ$, where $P^\circ := \{\xi \in H : \langle \xi, \eta \rangle \geqslant 0 \text{ for } \eta \in P\}$. Then (M, H, J, P) is called a *standard form* if the following conditions are satisfied:
(i) $JMJ = M'$;
(ii) $J\xi = \xi$ for any $\xi \in P$;
(iii) $aJaJP \subset P$ for any $a \in M$;
(iv) $JcJ = c^*$ for any $c \in \mathcal{Z}(M) := M \cap M'$.

REMARK 2.2. In [1], Ando and Haagerup proved that the condition (iv) in the above definition can be removed.

We explain how each f.n.s. weight φ gives a standard form. We refer readers to the book of Takesaki [42] for details. Let M be a von Neumann algebra with $\varphi \in W(M)$. We write

$$n_\varphi := \{x \in M : \varphi(x^* x) < \infty\}.$$

Then H_φ is the completion of n_φ with respect to the norm

$$\|x\|_\varphi := \varphi(x^*x)^{1/2} \quad \text{for } x \in n_\varphi.$$

We write the canonical injection $\Lambda_\varphi : n_\varphi \to H_\varphi$.

Then

$$\mathcal{A}_\varphi := \Lambda_\varphi(n_\varphi \cap n_\varphi^*)$$

is an achieved left Hilbert algebra with the multiplication

$$\Lambda_\varphi(x) \cdot \Lambda_\varphi(y) := \Lambda_\varphi(xy) \quad \text{for } x, y \in n_\varphi \cap n_\varphi^*$$

and the involution

$$\Lambda_\varphi(x)^\sharp := \Lambda_\varphi(x^*) \quad \text{for } x \in n_\varphi \cap n_\varphi^*.$$

Let π_φ be the corresponding representation of M on H_φ. We always identify M with $\pi_\varphi(M)$.

We denote by S_φ the closure of the conjugate-linear operator $\xi \mapsto \xi^\sharp$ on H_φ, which has the polar decomposition

$$S_\varphi = J_\varphi \Delta_\varphi^{1/2},$$

where J_φ is the modular conjugation and Δ_φ is the modular operator. Then the Tomita algebra $\mathcal{T}_\varphi$ consists of the elements $\xi \in H_\varphi$ for which $\xi \in D(\Delta_\varphi^\alpha)$ and $\Delta^\alpha \xi \in \mathcal{A}_\varphi$ for all $\alpha \in \mathbb{C}$. The modular automorphism group $(\sigma_t^\varphi)_{t\in\mathbb{R}}$ is given by

$$\sigma_t^\varphi(x) := \Delta_\varphi^{\mathrm{i}t} x \Delta_\varphi^{-\mathrm{i}t} \quad \text{for } x \in M.$$

We denote the centralizer of φ by

$$M_\varphi := \{x \in M : \sigma_t^\varphi(x) = x \text{ for } t \in \mathbb{R}\}.$$

Note that a self-dual positive cone

$$P_\varphi^\natural := \overline{\{\xi(J_\varphi \xi) : \xi \in \mathcal{A}_\varphi\}} \subset H_\varphi.$$

is given by the closure of the set of $\Lambda_\varphi(x\sigma_{\mathrm{i}/2}^\varphi(x)^*)$, where $x \in \mathcal{A}_\varphi$ is entire with respect to σ^φ. Therefore the quadruple $(M, H_\varphi, J_\varphi, P_\varphi^\natural)$ is a standard form. Thanks to Theorem 2.3 of [19], a standard form is, in fact, unique up to a spatial isomorphism, and so it is independent to the choice of an f.n.s. weight φ.

Let us consider the $n \times n$ matrix algebra $\mathbb{M}_m$ and the normalized trace tr_n. The algebra $\mathbb{M}_m$ becomes a Hilbert space with the inner product $\langle x, y\rangle := \mathrm{tr}_n(y^*x)$ for $x, y \in \mathbb{M}_m$. We write the canonical involution $J_{\mathrm{tr}_n} : x \mapsto x^*$ for $x \in \mathbb{M}_m$. Then the quadruple $(\mathbb{M}_m, \mathbb{M}_m, J_{\mathrm{tr}_n}, \mathbb{M}_m^+)$ is a standard form. In the following, for a Hilbert space H, $\mathbb{M}_m(H)$ denotes the tensor product Hilbert space $H \otimes \mathbb{M}_m$.

DEFINITION 2.3 ([37], Definition 2.2). Let (M, H, J, P) be a standard form and $n \in \mathbb{N}$. A matrix $[\xi_{i,j}] \in \mathbb{M}_m(H)$ is said to be *positive* if

$$\sum_{i,j=1}^n x_i J x_j J \xi_{i,j} \in P \quad \text{for all } x_1, \ldots, x_n \in M.$$

We denote by $P^{(n)}$ the set of all positive matrices $[\xi_{i,j}]$ in $\mathbb{M}_m(H)$.

Notice that for $n = 1$, we have $\xi \in H$ is positive if and only if $\xi \in P$.

PROPOSITION 2.4 ([37], Proposition 2.4; [40], Lemma 1.1). *Let (M, H, J, P) be a standard form and $n \in \mathbb{N}$. Then $(\mathbb{M}_m(M), \mathbb{M}_m(H), J \otimes J_{\mathrm{tr}_n}, P^{(n)})$ is a standard form.*

Next, we will introduce the complete positivity of a bounded operator between standard Hilbert spaces.

DEFINITION 2.5. Let (M_1, H_1, J_1, P_1) and (M_2, H_2, J_2, P_2) be two standard forms. We will say that a bounded linear (or conjugate-linear) operator $T : H_1 \to H_2$ is *completely positive* if $(T \otimes 1_{\mathbb{M}_m})P_1^{(n)} \subset P_2^{(n)}$ for all $n \in \mathbb{N}$.

DEFINITION 2.6 ([38], Definition 2.7). A W*-algebra M has the *Haagerup approximation property* (HAP) if there exists a standard form (M, H, J, P) and a net of contractive completely positive (c.c.p.) compact operators T_n on H such that $T_n \to 1_H$ in the strong topology.

Thanks to Theorem 2.3 of [19], this definition does not depend on the choice of a standard from. We also remark that the weak convergence of a net T_n in the above definition is sufficient. In fact, we can arrange a net T_n such that $T_n \to 1_H$ in the strong topology by taking suitable convex combinations.

Now we assume that M is σ-finite with a faithful state $\varphi \in M_*^+$. We denote by (H_φ, ξ_φ) the GNS Hilbert space with the cyclic and separating vector associated with (M, φ). If M has the HAP, then we can recover a net of c.c.p. maps on M approximating the identity with respect to φ such that the associated implementing operators on H_φ are compact.

THEOREM 2.7 ([38], Theorem 4.8). *Let M be a σ-finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. Then M has the* HAP *if and only if there exists a net of normal* c.c.p. *maps Φ_n on M such that:*

(i) $\varphi \circ \Phi_n \leqslant \varphi$;

(ii) $\Phi_n \to \mathrm{id}_M$ *in the point-ultraweak topology;*

(iii) *the operator defined below is* c.c.p. *compact on H_φ and $T_n \to 1_{H_\varphi}$ in the strong topology:*

$$T_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4} \Phi_n(x) \xi_\varphi \quad \textit{for } x \in M.$$

This translation of the HAP looks similar to the following HAP introduced by Caspers and Skalski in [9].

DEFINITION 2.8 ([9], Definition 3.1). Let M be a von Neumann algebra with $\varphi \in W(M)$. We will say that M has the *Haagerup approximation property with respect to φ* in the sense of [9] (CS-HAP$_\varphi$) if there exists a net of normal c.p. maps Φ_n on M such that:

(i) $\varphi \circ \Phi_n \leqslant \varphi$;

(ii) The operator T_n defined below is compact and $T_n \to 1_{H_\varphi}$ in the strong topology:

$$T_n \Lambda_\varphi(x) := \Lambda_\varphi(\Phi_n(x)) \quad \text{for } x \in n_\varphi.$$

Here are two apparent differences between Theorem 2.7 and Definition 2.8, that is, the appearance of $\Delta_\varphi^{1/4}$, of course, and the assumption on the contractivity of Φ_n's. Actually, it is possible to show that the notion of the CS-HAP$_\varphi$ does not depend on the choice of φ ([9], Theorem 4.3). Furthermore we can take contractive Φ_n's. (See Theorem 3.17.) The proof of the weight-independence presented in [9] relies on a crossed product technique. Here, let us present a direct proof of the weight-independence of the CS-HAP.

LEMMA 2.9 ([9], Theorem 4.3). *The* CS-HAP *is the weight-independent property. Namely, let $\varphi, \psi \in W(M)$. Then M has the* CS-HAP$_\varphi$ *if and only if M has the* CS-HAP$_\psi$.

Proof. Suppose that M has the CS-HAP$_\varphi$. Let Φ_n and T_n be as in the statement of Definition 2.8. Note that an arbitrary $\psi \in W(M)$ is obtained from φ by combining the following four operations:

(1) $\varphi \mapsto \varphi \otimes \mathrm{Tr}$, where Tr denotes the canonical tracial weight on $\mathbb{B}(\ell^2)$;

(2) $\varphi \mapsto \varphi_e$, where $e \in M_\varphi$ is a projection;

(3) $\varphi \mapsto \varphi \circ \alpha$, $\alpha \in \mathrm{Aut}(M)$;

(4) $\varphi \mapsto \varphi_h$, where h is a non-singular positive operator affiliated with M_φ and $\varphi_h(x) := \varphi(h^{1/2} x h^{1/2})$ for $x \in M^+$.

For the proof of this fact, see the proof of Théorème 1.2.3 in [11] or Corollary 5.8 in [41]. Hence it suffices to consider each operation.

(1) Let $\psi := \varphi \otimes \mathrm{Tr}$. Take an increasing net of finite rank projections p_n on ℓ^2. Then $\Phi_n \otimes (p_n \cdot p_n)$ does the job, where $p_n \cdot p_n$ means the map $x \mapsto p_n x p_n$.

(2) Let $e \in M_\varphi$ be a projection. Set $\psi := \varphi_e$ and $\Psi_n := e\Phi_n(e \cdot e)e$. Then we have $\psi \circ \Psi_n \leqslant \psi$. Indeed, for $x \in (eMe)_+$, we obtain

$$\psi(x) = \varphi(exe) \geqslant \varphi(\Phi_n(exe)) \geqslant \varphi(e\Phi_n(exe)e) = \psi(\Psi_n(x)).$$

Moreover for $x \in n_\varphi$, we have

$$\Lambda_{\varphi_e}(\Psi_n(x)) = eJeJ\Lambda_\varphi(\Phi_n(exe)) = eJeJT_n\Lambda_\varphi(exe) = eJeJT_neJeJ\Lambda_{\varphi_e}(exe).$$

Since $eJeJT_neJeJ$ is compact, we are done.

(3) Let $\psi := \varphi \circ \alpha$. Regard as $H_\psi = H_\varphi$ by putting $\Lambda_\psi = \Lambda_\varphi \circ \alpha$. Then we obtain the canonical unitary implementation U_α which maps $\Lambda_\varphi(x) \mapsto \Lambda_\psi(\alpha^{-1}(x))$ for $x \in n_\varphi$. Set $\Psi_n := \alpha^{-1} \circ \Phi_n \circ \alpha$. Then we have

$$\psi(x) = \varphi(\alpha(x)) \geqslant \varphi(\Phi_n(\alpha(x))) = \psi(\Psi_n(x)) \quad \text{for } x \in M^+,$$

and

$$U_\alpha T_n U_\alpha^* \Lambda_\psi(x) = U_\alpha T_n \Lambda_\varphi(\alpha(x)) = U_\alpha \Lambda_\varphi(\Phi_n(\alpha(x))) = \Lambda_\psi(\Psi(x)) \quad \text{for } x \in n_\varphi.$$

Since $U_\alpha T_n U_\alpha^*$ is compact, we are done.

(4) This case is proved in Proposition 4.2 of [9]. Let us sketch out its proof for readers' convenience. Let $e(\cdot)$ be the spectral resolution of h and put $e_n := e([1/n, n])$ for $n \in \mathbb{N}$. Considering φ_{he_n}, we may and do assume that h is bounded and invertible by Lemma 4.1 of [9]. Put $\Psi_n(x) := h^{-1/2}\Phi_n(h^{1/2}xh^{1/2})h^{-1/2}$ for $x \in M$. Then we have $\varphi_h \circ \Psi_n \leqslant \varphi_h$, and the associated implementing operator is given by $h^{-1/2}T_n h^{1/2}$, which is compact. ∎

3. HAAGERUP APPROXIMATION PROPERTY AND POSITIVE CONES

In this section, we generalize the HAP using a one-parameter family of positive cones parametrized by $\alpha \in [0, 1/2]$, which is introduced by Araki in [2]. Let M be a von Neumann algebra and $\varphi \in W(M)$.

3.1. COMPLETE POSITIVITY ASSOCIATED WITH POSITIVE CONES.

Recall that $\mathcal{A}_\varphi$ is the associated left Hilbert algebra. Let us consider the following positive cones:

$$P_\varphi^\sharp := \overline{\{\xi\xi^\sharp : \xi \in \mathcal{A}_\varphi\}}, \quad P_\varphi^\natural := \overline{\{\xi(J_\varphi\xi) : \xi \in \mathcal{A}_\varphi\}}, \quad P_\varphi^\flat := \overline{\{\eta\eta^\flat : \eta \in \mathcal{A}_\varphi'\}}.$$

Then $P_\varphi^\sharp$ is contained in $D(\Delta_\varphi^{1/2})$, the domain of $\Delta_\varphi^{1/2}$.

