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DECOMPOSITION OF BILINEAR FORMS AS SUMS OF BOUNDED FORMS

MOHAMED ELMURSI

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ABSTRACT. The problem of decomposition of bilinear forms which satisfy a certain condition has been studied by many authors, by example in [4]: Let H and K be Hilbert spaces and let $A, C \in B(H), B, D \in B(K)$. Assume that $u : H \times K \to \mathbb{C}$ a bilinear form satisfies $|u(x,y)| \leqslant \|Ax\|\|By\| + \|Cx\|\|Dy\|$ for all $x \in H$ and $y \in K$. Then u can be decomposed as a sum of two bilinear forms $u = u_1 + u_2$ where $|u_1(x,y)| \leqslant \|Ax\|\|By\|$, $|u_2(x,y)| \leqslant \|Cx\|\|Dy\|$, $\forall x \in H, y \in K$. U. Haagerup conjectured that an analogous decomposition as a sum of bounded bilinear forms is not always possible for more than two terms. In this paper we give a necessary and sufficient condition for such a decomposition to exist and use this criterion to show that indeed it is not always possible for more than two terms.

KEYWORDS: *Tensor products, bilinear forms, trace class, finite rank operators.*

MSC (2010): 46C99, 46L07.

INTRODUCTION

A bilinear form on a vector space V is a bilinear mapping $V \times V \to \mathbb{F}$, where $\mathbb{F}$ is the field of scalars. That is, a bilinear form is a function $B : V \times V \to \mathbb{F}$ which is linear in each argument separately. When $\mathbb{F}$ is the field of complex numbers $\mathbb{C}$, one is often more interested in sesquilinear forms, which are similar to bilinear forms but are conjugate linear in one argument. We focus here on the bounded ones. A bilinear form on a normed vector space is bounded if there is a constant C such that for all $u, v \in V$

$$|B(u,v)| \leqslant C\|u\|\|v\|.$$

Let E and F be real or complex vector spaces. In several places in the literature one meets the following situation. One is given a bilinear form $u : E \times F \to \mathbb{C}$ which can be majorized by the sum of the absolute values of two bounded forms b_1 and b_2. One then wants to show that u can be decomposed as a sum $u = u_1 + u_2$ with

$|u_1| \leqslant |b_1|$ and $|u_2| \leqslant |b_2|$. Pisier and Shlyakhtenko [6] proved such a result for completely bounded forms on exact operator spaces $E \subseteq A$ and $F \subseteq B$ sitting in C^*-algebras A and B. Let f_1,f_2 be states on A and g_1, g_2 be states on B such that for all $a \in E$ and $b \in F$,

$$|u(a,b)| \leqslant \|u\|_{ER}(f_1(aa^*)^{1/2}g_1(b^*b)^{1/2} + f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}).$$

Then u can be decomposed as

$$u = u_1 + u_2,$$

where u_1 and u_2 are bilinear forms satisfying the following inequalities, for all $a \in A$ and $b \in B$:

$$|u_1(a,b)| \leqslant \|u\|_{ER} f_1(aa^*)^{1/2}g_1(b^*b)^{1/2}, \tag{0.1}$$

$$|u_2(a,b)| \leqslant \|u\|_{ER} f_2(a^*a)^{1/2}g_2(bb^*)^{1/2}. \tag{0.2}$$

In particular,

$$\|u_1\|_{\mathrm{cb}} \leqslant \|u\|_{ER} \quad \text{and} \quad \|u_2^{\mathrm{t}}\|_{\mathrm{cb}} \leqslant \|u\|_{ER},$$

where $u_2^{\mathrm{t}}(b,a) := u_2(a,b)$, for all $a \in E$ and $b \in F$. Using a similar argument a strengthened version of this result was proved by Xu [8] for ordinary bilinear forms. Let H and K be Hilbert spaces and let $A, C \in B(H), B, D \in B(K)$. Assume that $u : H \times K \to \mathbb{C}$ is a bilinear form that satisfies

$$|u(x,y)| \leqslant \|Ax\|\|By\| + \|Cx\|\|Dy\|$$

for all $x \in H$ and $y \in K$. Then u can be decomposed as a sum of two bilinear forms

$$u = u_1 + u_2$$

where

$$|u_1(x,y)| \leqslant \|Ax\|\|By\|, \quad |u_2(x,y)| \leqslant \|Cx\|\|Dy\|, \quad \forall x \in H, y \in K.$$

The proof there was merely sketched. Later Haagerup–Musat needed the stronger version for bounded bilinear forms on operator spaces [4]. U. Haagerup conjectured that an analogous decomposition as a sum of bounded bilinear forms is not always possible for more than two terms. It is the aim of the present paper to analyze this problem.

The article is organized as follows. In Section 1, we come to the main subject of this paper. We study the problem of decomposition of bounded bilinear forms as in Xu's result. The proof for such a decomposition that we give (following Haagerup) depends on the isomorphism between the projective tensor product $H \widehat{\otimes}_\pi K$ where H and K are Hilbert spaces, and the space of trace class operators from the conjugate Hilbert space $\overline{H}$ into K [2]. We give in this section a complete proof of Xu's result.

Section 2 takes a systematic look at the question of decomposing into n bounded terms. Restricting to the finite dimensional case, we give a necessary and sufficient criterion for such a decomposition. The criterion and its proof uses the correspondence between bounded operators on a Hilbert space and sesquilinear forms as well as the trace duality theorem. The proof also uses the Hahn–Banach theorem applied to convex hulls of certain sets of finite rank operators (the Hahn–Banach theorem had also been used in the proof of Theorem 2.1, the main theorem of this section). Section 3 is the last section of this paper. Here we use the criterion established in Section 2 to give a counterexample to the decomposability into three terms. First we give a useful lemma concerning a positive finite rank operator on a finite dimensional Hilbert space. We are then in a position to construct an example of a sesquilinear form u, even on a two-dimensional Hilbert space, which is majorized by the sum of the moduli of three bounded forms b_1, b_2 and b_3, but cannot be decomposed as a sum of three sesquilinear forms u_i, where each u_i is majorized by the corresponding $|b_i|$. The counterexample depends on a suitable choice of operators without common eigenvectors.

