J. OPERATOR THEORY
75:2(2016), 387–408
doi: 10.7900/jot.2015may17.2070



# ON THE LIE IDEALS OF $C^*$-ALGEBRAS

LEONEL ROBERT

*Communicated by Kenneth R. Davidson*

ABSTRACT. Various questions on Lie ideals of $C^*$-algebras are investigated. They fall roughly under the following topics: relation of Lie ideals to closed two-sided ideals; Lie ideals spanned by special classes of elements such as commutators, nilpotents, and the range of polynomials; characterization of Lie ideals as similarity invariant subspaces.

KEYWORDS: *Lie ideals, $C^*$-algebras, commutators, nilpotents, polynomials, similarity invariant subspaces.*

MSC (2010): 46L05, 16W10.

## INTRODUCTION

This paper deals with Lie ideals in $C^*$-algebras. Like other investigations on this topic ([5], [16]), we use, and take inspiration from, Herstein's work on the Lie ideals of semiprime rings. The abundance of semiprime ideals in a $C^*$-algebra—e.g., the norm-closed ideals—plus a number of $C^*$-algebra techniques—approximate units, polar decompositions, functional calculus—make it possible to further develop the results of the purely algebraic setting in the $C^*$-algebraic setting.

The contributions in the present paper, though varied, revolve around the following themes: the commutator equivalence of Lie ideals to two-sided ideals; the study of Lie ideals generated by special elements such as nilpotents and projections and by the range of polynomials; the characterization of Lie ideals as subspaces invariant by similarities. These topics have been studied before, and this paper is a direct beneficiary of works such as [5], [6] and [14].

A selection of results in this paper follows: Let $A$ be a $C^*$-algebra. We show below that the following are true:

(i) The closed two-sided ideal generated by the commutators of $A$ is also the $C^*$-algebra generated by the commutators of $A$ (Theorem 1.3).

(ii) The closure of the linear span of the square zero elements agrees with the closure of the linear span of the commutators. If $A$ is unital and without

1-dimensional representations, then the linear span of the square zero elements agrees with the linear span of the commutators (Corollary 2.3 and Theorem 4.2).

(iii) If $A$ is unital and has no bounded traces and $f$ is a nonconstant polynomial in noncommuting variables with coefficients in $\mathbb{C}$, then there exists $N$ such that every element of $A$ is a linear combination of at most $N$ values of $f$ on $A$. If $f(\mathbb{C}) = \{0\}$ (e.g., $f(x,y) = [x,y]$), then there exist $C^*$-algebras where the least such $N$ can be arbitrarily large (Corollary 3.10 and Example 3.11).

(iv) If $A$ is unital and either simple, or without bounded traces, or a von Neumann algebra, then a subspace $U$ of $A$ is a Lie ideal of $A$ if and only if $(1+x)U(1-x) \subseteq U$ for all square zero elements $x$ in $A$ (Corollaries 4.3 and 4.6).

## 1. FROM PURE ALGEBRA TO $C^*$-ALGEBRAS

Let us fix some notation:

*Throughout the paper $A$ denotes a $C^*$-algebra.*

Let $x$ and $y$ be elements in $A$. Then $[x,y]$ denotes the element $xy-yx$ (the commutator of $x$ and $y$). Let $X$ and $Y$ be subsets of $A$. Then $X+Y$, $XY$, and $[X,Y]$ denote the linear spans of the elements of the form $x+y$, $xy$, and $[x,y]$, with $x \in X$ and $y \in Y$, respectively. The linear span of $X$ is denoted by $\mathrm{span}(X)$. The $C^*$-algebra and the closed two-sided ideal generated by $X$ are denoted by $C^*(X)$ and $\mathrm{Id}(X)$, respectively. (For the 2-sided ideal algebraically generated by $X$ we simply write $AXA$.) From the identity $[xy,a] = [x,ya] + [y,ax]$, used inductively, we deduce that

$$[X^n, A] \subseteq [X, A] \tag{1.1}$$

for any set $X \subseteq A$ and all $n \in \mathbb{N}$. We sometimes refer to this fact as the "linearizing property of $[\cdot, A]$".

A subspace $L$ of $A$ is called a Lie ideal if it satisfies that $[L,A] \subseteq L$. We will make frequent use of the following elementary lemma:

LEMMA 1.1. *Let $L$ be a Lie ideal of $A$. Then $A[L,L]A \subseteq L + L^2$.*

*Proof.* We have $[[L,L],A] \subseteq [L,L]$, by Jacobi's identity. Thus,

$$[L,L]A \subseteq A[L,L] + [[L,L],A] \subseteq A[L,L] + [L,L].$$

Multiplying by $A$ on the left we get $A[L,L]A \subseteq A[L,L]$. Finally, from the identity $a[l_1,l_2] = [al_1,l_2] - [a,l_2]l_1$ we deduce that $A[L,L] \subseteq L + L^2$, as desired. ∎

The following theorem of Herstein is the basis of many of our arguments in this section (it holds for semiprime rings without 2-torsion):

THEOREM 1.2 ([13], Theorem 1). *Let $L$ be a Lie ideal of $A$. Then $[t,[t,L]] = 0$ implies $[t,L] = 0$ for all $t \in A$.*

Combining Herstein's theorem and Lemma 1.1 we get the following theorem:

THEOREM 1.3. *The closed two-sided ideal generated by* $[A,A]$ *agrees with the* $C^*$*-algebra generated by* $[A,A]$. *In fact,* $\mathrm{Id}([A,A]) = \overline{[A,A]+[A,A]^2}$.

*Proof.* Let $I = \mathrm{Id}([[A,A],[A,A]])$. Then $[[x,y],[[x,y],A/I]] = 0$ for all $x,y \in A/I$. Herstein's theorem implies that $[[x,y],A/I] = 0$ for all $x,y \in A/I$. That is, $[[A/I,A/I],A/I] = 0$. Herstein's theorem again implies that $[A/I,A/I] = 0$; i.e., $[A,A] \subseteq I$. On the other hand, $I \subseteq \overline{[A,A]+[A,A]^2}$, by Lemma 1.1. So,

$$\mathrm{Id}([A,A]) \subseteq I \subseteq \overline{[A,A]+[A,A]^2} \subseteq C^*([A,A]).$$

Since $C^*([A,A]) \subseteq \mathrm{Id}([A,A])$, these inclusions must be equalities. ∎

The following lemma is easily derived from the existence of approximately central approximate units for the closed two-sided ideals of $A$:

LEMMA 1.4 ([16], Lemma 1, [5], Proposition 5.25). *Let* $I$ *be a closed two-sided ideal of* $A$. *Then*

$$\overline{[I,I]} = \overline{[I,A]} = I \cap \overline{[A,A]}.$$

Brešar, Kissin, and Shulman show in Theorem 5.27 of [5] that $\overline{[L,A]} = \overline{[\mathrm{Id}([L,A]),A]}$ for any Lie ideal $L$ of $A$. In the theorem below we give a short proof of this important theorem:

THEOREM 1.5. *Let* $L$ *be a Lie ideal of* $A$. *Then*

(i) $\mathrm{Id}([L,A]) = \overline{[L,A]+[L,A]^2}$.

(ii) $\overline{[\mathrm{Id}([L,A]),A]} = \overline{[L,A]} = \overline{[[L,A],A]}$.

*Proof.* (i) We follow a line of argument similar to the proof of Theorem 1.3. Let $M = [L,A]$ and $I = \mathrm{Id}([M,M])$. Let $\widetilde{L}$ and $\widetilde{M}$ denote the images of $L$ and $M$ in $A/I$ by the quotient map. Then $[\widetilde{M},[\widetilde{M},A/I]] = 0$. By Herstein's theorem, $[\widetilde{M},A/I] = 0$; i.e., $[[\widetilde{L},A/I],A/I]$. By Herstein's theorem again, $[\widetilde{L},A/I] = 0$; i.e., $[L,A] \subseteq I$. On the other hand, $I \subseteq \overline{M+M^2} = \overline{[L,A]+[L,A]^2}$, by Lemma 1.1. So,

$$\mathrm{Id}([L,A]) \subseteq I \subseteq \overline{[L,A]+[L,A]^2} \subseteq C^*([L,A]).$$

Since $C^*([L,A]) \subseteq \mathrm{Id}([L,A])$, all these inclusions must be equalities.

(ii) By (i) and the linearizing property of $[\cdot,A]$ recalled in (1.1), we have that

$$[\mathrm{Id}([L,A]),A] = [\overline{[L,A]+[L,A]^2},A] \subseteq \overline{[[L,A],A]}.$$

Thus, $\overline{[\mathrm{Id}([L,A]),A]} \subseteq \overline{[[L,A],A]} \subseteq \overline{[L,A]}$. On the other hand,

$$[L,A] \subseteq \mathrm{Id}([L,A]) \cap [A,A] \subseteq \overline{[\mathrm{Id}([L,A]),A]},$$

(the second inclusion by Lemma 1.4). This completes the proof. ∎

LEMMA 1.6. *Let* $L$ *be a closed Lie ideal of* $A$ *such that* $\mathrm{Id}(L) = \mathrm{Id}([L,A])$ *and* $L \subseteq \overline{[A,A]}$. *Then* $L = \overline{[\mathrm{Id}(L),A]}$.

*Proof.* The inclusion $L \subseteq \overline{[\mathrm{Id}(L), A]}$ follows from $L \subseteq \overline{[A, A]} \cap \mathrm{Id}(L)$ and Lemma 1.4. As for the opposite inclusion, we have $\overline{[\mathrm{Id}(L), A]} = \overline{[\mathrm{Id}([L, A]), A]}$, by assumption, and $\overline{[\mathrm{Id}([L, A]), A]} \subseteq L$, by Theorem 1.5. ∎

The following is an improvement on Theorem 1.5(ii) obtained by the same technique:

THEOREM 1.7. *Let K and L be Lie ideals of A. Then* $\overline{[K, L]} = \overline{[\mathrm{Id}([K, L]), A]}$.