DEFINITION 3.1 (cf. Section 4 of [2]). For $\alpha \in [0, 1/2]$, we will define the positive cone P_φ^α by the closure of $\Delta_\varphi^\alpha P_\varphi^\sharp$.

Then P_φ^α has the same properties as in Theorem 3 of [2]:

(i) P_φ^α is the closed convex cone invariant under $\Delta_\varphi^{\mathrm{i}t}$;
(ii) $P_\varphi^\alpha \subset D(\Delta_\varphi^{1/2-2\alpha})$ and $J_\varphi\xi = \Delta_\varphi^{1/2-2\alpha}\xi$ for $\xi \in P_\varphi^\alpha$;
(iii) $J_\varphi P_\varphi^\alpha = P_\varphi^{\widehat{\alpha}}$, where $\widehat{\alpha} := 1/2 - \alpha$;
(iv) $P_\varphi^{\widehat{\alpha}} = \{\eta \in H_\varphi : \langle \eta, \xi \rangle \geqslant 0 \text{ for } \xi \in P_\varphi^\alpha\}$;
(v) $P_\varphi^\alpha = \Delta_\varphi^{\alpha-1/4}(P_\varphi^{1/4} \cap D(\Delta_\varphi^{\alpha-1/4}))$;
(vi) $P_\varphi^\natural = P_\varphi^{1/4}$ and $P_\varphi^\flat = P_\varphi^{1/2}$.

The condition (iv) means the duality between P_φ^α and $P_\varphi^{\widehat{\alpha}}$. For the modular involution, we have $J_\varphi\xi = \Delta_\varphi^{1/2-2\alpha}\xi$ for $\xi \in P_\varphi^\alpha$. This shows that $J_\varphi P_\varphi^\alpha = P_\varphi^{\widehat{\alpha}}$, that is, J_φ induces an inversion in the middle point $1/4$. (See also [36] for details.)

We set $\mathbb{M}_m(\mathcal{A}_\varphi) := \mathcal{A}_\varphi \otimes \mathbb{M}_m$ and $\varphi_n := \varphi \otimes \mathrm{tr}_n$. Then $\mathbb{M}_m(\mathcal{A}_\varphi)$ is a full left Hilbert algebra in $\mathbb{M}_m(H_\varphi)$. The multiplication and the involution are given by

$$[\xi_{i,j}] \cdot [\eta_{i,j}] := \sum_{k=1}^n [\xi_{i,k}\eta_{k,j}] \quad \text{and} \quad [\xi_{i,j}]^\sharp := [\xi_{j,i}^\sharp]_{i,j}.$$

Then we have $S_{\varphi_n} = S_\varphi \otimes J_{\mathrm{tr}}$. Hence the modular operator is $\Delta_{\varphi_n} = \Delta_\varphi \otimes \mathrm{id}_{\mathbb{M}_m}$. Denote by $P_{\varphi_n}^\alpha$ the positive cone in $\mathbb{M}_m(H_\varphi)$ for $\alpha \in [0, 1/2]$. We generalize the complete positivity presented in Definition 2.5.

DEFINITION 3.2. Let $\alpha \in [0, 1/2]$. A bounded linear operator T on H_φ is said to be *completely positive with respect to* P_φ^α if $(T \otimes 1_{\mathbb{M}_m}) P_{\varphi_n}^\alpha \subset P_{\varphi_n}^\alpha$ for all $n \in \mathbb{N}$.

3.2. COMPLETELY POSITIVE OPERATORS FROM COMPLETELY POSITIVE MAPS. Let M be a von Neumann algebra and $\varphi \in W(M)$. Let $C > 0$ and Φ a normal c.p. map on M such that

$$\varphi \circ \Phi(x) \leqslant C\varphi(x) \quad \text{for } x \in M^+. \tag{3.1}$$

In this subsection, we will show that Φ extends to a c.p. operator on H_φ with respect to P_φ^α for each $\alpha \in [0, 1/2]$. We use the following result, which is folklore among specialists. (See, for example, Lemma 4 of [2] for its proof.)

LEMMA 3.3. *Let T be a positive self-adjoint operator on a Hilbert space. For $0 \leqslant r \leqslant 1$ and $\xi \in D(T)$, the domain of T, we have $\|T^r\xi\|^2 \leqslant \|\xi\|^2 + \|T\xi\|^2$.*

LEMMA 3.4. *For $\alpha \in [0, 1/2]$, one has*

$$\|\Delta_\varphi^\alpha \Lambda_\varphi(\Phi(x))\| \leqslant C^{1/2}\|\Phi\|^{1/2}\|\Delta_\varphi^\alpha \Lambda_\varphi(x)\| \quad \textit{for } x \in n_\varphi \cap n_\varphi^*.$$

Proof. Note that if $x \in n_\varphi$, then $\Phi(x) \in n_\varphi$ because

$$\varphi(\Phi(x)^*\Phi(x)) \leqslant \|\Phi\|\varphi(\Phi(x^*x)) \leqslant C\|\Phi\|\varphi(x^*x) < \infty.$$

Let $x, y \in n_\varphi$ be entire elements with respect to σ^φ. We define the entire function F by

$$F(z) := \langle \Lambda_\varphi(\Phi(\sigma_{\mathrm{i}z/2}^\varphi(x))), \Lambda_\varphi(\sigma_{-\mathrm{i}\bar{z}/2}^\varphi(y))\rangle \quad \text{for } z \in \mathbb{C}.$$

For any $t \in \mathbb{R}$, we have

$$\begin{aligned}|F(\mathrm{i}t)| &= |\langle \Lambda_\varphi(\Phi(\sigma_{-t/2}^\varphi(x))), \Lambda_\varphi(\sigma_{-t/2}^\varphi(y))\rangle| \leqslant \|\Lambda_\varphi(\Phi(\sigma_{-t/2}^\varphi(x)))\| \cdot \|\Lambda_\varphi(\sigma_{-t/2}^\varphi(y))\| \\ &= \varphi(\Phi(\sigma_{-t/2}^\varphi(x))^*\Phi(\sigma_{-t/2}^\varphi(x)))^{1/2} \cdot \|\Lambda_\varphi(y)\| \leqslant C^{1/2}\|\Phi\|^{1/2}\|\Lambda_\varphi(x)\|\|\Lambda_\varphi(y)\|,\end{aligned}$$

and

$$\begin{aligned}|F(1+\mathrm{i}t)| &= |\langle \Delta_\varphi^{1/2}\Lambda_\varphi(\Phi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x))), \Delta_\varphi^{-\mathrm{i}t/2}\Lambda_\varphi(y)\rangle| \\ &= |\langle J_\varphi \Lambda_\varphi(\Phi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x))^*), \Delta_\varphi^{-\mathrm{i}t/2}\Lambda_\varphi(y)\rangle| \\ &\leqslant \|\Lambda_\varphi(\Phi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x))^*)\| \cdot \|\Lambda_\varphi(y)\| \\ &= \varphi(\Phi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x))\Phi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x))^*)^{1/2} \cdot \|\Lambda_\varphi(y)\| \\ &\leqslant C^{1/2}\|\Phi\|^{1/2}\varphi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x)\sigma_{(\mathrm{i}-t)/2}^\varphi(x)^*)^{1/2} \cdot \|\Lambda_\varphi(y)\| \quad \text{by (3.1)} \\ &= C^{1/2}\|\Phi\|^{1/2}\|\Lambda_\varphi(\sigma_{(\mathrm{i}-t)/2}^\varphi(x)^*)\| \cdot \|\Lambda_\varphi(y)\| \\ &= C^{1/2}\|\Phi\|^{1/2}\|J_\varphi\Lambda_\varphi(\sigma_{-t/2}^\varphi(x))\| \cdot \|\Lambda_\varphi(y)\| \\ &= C^{1/2}\|\Phi\|^{1/2}\|\Lambda_\varphi(x)\|\|\Lambda_\varphi(y)\|.\end{aligned}$$

Hence the three-lines theorem implies the following inequality for $0 \leqslant s \leqslant 1$:

$$|\langle \Delta_\varphi^{s/2}\Lambda_\varphi(\Phi(\sigma_{\mathrm{i}s/2}^\varphi(x))), \Lambda_\varphi(y)\rangle| = |F(s)| \leqslant C^{1/2}\|\Phi\|^{1/2}\|\Lambda_\varphi(x)\|\|\Lambda_\varphi(y)\|.$$

By replacing x by $\sigma^{\varphi}_{-\mathrm{i}s/2}(x)$, we obtain

$$|\langle \Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x)), \Lambda_{\varphi}(y)\rangle| \leqslant C^{1/2}\|\Phi\|^{1/2}\|\Lambda_{\varphi}(\sigma^{\varphi}_{-\mathrm{i}s/2}(x))\|\|\Lambda_{\varphi}(y)\|.$$

Since y is an arbitrary entire element of M with respect to σ^{φ}, we have

$$\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x))\| \leqslant C^{1/2}\|\Phi\|^{1/2}\|\Lambda_{\varphi}(\sigma^{\varphi}_{-\mathrm{i}s/2}(x))\| = C^{1/2}\|\Phi\|^{1/2}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(x)\|. \tag{3.2}$$

For $x \in \mathcal{A}_{\varphi}$, take a sequence of entire elements x_n of M with respect to σ^{φ} such that

$$\|\Lambda_{\varphi}(x_n) - \Lambda_{\varphi}(x)\| \to 0 \quad \text{and} \quad \|\Lambda_{\varphi}(x_n^*) - \Lambda_{\varphi}(x^*)\| \to 0 \quad (n \to \infty).$$

Then we also have

$$\begin{aligned}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(x_n - x)\|^2 &\leqslant \|\Lambda_{\varphi}(x_n - x)\|^2 + \|\Delta_{\varphi}^{1/2}\Lambda_{\varphi}(x_n - x)\|^2 \quad \text{by Lemma 3.3}\\ &= \|\Lambda_{\varphi}(x_n - x)\|^2 + \|\Lambda_{\varphi}(x_n^* - x^*)\|^2 \to 0.\end{aligned}$$

Since

$$\begin{aligned}\|\Lambda_{\varphi}(\Phi(x_n)) - \Lambda_{\varphi}(\Phi(x))\|^2 &= \|\Lambda_{\varphi}(\Phi(x_n - x))\|^2\\ &\leqslant C\|\Phi\|\|\Lambda_{\varphi}(x_n - x)\|^2 \to 0,\end{aligned}$$

we have

$$\langle \Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x_n)), \Lambda_{\varphi}(y)\rangle \to \langle \Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x)), \Lambda_{\varphi}(y)\rangle \quad \text{for } y \in n_{\varphi}. \tag{3.3}$$

Moreover, since

$$\begin{aligned}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x_m)) - \Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x_n))\| &\leqslant C^{1/2}\|\Phi\|^{1/2}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(x_m - x_n)\| \quad \text{by (3.2)}\\ &\to 0 \quad (m, n \to \infty),\end{aligned}$$

the sequence $\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x_n))$ is a Cauchy sequence. Thus $\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x_n))$ converges to $\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x))$ in norm by (3.3). Therefore, we have

$$\begin{aligned}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x))\| &= \lim_{n\to\infty}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(\Phi(x_n))\| \leqslant C^{1/2}\|\Phi\|^{1/2}\lim_{n\to\infty}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(x_n)\|\\ &= C^{1/2}\|\Phi\|^{1/2}\|\Delta_{\varphi}^{s/2}\Lambda_{\varphi}(x)\|. \quad ∎\end{aligned}$$

LEMMA 3.5. *Let M be a von Neumann algebra with $\varphi \in W(M)$ and Φ be a normal* c.p. *map on M. Suppose $\varphi \circ \Phi \leqslant C\varphi$ as before. Then for $\alpha \in [0, 1/2]$, one can define the bounded operator T_{Φ}^{α} on H_{φ} with $\|T_{\Phi}^{\alpha}\| \leqslant C^{1/2}\|\Phi\|^{1/2}$ by*

$$T_{\Phi}^{\alpha}(\Delta_{\varphi}^{\alpha}\Lambda_{\varphi}(x)) := \Delta_{\varphi}^{\alpha}\Lambda_{\varphi}(\Phi(x)) \quad \textit{for } x \in n_{\varphi} \cap n_{\varphi}^*.$$

It is not hard to see that T_{Φ}^{α} in the above is c.p. with respect to P_{φ}^{α} since $T_{\Phi}^{\alpha} \otimes 1_{\mathbb{M}_m} = T_{\Phi\otimes \mathrm{id}_{\mathbb{M}_m}}^{\alpha}$ preserves $P_{\varphi_n}^{\alpha}$.

3.3. HAAGERUP APPROXIMATION PROPERTY ASSOCIATED WITH POSITIVE CONES. We will introduce the "interpolated" HAP for a von Neumann algebra.

DEFINITION 3.6. Let $\alpha \in [0, 1/2]$ and M a von Neumann algebra with $\varphi \in W(M)$. We will say that M has the *α-Haagerup approximation property with respect to φ* (α-HAP$_\varphi$) if there exists a net of compact contractive operators T_n on H_φ such that $T_n \to 1_{H_\varphi}$ in the strong topology and each T_n is c.p. with respect to P_φ^α.

We will show the above approximation property is actually a weight-free notion in what follows.