1. DECOMPOSITION OF BILINEAR FORMS INTO TWO TERMS

In this section, we show how to decompose a bilinear form into two terms by using the isomorphism between the projective tensor product $H\widehat{\otimes}_\pi K$ and the space $B_1(\overline{H}, K)$ of trace-class operators from $\overline{H}$ into K, see for instance [2]. We explain how to decompose a bilinear form which satisfies the condition (1.1) below into a sum of two bilinear forms satisfying certain boundedness conditions. We will start with the following theorem which gives the isometric isomorphism between the projective tensor product and the space of trace class operators, see [2].

THEOREM 1.1. *The map*

$$\begin{aligned} J : H' \otimes_\pi H &\to B_1(H) \\ x' \otimes y &\to x' \underline{\otimes} y \end{aligned}$$

gives an isometric isomorphism, where $x' \underline{\otimes} y$ denotes a rank one operator.

Now we present the main theorem in this section.

THEOREM 1.2. *Let H and K be Hilbert spaces and let $A, C \in B(H), B, D \in B(K)$. Assume that a bilinear form $u : H \times K \to \mathbb{C}$ satisfies*

$$|u(x,y)| \leqslant \|Ax\|\|By\| + \|Cx\|\|Dy\| \tag{1.1}$$

for all $x \in H$ and $y \in K$. Then u can be decomposed as a sum of two bilinear forms

$$u = u_1 + u_2$$

where

$$|u_1(x,y)| \leqslant \|Ax\|\|By\|, \quad |u_2(x,y)| \leqslant \|Cx\|\|Dy\|, \quad x \in H, y \in K.$$

Proof. We consider three cases to prove the claim.

Case 1. We suppose that $A = B = I$ and C, D are invertible operators. We will prove this case below.

Case 2. We suppose that all the operators A, B, C and D are invertible. We set

$$v(x,y) := u(A^{-1}x, B^{-1}y).$$

We can now apply Case 1 to the bilinear form v. We have

$$|v(x,y)| \leqslant \|x\|\|y\| + \|CA^{-1}x\|\|DB^{-1}y\|.$$

Thus by Case 1

$$v = v_1 + v_2,$$

where

$$|v_1(x,y)| \leqslant \|x\|\|y\|, \quad |v_2(x,y)| \leqslant \|CA^{-1}x\|\|DB^{-1}y\|.$$

Setting

$$u_i(x,y) := v_i(Ax, By),$$

Case 2 can be reduced to Case 1.

Case 3. We consider the general case that A is any linear operator in $B(H)$; then A^*A is positive i.e. $A^*A \geqslant 0$. If $\varepsilon > 0$ is any positive number, then $\varepsilon I \geqslant 0$ and $K = A^*A + \varepsilon I \geqslant 0$. Clearly $K \geqslant \varepsilon I$. Therefore K is invertible. In fact, $\mathrm{sp}(K) \subseteq [\varepsilon, \infty)$. This means $0 \notin \mathrm{sp}(K)$ which is equivalent to K being invertible. Thus $A^*A + \varepsilon I$ is invertible for $\varepsilon > 0$. So

$$A(\varepsilon) := (A^*A + \varepsilon I)^{1/2}$$

is invertible. Set

$$A(\varepsilon) := (A^*A + \varepsilon 1)^{1/2}, \quad B(\varepsilon) := (B^*B + \varepsilon 1)^{1/2},$$
$$C(\varepsilon) := (C^*C + \varepsilon 1)^{1/2}, \quad D(\varepsilon) := (D^*D + \varepsilon 1)^{1/2}.$$

Now from the polar decomposition, we can represent any operator $A \in B(H)$ by $A = u|A|$ where u is a partial isometry on H, so

$$\|Ax\| = \|u|A|x\| \leqslant \||A|x\|, \quad x \in H.$$

Since

$$0 \leqslant |A| = (A^*A)^{1/2} \leqslant (A^*A + \varepsilon I)^{1/2},$$

it follows that

$$\||A|x\| \leqslant \|A(\varepsilon)x\|, \quad x \in H.$$

Hence, we have

$$|u(x,y)| \leqslant \|A(\varepsilon)x\|\|B(\varepsilon)y\| + \|C(\varepsilon)x\|\|D(\varepsilon)y\|.$$

Then, from Case 2

$$u = u_1^{\varepsilon} + u_2^{\varepsilon}, \tag{1.2}$$

such that

$$|u_1^\varepsilon(x,y)| \leqslant \|A(\varepsilon)x\|\|B(\varepsilon)y\|, \tag{1.3}$$

$$|u_2^\varepsilon(x,y)| \leqslant \|C(\varepsilon)x\|\|D(\varepsilon)y\|. \tag{1.4}$$

Take $0 < \varepsilon < 1$, then $A(\varepsilon) \leqslant A(1)$. Therefore,

$$\|A(\varepsilon)\| \leqslant \|A(1)\|.$$

In fact, the norms of $\|A(\varepsilon)\|$ are uniformly bounded for all $0 < \varepsilon < 1$. So from the estimations in (1.3) and (1.4)

$$\|u_1^\varepsilon\| \leqslant N_1 := \|A(1)\|\|B(1)\|, \tag{1.5}$$

$$\|u_2^\varepsilon\| \leqslant N_2 := \|C(1)\|\|D(1)\|. \tag{1.6}$$

We know from the universal property of projective tensor product (see Proposition 1.4 from [7]) that,

$$\mathrm{Bil}(H,K) = (H \otimes_\pi K)',$$

so, there is $w \in (H \otimes_\pi K)'$ such that

$$u(x,y) = w(x \otimes y), \quad x \in H \text{ and } y \in K.$$

Set

$$M := N_1 + N_2,$$

and let

$$S = \{w \in (H \otimes_\pi K)' : \|w\| \leqslant M\}.$$

By Banach–Alaoglu theorem, S is weak*-compact. Choose two sequences $\{w_1^{(n)}\}$ and $\{w_2^{(n)}\}$ in $(H \otimes_\pi K)'$ such that

$$w_1^{(n)}(x \otimes y) = u_1^{(1/n)}(x,y) \quad n \in \mathbb{N},$$
$$w_2^{(n)}(x \otimes y) = u_2^{(1/n)}(x,y) \quad n \in \mathbb{N}.$$