*Proof.* Let $M = [K, L]$. Notice that $M$ is again a Lie ideal (by Jacobi's identity). We will deduce that $\overline{M} = \overline{[\mathrm{Id}(M), A]}$ from the previous lemma. We clearly have that $\overline{M} \subseteq \overline{[A, A]}$. Let $I = \mathrm{Id}([M, A])$ and let $\widetilde{K}$, $\widetilde{L}$, and $\widetilde{M}$ denote the images of $K$, $L$, and $M$ in the quotient by this ideal. From $[\widetilde{M}, A/I] = 0$ and $[\widetilde{K}, \widetilde{L}] = \widetilde{M}$ we get that $[[\widetilde{K}, \widetilde{L}], \widetilde{L}] = 0$. By Herstein's theorem, $[\widetilde{K}, \widetilde{L}] = 0$; i.e, $M = [K, L] \subseteq I$. It follows that $\mathrm{Id}(M) = \mathrm{Id}([M, A])$. By Lemma 1.6, $M = \overline{[\mathrm{Id}(M), A]}$, as desired. ∎

REMARK 1.8. The arguments in Theorems 1.3, 1.5, and 1.7 rely crucially on the fact that the closed two-ideals of a $C^*$-algebra are semiprime. This makes it possible to apply Herstein's theorem in the quotient by a closed two-sided ideal. Turning to non-closed Lie ideals, if we impose the semiprimeness of a suitable non-closed two-sided ideal at the outset, part of those same arguments still goes through. We may obtain in this way, for instance, the following result: *If L is a Lie ideal of A such that the two-sided ideal generated by* $[[L, A], [L, A]]$ *is semiprime then* (i) $A[L, A]A = [L, A] + [L, A]^2$, *and* (ii) $[A[L, A]A], A] = [[L, A], A]$. To get (i) we proceed as in Theorem 1.5(i): Setting $M = [L, A]$ and $I = A[M, M]A$ and applying Herstein's theorem in $A/I$ in much the same way as we did in Theorem 1.5(i) we arrive at $[L, A] \subseteq I$. We then have the inclusions $A[L, A]A \subseteq I \subseteq [L, A] + [L, A]^2$, which must in fact be equalities. To get (ii) we apply (i) and the linearizing property of $[\cdot, A]$:

$$[A[L, A]A, A] = [[L, A] + [L, A]^2, A] = [[L, A], A].$$

Next we discuss another variation on Theorem 1.5 for non-closed Lie ideals. This time we make use of the Pedersen ideal. Recall that the Pedersen ideal of a $C^*$-algebra is the smallest dense two-sided ideal of the algebra (see 5.6 of [17]). Given a $C^*$-algebra $B$, we denote its Pedersen ideal by $\mathrm{Ped}(B)$.

LEMMA 1.9. *Let I be a closed two-sided ideal of A. Then*

$$[\mathrm{Ped}(I), \mathrm{Ped}(I)] = [\mathrm{Ped}(I), A].$$

*Proof.* Let $P = \mathrm{Ped}(I)$. The subspace $P^2$ is a dense two-sided ideal of $I$. Since $P$ is the minimum such ideal, we must have that $P = P^2$. From $[P, A] = [P^2, A]$ and the identity $[xy, a] = [x, ya] + [y, ax]$ we get that $[P^2, A] \subseteq [P, P]$. ∎

THEOREM 1.10. *Let L be a Lie ideal of A and let* $P = \mathrm{Ped}(\mathrm{Id}([L, A]))$. *Then*

$$[P, P] = [L, P] = [[L, A], P].$$

*Furthermore, if $L \subseteq P$ then $[L, A] = [P, P]$.*

*Proof.* In the course of proving Theorem 1.5 we have shown that $\mathrm{Id}([L, A]) = \mathrm{Id}([[L, A], [L, A]])$. Therefore, the two-sided ideal $A[[L, A], [L, A]]A$ is dense in $\mathrm{Id}([L, A])$. Since $P$ is the smallest such ideal, $P \subseteq A[[L, A], [L, A]]A$. Hence,

$$[P, P] \subseteq [A[[L, A], [L, A]]A, P] \subseteq [[L, A] + [L, A]^2, P] \subseteq [[L, A], P] \subseteq [L, P].$$

But $[L, P] \subseteq [P, P]$, by Lemma 1.9 . Thus, the inclusions above must be equalities.

Suppose now that $L \subseteq P$. Then $[L, P] \subseteq [L, A] \subseteq [P, A] = [P, P]$, the latter equality by Lemma 1.9. Since $[L, P] = [P, P]$, these inclusions must be equalities. ∎

COROLLARY 1.11. *Among the Lie ideals $L$ such that $\overline{[L, A]} = \overline{[A, A]}$, the Lie ideal*

$$[\mathrm{Ped}(\mathrm{Id}([A, A])), \mathrm{Ped}(\mathrm{Id}([A, A]))]$$

*is the smallest.*

*Proof.* Let $P = \mathrm{Ped}(\mathrm{Id}([A, A]))$. Then

$$\begin{aligned}\overline{[[P, P], A]} &= \overline{[[\mathrm{Id}([A, A]), \mathrm{Id}([A, A])], A]} \\ &= \overline{[[\mathrm{Id}([A, A]), A], A]} = \overline{[\mathrm{Id}([A, A]), A]} = \overline{[A, A]}.\end{aligned}$$

The second equality holds by Lemma 1.4 and the third and fourth by Theorem 1.5. Thus, $[P, P]$ is a Lie ideal satisfying that $\overline{[L, A]} = \overline{[A, A]}$.

Suppose now that $L$ is a Lie ideal such that $\overline{[L, A]} = \overline{[A, A]}$. By Theorem 1.10, $[P, P] = [L, P] \subseteq L$. So $L$ contains $[P, P]$. ∎

It seems possible that under some $C^*$-algebra regularity condition, such as $A$ being pure (i.e, having almost unperforated and almost divisible Cuntz semigroup), it is the case that for every Lie ideal $L$ there exists a two-sided—possibly non-closed—ideal $I$ such that $[L, A] = [I, A]$ (in the language of [5], $L$ and $I$ are called commutator equal). At present, we do not even have an answer to the following question:

QUESTION 1.12. Is there a $C^*$-algebra $A$ and a Lie ideal $L$ of $A$, such that $[L, A] \neq [I, A]$ for all two-sided (possibly non-closed) ideals $I$ of $A$?

We turn now to Lie ideals of $[A, A]$. A linear subspace $U \subseteq A$ is called a Lie ideal of $[A, A]$ if $[U, [A, A]] \subseteq U$. Herstein's Theorem 1.12 of [12] implies that if $A$ is simple and unital then a Lie ideal of $[A, A]$ is automatically a Lie ideal of $A$ (this holds for simple rings without 2-torsion). In Theorem 1.15 below we show that the simplicity assumption can be dropped for *closed* Lie ideals of $[A, A]$. The key of the argument is again to apply a theorem of Herstein (Lemma 1.14 below) in the quotient by a suitable closed two-sided ideal.

LEMMA 1.13. *Let $U$ be a Lie ideal of $[A, A]$. Let $V = [U, U]$, $W = [V, V]$, and $X = [W, W]$. Then $A[X, X]A \subseteq [U, U] + [U, U]^2$.*

*Proof.* (Cf. Lemma 1.7 of [12].) In the following inclusions we make use of Jacobi's identity and the fact that $U$ is a Lie ideal of $[A, A]$:

$$[[U, U], A] \subseteq [U, [A, A]] \subseteq U,$$
$$[[U, U], [A, A]] \subseteq [[U, [A, A]], U] \subseteq [U, U].$$

That is, $[V, A] \subseteq U$ and $V$ is a Lie ideal of $[A, A]$. We deduce similarly that $[A, W] \subseteq V$ and that $W$ and $X$ are Lie ideals of $[A, A]$. Finally, since $V \subseteq [A, A]$ we have that $[V, V] \subseteq V$; i.e., $W \subseteq V$. We deduce similarly that $[X, X] \subseteq X$. Having made this preparatory remarks, we attack the lemma:

$$[X, X]A \subseteq A[X, X] + [[X, X], A] \subseteq A[X, X] + X \subseteq AX + X.$$

Hence, $A[X, X]A \subseteq AX = A[W, W]$. Using now that $a[w_1, w_2] = [aw_1, w_2] - [a, w_2]w_1$ we get that

$$A[W, W] \subseteq [A, W] + [A, W]W \subseteq V + VW \subseteq V + V^2.$$

Thus, $A[X, X]A \subseteq V + V^2$, as desired. ∎

LEMMA 1.14. *Let $U$ be a Lie ideal of $[A, A]$. If $[[U, U], A] = 0$ then $[U, A] = 0$.*

*Proof.* See Theorem 1.11 of [12] for the case of simple rings without 2-torsion. See Exercise 17, page 344 of [21] for the extension to semiprime rings without 2-torsion (e.g., $C^*$-algebras). ∎