LEMMA 3.7. *Let $\alpha \in [0, 1/2]$. Then the following statements hold:*

(i) *Let $e \in M_\varphi$ be a projection. If M has the α-*HAP$_\varphi$*, then eMe has the α-*HAP$_{\varphi_e}$*;*

(ii) *If there exists an increasing net of projections e_i in M_φ such that $e_i \to 1$ in the strong topology and e_iMe_i has the α-*HAP$_{\varphi_{e_i}}$ *for all i, then M has the α-*HAP$_\varphi$*.*

Proof. (i) We will regard $H_{\varphi_e} = eJ_\varphi eJ_\varphi H_\varphi$, $J_{\varphi_e} = eJ_\varphi e$ and $\Delta_{\varphi_e} = eJ_\varphi eJ_\varphi \Delta_\varphi$ as usual. Then it is not so difficult to show that $P_{\varphi_e}^\alpha = eJ_\varphi eJ_\varphi P_\varphi^\alpha$. Take a net T_n as in Definition 3.6. Then the net $eJ_\varphi eJ_\varphi T_n eJ_\varphi eJ_\varphi$ does the job.

(ii) Let $\mathcal{F}$ be a finite subset of H_φ and $\varepsilon > 0$. Take i such that

$$\|e_iJ_\varphi e_iJ_\varphi \xi - \xi\| < \frac{\varepsilon}{2} \quad \text{for all } \xi \in \mathcal{F}.$$

We identify $H_{\varphi_{e_i}}$ with $e_iJ_\varphi e_iJ_\varphi H_\varphi$ as usual. Then take a compact contractive operator T on $H_{\varphi_{e_i}}$ such that it is c.p. with respect to $P_{\varphi_{e_i}}^\alpha$ and satisfies

$$\|Te_iJ_\varphi e_iJ_\varphi \xi - e_iJ_\varphi e_iJ_\varphi \xi\| < \frac{\varepsilon}{2} \quad \text{for all } \xi \in \mathcal{F}.$$

Thus we have $\|Te_iJ_\varphi e_iJ_\varphi \xi - \xi\| < \varepsilon$ for $\xi \in \mathcal{F}$. One can show that $Te_iJ_\varphi e_iJ_\varphi$ is a compact contractive operator such that it is c.p. with respect to P_φ^α, and we are done. ∎

LEMMA 3.8. *The approximation property introduced in Definition* 3.6 *does not depend on the choice of an* f.n.s. *weight. Namely, let M be a von Neumann algebra and $\varphi, \psi \in W(M)$. If M has the α-*HAP$_\varphi$*, then M has the α-*HAP$_\psi$*.*

Proof. Similarly as in the proof of Lemma 2.9, it suffices to check that each operation below inherits the approximation property introduced in Definition 3.6.

(1) $\varphi \mapsto \varphi \otimes \mathrm{Tr}$, where Tr denotes the canonical tracial weight on $\mathbb{B}(\ell^2)$;
(2) $\varphi \mapsto \varphi_e$, where $e \in M_\varphi$ is a projection;
(3) $\varphi \mapsto \varphi \circ \alpha$, $\alpha \in \mathrm{Aut}(M)$;
(4) $\varphi \mapsto \varphi_h$, where h is a non-singular positive operator affiliated with M_φ.

(1) Let $N := M \otimes B(\ell^2)$ and $\psi := \varphi \otimes \mathrm{Tr}$. Take an increasing sequence of finite rank projections e_n on ℓ^2 such that $e_n \to 1$ in the strong topology. Then $f_n := 1 \otimes e_n$ belongs to N_ψ and $f_nNf_n = M \otimes e_n\mathbb{B}(\ell^2)e_n$, which has the α-HAP$_{\psi_{f_n}}$. By Lemma 3.7(ii), N has the α-HAP$_\psi$.

(2) This is nothing but Lemma 3.7(i).

(3). Let $\psi := \varphi \circ \alpha$. Regard as $H_\psi = H_\varphi$ by putting $\Lambda_\psi = \Lambda_\varphi \circ \alpha$. We denote by U_α the canonical unitary implementation, which maps $\Lambda_\varphi(x)$ to $\Lambda_\psi(\alpha^{-1}(x))$ for $x \in n_\varphi$. Then it is direct to see that $\Delta_\psi = U_\alpha \Delta_\varphi U_\alpha^*$, and $P_\psi^\alpha = U_\alpha P_\varphi^\alpha$. We can show M has the α-HAP$_\psi$ by using U_α.

(4). Our proof requires a preparation. We will give a proof after proving Lemma 3.10. ∎

Let $\alpha \in [0, 1/2]$ and $\varphi \in W(M)$. Recall that $\widehat{\alpha} = 1/2 - \alpha$. Note that for an entire element $x \in M$ with respect to σ^φ, an operator $x J_\varphi \sigma^\varphi_{i(\alpha-\widehat{\alpha})}(x) J_\varphi$ is c.p. with respect to P_φ^α.

LEMMA 3.9. *Let T be a* c.p. *operator with respect to P_φ^α and $\{e_i\}_{i=1}^m$ a partition of unity in M_φ. Then the operator $\sum_{i,j=1}^m e_i J_\varphi e_j J_\varphi T e_i J_\varphi e_j J_\varphi$ is* c.p. *with respect to P_φ^α.*

Proof. Let E_{ij} be the matrix unit of $\mathbb{M}_m$. Set $\rho := \sum_{i=1}^m e_i \otimes E_{1i}$. Note that ρ belongs to $(M \otimes \mathbb{M}_m)_{\varphi\otimes \mathrm{tr}_m}$. Then the operator

$$\rho J_{\varphi\otimes\mathrm{tr}_m} \rho J_{\varphi\otimes\mathrm{tr}_m} (T \otimes 1_{\mathbb{M}_m}) \rho^* J_{\varphi\otimes\mathrm{tr}_m} \rho^* J_{\varphi\otimes\mathrm{tr}_m}$$

on $H_\varphi \otimes \mathbb{M}_m$ is positive with respect to $P^\alpha_{\varphi\otimes\mathrm{tr}_m}$ since so is $T \otimes 1_{\mathbb{M}_m}$. By direct calculation, this operator equals $\sum_{i,j=1}^m e_i J_\varphi e_j J_\varphi T e_i J_\varphi e_j J_\varphi \otimes E_{ii} J_{\mathrm{tr}_m} E_{jj} J_{\mathrm{tr}_m}$. Thus we are done. ∎

Let $h \in M_\varphi$ be positive and invertible. We can put $\Lambda_{\varphi_h}(x) := \Lambda_\varphi(x h^{1/2})$ for $x \in n_{\varphi_h} = n_\varphi$. This immediately implies that $\Delta_{\varphi_h} = h J_\varphi h^{-1} J_\varphi \Delta_\varphi$, and $P^\alpha_{\varphi_h} = h^\alpha J_\varphi h^{\widehat{\alpha}} J_\varphi P_\varphi^\alpha$. Thus we have the following result.

LEMMA 3.10. *Let $h \in M_\varphi$ be positive and invertible. If T is a* c.p. *operator with respect to P_φ^α, then*

$$T_h := h^\alpha J_\varphi h^{\widehat{\alpha}} J_\varphi T h^{-\alpha} J_\varphi h^{-\widehat{\alpha}} J_\varphi$$

is c.p. *with respect to $P^\alpha_{\varphi_h}$.*

Resumption of Proof of Lemma 3.8. Let $\psi := \varphi_h$ and $e(\cdot)$ the spectral resolution of h. Put $e_n := e([1/n, n]) \in M_\varphi$ for $n \in \mathbb{N}$. Since $e_n \in M_\psi$ and $e_n \to 1$ in the strong topology, it suffices to show that $e_n M e_n$ has the α-HAP$_{\varphi_{he_n}}$ by Lemma 3.7. Thus we may and do assume that h is bounded and invertible.

Let us identify $H_\psi = H_\varphi$ by putting $\Lambda_\psi(x) := \Lambda_\varphi(x h^{1/2})$ for $x \in n_\varphi$ as usual, where we should note that $n_\varphi = n_\psi$. Then we have $\Delta_\psi = h J_\varphi h^{-1} J_\varphi \Delta_\varphi$ and $P_\psi^\alpha = h^\alpha J_\varphi h^{\widehat{\alpha}} J_\varphi P_\varphi^\alpha$ as well.

Let $\mathcal{F}$ be a finite subset of H_φ and $\varepsilon > 0$. Take $\delta > 0$ so that $1 - (1 + \delta)^{-1/2} < \varepsilon/2$. Let $\{e_i\}_{i=1}^m$ be spectral projections of h such that $\sum_{i=1}^m e_i = 1$ and

$he_i \leqslant \lambda_i e_i \leqslant (1+\delta)he_i$ for some $\lambda_i > 0$. Note that e_i belongs to $M_\varphi \cap M_{\varphi_h}$. For a c.p. operator T with respect to P_φ^α, we put

$$T_{h,\delta} := \sum_{i,j=1}^{m} e_i J_\varphi e_j J_\varphi T_h e_i J_\varphi e_j J_\varphi = \sum_{i,j=1}^{m} h^\alpha e_i J_\varphi h^{\widehat{\alpha}} e_j J_\varphi T h^{-\alpha} e_i J_\varphi h^{-\widehat{\alpha}} e_j J_\varphi,$$

which is c.p. with respect to $P_{\varphi_h}^\alpha$ by Lemma 3.9 and Lemma 3.10. Since $\{e_i J_\varphi e_j J_\varphi\}_{i,j}$ is a partition of unity, the norm of $T_{h,\delta}$ equals the maximum of

$$\|h^\alpha e_i J_\varphi h^{\widehat{\alpha}} e_j J_\varphi T h^{-\alpha} e_i J_\varphi h^{-\widehat{\alpha}} e_j J_\varphi\|.$$

Since we have

$$\begin{aligned}
\|h^\alpha e_i J_\varphi h^{\widehat{\alpha}} e_j J_\varphi T h^{-\alpha} e_i J_\varphi h^{-\widehat{\alpha}} e_j J_\varphi\| &\leqslant \|h^\alpha e_i\| \|h^{\widehat{\alpha}} e_j\| \|T\| \|h^{-\alpha} e_i\| \|h^{-\widehat{\alpha}} e_j\| \\
&\leqslant \lambda_i^\alpha \lambda_j^{\widehat{\alpha}} ((1+\varepsilon)\lambda_i^{-1})^\alpha ((1+\varepsilon)\lambda_j^{-1})^{\widehat{\alpha}} \|T\| \\
&= (1+\delta)^{1/2},
\end{aligned}$$

we get $\|T_{h,\delta}\| \leqslant (1+\delta)^{1/2}$.

Since M has the α-HAP$_\varphi$, we can find a c.c.p. compact operator T with respect to P_φ^α such that $\|T_{h,\delta}\xi - \xi\| < \varepsilon/2$ for all $\xi \in \mathcal{F}$. Then $\widetilde{T} := (1+\delta)^{-1/2} T_{h,\delta}$ is a c.c.p. operator with respect to $P_{\varphi_h}^\alpha$, which satisfies $\|\widetilde{T}\xi - \xi\| < \varepsilon$ for all $\xi \in \mathcal{F}$. Thus we are done. ∎

Therefore, the α-HAP$_\varphi$ does not depend on a choice of $\varphi \in W(M)$. So, we will simply say α-HAP for α-HAP$_\varphi$.

Now we are ready to introduce the main theorem in this section.

THEOREM 3.11. *Let M be a von Neumann algebra. Then the following statements are equivalent:*

(i) *M has the* HAP, *i.e., the* 1/4-HAP;

(ii) *M has the* 0-HAP;

(iii) *M has the* α-HAP *for any* $\alpha \in [0, 1/2]$;

(iv) *M has the* α-HAP *for some* $\alpha \in [0, 1/2]$.

(v) *M has the* CS-HAP;

We will prove the above theorem in several steps.

Proof of (i)⇒(ii) *in Theorem* 3.11. Suppose that M has the 1/4-HAP. Thanks to Lemma 3.7, we may and do assume that M is σ-finite. Let $\varphi \in M_*^+$ be a faithful state. By Theorem 2.7, we can take a net of normal c.c.p. maps Φ_n on M with $\varphi \circ \Phi_n \leqslant \varphi$ such that the following implementing operator T_n is compact and $T_n \to 1_{H_\varphi}$ in the strong topology:

$$T_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4} \Phi_n(x) \xi_\varphi \quad \text{for } x \in M.$$

Let $T_{\Phi_n}^0$ be the closure of $\Delta_\varphi^{-1/4} T_n \Delta_\varphi^{1/4}$ as in Lemma 3.5. Recall that $T_{\Phi_n}^0$ satisfies

$$T_{\Phi_n}^0 (x\xi_\varphi) = \Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

However, the compactness of $T^0_{\Phi_n}$ is not clear. Thus we will perturb Φ_n by averaging σ^φ. We put

$$g_\beta(t) := \sqrt{\frac{\beta}{\pi}}\exp(-\beta t^2) \quad \text{for } \beta > 0 \text{ and } t \in \mathbb{R}.$$

Then set

$$\sigma^\varphi_{g_\beta}(x) := \int_{\mathbb{R}} g_\beta(t)\sigma^\varphi_t(x)\mathrm{d}t \quad \text{for } x \in M, \quad \text{and} \quad U_\beta := \int_{\mathbb{R}} g_\beta(t)\Delta_\varphi^{\mathrm{i}t}\,\mathrm{d}t = \widehat{g}_\beta(-\log\Delta_\varphi),$$

where

$$\widehat{g}_\beta(t) := \int_{\mathbb{R}} g_\beta(s)\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s = \exp(-t^2/(4\beta)) \quad \text{for } t \in \mathbb{R}.$$

Then $U_\beta \to 1$ in the strong topology as $\beta \to \infty$.