So from the definition of S and (1.5), (1.6), theses sequences $\{w_1^{(n)}\}$ and $\{w_2^{(n)}\}$ are in S. Hence, they have convergent subsequences $\{w_1^{(n_k)}\}$ and $\{w_2^{(n_k)}\}$ respectively. Thus, when $k \to \infty$,

$$w_1^{(n_k)} \xrightarrow{w^*} w_1, \quad w_2^{(n_k)} \xrightarrow{w^*} w_2.$$

Also, $\varepsilon \to 0$ when $k \to \infty$. So from the inequalities in (1.3) and (1.4) we get:

$$\begin{aligned} |w_1(x \otimes y)| &= \Big| \lim_{k\to\infty} w_1^{(n_k)}(x \otimes y) \Big| = \lim_{k\to\infty} |w_1^{(n_k)}(x \otimes y)| \leqslant \lim_{\varepsilon\to 0} \|A(\varepsilon)x\|\|B(\varepsilon)y\| \\ &= \||A|x\|\||B|y\| = \|Ax\|\|By\|. \end{aligned}$$

Therefore, $w_1 \in (H \otimes_\pi K)'$. Similary for w_2, we have

$$\begin{aligned} |w_2(x \otimes y)| &= \Big| \lim_{k\to\infty} w_2^{(n_k)}(x \otimes y) \Big| = \lim_{k\to\infty} |w_2^{(n_k)}(x \otimes y)| \leqslant \lim_{\varepsilon\to 0} \|C(\varepsilon)x\|\|D(\varepsilon)y\| \\ &= \||C|x\|\||D|y\| = \|Cx\|\|Dy\|. \end{aligned}$$

Also, $w_2 \in (H \otimes_\pi K)'$. Set

$$w_1(x \otimes y) = u_1(x,y), \quad w_2(x \otimes y) = u_2(x,y).$$

From (1.2)

$$u(x,y) = w_1^{(n_k)}(x \otimes y) + w_2^{(n_k)}(x \otimes y).$$

Now, take the limit point when $k \to \infty$, we get

$$u(x,y) = w_1(x \otimes y) + w_2(x \otimes y).$$

By construction,

$$u(x,y) = u_1(x,y) + u_2(x,y)$$

such that

$$|u_1(x,y)| \leqslant \|Ax\|\|By\|, \quad |u_2(x,y)| \leqslant \|Cx\|\|Dy\|.$$

and

$$|u(x,y)| = |u_1(x,y) + u_2(x,y)| \leqslant \|Ax\|\|By\| + \|Cx\|\|Dy\|.$$

Hence Case 3 follows from Case 2. So

$$\text{Case 1} \Rightarrow \text{Case 2} \Rightarrow \text{Case 3}$$

therefore if we prove Case 1, we are done. Case 1 will follow from the next lemma, ending the proof of the theorem. ∎

LEMMA 1.3. *Let H and K be Hilbert spaces and let $C \in B(H)$ and $D \in B(K)$ be invertible. Assume that a bilinear form $u : H \times K \to \mathbb{C}$ satisfies*

$$|u(x,y)| \leqslant \|x\|\|y\| + \|Cx\|\|Dy\| \tag{1.7}$$

for all $x \in H$ and $y \in K$. Then there are bilinear forms $u_1, u_2 : H \times K \longrightarrow \mathbb{C}$ such that

$$u = u_1 + u_2$$

and

$$|u_1(x,y)| \leqslant \|x\|\|y\|, \quad |u_2(x,y)| \leqslant \|Cx\|\|Dy\|$$

for all $x \in H$ and $y \in K$

Proof. Let $H\widehat{\otimes}_\pi K$ be the projective tensor product of H and K. This space is isometrically isomorphic to the space $B_1(\overline{H}, K)$ of trace-class operators from $\overline{H}$ into K [2] . Here, $\overline{H}$ denotes the conjugate Hilbert space of H. Let

$$w = \sum_{i=1}^{m} x_i \otimes y_i$$

be a linear combination of elementary tensors in $H\widehat{\otimes}_\pi K$. As in Xu's paper, then the corresponding linear map $T_w : \overline{H} \to K$ given by

$$T_w\zeta = \sum_{i=1}^{m} \langle x_i|\zeta\rangle y_i, \quad \zeta \in H$$

is a finite rank operator and the projective norm π of w is given by

$$\|w\|_\pi = \|T_w\|_1 = \mathrm{Tr}((T_w^* T_w)^{1/2}).$$

Therefore, by Theorem 18.13 of [1], we can find orthogonal vectors $\{\xi_1, \dots, \xi_n\} \in H$ and $\{\eta_1, \dots, \eta_n\} \in K$ such that

$$w = \sum_{i=1}^{n} \xi_i \otimes \eta_i \quad \text{and} \quad \|w\|_\pi = \sum \|\xi_i\|^2 = \sum \|\eta_i\|^2,$$

where n is the rank of (T_w).

In the same way

$$(C \otimes D)w = \sum_{i=1}^{n} C\xi_i \otimes D\eta_i$$

can be written as

$$(C \otimes D)w = \sum_{i=1}^{n'} \rho_i \otimes \sigma_i.$$

By the invertibility of C and D,

$$n' = \operatorname{rank}(T_{(C\otimes D)w}) = \operatorname{rank}(T_w) = n$$

and

$$\|(C \otimes D)w\|_\pi = \Big(\sum_{i=1}^{n} \|\rho_i\|^2\Big) = \Big(\sum_{i=1}^{n} \|\sigma_i\|^2\Big),$$

for orthogonal vectors $\{\rho_1, \dots, \rho_n\} \in H$ and $\{\sigma_1, \dots, \sigma_n\} \in K$.