THEOREM 1.15. *A (norm) closed Lie ideal of $[A, A]$ is a Lie ideal of $A$.*

*Proof.* Let $U$ be a closed Lie ideal of $[A, A]$. Consider the sets $V = [U, U]$, $W = [V, V]$ and $X = [W, W]$. Let $I = \mathrm{Id}([X, X])$. Let $\widetilde{U}$ denote the image of $U$ in $A/I$ by the quotient map. Define $\widetilde{V}$, $\widetilde{W}$, and $\widetilde{X}$ similarly. Then $[\widetilde{X}, \widetilde{X}] = 0$, which, by Lemma 1.14, implies that $[\widetilde{X}, A/I] = 0$. That is, $[[\widetilde{W}, \widetilde{W}], A/I] = 0$. Again by Lemma 1.14 we get that $[\widetilde{W}, A/I] = 0$. That is, $[[\widetilde{V}, \widetilde{V}], A/I] = 0$. Two more applications of Lemma 1.14 then yield that $[\widetilde{U}, A/I] = 0$. That is, $[U, A] \subseteq I$. Hence,

$$\mathrm{Id}([U, A]) \subseteq I \subseteq \overline{[U, U] + [U, U]^2} \subseteq \mathrm{Id}([U, U]).$$

In the second inclusion we have used Lemma 1.13. Since $\mathrm{Id}([U, U]) \subseteq \mathrm{Id}([U, A])$, all these must be equalities. Taking commutators with $A$ and using (1.1) we get

$$\overline{[\mathrm{Id}([U, A]), A]} = \overline{[[U, U] + [U, U]^2, A]} = \overline{[[U, U], A]} \subseteq U.$$

Lemma 1.4, on the other hand, implies that

$$[U, A] \subseteq \mathrm{Id}([U, A]) \cap \overline{[A, A]} = \overline{[\mathrm{Id}([U, A]), A]}.$$

Hence, $[U, A] \subseteq U$; i.e., $U$ is a Lie ideal of $A$. ∎

## 2. NILPOTENTS AND POLYNOMIALS

In this section we look at closed Lie ideals spanned by nilpotents and by the range of polynomials.

For each natural number $k \geqslant 2$ let $N_k$ denote the set of nilpotent elements of $A$ of order exactly $k$. Since the set $N_k$ is invariant by unitary conjugation (and by similarity), the closed subspace $\overline{\mathrm{span}(N_k)}$ is a Lie ideal of $A$ (see [17] and Theorem 2.6 below).

The following lemma is surely well known:

LEMMA 2.1. *Every element of $N_k$ is a sum of $k-1$ commutators for all $k \geqslant 2$.*

*Proof.* Let $x$ be a nilpotent of order at most $k$ (i.e., in $\bigcup\limits_{j\leqslant k} N_j$). Let $x = v|x|$ be the polar decomposition of $x$ in $A^{**}$. Let $\widetilde{x} = |x|^{1/2}v|x|^{1/2}$ (the Aluthge transform of $x$). Observe that $x = [v|x|^{1/2}, |x|^{1/2}] + \widetilde{x}$. Also,

$$\widetilde{x}^{k-1}(\widetilde{x}^{k-1})^* = |x|^{1/2}x^{k-1}v^*(x^{k-2})^*|x|^{1/2} = 0,$$

where we have used that $|x|^{1/2}x^{k-1} = 0$ (since $|x|^{1/2} \in C^*(x^*x)$ and $(x^*x)x^{k-1} = 0$). Thus $\widetilde{x}$ is a nilpotent of order at most $k-1$. Continuing this process inductively we arrive at the desired result. ∎

For each $k \in \mathbb{N}$ let $I_k$ denote the intersection of the kernels of all representations of $A$ of dimension at most $k$. Notice that $I_1 = \mathrm{Id}([A,A])$ and that $I_1 \supseteq I_2 \supseteq \cdots$. It is not hard to show that $I_k$ is the smallest closed two-sided ideal the quotient by which is a $k$-subhomogeneous $C^*$-algebra (i.e., one whose irreducible representations are at most $k$-dimensional).

THEOREM 2.2. $\overline{\mathrm{span}(N_k)} = \overline{[I_{k-1}, A]}$ *for all* $k \geqslant 2$.

*Proof.* It is well known that $\mathrm{Id}(N_k) = I_{k-1}$ (e.g., see Lemma 6.1.3 of [3]). We must then show that $\overline{\mathrm{span}(N_k)} = \overline{[\mathrm{Id}(N_k), A]}$. Let $I = \mathrm{Id}([N_k, A])$. Let $x \in N_k$. Since $[x, A] \subseteq I$, the quotient map sends $x$ to the center of $A/I$. But the center, being a commutative $C^*$-algebra, cannot contain nonzero nilpotents. Thus, $x \in I$. This shows that $N_k \subseteq \mathrm{Id}([N_k, A])$. On the other hand, $N_k \subseteq [A, A]$ by Lemma 2.1. Thus, $\overline{\mathrm{span}(N_k)} = \overline{[\mathrm{Id}(N_k), A]}$ by Lemma 1.6. ∎

COROLLARY 2.3. $\overline{\mathrm{span}(N_2)} = \overline{[A, A]}$.

*Proof.* The previous theorem implies that $\overline{\mathrm{span}(N_2)} = \overline{[\mathrm{Id}([A,A]), A]}$. On the other hand, $\overline{[\mathrm{Id}([A,A]), A]} = \overline{[A,A]}$, by Theorem 1.5(ii) applied with $L = A$. ∎

The following corollary is merely a restatement of Corollary 2.3

COROLLARY 2.4. *A positive bounded functional on $A$ is a trace if and only if it vanishes on $N_2$.*

QUESTION 2.5. Is $[A, A] = \mathrm{span}(N_2)$? Is $\mathrm{span}(N_2)$ a Lie ideal?

We will return to these questions in Section 4.

Combining Corollary 2.3 and Theorem 1.15 of the previous section we can prove the following $C^*$-algebraic version of a theorem of Amitsur for simple rings ([1], Theorem 1):

THEOREM 2.6. *A closed subspace $U$ of $A$ is a Lie ideal if and only if $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$.*

*Proof.* Say $U$ is a Lie ideal. Let $u \in U$ and $x \in N_2$. Then

$$(1+x)u(1-x) = u + [x,u] + \frac{1}{2}[x,[x,u]] \in U.$$

Suppose now that $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$. Let $u \in U$ and $x \in N_2$. Then

$$\begin{aligned}[x,u] - xux &= (1+x)u(1-x) - u \in U,\\ [x,u] + xux &= -(1-x)u(1+x) + u \in U.\end{aligned}$$

Hence $[u,x] \in U$. That is, $[U, N_2] \subseteq U$. Passing to the span of $N_2$ and taking closure we get from Corollary 2.3 that $[U, [A,A]] \subseteq U$. That is, $U$ is a closed Lie ideal of $[A,A]$. By Theorem 1.15, $U$ is a Lie ideal of $A$. ∎

Let $f(x_1, \ldots, x_n)$ be a polynomial in noncommuting variables with coefficients in $\mathbb{C}$. Let us denote by $f(A, \ldots, A)$, or $f(A)$ for short, the range of $f$ on $A$. (If $A$ is non-unital we assume that $f$ has no independent term.) Since the set $f(A)$ is invariant by similarity, $\overline{\operatorname{span}(f(A))}$ is a Lie ideal. It is shown in Theorem 2.3 of [6] that even $\operatorname{span}(f(A))$ is Lie ideal.

In the sequel by a polynomial we always understand a polynomial in noncommuting variables with coefficients in $\mathbb{C}$.

Recall that for each $k \in \mathbb{N}$ we let $I_k$ denote the intersection of the kernels of all representations of $A$ of dimension at most $k$. In the following theorem we use the conventions $I_0 = A$ and $M_0(\mathbb{C}) = \{0\}$. We regard every polynomial as an identity on $M_0(\mathbb{C})$. By a nonconstant polynomial we mean one with positive degree in at least one of its variables.

THEOREM 2.7. *Let $f$ be a nonconstant polynomial. Suppose that $f(A) \subseteq \overline{[A,A]}$. Then $\overline{\operatorname{span}(f(A))} = \overline{[I_k, A]}$, where $k \geqslant 0$ is the largest number such that $f$ is an identity on $M_k(\mathbb{C})$ (such a number must exist since no polynomial is an identity on all matrix algebras).*

*Proof.* Let $I = \operatorname{Id}([f(A), A])$. Then $A/I$ is a subhomogeneous $C^*$-algebra, since it satisfies the (nontrivial) polynomial identity $[f(x_1, \ldots, x_n), y]$ (see Proposition IV.1.4.6 of [4]). The range of $f$ on $A/I$ is both in the center of $A/I$ and in $\overline{[A/I, A/I]}$, as $f(A) \subseteq \overline{[A,A]}$. But in a subhomogeneous $C^*$-algebra the center and the closure of the span of the commutators have zero intersection (since this is true in every finite dimensional representation). Hence, $f(A/I) = \{0\}$; i.e.,

$f(A) \subseteq I$. Thus, $\mathrm{Id}(f(A)) = I = \mathrm{Id}([f(A), A])$. By assumption, we also have that $f(A) \subseteq \overline{[A,A]}$. It follows that $\overline{\mathrm{span}(f(A))} = \overline{[I,A]}$ by Lemma 1.6.