For $\beta, \gamma > 0$, we define

$$\Phi_{n,\beta,\gamma}(x) := (\sigma^\varphi_{g_\beta} \circ \Phi_n \circ \sigma^\varphi_{g_\gamma})(x) \quad \text{for } x \in M.$$

Since $\int_{\mathbb{R}} g_\gamma(t)\,\mathrm{d}t = 1$ and $g_\gamma \geqslant 0$, the map $\Phi_{n,\beta,\gamma}$ is normal c.c.p. such that $\varphi \circ \Phi_{n,\beta,\gamma} \leqslant \varphi$. By Lemma 3.5, we obtain the associated operator $T^0_{\Phi_{n,\beta,\gamma}}$, which is given by

$$T^0_{\Phi_{n,\beta,\gamma}}(x\xi_\varphi) = \Phi_{n,\beta,\gamma}(x)\xi_\varphi \quad \text{for } x \in M.$$

Moreover, we have $T^0_{\Phi_{n,\beta,\gamma}} = U_\beta T^0_{\Phi_n} U_\gamma = U_\beta \Delta_\varphi^{-1/4} T_n \Delta_\varphi^{1/4} U_\gamma$. Hence $T^0_{\Phi_{n,\beta,\gamma}}$ is compact, because $\mathrm{e}^{-t/4}\widehat{g}_\beta(t)$ and $\mathrm{e}^{t/4}\widehat{g}_\gamma(t)$ are bounded functions on $\mathbb{R}$. Thus we have shown that $(T^0_{\Phi_{n,\beta,\gamma}})_{(n,\beta,\gamma)}$ is a net of contractive compact operators.

It is trivial that $T^0_{\Phi_{n,\beta,\gamma}} \to 1_{H_\varphi}$ in the weak topology, because $U_\beta, U_\gamma \to 1_{H_\varphi}$ as $\beta, \gamma \to \infty$ and $T_n \to 1_{H_\varphi}$ as $n \to \infty$ in the strong topology. ∎

In order to prove Theorem 3.11 (ii)⇒(iii), we need a few lemmas. In what follows, let M be a von Neumann algebra with $\varphi \in W(M)$.

LEMMA 3.12. *Let $\alpha \in [0, 1/2]$. Then M has the α-HAP$_\varphi$ if and only if M has the $\widehat{\alpha}$-HAP$_\varphi$.*

Proof. It immediately follows from the fact that T is c.p. with respect to P^α_φ if and only if $J_\varphi T J_\varphi$ is c.p. with respect to $P^{\widehat{\alpha}}_\varphi$. ∎

LEMMA 3.13. *Let $(U_t)_{t\in\mathbb{R}}$ be a one-parameter unitary group and T be a compact operator on a Hilbert space H. If a sequence (ξ_n) in H converges to 0 weakly, then $(TU_t\xi_n)$ converges to 0 in norm, compact uniformly for $t \in \mathbb{R}$.*

Proof. Since T is compact, the map $\mathbb{R} \ni t \mapsto TU_t \in \mathbb{B}(H)$ is norm continuous. In particular, for any $R > 0$, the set $\{TU_t : t \in [-R, R]\}$ is norm compact. Since (ξ_n) converges weakly, it is uniformly norm bounded. Thus the statement holds by using a covering of $\{TU_t : t \in [-R, R]\}$ by small balls. ∎

LEMMA 3.14. *Let* $\alpha \in [0, 1/4]$ *and* $\beta \in [\alpha, \widehat{\alpha}]$. *Then* $P_\varphi^\alpha \subset D(\Delta_\varphi^{\beta-\alpha})$ *and* $P_\varphi^\beta = \overline{\Delta_\varphi^{\beta-\alpha} P_\varphi^\alpha}$.

Proof. Since $P_\varphi^\alpha \subset D(\Delta_\varphi^{1/2-2\alpha})$ and $0 \leqslant \beta - \alpha \leqslant 1/2 - 2\alpha$, it turns out that $P_\varphi^\alpha \subset D(\Delta_\varphi^{\beta-\alpha})$. Let $\zeta \in P_\varphi^\alpha$ and take a sequence $\zeta_n \in P_\varphi^\sharp$ such that $\Delta_\varphi^\alpha \zeta_n \to \zeta$. Then we have

$$\begin{aligned}
\|\Delta_\varphi^\beta(\xi_m - \xi_n)\|^2 &= \|\Delta_\varphi^{\beta-\alpha}\Delta_\varphi^\alpha(\xi_m - \xi_n)\|^2 \\
&\leqslant \|\Delta_\varphi^0 \cdot \Delta_\varphi^\alpha(\xi_m - \xi_n)\|^2 + \|\Delta_\varphi^{1/2-2\alpha} \cdot \Delta_\varphi^\alpha(\xi_m - \xi_n)\|^2 \quad \text{by Lemma 3.3} \\
&= \|\Delta_\varphi^\alpha(\xi_m - \xi_n)\|^2 + \|J_\varphi \Delta_\varphi^\alpha S_\varphi(\xi_m - \xi_n)\|^2 \\
&= 2\|\Delta_\varphi^\alpha(\xi_m - \xi_n)\|^2 \to 0.
\end{aligned}$$

Hence $\Delta_\varphi^\beta \xi_n$ converges to a vector η which belongs to P_φ^β. Since $\Delta_\varphi^{\beta-\alpha}(\Delta_\varphi^\alpha \xi_n) = \Delta_\varphi^\beta \xi_n \to \eta$ and $\Delta_\varphi^{\beta-\alpha}$ is closed, $\Delta_\varphi^{\beta-\alpha}\zeta = \eta \in P_\varphi^\beta$. Hence $P_\varphi^\beta \supset \overline{\Delta_\varphi^{\beta-\alpha} P_\varphi^\alpha}$. The converse inclusion is obvious since $\Delta_\varphi^\beta P_\varphi^\sharp = \Delta_\varphi^{\beta-\alpha}(\Delta_\varphi^\alpha P_\varphi^\sharp)$. ∎

Note that the real subspace $R_\varphi^\alpha := P_\varphi^\alpha - P_\varphi^\alpha$ in H_φ is closed and the mapping

$$S_\varphi^\alpha : R_\varphi^\alpha + \mathrm{i}R_\varphi^\alpha \ni \xi + \mathrm{i}\eta \mapsto \xi - \mathrm{i}\eta \in R_\varphi^\alpha + \mathrm{i}R_\varphi^\alpha$$

is a conjugate-linear closed operator which has the polar decomposition

$$S_\varphi^\alpha = J_\varphi \Delta_\varphi^{1/2-2\alpha}.$$

(See Poposition 2.4 of [29] in the case where M is σ-finite.)

LEMMA 3.15. *Let* $\alpha \in [0, 1/4]$ *and* $T \in \mathbb{B}(H_\varphi)$ *be a* c.p. *operator with respect to* P_φ^α. *Let* $\beta \in [\alpha, \widehat{\alpha}]$. *Then the following statements hold:*

(i) *Then the operator* $\Delta_\varphi^{\beta-\alpha} T \Delta_\varphi^{\alpha-\beta}$ *extends to the bounded operator on* H_φ, *which is denoted by* T^β *in what follows, so that* $\|T^\beta\| \leqslant \|T\|$. *Also,* T^β *is a* c.p. *operator with respect to* P_φ^β.

(ii) *If a bounded net of* c.p. *operators* T_n *with respect to* P_φ^α *weakly converges to* 1_{H_φ}, *then so does the net* T_n^β.

(iii) *If* T *in* (i) *is non-zero compact, then so is* T^β.

Proof. (i) Let $\zeta \in P_\varphi^\sharp$ and $\eta := \Delta_\varphi^\beta \zeta$ which belongs to P_φ^β. We put $\xi := T\Delta_\varphi^{\alpha-\beta}\eta$. Since $\Delta_\varphi^{\alpha-\beta}\eta = \Delta_\varphi^\alpha \zeta \in P_\varphi^\alpha$ and T is c.p. with respect to P_φ^α, we obtain $\xi \in P_\varphi^\alpha$. By Lemma 3.14, we know that $\Delta_\varphi^{\beta-\alpha}\xi \in P_\varphi^\beta$. Thus $\Delta_\varphi^{\beta-\alpha} T \Delta_\varphi^{\alpha-\beta}$ maps $\Delta_\varphi^\beta P_\varphi^\sharp$ into P_φ^β.

Hence the complete positivity with respect to P_φ^β immediately follows when we prove the norm boundedness of that map. The proof given below is quite similar to the one of Lemma 3.4. Recall the associated Tomita algebra $\mathcal{T}_\varphi$. Let

$\xi, \eta \in \mathcal{T}_\varphi$. We define the entire function F by

$$F(z) := \langle T\Delta_\varphi^{-z}\xi, \Delta_\varphi^{\bar{z}}\eta\rangle \quad \text{for } z \in \mathbb{C}.$$

For any $t \in \mathbb{R}$, we have

$$|F(\mathrm{i}t)| = |\langle T\Delta_\varphi^{-\mathrm{i}t}\xi, \Delta_\varphi^{-\mathrm{i}t}\eta\rangle| \leqslant \|T\|\|\xi\|\|\eta\|.$$

Note that

$$\Delta_\varphi^{-(\widehat{\alpha}-\alpha+\mathrm{i}t)}\xi = \Delta_\varphi^{\alpha}\Delta_\varphi^{-(\widehat{\alpha}+\mathrm{i}t)}\xi = \Delta_\varphi^{\alpha}\xi_1 + \mathrm{i}\Delta_\varphi^{\alpha}\xi_2 \in R_\varphi^\alpha + \mathrm{i}R_\varphi^\alpha,$$

where $\xi_1, \xi_2 \in R_\varphi^\alpha$ satisfies $\Delta_\varphi^{-(\widehat{\alpha}+\mathrm{i}t)}\xi = \xi_1 + \mathrm{i}\xi_2$. Note that ξ_1 and ξ_2 also belong to $\mathcal{T}_\varphi$. Since T is c.p. with respect to P_φ^α, we see that $TR_\varphi^\alpha \subset R_\varphi^\alpha$. Then we have

$$\begin{aligned}\Delta_\varphi^{\widehat{\alpha}-\alpha}T\Delta_\varphi^{-(\widehat{\alpha}-\alpha+\mathrm{i}t)}\xi &= \Delta_\varphi^{1/2-2\alpha}(T\Delta_\varphi^\alpha\xi_1 + \mathrm{i}T\Delta_\varphi^\alpha\xi_2) = J_\varphi(T\Delta_\varphi^\alpha\xi_1 - \mathrm{i}T\Delta_\varphi^\alpha\xi_2)\\ &= J_\varphi TS_\varphi^\alpha(\Delta_\varphi^\alpha\xi_1 + \mathrm{i}\Delta_\varphi^\alpha\xi_2)\\ &= J_\varphi TJ_\varphi\Delta_\varphi^{1/2-2\alpha}\Delta_\varphi^{-(\widehat{\alpha}-\alpha+\mathrm{i}t)}\xi = J_\varphi TJ_\varphi\Delta_\varphi^{-\mathrm{i}t}\xi.\end{aligned}$$

In particular, $\Delta_\varphi^{\widehat{\alpha}-\alpha}T\Delta_\varphi^{-(\widehat{\alpha}-\alpha)}$ is norm bounded, and its closure is $J_\varphi TJ_\varphi$. Hence

$$\begin{aligned}|F(\widehat{\alpha}-\alpha+\mathrm{i}t)| &= |\langle T\Delta_\varphi^{-(\widehat{\alpha}-\alpha+\mathrm{i}t)}\xi, \Delta_\varphi^{\widehat{\alpha}-\alpha-\mathrm{i}t}\eta\rangle|\\ &= |\langle J_\varphi TJ_\varphi\Delta_\varphi^{-\mathrm{i}t}\xi, \Delta_\varphi^{\mathrm{i}t}\eta\rangle| \leqslant \|T\|\|\xi\|\|\eta\|.\end{aligned}$$

Applying the three-lines theorem to $F(z)$ at $z = \beta - \alpha \in [0, \widehat{\alpha}-\alpha]$, we obtain

$$|\langle \Delta_\varphi^{\beta-\alpha}T\Delta_\varphi^{\alpha-\beta}\xi, \eta\rangle| = |F(\beta-\alpha)| \leqslant \|T\|\|\xi\|\|\eta\|. \tag{3.4}$$

This implies

$$\|(\Delta_\varphi^{\beta-\alpha}T\Delta_\varphi^{\alpha-\beta})\xi\| \leqslant \|T\|\|\xi\| \quad \text{for all } \xi \in \mathcal{T}_\varphi.$$

Therefore $\Delta_\varphi^{\beta-\alpha}T\Delta_\varphi^{\alpha-\beta}$ extends to a bounded operator, which we denote by T^β, on H_φ such that $\|T^\beta\| \leqslant \|T\|$.