Since

$$\sum_{i=1}^{n} C\xi_i \otimes D\eta_i = \sum_{i=1}^{n} \rho_i \otimes \sigma_i,$$

we have by the linear independence of each of the sets $(C\xi_i)_{i=1}^n$, $(D\eta_i)_{i=1}^n$, $(\rho_i)_{i=1}^n$ and $(\sigma_i)_{i=1}^n$ that

$$C\xi_i = \sum_{j=1}^{n} \alpha_{ij}\rho_j \quad \text{and} \quad D\eta_i = \sum_{j=1}^{n} \beta_{ij}\sigma_j$$

for unique $\alpha_{ij}, \beta_{ij} \in \mathbb{C}$. Moreover, since

$$\sum_{i=1}^{n} \rho_i \otimes \sigma_i = \sum_{i=1}^{n} C\xi_i \otimes D\eta_i = \sum_{i,j,k=1}^{n} \alpha_{ij}\beta_{ik}\rho_j \otimes \sigma_k$$

and from linear independence, we must have

$$\sum_{j=1}^{n} \alpha_{ji}\beta_{jk} = \delta_{ik}.$$

Hence the matrices

$$\alpha = (\alpha_{ij}) \quad \text{where } i, j = \{1, \dots, n\}, \quad \text{and} \quad \beta = (\beta_{i,j}) \quad \text{where } i, j = \{1, \dots, n\},$$

are invertible and $\beta^{-1} = (\alpha^{\mathrm{t}})$ where α^{t} is the transpose of α. Write now

$$\alpha = UdV$$

where $U, V \in U(n)$ and $d = \mathrm{diag}(d_1, \ldots, d_n)$ is a diagonal matrix with strictly positive entries $d_1, \ldots, d_n$, see [8]. Set

$$\widehat{\xi}_i = \sum_{j=1}^{n} u^*_{ij}\xi_j = \sum_{j=1}^{n} \overline{u}_{ji}\xi_j \quad \text{and} \quad \widehat{\rho}_i = \sum_{i=1}^{n} v_{ij}\rho_j;$$

then we obtain

$$C\xi_i = \sum_{j=1}^{n} u_{ij} d_j \widehat{\rho}_j.$$

Now

$$\beta = (\alpha^{\mathrm{t}})^{-1} = (v^{\mathrm{t}} d u^{\mathrm{t}})^{-1} = \overline{u} d^{-1} \overline{v}.$$

Then setting

$$\widehat{\eta}_i = \sum_{j=1}^{n} \overline{u}^*_{ij}\eta_j = \sum_{j=1}^{n} u_{ji}\eta_j \quad \text{and} \quad \widehat{\sigma}_i = \sum_{i=1}^{n} \overline{v}_{ij}\sigma_j,$$

we obtain similarly

$$D\eta_i = \sum_{j=1}^{n} \overline{u}_{ij} d_j^{-1} \widehat{\sigma}_j.$$

Thus

$$C(\widehat{\xi}_i) = \sum_{j=1}^{n} \overline{u}_{ji} C(\xi_j) = \sum_{j,k=1}^{n} \overline{u}_{ji} u_{jk} d_k \widehat{\rho}_k = \sum_{k=1}^{n} \delta_{ik} d_k \widehat{\rho}_k = d_i \widehat{\rho}_i,$$

and we obtain similarly,

$$D(\widehat{\eta}_i) = \sum_{j=1}^{n} u_{ji} D(\eta_j) = \sum_{j,k=1}^{n} u_{ji} \overline{u}_{jk} d_k^{-1} \widehat{\sigma}_k = \sum_{k=1}^{n} \delta_{ik} d_k^{-1} \widehat{\sigma}_k = d_i^{-1} \widehat{\sigma}_i.$$

Since

$$\sum_{i=1}^{n} \widehat{\xi}_i \otimes \widehat{\eta}_i = \sum_{i,j,k=1}^{n} \overline{u}_{ji} \xi_j \otimes u_{ki} \eta_k = \sum_{j,k=1}^{n} \delta_{jk} \xi_j \otimes \eta_k = \sum_{j=1}^{n} \xi_j \otimes \eta_j = w,$$

this implies

$$(C \otimes D)w = \sum_{j=1}^{n} d_j \widehat{\rho}_j \otimes d_j^- 1 \widehat{\sigma}_j = \sum_{j=1}^{n} \widehat{\rho}_j \otimes \widehat{\sigma}_j.$$

Now

$$\sum_{i=1}^{n} \|\widehat{\xi}_j\|^2 = \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} \overline{u}_{ji} \xi_j \Big\| = \sum_{i=1}^{n} \sum_{j=1}^{n} |u_{ij}|^2 \|\xi_i\|^2 = \sum_{i=1}^{n} \|\xi_i\|^2,$$

and similarly,

$$\sum_{i=1}^{n} \|\widehat{\eta}_j\|^2 = \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} u_{ji} \eta_j \Big\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |u_{ji}|^2 \|\eta_j\|^2 = \sum_{i=1}^{n} \|\eta_i\|^2.$$

Also

$$\sum_{i=1}^{n} \|\widehat{\rho}_i\|^2 = \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} v_{ij} \rho_j \Big\|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} |v_{ij}|^2 \|\rho_j\|^2 = \sum_{i=1}^{n} \|\rho_i\|^2$$

and similarly,

$$\sum_{i=1}^{n} \|\widehat{\sigma}_i\|^2 = \sum_{i=1}^{n} \Big\| \sum_{j=1}^{n} \overline{v}_{ij}\sigma_j \Big\|^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} |\overline{v}_{ij}|^2 \|\sigma_j\|^2 = \sum_{i=1}^{n} \|\sigma_i\|^2.$$

Therefore

$$\|w\|_\pi = \sum_{j=1}^{n} \|\widehat{\xi}_j\|^2 = \sum_{j=1}^{n} \|\widehat{\eta}_j\|^2, \quad \|(C\otimes D)w\|_\pi = \sum_{j=1}^{n} \|\widehat{\rho}_j\|^2 = \sum_{j=1}^{n} \|\widehat{\sigma}_j\|^2.$$