Let us now show that $I = I_k$, with $k \geqslant 0$ as in the statement of the theorem. Let $\pi : A \to M_l(\mathbb{C})$ be a representation of $A$ with $l \leqslant k$. By assumption, $f(M_l(\mathbb{C})) = \{0\}$. Hence, $f(A) \subseteq \ker \pi$, and so $I = \mathrm{Id}(f(A)) \subseteq \ker \pi$. Since, by definition, $I_k$ is the intersection of the kernels of all such $\pi$, we get that $I \subseteq I_k$. To prove the opposite inclusion notice first that $A/I$ must be a $k$-subhomogeneous $C^*$-algebra. For suppose that there exists an irreducible representation $\pi : A/I \to M_m(\mathbb{C})$, with $m > k$. Since $f$ is an identity on $A/I$ and $\pi$ is onto, we get that $f$ is an identity on $M_m(\mathbb{C})$. This contradicts our choice of $k$. Hence, every irreducible representation of $A/I$ has dimension at most $k$; i.e., $A/I$ is $k$-subhomogeneous. Since $I_k$ may be alternatively described as the smallest closed two-sided ideal the quotient by which is $k$-subhomogeneous, $I_k \subseteq I$. ∎

Let $s_k$ denote the standard polynomial in $k$ noncommuting variables. That is,

$$s_k(x_1, \ldots, x_k) = \sum_{\sigma \in S_k} \mathrm{sign}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(k)},$$

where $S_k$ denotes the symmetric group on $k$ elements. The Amitsur–Levitzky theorem states that $s_{2k}$ is a polynomial identity of minimal degree on $M_k(\mathbb{C})$ [2]. Define $\pi_1(x,y) = [x,y]$ and

$$\pi_{k+1}(x_1, \ldots, x_{2^{k+1}}) = [\pi_k(x_1, \ldots, x_{2^k}), \pi_k(x_{2^k+1}, \ldots, x_{2^{k+1}})]$$

for all $k \geqslant 1$. The following two special cases of the previous theorem are worth remarking upon:

COROLLARY 2.8. *$\overline{\mathrm{span}(\sigma_{2k}(A))} = \overline{[I_k, A]}$ and $\overline{\mathrm{span}(\pi_k(A))} = \overline{[A,A]}$ for all $k \geqslant 1$.*

*Proof.* Let $k \in \mathbb{N}$. It is well known that $s_{2k}$ is expressible as a sum of commutators in the algebra of polynomials in $2k$ noncommuting variables. Hence, $s_{2k}(A) \subseteq [A,A]$. We can thus apply Theorem 2.7 to $s_{2k}$. By the Amitsur–Levitsky theorem, $s_{2k}$ is a polynomial identity of $M_k(\mathbb{C})$ but not of $M_{k+1}(\mathbb{C})$. Thus, by Theorem 2.7, $\overline{\mathrm{span}(\sigma_{2k}(A))} = \overline{[I_k, A]}$.

The polynomial $\pi_k$ is an identity on $\mathbb{C}$ but not on $M_2(\mathbb{C})$. (In fact, by Theorem 2 of [13], if $\pi_k$ is a polynomial identity on a semiprime ring without 2-torsion then the ring must be commutative.) Thus, by Theorem 2.7, $\overline{\mathrm{span}(\pi_k(A))} = \overline{[A,A]}$. ∎

Let's now give a characterization of the polynomials whose range is contained in $\overline{[A,A]}$. Following [6], we say that two polynomials $f$ and $g$ (in noncommuting variables, with coefficients in $\mathbb{C}$) are cyclically equivalent if $f - g$ is a sum of commutators in the ring $\mathbb{C}\langle X_1, X_2, \ldots \rangle$ of polynomials in noncommuting variables. If a polynomial is cyclically equivalent to 0 then its range is clearly in $\overline{[A,A]}$. On the other hand, if $A$ has no bounded traces then $A = \overline{[A,A]}$ (see

[8]) and so any polynomial has range in $\overline{[A,A]}$. The general case is a mixture of these two. In the following theorem we maintain the conventions that $I_0 = A$, $M_0(\mathbb{C}) = \{0\}$, and that every polynomial is an identity on $M_0(\mathbb{C})$.

THEOREM 2.9. *Let $k \geqslant 0$ be the smallest number such that the closed two-sided ideal $I_k$ has no bounded traces (set $k = \infty$ if this is never the case). Let $f$ be a nonconstant polynomial.*

(i) *If $k = \infty$ then $f(A) \subseteq \overline{[A,A]}$ if and only if $f$ is cyclically equivalent to 0.*

(ii) *If $k < \infty$ then $f(A) \subseteq \overline{[A,A]}$ if and only if $f$ is cyclically equivalent to a polynomial identity on $M_k(\mathbb{C})$.*

*Proof.* Let us first prove the forward implications. If $f$ is cyclically equivalent to 0 then clearly $f(A) \subseteq \overline{[A,A]}$. Suppose that $k < \infty$ and that $f$ is cyclically equivalent to a polynomial $g$ which is an identity on $M_k(\mathbb{C})$. Then $g(A) \subseteq I_k$ and $I_k = \overline{[I_k, I_k]}$, since $I_k$ has no bounded traces. Thus, $g(A) \subseteq \overline{[A,A]}$. But $(f-g)(A) \subseteq [A,A]$. Thus, $f(A) \subseteq \overline{[A,A]}$, as desired.

Let us suppose now that $f(A) \subseteq \overline{[A,A]}$. We will follow closely the proof of Theorem 4.5 in [6] where the result is obtained for the range of polynomials on matrix algebras. If the independent term of $f$ is nonzero then $1 \in f(A) \subseteq \overline{[A,A]}$. Hence, $A$ has no bounded traces; i.e., $k = 0$. Since, by convention, any polynomial is an identity on $M_0(\mathbb{C})$, we are done. Let us assume now that $f$ has no independent term. Let $f = \sum_{i=1}^{m} f_i$ be the decomposition of $f$ into multihomogeneous polynomials. Then, by the proof of Theorem 2.3 in [6], $f_i(A) \subseteq \operatorname{span}(f(A))$ for all $i$. This reduces the proof to the case that $f$ is multihomogeneous. We prove the theorem for multihomogeneous polynomials by induction on the smallest degree of its variables. Suppose that the degree of $f$ on $x_1$ is 1. Then $f$ is cyclically equivalent to a polynomial of the form $x_1 g(x_2, \ldots, x_n)$. Hence $Ag(A) \subseteq \overline{[A,A]}$, which in turn implies that $\operatorname{Id}(g(A)) \subseteq \overline{[A,A]}$. If $g$ is 0, then $f$ is cyclically equivalent to 0 and we are done. If $g$ is constant and nonzero, then $A = \operatorname{Id}(g(A)) \subseteq \overline{[A,A]}$. That is, $A = \overline{[A,A]}$, $k = 0$, and $f$ is an identity on $M_0(\mathbb{C})$; again we are done. If $g$ is nonconstant then $\operatorname{Id}(g(A)) = I_{k'}$ for some $k'$ and furthermore $g$ is an identity on $M_{k'}(\mathbb{C})$ (see the proof of Theorem 2.7). From $I_{k'} \subseteq \overline{[A,A]}$ and Lemma 1.4 we deduce that $I_{k'} = \overline{[I_{k'}, I_{k'}]}$. Hence, $I_{k'}$ has no bounded traces; i.e., $k' \geqslant k$. It follows that $g$ is an identity on $M_k(\mathbb{C})$, and since $f = x_1 g$, so is $f$. This completes the first step of the induction.

Suppose now that $f(x_1, \ldots, x_n)$ is a multihomogeneous polynomial whose variable of smallest degree, $x_n$, has degree $d$, with $d > 1$. Consider the polynomial

$$\begin{aligned} g(x_1, \ldots, x_n, x_{n+1}) \\ = f(x_1, \ldots, x_{n-1}, x_n + x_{n+1}) - f(x_1, \ldots, x_{n-1}, x_{n+1}) - f(x_1, \ldots, x_n). \end{aligned}$$

Then $g(A) \subseteq \overline{[A,A]}$ and the degree of $g$ on $x_n$ is less than $d$. By induction, $g$ is cyclically equivalent to a polynomial identity on $M_k(\mathbb{C})$ (if $k < \infty$) or cyclically equivalent to 0 (if $k = \infty$). Since $f(x_1, \ldots, x_n) = \frac{1}{2^d-2} g(x_1, \ldots, x_n, x_n)$, the same holds for $f$. ∎

## 3. FINITE SUMS AND SUMS OF PRODUCTS

Recall the following basic fact: a dense two-sided ideal in a unital $C^*$-algebra must agree with the whole $C^*$-algebra (because it would intersect the ball of radius one centered at the unit, all whose elements are invertible). It follows that if $A$ is unital and $A = \mathrm{Id}(X)$ then $A = AXA$. Here we exploit this fact to obtain quantitative versions of some of the results from the previous sections.

THEOREM 3.1. *Let $A$ be unital and let $L$ be a Lie ideal of $A$ such that* $\mathrm{Id}([L, A]) = A$. *Suppose that $L$ is linearly spanned by a set $\Gamma \subseteq A$; i.e., $L = \mathrm{span}(\Gamma)$. Suppose furthermore that there exists $M \in \mathbb{N}$ such that for all $l \in \Gamma$ and $z \in A$ the commutator $[l, z]$ is a linear combination of at most $M$ elements of the set $\Gamma$. The following are true:*

(i) *There exists $N$ such that every element of $A$ is expressible as a linear combination of $N$ elements of $\Gamma$ and $N$ products of two elements of $\Gamma$.*

(ii) *There exists $K$ such that every single commutator $[x, y]$ in $A$ is expressible as a linear combination of $K$ elements of $\Gamma$.*

*Proof.* We have shown that $\mathrm{Id}([L, A]) = \mathrm{Id}([L, L])$ in the proof of Theorem 1.5(i). (Indeed, after setting $I = \mathrm{Id}([[L, A], [L, A]])$, we proceeded to show that $[L, A] \subseteq I$, which implies that $\mathrm{Id}([L, A]) \subseteq I \subseteq \mathrm{Id}([L, L])$. Clearly, these inclusions must be equalities.) Therefore, $A = \mathrm{Id}([L, L]) = \mathrm{Id}([\Gamma, \Gamma])$. Since $A$ is unital, it is algebraically generated as a two-sided ideal by $[\Gamma, \Gamma]$. Hence,

$$1 = \sum_{i=1}^{n} x_i [k_i, l_i] y_i,$$

for some $x_i, y_i \in A$ and $k_i, l_i \in \Gamma$. Let $a \in A$. Then

$$a = \sum_{i=1}^{n} (a x_i) [k_i, l_i] y_i.$$

It suffices to show that each term of the sum on the right is a linear combination of a fixed number of elements of $\Gamma$ and of products of two elements of $\Gamma$. We have the following identity (derived from the arguments in the proof of Lemma 1.1(i)):

$$x[l, m]y = [xyl, m] - [xy, m]l + [xm, [y, l]] - [x, [y, l]]m + [xl, [m, y]] - [x, [m, y]]l,$$

for all $x, y \in A$ and $l, m \in \Gamma$. Observe that each of the terms on the right side are of either one of the following forms: $[z, l]$, $[z, l]l'$, $[z, [z', l]]$, or $[z, [z', l]]l'$, where $z, z' \in A$ and $l, l' \in \Gamma$. Recall now that, by assumption, the commutators $[z, l]$, with $z \in A$ and $l \in \Gamma$, are expressible as linear combinations of at most $M$ elements of $\Gamma$. This implies that elements of either one of the forms mentioned before are linear combinations of either $M$ or $M^2$ elements of $\Gamma$ or products of two elements of $\Gamma$.