(ii) By (i), we have $\|T_n^\beta\| \leqslant \|T_n\|$, and thus the net $(T_n^\beta)_n$ is also bounded. Hence the statement follows from the following equality for all $\xi, \eta \in \mathcal{T}_\varphi$:

$$|\langle (T_n^\beta - 1_{H_\varphi})\xi, \eta\rangle| = |\langle (T_n - 1_{H_\varphi})\Delta_\varphi^{\alpha-\beta}\xi, \Delta_\varphi^{\beta-\alpha}\eta\rangle|.$$

(iii) Suppose that T is compact. Let η_n be a sequence in H_φ with $\eta_n \to 0$ weakly. Take $\xi_n \in \mathcal{T}_\varphi$ such that $\|\xi_n - \eta_n\| < 1/n$ for $n \in \mathbb{N}$. It suffices to check that $\|T^\beta\xi_n\| \to 0$. Since the sequence ξ_n is weakly converging, there exists $D > 0$ such that

$$\|\xi_n\| \leqslant D \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Let $\eta \in \mathcal{T}_\varphi$. For each $n \in \mathbb{N}$, we define the entire function F_n by

$$F_n(z) := \exp(z^2)\langle T\Delta_\varphi^{-z}\xi_n, \Delta_\varphi^{\bar{z}}\eta\rangle.$$

Let $\varepsilon > 0$. Take $t_0 > 0$ such that

$$\mathrm{e}^{-t^2} \leqslant \frac{\varepsilon}{D\|T\|} \quad \text{for } |t| > t_0. \tag{3.6}$$

We let $I := [-t_0, t_0]$. Since T is compact, there exists $n_0 \in \mathbb{N}$ such that

$$\|T\Delta_\varphi^{-\mathrm{i}t}\xi_n\| \leqslant \varepsilon \quad \text{and} \quad \|J_\varphi T J_\varphi \Delta_\varphi^{-\mathrm{i}t}\xi_n\| \leqslant \varepsilon \quad \text{for } n \geqslant n_0 \text{ and } t \in I. \tag{3.7}$$

Then for $n \geqslant n_0$ we have

$$|F_n(\mathrm{i}t)| = \mathrm{e}^{-t^2}|\langle T\Delta_\varphi^{-\mathrm{i}t}\xi_n, \Delta_\varphi^{-\mathrm{i}t}\eta\rangle| \leqslant \mathrm{e}^{-t^2}\|T\Delta_\varphi^{-\mathrm{i}t}\xi_n\|\|\eta\|.$$

Hence if $t \notin I$, then

$$\begin{aligned}|F_n(\mathrm{i}t)| &\leqslant \mathrm{e}^{-t^2}\|T\|\|\xi_n\|\|\eta\| \\ &\leqslant \mathrm{e}^{-t^2} D\|T\|\|\eta\| \quad \text{by (3.5)} \\ &\leqslant \varepsilon\|\eta\| \quad \text{by (3.6)},\end{aligned}$$

and if $t \in I$, then

$$\begin{aligned}|F_n(\mathrm{i}t)| &\leqslant \|T\Delta_\varphi^{-\mathrm{i}t}\xi_n\|\|\eta\| \\ &\leqslant \varepsilon\|\eta\| \quad \text{by (3.7)}.\end{aligned}$$

Similarly, by using the fact that the closure of $\Delta_\varphi^{\widehat{\alpha}-\alpha} T \Delta_\varphi^{-(\widehat{\alpha}-\alpha)}$ is $J_\varphi T J_\varphi$, we obtain

$$|F_n(\widehat{\alpha} - \alpha + \mathrm{i}t)| \leqslant \varepsilon\|\eta\| \quad \text{for } n \geqslant n_0 \text{ and } t \in \mathbb{R}.$$

Therefore the three-lines theorem implies

$$\mathrm{e}^{(\beta-\alpha)^2}|\langle T^\beta \xi_n, \eta\rangle| = |F_n(\beta - \alpha)| \leqslant \varepsilon\|\eta\| \quad \text{for } n \geqslant n_0.$$

Hence we have $\|T^\beta \xi_n\| \leqslant \varepsilon$ for $n \geqslant n_0$. Therefore T^β is compact. ∎

LEMMA 3.16. *Let M be a von Neumann algebra and $\alpha \in [0, 1/4]$. If M has the α-HAP, then M also has the β-HAP for all $\beta \in [\alpha, \widehat{\alpha}]$.*

Proof. Take a net of c.c.p. compact operators T_n with respect to P_φ^α as before. By Lemma 3.15, we obtain a net of c.c.p. compact operators T_n^β with respect to P_φ^β such that T_n^β is converging to 1_{H_φ} in the weak topology. Thus we are done. ∎

Now we resume to prove Theorem 3.11.

Proof of (ii)⇒(iii) *in Theorem* 3.11. It follows from Lemma 3.16. ∎

Proof of (iii)⇒(iv) *in Theorem* 3.11. This is a trivial implication. ∎

Proof of (iv)⇒(i) *in Theorem* 3.11. Suppose that M has the α-HAP for some $\alpha \in [0, 1/2]$. By Lemma 3.12, we may and do assume that $\alpha \in [0, 1/4]$. By Lemma 3.16, M has the 1/4-HAP. ∎

Therefore we prove the conditions from (i) to (iv) are equivalent. Finally we check the condition (v) and the others are equivalent.

Proof of (i)⇒(v) *in Theorem* 3.11. It also follows from the proof of (i)⇒(ii). ∎

Proof of (v)⇒(i) *in Theorem* 3.11. We may assume that M is σ-finite by Lemma 3.7 and Proposition 3.5 of [38]. Let $\varphi \in M_*^+$ be a faithful state. For every finite subset $F \subset M$, we denote by M_F the von Neumann subalgebra generated by 1 and

$$\{\sigma_t^\varphi(x) : x \in F, t \in \mathbb{Q}\}.$$

Then M_F is a separable σ^φ-invariant and contains F. By Theorem IX.4.2 of [42], there exists a normal conditional expectation $\mathcal{E}_F$ of M onto M_F such that $\varphi \circ \mathcal{E}_F = \varphi$. As in the proof of Theorem 3.6 of [38], the projection E_F on H_φ defined below is a c.c.p. operator:

$$E_F(x\xi_\varphi) = \mathcal{E}_F(x)\xi_\varphi \quad \text{for } x \in M.$$

It is easy to see that M_F has the CS-HAP. It also can be checked that if M_F has the HAP for every F, then M has the HAP. Hence we can further assume that M is separable.

Since M has the CS-HAP, there exists a *sequence* of normal c.p. maps Φ_n with $\varphi \circ \Phi_n \leqslant \varphi$ such that the following implementing operator T_n^0 is compact and $T_n^0 \to 1_{H_\varphi}$ strongly:

$$T_n^0(x\xi_\varphi) := \Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

In particular, T_n^0 is a c.p. operator with respect to $P_\varphi^\sharp$. By the principle of uniform boundedness, the sequence (T_n^0) is uniformly norm-bounded. By Lemma 3.15, we have a uniformly norm-bounded sequence of compact operators T_n such that each T_n is c.p. with respect to $P_\varphi^{1/4}$ and T_n weakly converges to 1_{H_φ}. By a convexity argument, we may assume that $T_n \to 1_{H_\varphi}$ strongly. It turns out from Theorem 4.9 of [38] that M has the HAP. ∎

Therefore we have finished proving Theorem 3.11. We will close this section with the following result that is the contractive map version of Definition 2.8.

THEOREM 3.17. *Let M be a von Neumann algebra. Then the following statements are equivalent:*

(i) *M has the* HAP.

(ii) *For any $\varphi \in W(M)$, there exists a net of normal* c.c.p. *maps Φ_n on M such that:*

(a) $\varphi \circ \Phi_n \leqslant \varphi$;

(b) $\Phi_n \to \mathrm{id}_M$ *in the point-ultraweak topology;*

(c) *for all $\alpha \in [0, 1/2]$, the associated* c.c.p. *operators T_n^α on H_φ defined below are compact and $T_n^\alpha \to 1_{H_\varphi}$ in the strong topology:*

$$T_n^\alpha \Delta_\varphi^\alpha \Lambda_\varphi(x) = \Delta_\varphi^\alpha \Lambda_\varphi(\Phi_n(x)) \quad \textit{for all } x \in n_\varphi.$$

(iii) *For some $\varphi \in W(M)$ and some $\alpha \in [0, 1/2]$, there exists a net of normal* c.c.p. *maps Φ_n on M such that:*

(a) $\varphi \circ \Phi_n \leqslant \varphi$;

(b) $\Phi_n \to \mathrm{id}_M$ *in the point-ultraweak topology;*

(c) *the associated* c.c.p. *operators* T_n^α *on* H_φ *defined below are compact and* $T_n^\alpha \to 1_{H_\varphi}$ *in the strong topology:*

$$T_n^\alpha \Delta_\varphi^\alpha \Lambda_\varphi(x) = \Delta_\varphi^\alpha \Lambda_\varphi(\Phi_n(x)) \quad \text{for all } x \in n_\varphi.$$

First, we will show that the second statement does not depend on the choice of φ. So let us denote here the approximation property therein by "approximation property (α, φ)"; afterwards we will simply denote it by "approximation property (α)".

LEMMA 3.18. *The approximation property* (α, φ) *does not depend on the choice of* $\varphi \in W(M)$.

Proof. Suppose that M has the approximation property (α, φ). It suffices to show that each operation listed in the proof of Lemma 2.9 inherits the property (α, φ). It is relatively easy to treat the first three operations, and let us omit proofs for them. Also, we can show that if e_i is a net as in statement of Lemma 3.7(ii) and $e_i M e_i$ has the approximation property (α, φ_{e_i}) for each i, then M has the approximation property (α, φ).

Thus it suffices to treat $\psi := \varphi_h$ for a positive invertible element $h \in M_\varphi$. Our idea is similar to the one of the proof of Lemma 3.8.

Let $\varepsilon > 0$. Take $\delta > 0$ so that $2\delta/(1+\delta) < \varepsilon$. Let $\{e_i\}_{i=1}^m$ be a spectral projections of h such that $\sum_{i=1}^m e_i = 1$ and $he_i \leqslant \lambda_i e_i \leqslant (1+\delta)he_i$ for some $\lambda_i > 0$.

For a normal c.c.p. map Φ on M such that $\varphi \circ \Phi \leqslant \varphi$, we let $\Phi_h(x) := h^{-1/2}\Phi(h^{1/2}xh^{1/2})h^{-1/2}$ for $x \in M$. Then Φ_h is a normal c.p. map satisfying $\psi \circ \Phi_h \leqslant \psi$. Next we let $\Phi_{(h,\delta)}(x) := \sum_{i,j=1}^m e_i \Phi_h(e_i x e_j) e_j$ for $x \in M$. For $x \in M^+$, we have

$$\psi(\Phi_{(h,\delta)}(x)) = \sum_{i=1}^m \psi(e_i \Phi_h(e_i x e_i)) \leqslant \sum_{i=1}^m \psi(\Phi_h(e_i x e_i)) \leqslant \sum_{i=1}^m \psi(e_i x e_i) = \psi(x).$$

Also, we obtain

$$\Phi_{(h,\delta)}(1) = \sum_{i=1}^m e_i \Phi_h(e_i) e_i = \sum_{i=1}^m e_i h^{-1/2} \Phi(he_i) h^{-1/2} e_i,$$

and the norm of $\Phi_{(h,\delta)}(1)$ equals the maximum of that of $e_i h^{-1/2}\Phi(he_i)h^{-1/2}e_i$. Since

$$\|e_i h^{-1/2}\Phi(he_i)h^{-1/2}e_i\| \leqslant \|e_i h^{-1/2}\|^2 \|he_i\| \leqslant (1+\delta)\lambda_i^{-1} \cdot \lambda_i = 1+\delta,$$

we have $\|\Psi_\delta\| \leqslant 1+\delta$.

Now let $\mathcal{F}$ be a finite subset in the norm unit ball of M and $\mathcal{G}$ a finite subset in M_*. Let $\alpha \in [0, 1/2]$. By the property (α, φ), we can take a normal c.c.p. map Φ on M with $\varphi \circ \Phi \leqslant \varphi$ such that $|\omega(\Phi_{(h,\delta)}(x) - x)| < \delta$ for all $x \in \mathcal{F}$ and $\omega \in \mathcal{G}$ and the implementing operator T^α of Φ with respect to P_φ^α is compact. Put

$\Psi_{(h,\delta)} := (1+\delta)^{-1}\Phi_{(h,\delta)}$ that is a normal c.c.p. map satisfying $\psi \circ \Psi_{(h,\delta)} \leqslant \psi$. Then we have $|\omega(\Psi_{(h,\delta)}(x) - x)| < 2\delta/(1+\delta) < \varepsilon$ for all $x \in \mathcal{F}$ and $\omega \in \mathcal{G}$.

By direct computation, we see that the implementing operator of $\Psi_{(h,\varepsilon)}$ with respect to P_φ^α is equal to the following operator:

$$\widetilde{T} := (1+\delta)^{-1} \sum_{i,j=1}^{m} h^\alpha e_i J_\varphi h^{\widehat{\alpha}} e_j J_\varphi T h^{-\alpha} e_i J_\varphi h^{-\widehat{\alpha}} e_j J_\varphi.$$

Thus $\widetilde{T}$ is compact, and we are done. (See also $\widetilde{T}$ in the proof of Lemma 3.8.) ∎

Proof of Theorem 3.17. (i)⇒(ii). Take $\varphi_0 \in W(M)$ such that there exists a partition of unity $\{e_i\}_{i\in I}$ of projections in M_{φ_0}, the centralizer of φ_0, such that $\psi_i := \varphi_0 e_i$ is a faithful normal state on $e_i M e_i$ for each $i \in I$. Then we have an increasing net of projections f_j in M_{φ_0} with $f_j \to 1$ such that $f_j M f_j$ is σ-finite for all j. Thus we may and do assume that M is σ-finite as usual. Employing Theorem 2.7, we obtain a net of normal c.c.p. maps Φ_n on M such that:

- $\varphi \circ \Phi \leqslant \varphi$;
- $\Phi_n \to \mathrm{id}_M$ in the point-ultraweak topology;
- the operator defined below is c.c.p. compact on H_φ:

$$T_n(\Delta_\varphi^{1/4} x \xi_\varphi) = \Delta_\varphi^{1/4}\Phi_n(x)\xi_\varphi \quad \text{for } x \in M.$$

Now recall our proof of Theorem 3.11 (i)⇒(ii). After averaging Φ_n by $g_\beta(t)$ and $g_\gamma(t)$, we obtain a normal c.c.p. map $\Phi_{n,\beta,\gamma}$ which satisfies $\varphi \circ \Phi_{n,\beta,\gamma} \leqslant \varphi$ and $\Phi_{n,\beta,\gamma} \to \mathrm{id}_M$ in the point-ultraweak topology. For $\alpha \in [0, 1/2]$, we define the following operator:

$$T^\alpha_{n,\beta,\gamma}\Delta_\varphi^\alpha \Lambda_\varphi(x) := \Delta_\varphi^\alpha \Lambda_\varphi(\Phi_{n,\beta,\gamma}(x)) \quad \text{for } x \in n_\varphi.$$

Then we can show the compactness of $T^\alpha_{n,\beta,\gamma}$ in a similar way to the proof of Theorem 3.11 (i)⇒(ii), and we are done.