Hence

$$\begin{aligned}
\Big|\sum_{i=1}^{n} u(x_i,y_i)\Big| &= \Big|\sum_{i=1}^{n} u(\widehat{\xi}_i,\widehat{\eta}_i)\Big| \leqslant \sum_{i=1}^{n} \|\widehat{\xi}_i\|\|\widehat{\eta}_i\| + \sum_{i=1}^{n} \|C\widehat{\xi}_i\|\|D\widehat{\eta}_i\| \\
&= \sum_{i=1}^{n} \|\widehat{\xi}_i\|\|\widehat{\eta}_i\| + \sum_{i=1}^{n} \|\widehat{\rho}_i\|\|\widehat{\sigma}_i\| \\
&\leqslant \Big(\sum_{i=1}^{n}(\|\widehat{\xi}_i\|)^2\Big)^{1/2}\Big(\sum_{i=1}^{n}\|\eta_i\|^2\Big)^{1/2} + \Big(\sum_{i=1}^{n}(\|\widehat{\rho}_i\|)^2\Big)^{1/2}\Big(\sum_{i=1}^{n}\|\sigma_i\|^2\Big)^{1/2} \\
&= \Big\|\sum x_i\otimes y_i\Big\|_\pi + \Big\|\sum Cx_i\otimes Dy_i\Big\|_\pi. \qquad (1.8)
\end{aligned}$$

If V and W are Banach spaces we denote by $V\oplus_1 W$ the direct sum of V and W endowed with the norm

$$\|(v,w)\| = \|v\| + \|w\|.$$

Let E be the linear span of all vectors $(x\otimes y, C(x)\otimes D(y))$ in $(H\widehat{\otimes}_\pi K)\oplus_1(H\widehat{\otimes}_\pi K)$ where $x\in H$ and $y\in K$. According to the above estimate in (1.8) we find a bounded linear functional $w\in E^*$ with $\|w\|\leqslant 1$ such that

$$u(x,y) = w((x\otimes y, C(x)\otimes D(y)))$$

for all $x\in H$ and $y\in K$. By the Hahn–Banach theorem there exists a bounded linear functional $\widetilde{w}$ on $(H\widehat{\otimes}_\pi K)\oplus_1(H\widehat{\otimes}_\pi K)$ with $\|\widetilde{w}\| = \|w\|\leqslant 1$ extending w. We set

$$u_1(x,y) = \widetilde{w}((x\otimes y,0)), \quad u_2(x,y) = \widetilde{w}(0, C(x)\otimes D(y)).$$

By construction we have

$$u = u_1 + u_2.$$

Moreover

$$\begin{aligned}
&|u_1(x,y)| \leqslant \|\widetilde{w}\|\|x\|\|y\| \leqslant \|x\|\|y\| \quad \text{and} \\
&|u_2(x,y)| \leqslant \|\widetilde{w}\|\|C(x)\|\|D(y)\| \leqslant \|C(x)\|\|D(y)\|.
\end{aligned}$$

This yields the claim. ∎

2. DECOMPOSITION OF BILINEAR FORMS INTO n TERMS

In this section, we discuss the problem of decomposing into n bounded terms. Turning to the finite dimensional case, we find a criterion to make the decomposition possible for n terms. We will work with sesquilinear forms instead of bilinear ones. We begin with a lemma which provides a bijective correspondence between bounded operators on H and bounded sesquilinear forms. This is well known, see [5].

LEMMA 2.1 ([5]). *There is a bijective correspondence $A \mapsto b_A$ between bounded operators on H and bounded sesquilinear forms given by*

$$b_A(x,y) = \langle Ax|y\rangle \quad x,y \in H.$$

One has

$$\|A\| = \sup\{|b_A(x|y)| : \|x\|, \|y\| \leqslant 1\}.$$

Now we come to the main theorem in this section.

THEOREM 2.2. *Let H be a finite-dimensional Hilbert space and let $A_2,\dots,A_n$ and $B_2,\dots,B_n$ be invertible operators in $B(H)$. Assume that $U \in B(H)$ is a bounded operator which satisfies*

$$|\langle Ux|y\rangle| \leqslant \|x\|\|y\| + \|A_2x\|\|B_2y\| + \cdots + \|A_nx\|\|B_ny\|$$

for all $x,y \in H$.

Then the following two conditions are equivalent:

(i) *U can be split into a sum of n-terms*

$$U = U_1 + U_2 + \cdots + U_n, \quad U_i \in B(H),$$

such that

$$\begin{aligned} |\langle U_1x|y\rangle| &\leqslant \|x\|\|y\|, \\ |\langle U_2x|y\rangle| &\leqslant \|A_2x\|\|B_2y\|, \\ &\vdots \\ |\langle U_nx|y\rangle| &\leqslant \|A_nx\|\|B_ny\|, \end{aligned}$$

for all $x,y \in H$.

(ii) *If we set*

$$K = \{x\underline{\otimes}y : \|x\|\|y\| + \|A_2x\|\|B_2y\| + \cdots + \|A_nx\|\|B_ny\| \leqslant 1\}$$

(where $x\underline{\otimes}y$ denotes a rank one operator) and

$$\Delta = \{T \in B(H) : \|T\|_1 + \|A_2TB_2^*\|_1 + \cdots + \|A_nTB_n^*\|_1 \leqslant 1\},$$

(where $\|S\|_1 = \mathrm{tr}|S|$ denotes the trace class norm of S), then

$$\mathrm{conv}(K) = \Delta$$

(conv denotes the convex hull).

Proof. (ii)⇒(i) For any bounded operator $U \in B(H)$, by trace duality (see [3]) we can associate a linear functional ϕ on $B(H)$ such that,

$$\phi(T) = \mathrm{tr}(UT), \quad T \in B(H).$$

Hence,

$$\phi(x\underline{\otimes}y) = \mathrm{tr}(Ux\underline{\otimes}y) = \langle Ux|y\rangle,$$

and therefore,

$$|\langle Ux|y\rangle| = |\phi(x\underline{\otimes}y)| \leqslant \|x\underline{\otimes}y\|_1 + \|A_2x\underline{\otimes}B_2y\|_1 + \cdots + \|A_nx\underline{\otimes}B_ny\|_1 \leqslant 1$$

for all $x\underline{\otimes}y \in K$. By assumption,

$$\mathrm{conv}(K) = \Delta.$$

So any $T \in \Delta$ has the form $T = \sum_{i=1}^{n} \lambda_i x_i\underline{\otimes}y_i$, where

$$\sum_{i=1}^{n} \lambda_i = 1 \quad \text{and} \quad x_i\underline{\otimes}y_i \in K.$$