(ii) Let $x \in A$. By (i), $x = \sum_{i=1}^{N} \lambda_i l_i + \sum_{i=1}^{N} \mu_i m_i n_i$ for some scalars $\lambda_i, \mu_i$ and some $l_i, m_i, n_i \in \Gamma$. Let $y \in A$. Then,

$$[x,y] = \sum_{i=1}^{N} \lambda_i [l_i, y] + \sum_{i=1}^{N} \mu_i [m_i, n_i y] + \sum_{i=1}^{N} \mu_i [n_i, y m_i].$$

Appealing to the fact that every commutator of the form $[l,z]$, with $l \in \Gamma$ and $z \in A$ is a linear combination of at most $M$ elements of $\Gamma$, we deduce that the right side is a linear combination of $3MN$ elements of $\Gamma$. ∎

THEOREM 3.2. *Let $A$ be unital and without 1-dimensional representations. Then there exists $N \in \mathbb{N}$ such that every element of $A$ is expressible as a sum of the form*

$$\sum_{i=1}^{N} [a_i, b_i] + \sum_{i=1}^{N} [c_i, d_i] \cdot [c_i', d_i'].$$

*Proof.* The quotient $A/\mathrm{Id}([A,A])$ is a commutative $C^*$-algebra. If it were nonzero, it would have non-trivial 1-dimensional representations. But we have asssumed that $A$ has no 1-dimensional representations, Thus, $A = \mathrm{Id}([A,A])$. The previous theorem is then applicable to $L = [A,A]$ and $\Gamma = \{[x,y] : x, y \in A\}$, yielding the desired result. ∎

We can link the constant $N$ in Theorem 3.2 to a certain notion of "divisibility" studied in [19]. A unital $C^*$-algebra $A$ is called weakly $(2,N)$-divisible if there exist $x_1, \ldots, x_N \in N_2$ and $d_1, \ldots, d_N \in A$ such that

$$1 = \sum_{i=1}^{N} d_i^* x_i^* x_i d_i.$$

(The definition of weakly $(2,N)$-divisible in [19] is in terms of the Cuntz semigroup of $A$ but can be seen to be equivalent to this one.) A unital $C^*$-algebra without 1-dimensional representations must be weakly $(2,N)$-divisible for some $N$ ([19], Corollary 5.4). This fact, combined with the following proposition, gives another proof of Theorem 3.2.

PROPOSITION 3.3. *If $A$ is unital and weakly $(2,N)$-divisible then every element of $A$ is expressible as a sum of the form $\sum_{i=1}^{N} [a_i, b_i] + \sum_{i=1}^{N} [c_i, d_i] \cdot [c_i', d_i']$.*

*Proof.* Suppose that $1 = \sum_{i=1}^{N} d_i^* x_i^* x_i d_i$, with $x_i \in N_2$ for all $i$. Let $a \in A$. Then

$$a = \Big( \sum_{i=1}^{N} d_i^* x_i^* x_i d_i \Big) \cdot a = \sum_{i=1}^{N} [d_i^* x_i^*, x_i d_i a] + \sum_{i=1}^{N} x_i d_i a d_i^* x_i^*.$$

It thus suffices to show that $xbx^*$ is a product of 2 commutators for all $x \in N_2$ and $b \in A$. Say $x = v|x|$ is the polar decomposition of $x$ in $A^{**}$. Then $xbx^* = (xb|x|^{1/2}) \cdot |x|^{1/2}v^*$. But both $xb|x|^{1/2}$ and $|x|^{1/2}v^*$ belong to $N_2$. (Let us prove this for the latter: We have $|x|^{1/2} \in C^*(x^*x) \subseteq \overline{|x|Ax}$. Multipliying by $v$ on the left we get that $v|x|^{1/2} \in \overline{xAx}$. Since $x$ is a square zero element, we deduce that $v|x|^{1/2}$, and its adjoint, are square zero elements as well.) By Lemma 2.1, both $xb|x|^{1/2}$ and $|x|^{1/2}v^*$ are commutators. ∎

REMARK 3.4. If $1 \in B \subseteq A$ and $B$ is weakly $(2, N)$-divisible then so is $A$. This observation can be used to find upper bounds on $N$ for specific examples (e.g., when $B$ is a dimension drop $C^*$-algebra; see Example 3.12 of [19]).

Let $P \subseteq A$ denote the set of projections of $A$. Let us apply Theorem 3.1 to $\mathrm{span}(P)$. To see that this is a Lie ideal, recall that the linear span of the idempotents is Lie ideal and that, by a theorem of Davidson (see paragraph after Theorem 4.2 of [15]), every idempotent is a linear combination of five projections. In Davidson's theorem, the number of projections can be reduced to four:

LEMMA 3.5. *Every idempotent of $A$ is a linear combination of four projections.*

*Proof.* Let $e \in A$ be an idempotent and let $p \in A$ denote its range projection. Then $e = p + x$, with $x \in pA(1-p)$. Let us show that $x$ is a linear combination of three projections. It suffices to assume that $\|x\| < \frac{1}{2}$. For each $x \in pA(1-p)$ such that $\|x\| < \frac{1}{2}$ let us define

$$q(x) = \begin{pmatrix} \frac{1+\sqrt{1-4xx^*}}{2} & x \\ x^* & \frac{1-\sqrt{1-4x^*x}}{2} \end{pmatrix} \in \begin{pmatrix} pAp & pA(1-p) \\ (1-p)Ap & (1-p)A(1-p) \end{pmatrix}.$$

A straightforward computation shows that $q(x)$ is a projection and, furthermore, that

$$x = \frac{1+\mathrm{i}}{4}q(x) + \frac{-1+\mathrm{i}}{4}q(-x) - \frac{\mathrm{i}}{2}q(\mathrm{i}x). \quad ∎$$

THEOREM 3.6. *Suppose that the $C^*$-algebra $A$ is unital and that* $\mathrm{Id}([P, A]) = A$. *The following are true:*

(i) *There exists $N$ such that every element of $A$ is expressible as a linear combination of $N$ projections and $N$ products of two projections.*

(ii) *There exists $K$ such that every commutator $[x, y]$, with $x, y \in A$, is expressible as a linear combination of $K$ projections.*

*Proof.* Both (i) and (ii) will follow once we show that Theorem 3.1 is applicable to the Lie ideal $\mathrm{span}(P)$ and the generating set $P$. It suffices to show that a commutator of the form $[p, z]$, with $p$ a projection, is a linear combination of projections with a uniform bound on the number of terms. But

$$[p, z] = (p + pz(1-p)) - (p + (1-p)zp),$$

where $p + pz(1-p)$ and $p + (1-p)zp$ are idempotents. Each of them is a linear combination of four projections by Lemma 3.5. ∎

REMARK 3.7. If $B$ is a unital $C^*$-subalgebra of $A$ and $\mathrm{Id}([P_B, B]) = B$, then

$$1 = \sum_{i=1}^{n} x_i[p_i, q_i]z_i,$$

for $x_i, y_i, z_i \in B$ and projections $p_i, q_i \in P_B$. It follows that the constants $N$ and $K$ that one finds for $B$ following the proof of Theorem 3.1 applied to $L = \mathrm{span}(P_B)$ also work for the $C^*$-algebra $A$. This observation can be used to obtain concrete estimates of these constants in cases where $B$ is rather simple.

An element of a $C^*$-algebra is called full if it generates the $C^*$-algebra as a closed two-sided ideal. Recall also that a unital $C^*$-algebra is said to have real rank zero if its invertible selfadjoint elements are dense in the set of selfadjoint elements. By Theorem V.7.3 of [9], this is equivalent to asking that every hereditary $C^*$-subalgebra of $A$ has an approximate unit consisting of projections.

COROLLARY 3.8. *Suppose that $A$ is unital and either contains two full orthogonal projections or has real rank zero and no* 1*-dimensional representations. Then there exist $N$ and $K$ such that* (i) *and* (ii) *of the previous theorem hold for $A$.*

*Proof.* Let us show in both cases that $\mathrm{Id}([P, A]) = A$.

Say $p$ is a projection such that $p$ and $1-p$ are full; i.e, $A = \mathrm{Id}(p) = \mathrm{Id}(1-p)$. Then

$$A = \mathrm{Id}(p) \cdot \mathrm{Id}(1-p) = \overline{ApA(1-p)A} = \mathrm{Id}(pA(1-p)).$$

On the other hand, $\mathrm{Id}(pA(1-p)) = \mathrm{Id}([p, A])$. Indeed,

$$pA(1-p) = [p, A](1-p) \subseteq \mathrm{Id}([p, A]),$$

and conversely

$$[p, A] = \{pa(1-p) - (1-p)ap : a \in A\} \subseteq \mathrm{Id}(pA(1-p)).$$

(We have $(1-p)ap \in \mathrm{Id}(pA(1-p))$ since closed two-sided ideals are selfadjoint.) Hence, $A = \mathrm{Id}(pA(1-p)) = \mathrm{Id}([p, A])$, as desired.