(ii)⇒(iii). This implication is trivial.

(iii)⇒(i). By our assumption, we have a net of c.c.p. compact operators T_n^α with respect to some P_φ^α such that $T_n^\alpha \to 1$ in the strong operator topology. Namely M has the α-HAP, and thus M has the HAP by Theorem 3.11. ∎

4. HAAGERUP APPROXIMATION PROPERTY AND NON-COMMUTATIVE L^p-SPACES

In this section, we study some relations between the Haagerup approximation property and non-commutative L^p-spaces associated with a von Neumann algebra.

4.1. HAAGERUP'S L^p-SPACES. We begin with Haagerup's L^p-spaces in [20]. (See also [43].) Throughout this subsection, we fix an f.n.s. weight φ on a von Neumann algebra M. We denote by R the crossed product $M \rtimes_\sigma \mathbb{R}$ of M by the $\mathbb{R}$-action $\sigma := \sigma^\varphi$. Via the natural embedding, we have the inclusion $M \subset R$. Then R admits the canonical faithful normal semifinite trace τ and there exists the dual action θ satisfying $\tau \circ \theta_s = \mathrm{e}^{-s}\tau$ for $s \in \mathbb{R}$. Note that M is equal to the fixed point algebra R^θ, that is, $M = \{y \in R : \theta_s(y) = y \text{ for } s \in \mathbb{R}\}$.

We denote by $\widetilde{R}$ the set of all τ-measurable closed densely defined operators affiliated with R. The set of positive elements in $\widetilde{R}$ is denoted by $\widetilde{R}^+$. For $\psi \in M_*^+$, we denote by $\widehat{\psi}$ its dual weight on R and by h_ψ the element of $\widetilde{R}^+$ satisfying $\widehat{\psi}(y) = \tau(h_\psi y)$ for all $y \in R$.

Then the map $\psi \mapsto h_\psi$ is extended to a linear bijection of M_* onto the subspace

$$\{h \in \widetilde{R} : \theta_s(h) = \mathrm{e}^{-s}h \text{ for } s \in \mathbb{R}\}.$$

Let $1 \leqslant p < \infty$. The L^p-space of M due to Haagerup is defined as follows:

$$L^p(M) := \{a \in \widetilde{R} : \theta_s(a) = \mathrm{e}^{-s/p}a \text{ for } s \in \mathbb{R}\}.$$

Note that the spaces $L^p(M)$ and their relations are independent of the choice of φ, and thus canonically associated with a von Neumann algebra M. Denote by $L^p(M)^+$ the cone $L^p(M) \cap \widetilde{R}^+$. Recall that if $a \in \widetilde{R}$ with the polar decomposition $a = u|a|$, then $a \in L^p(M)$ if and only if $|a|^p \in L^1(M)$. The linear functional tr on $L^1(M)$ is defined by

$$\operatorname{tr}(h_\psi) := \psi(1) \quad \text{for } \psi \in M_*.$$

Then $L^p(M)$ becomes a Banach space with the norm

$$\|a\|_p := \operatorname{tr}(|a|^p)^{1/p} \quad \text{for } a \in L^p(M).$$

In particular, $M_* \simeq L^1(M)$ via the isometry $\psi \mapsto h_\psi$. For non-commutative L^p-spaces, the usual Hölder inequality also holds. Namely, let $q > 1$ with $1/p + 1/q = 1$, and we have

$$|\operatorname{tr}(ab)| \leqslant \|ab\|_1 \leqslant \|a\|_p\|b\|_q \quad \text{for } a \in L^p(M), b \in L^q(M).$$

Thus the form $(a, b) \mapsto \operatorname{tr}(ab)$ gives a duality between $L^p(M)$ and $L^q(M)$. Moreover the functional tr has the "tracial" property:

$$\operatorname{tr}(ab) = \operatorname{tr}(ba) \quad \text{for } a \in L^p(M), b \in L^q(M).$$

Among non-commutative L^p-spaces, $L^2(M)$ becomes a Hilbert space with the inner product

$$\langle a, b\rangle := \operatorname{tr}(b^*a) \quad \text{for } a, b \in L^2(M).$$

The Banach space $L^p(M)$ has the natural M-M-bimodule structure as defined below:

$$x \cdot a \cdot y := xay \quad \text{for } x, y \in M,\ a \in L^p(M).$$

The conjugate-linear isometric involution J_p on $L^p(M)$ is defined by $a \mapsto a^*$ for $a \in L^p(M)$. Then the quadruple $(M, L^2(M), J_2, L^2(M)^+)$ is a standard form.

4.2. HAAGERUP APPROXIMATION PROPERTY FOR NON-COMMUTATIVE L^p-SPACES. We consider the f.n.s. weight $\varphi^{(n)} := \varphi \otimes \mathrm{tr}_n$ on $\mathbb{M}_m(M) := M \otimes \mathbb{M}_m$. Since $\sigma_t^{(n)} := \sigma_t^{\varphi^{(n)}} = \sigma_t \otimes \mathrm{id}_n$, we have

$$R^{(n)} := \mathbb{M}_m(M) \rtimes_{\sigma^{(n)}} \mathbb{R} = (M \rtimes_\sigma \mathbb{R}) \otimes \mathbb{M}_m = \mathbb{M}_m(R).$$

The canonical f.n.s. trace on $R^{(n)}$ is given by $\tau^{(n)} = \tau \otimes \mathrm{tr}_n$. Moreover $\theta^{(n)} := \theta \otimes \mathrm{id}_n$ is the dual action on $R^{(n)}$. Since $\widetilde{R^{(n)}} = \mathbb{M}_m(\widetilde{R})$, we have

$$L^p(\mathbb{M}_m(M)) = \mathbb{M}_m(L^p(M)) \quad \text{and} \quad \mathrm{tr}^{(n)} = \mathrm{tr} \otimes \mathrm{tr}_n.$$

DEFINITION 4.1. Let M and N be two von Neumann algebras with f.n.s. weights φ and ψ, respectively. For $1 \leqslant p, q \leqslant \infty$, a bounded linear operator $T : L^p(M) \to L^q(N)$ is *completely positive* if $T^{(n)} : L^p(\mathbb{M}_m(M)) \to L^q(\mathbb{M}_m(N))$ is positive for every $n \in \mathbb{N}$, where $T^{(n)}[a_{i,j}] = [Ta_{i,j}]$ for $[a_{i,j}] \in L^p(\mathbb{M}_m(M)) = \mathbb{M}_m(L^p(M))$.

In the case where M is σ-finite, the following result gives a construction of a c.p. operator on $L^p(M)$ from a c.p. map on M.

THEOREM 4.2 (cf. Theorem 5.1 of [22]). *If Φ is a* c.c.p. *map on M with $\varphi \circ \Phi \leqslant C\varphi$, then one obtain a* c.p. *operator T_Φ^p on $L^p(M)$ with $\|T_\Phi^p\| \leqslant C^{1/p}\|\Phi\|^{1-1/p}$, which is defined by*

$$T_\Phi^p(h_\varphi^{1/2p} x h_\varphi^{1/2p}) := h_\varphi^{1/2p}\Phi(x)h_\varphi^{1/2p} \quad \textit{for } x \in M. \tag{4.1}$$

Let M be a σ-finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. Since

$$\|h_\varphi^{1/4} x h_\varphi^{1/4}\|_2^2 = \mathrm{tr}(h_\varphi^{1/4} x^* h_\varphi^{1/2} x h_\varphi^{1/4}) = \|\Delta_\varphi^{1/4} x \xi_\varphi\|^2 \quad \text{for } x \in M,$$

we have the isometric isomorphism $L^2(M) \simeq H_\varphi$ defined by $h_\varphi^{1/4} x h_\varphi^{1/4} \mapsto \Delta_\varphi^{1/4} x \xi_\varphi$ for $x \in M$. Therefore under this identification, the above operator T_Φ^2 is nothing but $T_\Phi^{1/4}$, which is given in Lemma 3.5.

DEFINITION 4.3. Let $1 < p < \infty$ and M be a von Neumann algebra. We will say that M has the *L^p-Haagerup approximation property* (L^p-HAP) if there exists a net of c.c.p. compact operators T_n on $L^p(M)$ such that $T_n \to 1_{L^p(M)}$ in the strong topology.

Note that a von Neumann algebra M has the HAP if and only if M has the L^2-HAP, because $(M, L^2(M), J_2, L^2(M)^+)$ is a standard form as mentioned previously.

4.3. KOSAKI'S L^p-SPACES. We assume that φ is a faithful normal state on a σ-finite von Neumann algebra M. For each $\eta \in [0,1]$, M is embedded into $L^1(M)$ by $M \ni x \mapsto h_\varphi^\eta x h_\varphi^{1-\eta} \in L^1(M)$. We define the norm $\|h_\varphi^\eta x h_\varphi^{1-\eta}\|_{\infty,\eta} := \|x\|$ on $h_\varphi^\eta M h_\varphi^{1-\eta} \subset L^1(M)$, i.e., $M \simeq h_\varphi^\eta M h_\varphi^{1-\eta}$. Then $(h_\varphi^\eta M h_\varphi^{1-\eta}, L^1(M))$ becomes a pair of compatible Banach spaces in the sense of A.P. Calderón [7]. For $1 < p < \infty$, Kosaki's L^p-space $L^p(M;\varphi)_\eta$ is defined as the complex interpolation space $C_\theta(h_\varphi^\eta M h_\varphi^{1-\eta}, L^1(M))$ equipped with the complex interpolation norm $\|\cdot\|_{p,\eta} := \|\cdot\|_{C_\theta}$, where $\theta = 1/p$. In particular, $L^p(M;\varphi)_0$, $L^p(M;\varphi)_1$ and $L^p(M;\varphi)_{1/2}$ are called the left, the right and the symmetric L^p-spaces, respectively. Note that the symmetric L^p-space $L^p(M;\varphi)_{1/2}$ is exactly the L^p-space studied in [44].

From now on, we assume that $\eta = 1/2$, and we will use the notation $L^p(M;\varphi)$ for the symmetric L^p-space $L^p(M;\varphi)_{1/2}$.

Note that $L^p(M;\varphi)$ is exactly $h_\varphi^{1/2q} L^p(M) h_\varphi^{1/2q}$, where $1/p + 1/q = 1$, and

$$\|h_\varphi^{1/2q} a h_\varphi^{1/2q}\|_{p,1/2} = \|a\|_p \quad \text{for } a \in L^p(M).$$

Namely, we have $L^p(M;\varphi) = h_\varphi^{1/2q} L^p(M) h_\varphi^{1/2q} \simeq L^p(M)$. Furthermore, we have

$$h_\varphi^{1/2} M h_\varphi^{1/2} \subset L^p(M;\varphi) \subset L^1(M),$$

and $h_\varphi^{1/2} M h_\varphi^{1/2}$ is dense in $L^p(M;\varphi)$.

Let Φ be a c.p. map on M with $\varphi \circ \Phi \leqslant \varphi$. Note that T_Φ^2 in Theorem 4.2 is equal to $T_\Phi^{1/4}$ in Lemma 3.5 under the identification $L^2(M;\varphi) = H_\varphi$. By the reiteration theorem for the complex interpolation method in [4], [7], we have

$$L^p(M;\varphi) = C_{2/p}(h_\varphi^{1/2} M h_\varphi^{1/2}, L^2(M;\varphi)) \quad \text{for } 2 < p < \infty, \quad \text{and} \tag{4.2}$$

$$L^p(M;\varphi) = C_{2/p-1}(L^2(M;\varphi), L^1(M)) \quad \text{for } 1 < p < 2. \tag{4.3}$$

(See also Section 4 of [31].) Thanks to [13], if $T_\Phi^2 = T_\Phi^{1/4}$ is compact on $L^2(M;\varphi) = H_\varphi$, then T_Φ^p is also compact on $L^p(M;\varphi)$ for $1 < p < \infty$.

4.4. THE EQUIVALENCE BETWEEN THE HAP AND THE L^p-HAP. We first show that the HAP implies the L^p-HAP in the case where M is σ-finite.