Therefore,

$$\begin{aligned} \Big|\sum_{i=1}^{n}\langle Ux_i|y_i\rangle\Big| &= \Big|\phi\Big(\sum_{i=1}^{n} x_i\underline{\otimes}y_i\Big)\Big| = \Big|\phi\Big(\sum_{i=1}^{n} \lambda_i x_i\underline{\otimes}y_i\Big)\Big| \\ &= \Big|\sum_{i=1}^{n}\lambda_i\langle Ux_i|y_i\rangle\Big| \leqslant \sum_{i=1}^{n}\lambda_i|\langle Ux_i|y_i\rangle| \\ &\leqslant \sum_{i=1}^{n}\lambda_i[\|x_i\underline{\otimes}y_i\|_1 + \|A_2x_i\underline{\otimes}B_2y_i\|_1 + \cdots + \|A_nx_i\underline{\otimes}B_ny_i\|_1] \\ &\leqslant \sum_{i=1}^{n}\lambda_i = 1. \end{aligned} \tag{2.1}$$

Let $E = \mathrm{span}\{(x \otimes y, A_2x \otimes B_2y, \ldots, A_nx \otimes B_ny) : x, y \in H\} \subseteq H \otimes H \oplus H \otimes H \oplus \cdots \oplus H \otimes H$.

By (2.1), we can find a bounded linear functional ϕ on E with

$$\|\phi\| = \sup\Big\{\frac{|\phi(t)|}{\|t\|} : t \in E, t \neq 0\Big\} = \sup\{|\phi(t)| : t \in E, \|t\| \leqslant 1\} \leqslant 1,$$

such that,

$$\langle Ux|y\rangle = \phi((x \otimes y, A_2x \otimes B_2y, \ldots, A_nx \otimes B_ny)), \quad x, y \in H.$$

Hence by the Hahn–Banach theorem there is an extension $\widetilde{\phi}$ of ϕ to all $H \otimes H \oplus H \otimes H \oplus \cdots \oplus H \otimes H$ with $\|\widetilde{\phi}\| = \|\phi\| \leqslant 1$. If we set

$$\begin{aligned}
\langle U_1 x|y\rangle &= \widetilde{\phi}((x \otimes y, 0, \ldots, 0)),\\
\langle U_2 x|y\rangle &= \widetilde{\phi}(0, A_2(x) \otimes B_2(y), \ldots, 0),\\
&\vdots\\
\langle U_n x|y\rangle &= \widetilde{\phi}(0, 0, \ldots, A_n(x) \otimes B_n(y)),
\end{aligned}$$

then by construction we have

$$U = U_1 + U_2 + \cdots + U_n$$

and

$$\begin{aligned}
|\langle U_1 x|y\rangle| &\leqslant \|\widetilde{\phi}\|\|x\|\|y\| \leqslant \|x\|\|y\|,\\
|\langle U_2 x|y\rangle| &\leqslant \|\widetilde{\phi}\|\|A_2 x\|\|B_2 y\| \leqslant \|A_2 x\|\|B_2 y\|,\\
&\vdots\\
|\langle U_n x|y\rangle| &\leqslant \|\widetilde{\phi}\|\|A_n x\|\|B_n y\| \leqslant \|A_n x\|\|B_n y\|.
\end{aligned}$$

(i)$\Rightarrow$(ii) Assume (i). If (ii) does not hold, we can choose $T_0 \in \Delta \setminus \mathrm{conv}(K)$. Since $\mathrm{conv}(K)$ is closed, there exists by the Hahn–Banach theorem a functional ϕ on $B_1(H) = B(H)^*$, such that

$$\sup\{\mathrm{Re}\phi(T) : T \in \mathrm{conv}(K)\} < \mathrm{Re}\phi(T_0).$$

Since $T \in K \Rightarrow \gamma T \in K$ for all $\gamma \in \mathbb{C}$ with $|\gamma| = 1$, we have

$$\sup\{(\mathrm{Re}\phi(T)) : T \in \mathrm{conv}(K)\} = \sup\{|\phi(T)| : T \in \mathrm{conv}(K)\} \geqslant 0.$$

Moreover,

$$\mathrm{Re}\phi(T_0) \leqslant |\phi(T_0)|.$$

Hence,

$$\sup\{|\phi(T)| : T \in \mathrm{conv}(K)\} < |\phi(T_0)|.$$

By replacing ϕ by a positive multiple of ϕ we can without loss of generality, assume that

$$\sup\{|\phi(T)| : T \in \mathrm{conv}(K)\} \leqslant 1 < |\phi(T_0)|. \tag{2.2}$$

Using the standard duality $B_1(H)^* = B(H)$ there is a unique $U \in B(H)$, such that

$$\phi(T) = \mathrm{Tr}(UT), \quad \forall T \in B_1(H).$$

By (2.2), we have for $x, y \in H$ satisfying

$$\|x\|\|y\| + \|A_2 x\|\|B_2 y\| + \cdots + \|A_n x\|\|B_n y\| = 1, \tag{2.3}$$

that $T = x \underline{\otimes} \overline{y} \in K$ and thus

$$|\langle Ux|y\rangle| = |\mathrm{Tr}(U(x \underline{\otimes} \overline{y}))| = |\phi(x \otimes \overline{y})| \leqslant 1, \tag{2.4}$$

and since (2.3) $\Rightarrow$ (2.4) we have by linearity in x, that

$$|\langle Ux|y\rangle| \leqslant \|x\|\|y\| + \|A_2x\|\|B_2x\| + \cdots + \|A_ny\|\|B_ny\|, \quad \forall x, y \in H.$$

However,

$$|\mathrm{Tr}(UT_0)| = |\phi(T_0)| > 1. \tag{2.5}$$

Now by (i), U has a decomposition:

$$U = U_1 + U_2 + \cdots + U_n.$$

Therefore

$$|\mathrm{Tr}(UT_0)| \leqslant |\mathrm{Tr}(U_1T_0)| + |\mathrm{Tr}(U_2T_0)| + \cdots + |\mathrm{Tr}(U_nT_0)| \tag{2.6}$$

also from (i):

$$|\langle U_1x|y\rangle| \leqslant \|x\|\|y\|$$

for all x, y in H.