Suppose now that $A$ has real rank zero and no 1-dimensional representations, i.e., $A = \mathrm{Id}([A, A])$. Since $\mathrm{Id}([A, A]) = \mathrm{Id}(N_2)$ (where, as before, $N_2$ denotes the set of nilpotents of order two), $A = \mathrm{Id}(N_2)$. Furthermore, since $A$ is unital there exist $x_1, \ldots, x_n \in N_2$ such that $A = \mathrm{Id}(x_1, \ldots, x_n)$, for it suffices to choose these elements such that $\sum_{i=1}^{n} a_i x_i b_i$ is invertible for some $a_i, b_i \in A$. Since $A$ has real rank-zero, the hereditary subalgebras $\overline{x_i^* A x_i}$ have approximate units consisting of projections for all $i$. Using this, we can find projections $p_i \in \overline{x_i^* A x_i}$ for $i = 1, \ldots, n$ such that $A = \mathrm{Id}(p_1, \ldots, p_n)$. We claim that $p_i$ is Murray–von Neumann subequivalent to $1 - p_i$ for all $i$. To prove this, let $x_i = v_i|x_i|$ be the polar decomposition of $x_i$ in $A^{**}$. Since $p_i \in \overline{x_i^* A x_i}$ we have $p_i \leqslant v_i^* v_i$. On the other hand,

$x_i^2 = 0$ implies that $v_i^* v_i$ and $v_i v_i^*$ are orthogonal projections. Hence, $v_i v_i^* \leqslant 1 - p_i$. It follows that $p_i = (p_i v_i^*)(v_i p_i)$ and $(v_i p_i)(p_i v_i^*) = v_i p_i v_i^* \leqslant v_i v_i^* \leqslant 1 - p_i$. This proves the claim. We now have that $\mathrm{Id}(p_i) \subseteq \mathrm{Id}(1 - p_i)$ for all $i = 1, \ldots, n$. Hence,

$$\mathrm{Id}(p_i) = \mathrm{Id}(p_i) \cdot \mathrm{Id}(1 - p_i) = \mathrm{Id}(p_i A(1 - p_i)) = \mathrm{Id}([p_i, A]),$$

for all $i = 1, \ldots, n$. So $A = \mathrm{Id}(p_1, \ldots, p_n) = \mathrm{Id}([p_1, A], \ldots, [p_n, A])$, as desired. ∎

Next we turn to the Lie ideals generated by polynomials already investigated in the previous section. As before, by a polynomial we understand a polynomial in noncommuting variables with coefficients in $\mathbb{C}$.

THEOREM 3.9. *Let $k \in \mathbb{N}$. Suppose that the $C^*$-algebra $A$ is unital and has no representations of dimension less than or equal to $k$. Let $f$ be a nonconstant polynomial such that $f(A) \subseteq \overline{[A, A]}$ and which is not a polynomial identity on $M_k(\mathbb{C})$. The following are true:*

(i) *There exists $N$ such that each element of $A$ is expressible as a linear combination of $N$ values of $f$ on $A$ and $N$ products of two values of $f$ on $A$.*

(ii) *There exists $K$ such that each commutator $[x, y]$ in $A$ is expressible as a linear combination of $K$ values of $f$ on $A$.*

*Proof.* Both (i) and (ii) will follow from Theorem 3.1 applied to the Lie ideal $\mathrm{span}(f(A))$, with generating set $f(A)$, once we show the hypotheses of that theorem are valid in this case.

Since all representations of $A$ have dimension at least $k + 1$, we have $A = I_k$, where $I_k$ is as defined in the previous section. Also, by the proof of Theorem 2.7, $\mathrm{Id}(f(A)) = I_{k'}$, where $k'$ is the largest number such that $f$ is an identity on $M_{k'}(\mathbb{C})$. But $f$ is not an identity on $M_k(\mathbb{C})$, so we must have that $k' \leqslant k$. Hence $\mathrm{Id}(f(A)) = A$. Furthermore, as argued in the proof of Theorem 2.7, $\mathrm{Id}([f(A), A]) = \mathrm{Id}(f(A))$. Thus, $A = \mathrm{Id}([f(A), A])$.

To complete the proof, it remains to show that there is a uniform bound on the number of terms expressing a commutator $[f(\overline{a}), y]$ as a linear combination of elements of $f(A)$. This is indeed true, and can be derived from the proof of Theorem 2.3 in [6] (showing that $\mathrm{span}(f(A))$ is a Lie ideal). We only sketch the argument here: Say $f = \sum_{i=1}^{m} f_i$ is the decomposition of $f$ into a sum of multihomogeneous polynomials. Then, as argued in the proof of Theorem 2.3 in [6], relying on Lemma 2.2 of [6], each evaluation $f_i(\overline{a})$ is expressible as a linear combination of at most $(d + 1)^n$ values of $f$. Here $d$ is the maximum of the degrees of $f$ on its variables and $n$ the number of variables. It thus suffices to prove the desired result for each $f_i$, or put differently, to assume that $f$ is multihomogeneous. If $f$ is a constant polynomial then $[f(\overline{a}), y] = 0$ and the desired conclusion holds trivially. Let us assume that $f$ is multihomogeneous and has nonzero degree. We can furthermore reduce ourselves to the multilinear case. For suppose that $f$ has degree $d > 1$ on $x_n$. Let

$$g(x_1,\dots,x_n,x_{n+1}) = f(x_1,\dots,x_{n-1},x_n+x_{n+1}) - f(x_1,\dots,x_{n-1},x_{n+1}) - f(x_1,\dots,x_n).$$

Then the degree of $g$ on $x_n$ is less than $d$ and $f(x_1,\dots,x_n) = \frac{1}{2^d-2}g(x_1,\dots,x_n,x_n)$. This reduces the proof to $g$. Continuing in this way, we arrive at a multilinear polynomial. Finally, if $f$ is multilinear then the identity

$$[f(a_1,\dots,a_n),y] = f([a_1,y],\dots,a_n) + f(a_1,[a_2,y],\dots,a_n) + \cdots + f(a_1,\dots,[a_n,y])$$

shows that there is a uniform bound on the number of terms expressing $[f(\overline{a}),y]$ as a linear combination of values of $f$. ∎

A theorem of Pop ([18], Theorem 1) says that if $A$ is unital and without bounded traces then there exists $M \in \mathbb{N}$ such every element of $A$ is a sum of $M$ commutators. Combining this with the previous theorem yields the following corollary:

COROLLARY 3.10. *Let $A$ be unital and without bounded traces. Let $f$ be a non-constant polynomial. Then there exists $N \in \mathbb{N}$ such that each element of $A$ is expressible as a linear combination of $N$ values of $f$ on $A$.*

*Proof.* Since $A$ has no bounded traces it has no finite dimensional representations. Hence $I_k = A$ for all $k \in \mathbb{N}$. Furthermore, $f(A) \subseteq A = \overline{[A,A]}$ by Pop's theorem. Thus, by the preceding theorem, every commutator is a linear combination of $K$ values of $f$. On the other hand, every element of $A$ is a sum of $M$ commutators (by Pop's theorem). So every element of $A$ is a linear combination of $KM$ values of $f$. ∎

In [7], Brešar and Klep reach the conclusion of the preceding corollary for $K(H)$ and $B(H)$ (the compact and bounded operators on a Hilbert space) and for certain rings obtained as tensor products.

Next we construct examples showing that if $f(\mathbb{C}) = \{0\}$ then the number $N$ in Corollary 3.10 can be arbitrarily large. Taking $f(x,y) = [x,y]$ this shows that in Pop's theorem the number of commutators can be arbitrarily large. Taking $f(x_1,\dots,x_6) = [x_1,x_2] + [x_3,x_4]\cdot[x_5,x_6]$ this shows that the $N$ in Theorem 3.2 can be arbitrarily large as well.

EXAMPLE 3.11. Let $f$ be a polynomial in $n$ noncommuting variables such that $f(\mathbb{C}) = \{0\}$. Let $K \in \mathbb{N}$. We will construct a $C^*$-algebra $A$, unital and without bounded traces, and an element $e \in A$ not expressible as a linear combination of $K$ values of $f$. Let $S^2$ denote the 2-dimensional sphere. Let $\eta \in M_2(C(S^2))$ be a rank one non-trivial projection (i.e, one not Murray–von Neumann equivalent to a constant rank one projection). Choose $N \geqslant 2Kn$. Let $\eta_N = \eta^{\otimes N} \in M_{2^N}(C((S^2)^N))$. It is well know that the vector bundle associated to $\eta_N^{\otimes N}$ has non-trivial Euler class. In particular, any $N$ sections of the vector bundle associated to $\eta_N$ have a common vanishing point.

Let $X = \prod_{i=1}^{\infty} (S^2)^N$. Let $1_X$ denote the unit of $C(X)$. Let $e$, $p$, and $q$ be projections in $C(X, B(\ell^2(\mathbb{N})))$ defined as follows:

$$\begin{aligned} e &= \mathrm{diag}(1_X, 0, 0, \ldots), \\ q(x_1, x_2, \ldots) &= \mathrm{diag}(0, \eta_N(x_1), \eta_N(x_2), \ldots), \\ p(x_1, x_2, \ldots) &= \mathrm{diag}(1_X, \eta_N(x_1), \eta_N(x_2), \ldots), \end{aligned}$$

where $x_i \in (S^2)^N$ for all $i = 1, 2, \ldots$. The following facts are known (see Théorème 6 of [10] and Section 4 of [20]):

(i) $q^{\oplus N+1}$ is a properly infinite projection (i.e., $q^{\oplus N+2}$ is Murray–von Neumann subequivalent to $q^{\oplus N+1}$),

(ii) $e$ is not Murray–von Neumann subequivalent to $q^{\oplus N}$. Thus, for any $N$ elements of $qC(X, B(\ell^2(\mathbb{N})))e$ (i.e., "sections" of $q$) there exists $x \in X$ on which they all vanish.