THEOREM 4.4. *Let M be a σ-finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. Suppose that M has the* HAP, *i.e., there exists a net of normal* c.c.p. *maps Φ_n on M with $\varphi \circ \Phi_n \leqslant \varphi$ satisfying the following:*

(i) $\Phi_n \to \mathrm{id}_M$ *in the point-ultraweak topology;*

(ii) *the associated operators $T_{\Phi_n}^2$ on $L^2(M)$ defined below are compact and $T_{\Phi_n}^2 \to 1_{L^2(M)}$ in the strong topology:*

$$T_{\Phi_n}^2(h_\varphi^{1/4} x h_\varphi^{1/4}) = h_\varphi^{1/4} \Phi_n(x) h_\varphi^{1/4} \quad \textit{for } x \in M.$$

*Then $T_{\Phi_n}^p \to 1_{L^p(M)}$ in the strong topology on $L^p(M)$ for $1 < p < \infty$. In particular, M has the L^p-*HAP *for all $1 < p < \infty$.*

Proof. We will freely use notations and results in [31]. First we consider the case where $p > 2$. By (4.2) we have

$$L^p(M;\varphi) = C_\theta(h_\varphi^{1/2} M h_\varphi^{1/2}, L^2(M;\varphi)) \quad \text{with } \theta := \frac{2}{p}.$$

Let $a \in L^p(M;\varphi)$ with $\|a\|_{L^p(M;\varphi)} = \|a\|_{C_\theta} \leqslant 1$ and $0 < \varepsilon < 1$. By the definition of the interpolation norm, there exists $f \in F(h_\varphi^{1/2} M h_\varphi^{1/2}, L^2(M;\varphi))$ such that $a = f(\theta)$ and $|||f|||_F \leqslant 1 + \varepsilon/3$. By Lemma 4.2.3 of [4] (or Lemma 1.3 of [31]), there exists $g \in F_0(h_\varphi^{1/2} M h_\varphi^{1/2}, L^2(M;\varphi))$ such that $|||f - g|||_F \leqslant \varepsilon/3$ and $g(z)$ is of the form

$$g(z) = \exp(\lambda z^2) \sum_{k=1}^{K} \exp(\lambda_k z) h_\varphi^{1/2} x_k h_\varphi^{1/2},$$

where $\lambda > 0$, $K \in \mathbb{N}$, $\lambda_1, \dots, \lambda_K \in \mathbb{R}$ and $x_1, \dots, x_K \in M$. Then

$$\|f(\theta) - g(\theta)\|_\theta \leqslant |||f - g|||_F \leqslant \frac{\varepsilon}{3}.$$

Since

$$\lim_{t \to \pm\infty} \|g(1 + \mathrm{i}t)\|_{L^2(M;\varphi)} = 0,$$

the subset $\{g(1 + \mathrm{i}t) : t \in \mathbb{R}\}$ of $L^2(M;\varphi)$ is compact in norm. Hence there exists $n_0 \in \mathbb{N}$ such that

$$\|T_{\Phi_n}^2 g(1 + \mathrm{i}t) - g(1 + \mathrm{i}t)\|_{L^2(M;\varphi)} \leqslant \Big(\frac{\varepsilon}{4^{1-\theta}3}\Big)^{1/\theta} \quad \text{for } n \geqslant n_0 \text{ and } t \in \mathbb{R}.$$

Moreover,

$$\|\Phi_n(g(\mathrm{i}t)) - g(\mathrm{i}t)\| \leqslant \|\Phi_n - \mathrm{id}_M\| \, \|g(\mathrm{i}t)\| \leqslant 2|||g|||_F \leqslant 2\Big(|||f|||_F + \frac{\varepsilon}{3}\Big) \leqslant 2\Big(1 + 2\frac{\varepsilon}{3}\Big) < 4.$$

Hence by Lemma 4.3.2 of [4] (or Lemma A.1 of [31]), we have

$$\begin{aligned} \|T_{\Phi_n}^p g(\theta) - g(\theta)\|_\theta &\leqslant \Big(\int_{\mathbb{R}} \|\Phi_n(g(\mathrm{i}t)) - g(\mathrm{i}t)\| P_0(\theta, t) \, \frac{\mathrm{d}t}{1-\theta}\Big)^{1-\theta} \\ &\quad \times \Big(\int_{\mathbb{R}} \|T_{\Phi_n}^2 g(1 + \mathrm{i}t) - g(1 + \mathrm{i}t)\|_{L^2(M;\varphi)} P_1(\theta, t) \, \frac{\mathrm{d}t}{\theta}\Big)^{\theta} \\ &\leqslant 4^{1-\theta} \cdot \frac{\varepsilon}{4^{1-\theta}3} = \frac{\varepsilon}{3}. \end{aligned}$$

Therefore since $T_{\Phi_n}^p$ are contractive on $L^p(M;\varphi)$, we have

$$\|T_{\Phi_n}^p f(\theta) - f(\theta)\|_\theta \leqslant \|T_{\Phi_n}^p f(\theta) - T_{\Phi_n}^p g(\theta)\|_\theta + \|T_{\Phi_n}^p g(\theta) - g(\theta)\|_\theta + \|g(\theta) - f(\theta)\|_\theta < \varepsilon.$$

Hence $T_{\Phi_n}^p \to 1_{L^p(M;\varphi)}$ in the strong topology.

In the case where $1 < p < 2$, the same argument also works. ∎

We continue further investigation of the L^p-HAP.

LEMMA 4.5. *Let $1 < p, q < \infty$ with $1/p + 1/q = 1$. Then M has the L^p-HAP if and only if M has the L^q-HAP.*

Proof. Suppose that M has the L^p-HAP, i.e., there exists a net of c.c.p. compact operators T_n on $L^p(M)$ such that $T_n \to 1_{L^p(M)}$ in the strong topology. Then we consider the transpose operators ${}^{\mathrm{t}}T_n$ on $L^q(M)$, which are defined by

$$\mathrm{tr}({}^{\mathrm{t}}T_n(b)a) = \mathrm{tr}(bT_n(a)) \quad \text{for } a \in L^p(M), b \in L^q(M).$$

It is easy to check that ${}^{\mathrm{t}}T_n$ is c.c.p. compact and ${}^{\mathrm{t}}T_n \to 1_{L^q(M)}$ in the weak topology. By taking suitable convex combinations, we have a net of c.c.p. compact operators $\widetilde{T}_n$ on $L^q(M)$ such that $\widetilde{T}_n \to 1_{L^q(M)}$ in the strong topology. Hence M has the L^q-HAP. ∎

We will use the following results.

LEMMA 4.6 (cf. Theorem 1.7 of [18]). *Let $1 \leqslant p < \infty$ and M be a σ-finite von Neumann algebra with non-singular $h_0 \in L^1(M)^+$. Then the embedding $\Theta^p_{h_0} : M \ni x \mapsto h_0^{1/2p} x h_0^{1/2p} \in L^p(M)$, induces an order isomorphism between $\{x \in M_{\mathrm{sa}} : -c1 \leqslant x \leqslant c1\}$ and $K^p_{h_0} := \{h \in L^p(M)_{\mathrm{sa}} : -ch_0^{1/p} \leqslant a \leqslant ch_0^{1/p}\}$ for each $c > 0$. Moreover $\Theta^p_{\xi_0}$ is $\sigma(M, M_*)$-$\sigma(L^p(M), L^q(M))$ continuous.*

LEMMA 4.7 ([32], Theorem 4.2). *For $1 \leqslant p, q < \infty$, the map*

$$L^p(M)^+ \ni a \mapsto a^{p/q} \in L^q(M)^+$$

is a homeomorphism with respect to the norm topologies.

In [33], it was proved that Furuta's inequality [17] remains valid for τ-measurable operators. In particular, the Löwner–Heinz inequality holds for τ-measurable operators.

LEMMA 4.8. *If τ-measurable positive self-adjoint operators a and b satisfy $a \leqslant b$, then $a^r \leqslant b^r$ for $0 < r < 1$.*

The following lemma can be proved similarly as in the proof of Lemma 4.2 in [38].

LEMMA 4.9. *Let $1 \leqslant p, q \leqslant \infty$ with $1/p + 1/q = 1$. If $a \in L^p(M)^+$, then*
(i) *the functional $f_a : L^q(M) \to \mathbb{C} = L^q(\mathbb{C})$, $b \mapsto \mathrm{tr}(ba)$ is a* c.p. *operator;*
(i) *the operator $g_a : \mathbb{C} = L^p(\mathbb{C}) \to L^p(M)$, $z \mapsto za$ is a* c.p. *operator.*

In the case where $p = 2$, the following lemma is also proved in Lemma 4.3 of [38]. We give a proof for reader's convenience.

LEMMA 4.10. *Let $1 < p < \infty$ and M be a σ-finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. If M has the L^p-*HAP*, then there exists a net of* c.c.p. *compact operators T_n on $L^p(M)$ such that $T_n \to 1_{L^p(M)}$ in the strong topology, and $(T_n h_\varphi^{1/p})^{p/2} \in L^2(M)^+$ is cyclic and separating for all n.*

Proof. Since M has the L^p-HAP, there exists a net of c.c.p. compact operators T_n on $L^p(M)$ such that $T_n \to 1_{L^p(M)}$ in the strong topology. Set $a_n^{1/p} := T_n h_\varphi^{1/p} \in$

$L^p(M)^+$. Then $a_n \in L^1(M)^+$. Take $\varphi_n \in M_*^+$ such that $a_n = h_{\varphi_n} \in L^1(M)^+$. If we set

$$\psi_n := \varphi_n + (\varphi_n - \varphi)_- \in M_*^+,$$

then $h_n := h_{\psi_n} \geqslant h_\varphi$. By Lemma 4.8, we obtain $h_n^{1/2} \geqslant h_\varphi^{1/2}$. It follows from Lemma 4.3 of [12] that $h_n^{1/2} \in L^2(M)^+$ is cyclic and separating. Now we define a compact operator T_n' on $L^p(M)$ by

$$T_n'a := T_n a + \operatorname{tr}(a h_\varphi^{1/q})(h_n^{1/p} - a_n^{1/p}) \quad \text{for } a \in L^p(M).$$

Since $h_n^{1/p} \geqslant a_n^{1/p}$ by Lemma 4.8, each T_n' is a c.p. operator, because of Lemma 4.9. Note that

$$T_n' h_\varphi^{1/p} = T_n h_\varphi^{1/p} + \operatorname{tr}(h_\varphi)(h_n^{1/p} - a_n^{1/p}) = h_n^{1/p}.$$

Since $a_n^{1/p} = T_n h_\varphi^{1/p} \to h_\varphi^{1/p}$ in norm, we have $a_n = h_{\varphi_n} \to h_\varphi$ in norm by Lemma 4.7. Since

$$\|h_n - a_n\|_1 = \|\psi_n - \varphi_n\| \leqslant \|\varphi_n - \varphi\| = \|h_{\varphi_n} - h_\varphi\|_1 \to 0,$$

we obtain $\|h_n^{1/p} - a_n^{1/p}\|_p \to 0$ by Lemma 4.7. Therefore $\|T_n'a - a\|_p \to 0$ for any $a \in L^p(M)$. Since $\|T_n' - T_n\| \leqslant \|h_n^{1/p} - a_n^{1/p}\|_p \to 0$, we get $\|T_n'\| \to 1$. Then operators $\widetilde{T}_n := \|T_n'\|^{-1} T_n'$ give a desired net. ∎

If M is σ-finite and the L^p-HAP for some $1 < p < \infty$, then we can recover a net of normal c.c.p. maps on M approximating to the identity such that the associated implementing operators on $L^p(M)$ are compact. In the case where $p = 2$, this is nothing but Theorem 4.8 of [38] (or Theorem 3.17).

THEOREM 4.11. *Let $1 < p < \infty$ and M a σ-finite von Neumann algebra with a faithful state $\varphi \in M_*^+$. If M has the L^p-*HAP*, then there exists a net of normal* c.c.p. *map Φ_n on M with $\varphi \circ \Phi_n \leqslant \varphi$ satisfying the following:*

(i) $\Phi_n \to \mathrm{id}_M$ *in the point-ultraweak topology;*

(ii) *the associated* c.c.p. *operator $T_{\Phi_n}^p$ on $L^p(M)$ defined below are compact and $T_{\Phi_n}^p \to 1_{L^p(M)}$ in the strong topology:*

$$T_{\Phi_n}^p (h_\varphi^{1/2p} x h_\varphi^{1/2p}) = h_\varphi^{1/2p} \Phi_n(x) h_\varphi^{1/2p} \quad \textit{for } x \in M.$$

Proof. The case where $p = 2$ is nothing but Theorem 4.8 of [38]. Let $p \neq 2$. Take $q > 1$ such that $1/p + 1/q = 1$. By Lemma 4.10, there exists a net of c.c.p. compact operators T_n on $L^p(M)$ such that $T_n \to 1_{L^p(M)}$ in the strong topology, and $h_n^{1/2} := (T_n h_\varphi^{1/p})^{p/2}$ is cyclic and separating on $L^2(M)$ for all n.

Let $\Theta_{h_\varphi}^p$ and $\Theta_{h_n}^p$ be the maps given in Lemma 4.6. For each $x \in M_{\mathrm{sa}}$, take $c > 0$ such that $-c1 \leqslant x \leqslant c1$. Then

$$-c h_\varphi^{1/p} \leqslant h_\varphi^{1/2p} x h_\varphi^{1/2p} \leqslant c h_\varphi^{1/p}.$$

Since T_n is positive, we have

$$-c h_n^{1/p} \leqslant T_n(h_\varphi^{1/2p} x h_\varphi^{1/2p}) \leqslant c h_n^{1/p}.$$

From Lemma 4.6, the operator $(\Theta^p_{h_n})^{-1}(T_n(h_\varphi^{1/2p} x h_\varphi^{1/2p}))$ in M is well-defined. Hence we can define a linear map Φ_n on M by

$$\Phi_n := (\Theta^p_{h_n})^{-1} \circ T_n \circ \Theta^p_{h_\varphi}.$$

In other words,

$$T_n(h_\varphi^{1/2p} x h_\varphi^{1/2p}) = h_n^{1/2p} \Phi_n(x) h_n^{1/2p} \quad \text{for } x \in M.$$

One can easily check that Φ_n is a normal u.c.p. map.