Hence

$$\|U_1\| = \sup\{|\langle U_1x|y\rangle| : \|x\| \leqslant 1, \|y\| \leqslant 1\} \leqslant 1.$$

Now by using Theorem 1.51(e) from [1], we get

$$|\mathrm{Tr}(U_1T_0)| \leqslant \|U_1\|\|T_0\|_1 \leqslant \|T_0\|_1 \tag{2.7}$$

where $U_1 \in B(H)$ and $T_0 \in L^1(H)$.

For the second term i.e. $|\mathrm{Tr}(U_2T_0)|$ in (2.6), if we define a new sesquilinear form v as

$$v(x, y) := u_2(A_2^{-1}x, B_2^{-1}y),$$

then from Lemma 2.1 there is $V \in B(H)$ such that

$$v(x, y) = u_2(A_2^{-1}x, B_2^{-1}y) = \langle Vx|y\rangle$$

for all $x, y \in H$.

Therefore also by Lemma 2.1, there is $U_2 \in B(H)$ satisfying

$$u_2(x, y) = \langle U_2x|y\rangle = v(A_2x, B_2y) = \langle VA_2x|B_2y\rangle$$

for all $x, y \in H$.

Hence,

$$\begin{aligned}\mathrm{Tr}(U_2T_0) &= \mathrm{Tr}\Big(\sum U_2x_j \underline{\otimes} y_j\Big) = \Big(\sum \langle U_2x_j|y_j\rangle\Big) = \sum \langle VA_2x_j|B_2y_j\rangle \\ &= \mathrm{Tr}\Big(\sum VA_2x_j \underline{\otimes} B_2y_j\Big) = \mathrm{Tr}\Big(V\Big[\sum A_2x_j \underline{\otimes} B_2y_j\Big]\Big) = \mathrm{Tr}(VA_2T_0B_2^*).\end{aligned}$$

Therefore,

$$|\mathrm{Tr}(U_2T_0)| = |\mathrm{Tr}(VA_2T_0B_2^*)| \leqslant \|V\|\|A_2T_0B_2^*\|_1.$$

From the definition of v, we can easily get that

$$\begin{aligned}\|V\| &= \sup\{|\langle VA_2x|B_2y\rangle| : \|A_2x\| \leqslant 1, \|B_2y\| \leqslant 1\} \\ &= \sup\{|\langle U_2x|y\rangle| : \|A_2x\| \leqslant 1, \|B_2y\| \leqslant 1\} \leqslant 1\end{aligned}$$

(where we use $|\langle U_2x|y\rangle| \leqslant \|A_2x\|\|B_2y\|$, from (i)).

Thus,

$$|\mathrm{Tr}(U_2T_0)| \leqslant \|A_2T_0B_2^*\|_1. \tag{2.8}$$

Similarly for the rest of the terms (for $n \geqslant 3$) in the above inequality (2.6). We can define a new sesquilinear form w by,

$$w(x,y) := u_n(A_n^{-1}x, B_n^{-1}y).$$

Also, from Lemma 2.1 there is $W \in B(H)$ such that,

$$w(x,y) = u_n(A_n^-1x, B_n^-1y) = \langle Wx|y\rangle$$

for all $x, y \in H$.

So, also by Lemma 2.1 there is $U_n \in B(H)$ such that

$$\langle U_nx|y\rangle = u_n(x,y) = w(A_nx, B_ny) = \langle WA_nx|B_ny\rangle.$$

Hence,

$$\begin{aligned}\mathrm{Tr}(U_nT_0) &= \mathrm{Tr}\Big(\sum U_nx_j\underline{\otimes}y_j\Big) = \Big(\sum\langle U_nx_j|y_j\rangle\Big) = \sum\langle WA_nx_j|B_ny_j\rangle\\ &= \mathrm{Tr}\Big(\sum WA_nx_j\underline{\otimes}B_ny_j\Big) = \mathrm{Tr}\Big(W\Big[\sum A_nx_j\underline{\otimes}B_ny_j\Big]\Big) = \mathrm{Tr}(WA_nT_0B_n^*).\end{aligned}$$

Therefore,

$$|\mathrm{Tr}(U_nT_0)| = |\mathrm{Tr}(WA_nT_0B_n^*)| \leqslant \|W\|\|A_nT_0B_n^*\|_1.$$

Also from the definition of w, we can easily see that

$$\begin{aligned}\|W\| &= \sup\{|\langle WA_nx|B_ny\rangle| : \|A_nx\| \leqslant 1, \|B_ny\| \leqslant 1\}\\ &= \sup\{|\langle U_nx|y\rangle| : \|A_nx\| \leqslant 1, \|B_ny\| \leqslant 1\} \leqslant 1,\end{aligned}$$

(where we use $|\langle U_nx|y\rangle| \leqslant \|A_nx\|\|B_ny\|$, from (i)).

Thus,

$$|\mathrm{Tr}(U_nT_0)| \leqslant \|A_nT_0B_n^*\|_1. \tag{2.9}$$

Finally from inequality (2.6),

$$\begin{aligned}|\mathrm{Tr}(UT_0)| &\leqslant |\mathrm{Tr}(U_1T_0)| + |\mathrm{Tr}(U_2T_0)| + \cdots + |\mathrm{Tr}(U_nT_0)|\\ &\leqslant \|T_0\|_1 + \|A_2T_0B_2^*\|_1 + \cdots + \|A_nT_0B_n^*\|_1 \leqslant 1.\end{aligned}$$

But this contradicts (2.5). Thus (i) does not hold and we have proved that (i)⇒(ii). ∎

3. A COUNTEREXAMPLE TO DECOMPOSING BILINEAR FORMS INTO THREE BILINEAR FORMS

In this section, we will use the criterion established in the previous section to give a counterexample which will show that the decomposition of a bilinear form into three bounded terms is not always possible. We will start with a useful lemma.

LEMMA 3.1. *Let H be a finite dimensional Hilbert space. If S is in $B_1(H)_+ = B(H)_+$ and*

$$S = \sum_{j=1}^{m} x_j \underline{\otimes} y_j,$$

such that

$$\|S\|_1 = \sum_{j=1}^{m} \|x_j\|\|y_j\| \tag{3.1}$$

then each y_j is a positive multiple of x_j.