Since $p = e \oplus q$, we have that $p^{\oplus N+1}$ is also a properly infinite projection. Let us define $A = pC(X, B(\ell^2(\mathbb{N})))p$. Notice first that $A$ cannot have bounded traces, since its unit is stably properly infinite. Let us show that $e \in A$ cannot be approximated within a distance less than one by a linear combination of $K$ elements of $f(A)$. Suppose, for the sake of contradiction, that

$$\Big\| e - \sum_{i=1}^{K} \lambda_i f(\bar{a}_i) \Big\| < 1.$$

Multiplying by $e$ on the left and on the right we get

$$\Big\| e - \sum_{i=1}^{K} \lambda_i e f(\bar{a}_i) e \Big\| < 1. \tag{3.1}$$

Say $\bar{a}_i = (a_{i,1}, \ldots, a_{i,n})$ for $i = 1, \ldots, K$. Since $p = e \oplus q$, we may regard each $a_{i,j} \in A$ as an "$e \times q$" matrix:

$$a_{i,j} = \begin{pmatrix} b_{i,j} & c_{i,j} \\ d_{i,j} & e_{i,j}, \end{pmatrix} \in \begin{pmatrix} eC(X, B(\ell^2))e & eC(X, B(\ell^2))q \\ qC(X, B(\ell^2))e & qC(X, B(\ell^2))q \end{pmatrix}$$

for all $i = 1, \ldots, K$ and $j = 1, \ldots, n$. Since $N \geqslant 2nK$, there exists $x \in X$ such that $c_{i,j}(x) = d_{i,j}(x) = 0$ for all $i, j$. But $eC(X, B(\ell^2))e \cong \mathbb{C}$ and $f(\mathbb{C}) = 0$. So $ef(\bar{a}_i(x))e = f(\bar{b}_i(x)) = 0$ for all $i = 1, \ldots, K$. Evaluating at $x \in X$ in (3.1) we then get $\|e(x) - 0\| < 1$, which is clearly impossible.

REMARK 3.12. The previous example shows also that the existence of a unit cannot be dropped neither in Theorem 3.2 nor in Corollary 3.10. Indeed, consider $A = \bigoplus_{N=1}^{\infty} A_N$, with $A_N$ as in the example above. Then $A$ has no bounded traces (whence no 1-dimensional representations) but $A \neq \mathrm{span}(f(A))$ for any polynomial $f$ in noncommuting variables such that $f(\mathbb{C}) = 0$.

## 4. SIMILARITY INVARIANCE AND THE SPAN OF $N_2$

Let $U \subseteq A$ be a linear subspace. In this section we investigate the equivalence between the following two properties of $U$:

(i) $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$,

(ii) $U$ is a Lie ideal.

We have seen in Theorem 2.6 that if $U$ is closed then (i) and (ii) are indeed equivalent. Furthermore, the proof of (ii) $\Rightarrow$ (i) in Theorem 2.6 is valid for any subspace $U$ of $A$. Thus, we are interested in the implication (i) $\Rightarrow$ (ii) when $U$ is not necessarily closed. In the closed case, the proof of (i) $\Rightarrow$ (ii) in Theorem 2.6 can be split into two steps: In the first step we showed that $U$ is a Lie ideal of $[A, A]$. This was done as follows: (i) readily implies that $[U, N_2] \subseteq U$. Then using that $[A, A] \subseteq \overline{\mathrm{span}(N_2)}$ (by Corollary 2.3) and that $U$ is closed, we arrived at $[U, [A, A]] \subseteq U$. In the second step we appealed to Theorem 1.15, showing that a closed Lie ideal of $[A, A]$ is a Lie ideal of $A$.

Let us first address the passage from $[U, N_2] \subseteq U$ to $[U, [A, A]] \subseteq U$ in the non-closed case. Let $A_+$ denote the positive elements of $A$. Let us define

$$N_2^c = \{x \in A : xe = fx = x \text{ for some } e, f \in A_+ \text{ such that } ef = 0\}.$$

One readily checks that $N_2^c \subseteq N_2$. Let us show that $N_2^c$ is dense in $N_2$. Let $x \in N_2$. Observe that for each $\phi \in C_0(0,1]$ we have $\phi(|x|)\phi(|x^*|) = 0$, since $\phi(|x|) \in C^*(x^*x)$ and $\phi(|x^*|) \in C^*(xx^*)$. Let us choose $\phi_1, \phi_2, \ldots \in C_0(0,1]$, an approximate unit of $C_0(0,1]$ such that $\phi_{n+1}\phi_n = \phi_n$ for all $n$. Then $\phi_n(|x^*|)x\phi_n(|x|) \in N_2^c$ for all $n$, since we can set $e = \phi_{n+1}(|x|)$ and $f = \phi_{n+1}(|x^*|)$. Furthermore, $\phi_n(|x^*|)x\phi_n(|x|) \to x$. Thus, $N_2^c$ is dense in $N_2$.

Let us define

$$SN_2^c = \bigcup_{x \in N_2 \cup \{0\}} (1+x)N_2^c(1-x).$$

Notice that we still have $SN_2^c \subseteq N_2$.

LEMMA 4.1. $\mathrm{span}(SN_2^c)$ *is a Lie ideal.*

*Proof.* It suffices to show that $[A, x] \subseteq \mathrm{span}(SN_2^c)$ for all $x \in N_2^c$. For then, conjugating by the algebra automorphism $a \mapsto (1+y)a(1-y)$, with $y \in N_2$, and using the invariance of $SN_2^c$ under such automorphisms, we get that $[A, (1+y)x(1-x)] \subseteq \mathrm{span}(SN_2^c)$ for all $x \in N_2^c$ and $y \in N_2$, as desired.

Let $x \in N_2^c$. Let $e$ and $f$ be positive elements such that $xe = fx = x$ and $ef = 0$. Using functional calculus on $e$, let us find positive contractions $e_0, e_1, e_2, e_3 \in C^*(e)$ such that $e_0e_1 = e_1$, $e_1e_2 = e_2$, $e_2e_3 = e_3$ and $xe_3 = x$. Similarly, let us find positive contractions $f_0, f_1, f_2, f_3 \in C^*(f)$ such that $f_0f_1 = f_1$, $f_1f_2 = f_2$, $f_2f_3 = f_3$ and $f_3x = x$. Note that $xe_i = f_jx = x$ and $e_if_j = 0$ for all $i, j = 0, 1, 2, 3$. Now let $a \in A$. Then

$$ax - xa = ax - e_1ax + e_1ax - xaf_1 + xaf_1 - xa = (1-e_1)ax + [e_1af_1, x] - xa(1-f_1).$$

The term $(1-e_1)ax$ is in $N_2^c$. Indeed, $1-e_2$ and $e_3$ act as multiplicative units on the left and on the right of $(1-e_1)ax$ and $(1-e_2)e_3 = 0$. We check similarly that $xa(1-f_1)$ is in $N_2^c$. As for $[e_1af_1, x]$ (a commutator of elements in $N_2^c$), we have that

$$[e_1af_1, x] = (1+e_1af_1)x(1-e_1af_1) + (e_1af_1)x(e_1af_1) - x.$$

The first term on the right belongs to $SN_2^c$. The other two have multiplicative units $e_0$ and $f_0$ on the left and on the right and thus belong to $N_2^c$. ∎

The following theorem answers Question 2.5 affirmatively when $A$ is unital and without 1-dimensional representations.

THEOREM 4.2. *Suppose that $A$ has no 1-dimensional representations. Then*

$$\operatorname{span}(SN_2^c) = [\operatorname{Ped}(A), \operatorname{Ped}(A)].$$

*If in addition $A$ is unital, then* $\operatorname{span}(N_2) = [A, A]$. *Furthermore, in the unital case there exists $K \in \mathbb{N}$ such that every single commutator $[x, y]$ in $A$ is a sum of at most $K$ square zero elements.*

*Proof.* Let $P = \operatorname{Ped}(A)$. Let us first show that $SN_2^c \subseteq [P, P]$. By the similarity invariance of $[P, P]$, it suffices to show that $N_2^c \subseteq [P, P]$. Let $x \in N_2^c$ and let $e, f \in A_+$ be such that $xe = x = fx$ and $ef = 0$. From the description of the Pedersen ideal in Theorem 5.6.1 of [17] we know that $g(e) \in P$ for any $g \in C_0(0, \infty)_+$ of compact support, and since $xg(e) = xg(1)$, we deduce that $x \in P$. Hence, $x = [x, e] \in [P, A]$. Since $[P, A] = [P, P]$ by Lemma 1.9, $x \in [P, P]$. This shows that $\operatorname{span}(SN_2^c) \subseteq [P, P]$. Notice now that

$$\overline{[\operatorname{span}(SN_2^c), A]} = \overline{[\operatorname{span}(N_2), A]} = \overline{[[A, A], A]} = \overline{[A, A]}.$$

But $[P, P]$ is the smallest Lie ideal such that $\overline{[L, A]} = \overline{[A, A]}$, by Corollary 1.11. (To apply Corollary 1.11 we have used that $\operatorname{Id}([A, A]) = A$, since $A$ has no 1-dimensional representations.) Thus, $[P, P] \subseteq \operatorname{span}(SN_2^c)$.