Step 1. We will show that $\Phi_n \to \mathrm{id}_M$ in the point-ultraweak topology.

Since $h_\varphi^{1/2} M h_\varphi^{1/2}$ is dense in $L^1(M)$, it suffices to show that

$$\mathrm{tr}(\Phi_n(x) h_\varphi^{1/2} y h_\varphi^{1/2}) \to \mathrm{tr}(x h_\varphi^{1/2} y h_\varphi^{1/2}) \quad \text{for } x, y \in M.$$

Moreover, thanks to Lemma 4.7, we have $\|h_n^{1/2p} - h_\varphi^{1/2p}\|_p \to 0$. Therefore it suffices to check that

$$\mathrm{tr}(\Phi_n(x) h_n^{1/2p} h_\varphi^{1/2q} y h_\varphi^{1/2q} h_n^{1/2p}) \to \mathrm{tr}(x h_\varphi^{1/2} y h_\varphi^{1/2}) \quad \text{for } x, y \in M.$$

However

$$\begin{aligned}
&|\mathrm{tr}(\Phi_n(x) h_n^{1/2p} h_\varphi^{1/2q} y h_\varphi^{1/2q} h_n^{1/2p}) - \mathrm{tr}(x h_\varphi^{1/2} y h_\varphi^{1/2})| \\
&\qquad = |\mathrm{tr}((h_n^{1/2p} \Phi_n(x) h_n^{1/2p} - h_\varphi^{1/2p} x h_\varphi^{1/2p}) \cdot h_\varphi^{1/2q} y h_\varphi^{1/2q})| \\
&\qquad = |\mathrm{tr}((T_n - 1_{L_p(M)})(h_\varphi^{1/2p} x h_\varphi^{1/2p}) \cdot h_\varphi^{1/2q} y h_\varphi^{1/2q})| \\
&\qquad \leqslant \|(T_n - 1_{L_p(M)})(h_\varphi^{1/2p} x h_\varphi^{1/2p})\|_p \|h_\varphi^{1/2q} y h_\varphi^{1/2q}\|_q \to 0.
\end{aligned}$$

Step 2. We will make a small perturbation of Φ_n.

By Lemma 4.7, we have $\|h_n - h_\varphi\|_1 \to 0$, i.e., $\|\varphi_n - \varphi\| \to 0$, where $\varphi_n \in M_*^+$ is the unique element with $h_n = h_{\varphi_n}$. By a similar argument as in the proof of Theorem 4.8 of [38], one can obtain normal c.c.p. maps $\widetilde{\Phi}_n$ on M with $\widetilde{\Phi}_n \to \mathrm{id}_M$ in the point-ultraweak topology, and c.c.p. compact operators $\widetilde{T}_n$ on $L^p(M)$ with $\widetilde{T}_n \to 1_{L^p(M)}$ in the strong topology such that $\varphi \circ \widetilde{\Phi}_n \leqslant \varphi$ and

$$\widetilde{T}_n(h_\varphi^{1/2p} x h_\varphi^{1/2p}) = h_\varphi^{1/2p} \widetilde{\Phi}_n(x) h_\varphi^{1/2p} \quad \text{for } x \in M.$$

Moreover operators $\widetilde{T}_n$ are nothing but $T^p_{\widetilde{\Phi}_n}$. ∎

Recall that M has the completely positive approximation property (CPAP) if and only if $L^p(M)$ has the CPAP for some/all $1 \leqslant p < \infty$. This result is proved in Theorem 3.2 of [27]. The following is the HAP version of this fact.

THEOREM 4.12. *Let M be a von Neumann algebra. Then the following are equivalent:*

(i) *M has the* HAP*;*

(ii) *M has the L^p-*HAP *for all $1 < p < \infty$;*

(iii) *M has the L^p-*HAP *for some $1 < p < \infty$.*

Proof. We first reduce the case where M is σ-finite by the following elementary fact similarly as in the proof of Theorem 3.2 in [27]. Take an f.n.s. weight φ on M and an increasing net of projection e_n in M with $e_n \to 1_M$ in the strong topology such that $\sigma_t^{\varphi}(e_n) = e_n$ for all $t \in \mathbb{R}$ and $e_n M e_n$ is σ-finite for all n. Then we can identify $L^p(e_n M e_n)$ with a subspace of $L^p(M)$ and there exists a completely positive projection from $L^p(M)$ onto $L^p(e_n M e_n)$ via $a \mapsto e_n a e_n$. Moreover the union of these subspaces is norm dense in $L^p(M)$. Therefore it suffices to prove the theorem in the case where M is σ-finite.

(i)⇒(ii) It is nothing but Theorem 4.4.

(ii)⇒(iii) It is trivial.

(iii)⇒(i) Suppose that M has the L^p-HAP for some $1 < p < \infty$. We may and do assume that $p < 2$ by Lemma 4.5. Let $\varphi \in M_*$ be a faithful state. By Theorem 4.11, there exists a net of normal c.c.p. maps Φ_n on M with $\varphi \circ \Phi_n \leqslant \varphi$ such that $\Phi_n \to \mathrm{id}_M$ in the point-ultraweak topology and a net of the associated compact operators $T^p_{\Phi_n}$ converges to $1_{L^p(M)}$ in the strong topology. By the reiteration theorem for the complex interpolation method, we have $L^2(M;\varphi) = C_\theta(h_\varphi^{1/2} M h_\varphi^{1/2}, L^p(M;\varphi))$ for some $0 < \theta < 1$. By [13], the operators $T^2_{\Phi_n}$ are also compact. Moreover, by the same argument as in the proof of Theorem 4.4, we have $T^2_{\Phi_n} \to 1_{L^2(M)}$ in the strong topology. ∎

REMARK 4.13. In the proof of Theorem 3.2 in [27], it is shown that c.p. operators on $L^p(M)$ give c.p. maps on M by using the result of L.M. Schmitt in [39]. See Theorem 3.1 of [27] for more details. However our approach is much different and based on the technique of A.M. Torpe in [45].

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REFERENCES

[1] H. ANDO, U. HAAGERUP, Ultraproducts of von Neumann algebras, *J. Funct. Anal.* **266**(2014), 6842–6913.

[2] H. ARAKI, Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon–Nikodym theorem with a chain rule, *Pacific J. Math.* **50**(1974), 309–354.

[3] H. ARAKI, T. MASUDA, Positive cones and L^p-spaces for von Neumann algebras, *Publ. Res. Inst. Math. Sci.* **18**(1982), 339–411.

[4] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Grundlehren Math. Wiss., vol. 223, Springer-Verlag, Berlin-New York 1976.

[5] M. BRANNAN, Approximation properties for free orthogonal and free unitary quantum groups, *J. Reine Angew. Math.* **672**(2012), 223–251.

[6] M. BRANNAN, Reduced operator algebras of trace-preserving quantum automorphism groups, *Doc. Math.* **18**(2013), 1349–1402.

[7] A.-P. CALDERÓN, Intermediate spaces and interpolation, the complex method, *Studia Math.* **24**(1964) 113–190.

[8] M. CASPERS, R. OKAYASU, A. SKALSKI, R. TOMATSU, Generalisations of the Haagerup property to arbitrary von Neumann algebras, *C. R. Math. Acad. Sci. Paris* **352**(2014), 507–510.

[9] M. CASPERS, A. SKALSKI, The Haagerup property for arbitrary von Neumann algebras, *Int. Math. Res. Notices* **19**(2015), 9857–9887.

[10] M. CHODA, Group factors of the Haagerup type, *Proc. Japan Acad. Ser. A Math. Sci.* **59**(1983), 174–177.

[11] A. CONNES, Une classification des facteurs de type. III, *Ann. Sci. École Norm. Sup. (4)* **6**(1973), 133–252.

[12] A. CONNES, Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann, *Ann. Inst. Fourier (Grenoble)* **24**(1974), 121–155.

[13] M. CWIKEL, N.J. KALTON, Interpolation of compact operators by the methods of Calderón and Gastavsson–Peetre, *Proc. Edinburgh Math. Soc.* **38**(1995), 261–276.

[14] M. DAWS, P. FIMA, A. SKALSKI, S. WHITE, The Haagerup property for locally compact quantum groups, *J. Reine Angew. Math.*, to appear.

[15] K. DE COMMER, A. FRESLON, M. YAMASHITA, CCAP for universal discrete quantum groups, *Comm. Math. Phys.* **331**(2014), 677–701.

[16] E.G. EFFROS, E.C. LANCE, Tensor products of operator algebras, *Adv. Math.* **25**(1977), 1–34.

[17] T. FURUTA, $A \geqslant B \geqslant 0$ assures $(B^r A^p B^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$, *Proc. Amer. Math. Soc.* **101**(1987), 85–88.

[18] S. GOLDSTEIN, J.M. LINDSAY, KMS-symmetric Markov semigroups, *Math. Z.* **219**(1995), 591–608.

[19] U. HAAGERUP, The standard form of von Neumann algebras, *Math. Scand.* **37**(1975), 271–283.

[20] U. HAAGERUP, L^p-spaces associated with an arbitrary von Neumann algebra, in *Algèbres d'opérateurs et leurs applications en physique mathématique (Proc. Colloq., Marseille, 1977)*, Colloq. Internat. CNRS, vol. 274, CNRS, Paris 1979, pp. 175–184.

[21] U. HAAGERUP, An example of a nonnuclear C^*-algebra, which has the metric approximation property, *Invent. Math.* **50**(1978/79), 279–293.

[22] U. HAAGERUP, M. JUNGE, Q. XU, A reduction method for noncommutative L_p-spaces and applications, *Trans. Amer. Math. Soc.* **362**(2010), 2125–2165.

[23] F. HANSEN, Les espaces L^p d'une algébre de von Neumann, *J. Funct. Anal.* **40**(1981), 151–169.

[24] H. IZUMI, Constructions of non-commutative L^p-spaces with a complex parameter arising from modular actions, *Internat. J. Math.* **8**(1997), 1029–1066.

[25] P. JOLISSAINT, Haagerup approximation property for finite von Neumann algebras, *J. Operator Theory* **48**(2002), suppl., 549–571.

[26] M. JUNGE, Z.-J. RUAN, Approximation properties for noncommutative L_p-spaces associated with discrete groups, *Duke Math. J.* **117**(2003), 313–341.

[27] M. JUNGE, Z.-J. RUAN, Q. XU, Rigid $\mathcal{OL}_p$ structures of non-commutative L_p-spaces associated with hyperfinite von Neumann algebras, *Math. Scand.* **96**(2005), 63–95.

[28] M. JUNGE, Q. XU, Noncommutative Burkholder/Rosenthal inequalities, *Ann. Probab.* **31**(2003), 948–995.

[29] H. KOSAKI, Positive cones associated with a von Neumann algebra, *Math. Scand.* **47**(1980), 295–307.

[30] H. KOSAKI, Positive cones and L^p-spaces associated with a von Neumann algebra, *J. Operator Theory* **6**(1981), 13–23.

[31] H. KOSAKI, Applications of the complex interpolation method to a von Neumann algebra: noncommutative L^p-spaces, *J. Funct. Anal.* **56**(1984), 29–78.

[32] H. KOSAKI, Applications of uniform convexity of noncommutative L^p-spaces, *Trans. Amer. Math. Soc.* **283**(1984), 265–282.

[33] H. KOSAKI, A remark on Sakai's quadratic Radon–Nikodým theorem, *Proc. Amer. Math. Soc.* **116**(1992), 783–786.

[34] J. KUSTERMANS, S. VAES, Locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92**(2003), 68–92.

[35] F. LEMEUX, Haagerup approximation property for quantum reflection groups, *Proc. Amer. Math. Soc.* **143**(2015), 2017–2031.

[36] Y. MIURA, Some properties of the convex cones in a Hilbert space, *Artes liberales* **25**(1979), 165–176.

[37] Y. MIURA, J. TOMIYAMA, On a characterization of the tensor product of self-dual cones associated to the standard von Neumann algebras, *Sci. Rep. Niigata Univ. Ser. A* **20**(1984), 1–11.

[38] R. OKAYASU, R. TOMATSU, Haagerup approximation property for arbitrary von Neumann algebras, *Publ. Res. Inst. Math. Sci.* **51**(2015), 567–603.

[39] L.M. SCHMITT, Facial structures on L^p-spaces of W^*-algebras, Ph.D. Dissertation, Universität des Saarlandes, Saarbrücken 1985.

[40] L.M. SCHMITT, G. WITTSTOCK, Characterization of matrix-ordered standard forms of W*-algebras, *Math. Scand.* **51**(1982), 241–260.

[41] Ş. STRĂTILĂ, *Modular Theory in Operator Algebras*, Editura Academiei Republicii Socialiste România, Bucharest; Abacus Press, Tunbridge Wells 1981.

[42] M. TAKESAKI, *Theory of Operator Algebras*. II, Encyclopaedia Math. Sci., vol. 125, Operator Algebras and Non-commutative Geometry, vol. 6, Springer-Verlag, Berlin 2003.

[43] M. TERP, L^p-spaces associated with von Neumann algebras, notes, Københavns Universitets Matematiske Institut, København 1981.

[44] M. TERP, Interpolation spaces between a von Neumann algebra and its predual, *J. Operator Theory* **8**(1982), 327–360.

[45] A.M. TORPE, A characterization of semidiscrete von Neumann algebras in terms of matrix ordered Hilbert spaces, preprint, Matematisk Institut, Odense Universitet, Odense 1981.

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