3.1. A COUNTEREXAMPLE. We use Theorem 2.2 to build the counterexample. We prove that the condition

$$\operatorname{conv}(K) = \Delta$$

with

$$K = \{x \underline{\otimes} y : \|x\|\|y\| + \|Ax\|\|By\| + \|Cx\|\|Dy\| \leqslant 1\}$$

and

$$\Delta = \{T \in B(H) : \|T\|_1 + \|ATB^*\|_1 + \|CTD^*\|_1 \leqslant 1\}$$

is not always true. Therefore the decomposition into three terms fails. Consider the Hilbert space $H = \mathbb{C}^2$. Consider the operators $B = D = I$ and A, C positive invertible and not commuting. In particular A and C do not have any common eigenvectors.

Put $c := (\|1\|_1 + \|A\|_1 + \|C\|_1)^{-1}$ and take $T = c1$. Now we will show that $T \in \Delta$ but $T \notin \operatorname{conv} K$. From the definition of Δ,

$$\Delta = \{T \in B(H) : \|T\|_1 + \|ATB^*\|_1 + \|CTD^*\|_1 \leqslant 1\}.$$

For $T = c1$, we find

$$\begin{aligned} \|T\|_1 + \|ATB^*\|_1 + \|CTD^*\|_1 &= c\|1\|_1 + c\|A \cdot 1\|_1 + c\|C \cdot 1\|_1 \\ &= c(\|1\|_1 + \|A\|_1 + \|C\|_1) \\ &= (\|1\|_1 + \|A\|_1 + \|C\|_1)^{-1}(\|1\|_1 + \|A\|_1 + \|C\|_1) = 1. \end{aligned}$$

Therefore,

$$T \in \Delta.$$

It is not difficult to see that the operators T, AT and CT are positive. In fact

$$T = |T| = cI$$

and

$$AT = A(cI) = cA \quad \text{and} \quad CT = C(cI) = cC.$$

Now suppose that,

$$T = \sum_{j=1}^{n} \lambda_j x_j \underline{\otimes} y_j \in \operatorname{conv}(K),$$

i.e.

$$\sum_{j=1}^{n} \lambda_j = 1 \quad \text{and} \quad x_j \underline{\otimes} y_j \in K \quad \text{for all } 1 \leqslant j \leqslant n.$$

We know $T \in \Delta$, in fact

$$\|T\|_1 + \|AT\|_1 + \|CT\|_1 = 1.$$

Moreover,

$$\|T\|_1 \leqslant \sum_{j=1}^{n} \lambda_j \|x_j\| \|y_j\| := M_1, \tag{3.2}$$

$$\|AT\|_1 \leqslant \sum_{j=1}^{n} \lambda_j \|Ax_j\| \|y_j\| := M_2, \tag{3.3}$$

$$\|CT\|_1 \leqslant \sum_{j=1}^{n} \lambda_j \|Cx_j\| \|y_j\| := M_3. \tag{3.4}$$

Also,

$$x_j \underline{\otimes} y_j \in K,$$

whence

$$\|x_j\| \|y_j\| + \|Ax_j\| \|y_j\| + \|Cx_j\| \|y_j\| \leqslant 1.$$

Therefore,

$$\lambda_j \|x_j\| \|y_j\| + \lambda_j \|Ax_j\| \|y_j\| + \lambda_j \|Cx_j\| \|y_j\| \leqslant \lambda_j \Longrightarrow$$
$$\sum_{j=1}^{n} \lambda_j \|x_j\| \|y_j\| + \sum_{j=1}^{n} \lambda_j \|Ax_j\| \|y_j\| + \sum_{j=1}^{n} \lambda_j \|Cx_j\| \|y_j\| \leqslant \sum_{j=1}^{n} \lambda_j = 1.$$

Hence

$$\sum_{j=1}^{n} \lambda_j \|x_j\| \|y_j\| + \sum_{j=1}^{n} \lambda_j \|Ax_j\| \|y_j\| + \sum_{j=1}^{n} \lambda_j \|Cx_j\| \|y_j\| = 1.$$

All the above inequalities (3.2), (3.3) and (3.4) are equalities since the system,

$$(M_1 - N_1) + (M_2 - N_2) + (M_3 - N_3) = 0,$$

and

$$N_1 \leqslant M_1, \quad N_2 \leqslant M_2, \quad N_3 \leqslant M_3,$$

has only the trivial solution,

$$M_1 = N_1, \quad M_2 = N_2, \quad M_3 = N_3,$$

i.e.

$$\|T\|_1 = \sum_{j=1}^{n} \lambda_j \|x_j\| \|y_j\|, \quad \|AT\|_1 = \sum_{j=1}^{n} \lambda_j \|Ax_j\| \|y_j\|, \quad \|CT\|_1 = \sum_{j=1}^{n} \lambda_j \|Cx_j\| \|y_j\|.$$

Applying Lemma 3.1 to the positive operator T, we find

$$y_j = \alpha_j(\lambda_j x_j) \tag{3.5}$$

where α_j is a positive scalar. We can also apply Lemma 3.1 to the positive operators AT and CT to get

$$y_j = \beta_j(\lambda_j A x_j) \tag{3.6}$$

where β_j is also a positive scalar, and

$$y_j = \gamma_j(\lambda_j B x_j) \tag{3.7}$$

for another positive scalar γ_j.

Now from (3.5), (3.6) and (3.7), we have

$$y_j = \lambda_j \alpha_j x_j = \lambda_j \beta_j A x_j = \lambda_j \gamma_j C x_j.$$

Hence

$$A x_j = \Big(\frac{\alpha_j}{\beta_j}\Big) x_j \quad \text{and} \quad C x_j = \Big(\frac{\alpha_j}{\gamma_j}\Big) x_j.$$

Therefore x_j is a common eigenvector for operators A and C but this contradicts our assumption on A, C.

Thus

$$T \notin \mathrm{conv}(K).$$

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MOHAMED ELMURSI, MATHEMATICS DEPARTMENT, SOHAG UNIVERSITY, SOHAG, EGYPT
E-mail address: m_elmursi@yahoo.com

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