Let us now assume that $A$ is unital. In this case $P = A$, so $[A, A] = \operatorname{span}(SN_2^c)$. But $\operatorname{span}(SN_2^c) \subseteq \operatorname{span}(N_2) \subseteq [A, A]$. Thus, $\operatorname{span}(N_2) = [A, A]$.

To deduce the existence of $K$ we will apply Theorem 3.1 to the Lie ideal $[A, A]$, with generating set $SN_2^c$. Notice first that $\operatorname{Id}([[A, A], A]) = \operatorname{Id}([A, A]) = \operatorname{Id}(A)$, since $A$ has no 1-dimensional representations. It remains to show that there is a uniform bound on the number of terms expressing a commutator of the form $[x, a]$, with $x \in SN_2^c$ and $a \in A$, as a linear combination of elements of $SN_2^c$. The proof of Lemma 4.1 shows that such commutators are sums of at most five elements of $SN_2^c$. ∎

For infinite von Neumann algebras, the following corollary is Theorem 2 of [16]. (Miers also considered closed subspaces of von Neumann algebras, which we have already dealt with in Theorem 2.6.)

COROLLARY 4.3. *Suppose that $A$ is either unital and without bounded traces or a von Neumann algebra. Then a subspace $U$ of $A$ is a Lie ideal if and only if $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$.*

*Proof.* That a Lie ideal satisfies the similarity invariance of the statement has already been shown in the proof of Theorem 2.6. So let us suppose that $U$ is a subspace such that $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$. As remarked at the start of this section, this implies that $[U, N_2] \subseteq U$. Let us consider first the case that $A$ is unital and without bounded traces. Then $\mathrm{span}(N_2) = [A, A]$, since $A$ is unital and has no 1-dimensional representations (since it has no bounded traces). Thus, $[U, [A, A]] \subseteq U$. Furthermore, $[A, A] = A$, by Pop's theorem. Hence, $[U, A] \subseteq U$; i.e., $U$ is a Lie ideal.

Suppose now that $A$ is a von Neumann algebra. Let us show again that $\mathrm{span}(N_2) = [A, A]$ and that if $U$ is a Lie ideal of $[A, A]$ then it is a Lie ideal of $A$. The latter is Lemma 3 of [16] and can be proven as follows: In a von Neumann algebra we have $A = Z(A) + [A, A]$, where $Z(A)$ denotes the center of $A$ (if $A$ is infinite, because $A = [A, A]$, and if $A$ is finite, by Theorem 3.2 of [11]); so $[U, A] = [U, [A, A]]$ for any subset $U$ of $A$. Let us now show that $\mathrm{span}(N_2) = [A, A]$. The ideal $\mathrm{Id}([A, A])$ is also a von Neumann algebra. (If $p$ is the unit of the type $\mathrm{I}_1$ direct summand of $A$, then $\mathrm{Id}([A, A]) = (1-p)A$; see Section 2.2 of [22]). Thus, $\mathrm{Id}([A, A])$ is unital and without 1-dimensional representations. Hence,

$$\mathrm{span}(N_2) = [\mathrm{Id}([A, A]), \mathrm{Id}([A, A])] \supseteq [[A, A], [A, A]].$$

From $A = [A, A] + Z(A)$ we get that $[A, A] = [[A, A], [A, A]]$. Hence, $\mathrm{span}(N_2) = [A, A]$. ∎

The passage from $U$ being a Lie ideal of $[A, A]$ to being a Lie ideal of $A$ can also be made assuming that $A$ is unital and that $[U, A]$ is full:

LEMMA 4.4. *Suppose that $A$ is unital. If $U$ is a Lie ideal of $[A, A]$ such that $\mathrm{Id}([U, A]) = A$ then $[A, A] \subseteq U$ (so $U$ is a Lie ideal of $A$).*

*Proof.* Let $V = [U, U]$, $W = [V, V]$, and $X = [W, W]$. We have shown in the proof of Theorem 1.15 that $\mathrm{Id}([U, A]) = \mathrm{Id}([X, X])$. So $A = \mathrm{Id}([X, X])$. Since $A$ is unital, the set $[X, X]$ generates $A$ algebraically as a two-sided ideal. But $A[X, X]A \subseteq [U, U] + [U, U]^2$, by Lemma 1.13. Hence, $A = [U, U] + [U, U]^2$. Then,

$$[A, A] = [[U, U] + [U, U]^2, A] = [[U, U], A] \subseteq [U, [U, A]] \subseteq U. \quad ∎$$

THEOREM 4.5. *Suppose that $A$ is unital and without* 1*-dimensional representations. Let $U$ be a subspace of $A$ such that* $\mathrm{Id}([U, A]) = A$. *If $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$ then $[A, A] \subseteq U$.*

*Proof.* The similarity invariance of $U$ implies that $[U, N_2] \subseteq U$ and by Theorem 4.2 we get that $[U, [A, A]] \subseteq U$. The previous lemma then shows that $[A, A] \subseteq U$. ∎

COROLLARY 4.6. *Let $A$ be simple and unital. A subspace $U$ of $A$ is a Lie ideal if and only if $(1+x)U(1-x) \subseteq U$ for all $x \in N_2$.*

*Proof.* Since $A$ is simple we have either $\mathrm{Id}([U,A]) = 0$ or $\mathrm{Id}([U,A]) = A$. If $\mathrm{Id}([U,A]) = 0$ then $U$ is a subset of the center, which by the simplicity of $A$ is $\mathbb{C}$. If $\mathrm{Id}([U,A]) = A$ then by the previous theorem $[A,A] \subseteq U$. In either case it follows that $U$ is a Lie ideal of $A$. ∎

Amitsur's Theorem 1 of [1] (that a similarity invariant subspace of a simple algebra must be a Lie ideal) requires the existence of a nontrivial idempotent in the algebra. An example in [1] shows that this hypothesis cannot be dropped. Corollary 4.6 shows, however, that for simple unital $C^*$-algebras this assumption is not necessary (even though they may well fail to have any nontrivial idempotents).

## REFERENCES

[1] S.A. AMITSUR, Invariant submodules of simple rings, *Proc. Amer. Math. Soc.* **7**(1956), 987–989.

[2] S.A. AMITSUR, J. LEVITZKI, Minimal identities for algebras, *Proc. Amer. Math. Soc.* **1**(1950), 449–463.

[3] P. ARA, M. MATHIEU, *Local Multipliers of $C^*$-Algebras*, Springer-Verlag London, Ltd., London 2003.

[4] B. BLACKADAR, *Operator Algebras*, Encyclopaedia Math. Sci., vol. 122, Springer-Verlag, Berlin 2006.

[5] M. BREŠAR, E. KISSIN, V.S. SHULMAN, Lie ideals: from pure algebra to $C^*$-algebras, *J. Reine Angew. Math.* **623**(2008), 73–121.

[6] M BREŠAR, I. KLEP, Values of noncommutative polynomials, Lie skew-ideals and tracial Nullstellensätze, *Math. Res. Lett.* **16**(2009), 605–626.

[7] M. BREŠAR, I. KLEP, A note on values of noncommutative polynomials, *Proc. Amer. Math. Soc.* **138**(2010), 2375–2379.

[8] J. CUNTZ, G.A. PEDERSEN, Equivalence and traces on $C^*$-algebras, *J. Funct. Anal.* **33**(1979), 135–164.

[9] K.R. DAVIDSON, *$C^*$-Algebras by Example*, Fields Inst. Monogr., vol. 6, Amer. Math. Soc., Providence, RI 1996.

[10] J. DIXMIER, A. DOUADY, Champs continus d'espaces hilbertiens et de $C^*$-algèbres, *Bull. Soc. Math. France* **91**(1963), 227–284.

[11] TH. FACK, P. DE LA HARPE, Sommes de commutateurs dans les algèbres de von Neumann finies continues, *Ann. Inst. Fourier (Grenoble)* **30**(1980), 49–73.

[12] I.N. HERSTEIN, *Topics in Ring Theory*, The University of Chicago Press, Chicago, Ill.-London 1969.

[13] I.N. HERSTEIN, On the Lie structure of an associative ring, *J. Algebra* **14**(1970), 561–571.

[14] L.W. MARCOUX, Projections, commutators and Lie ideals in $C^*$-algebras, *Math. Proc. R. Ir. Acad.* **110A**(2010), 31–55.

[15] L.W. MARCOUX, G.J. MURPHY, Unitarily-invariant linear spaces in $C^*$-algebras, *Proc. Amer. Math. Soc.* **126**(1998), 3597–3605.

[16] C.R. MIERS, Closed Lie ideals in operator algebras, *Canad. J. Math.* **33**(1981), 1271–1278.

[17] G.K. PEDERSEN, The linear span of projections in simple $C^*$-algebras, *J. Operator Theory* **4**(1980), 289–296.

[18] C. POP, Finite sums of commutators, *Proc. Amer. Math. Soc.* **130**(2002), 3039–3041 (electronic).

[19] L. ROBERT, M. RØRDAM, Divisibility properties for $C^*$-algebras, *Proc. London Math. Soc. (3)* **106**(2013), 1330–1370.

[20] M. RØRDAM, A simple $C^*$-algebra with a finite and an infinite projection, *Acta Math.* **191**(2003), 109–142.

[21] L.H. ROWEN, *Ring Theory. Vol. I*, Pure Appl. Math., vol. 127, Academic Press, Inc., Boston, MA 1988.

[22] S. SAKAI, *$C^*$-Algebras and $W^*$-Algebras*, Ergeb. Math. Grenzgeb., vol. 60, Springer-Verlag, New York-Heidelberg 1971.

LEONEL ROBERT, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF LOUISIANA AT LAFAYETTE, LAFAYETTE, 70506, U.S.A.

*E-mail address*: lrobert@louisiana.edu

Received May 17, 2015; revised September 21, 2015.