J. OPERATOR THEORY
75:2(2016), 475–495
doi: 10.7900/jot.2015jul07.2082

INVERTIBILITY OF TOEPLITZ OPERATORS VIA BEREZIN TRANSFORMS

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Communicated by Nikolai K. Nikolski

ABSTRACT. We obtain a sufficient condition for a Toeplitz operator to be invertible on the Bergman space via the n-th Berezin transforms of its symbol. For a harmonic symbol, we obtain a sufficient condition for a Toeplitz operator to be invertible on the Bergman space via only the Berezin transform of the symbol, which is analogous to the Chang–Tolokonnikov–Nikolski conditions on the Hardy space. For a nonnegative symbol, we prove that the Toeplitz operator is invertible on the Bergman space if and only if its Berezin transform is bounded below by a fixed positive constant on the unit disk.

KEYWORDS: *Bergman space, invertible Toeplitz operators, Berezin transform.*

MSC (2010): 47B 35, 47B 65.

1. INTRODUCTION

Let $\mathrm{d}A$ denote the Lebesgue area measure on $\mathbb{D}$, normalized so that the measure of the disk $\mathbb{D}$ is 1. The Bergman space $L_a^2(\mathbb{D})$ is the Hilbert space consisting of the analytic functions on $\mathbb{D}$ that are square integrable with respect to the measure $\mathrm{d}A$. For $\varphi \in L^\infty(\mathbb{D})$, the Toeplitz operator T_φ and the Hankel operator H_φ with symbol φ are defined on $L_a^2(\mathbb{D})$ by

$$T_\varphi f = P(\varphi f) \quad \text{and} \quad H_\varphi f = (I-P)(\varphi f),$$

where $P : L^2(\mathbb{D}, \mathrm{d}A) \to L_a^2(\mathbb{D})$ is the orthogonal projection. Using the reproducing kernel

$$K_z(w) = \frac{1}{(1-\bar{z}w)^2} \quad (z, w \in \mathbb{D}),$$

we express the Toeplitz operator and Hankel operator to be the integral operators:

$$T_\varphi f(z) = \int_{\mathbb{D}} \varphi(w) f(w) \overline{K_z(w)} \mathrm{d}A(w) = \int_{\mathbb{D}} \frac{\varphi(w) f(w)}{(1-\overline{w}z)^2} \mathrm{d}A(w)$$

and

$$H_\varphi f(z) = \int_{\mathbb{D}} (\varphi(z) - \varphi(w)) f(w) \overline{K_z(w)} \mathrm{d}A(w) = \int_{\mathbb{D}} \frac{(\varphi(z) - \varphi(w)) f(w)}{(1 - \overline{w}z)^2} \mathrm{d}A(w)$$

for f in $L_a^2(\mathbb{D})$.

A fundamental problem is to determine when a Toeplitz operator is invertible on the Bergman space. The problem has been investigated by many people ([5], [6], [7], [8]). Luecking [8] obtained a necessary and sufficient condition for T_φ to be invertible on $L_a^2(\mathbb{D})$ in the case when φ is nonnegative on $\mathbb{D}$. Based on Luecking's results, Faour [5] gave a necessary condition for T_φ to be invertible if φ is a continuous function on the closed disk and satisfies that $|\varphi(z_1)| \geqslant |\varphi(z_2)|$ whenever $|z_1| \leqslant |z_2|$. In general, Karaev [7] obtained some sufficient conditions on the invertibility of a linear bounded operator via the Berezin transform and atomic decomposition. Using Karaev's results, Gürdal and Söhret [6] gave a sufficient condition on the invertibility of Toeplitz operators with bounded symbols.

On the Hardy space, the invertible Toeplitz operators are completely characterized [2]. But the spectrum of a Toeplitz operator on the Hardy space is hardly determined by the geometric and analytic properties of the symbol of the operator. In [3], using homotopy, Douglas showed that for a continuous function φ on the unit circle, T_ϕ is invertible if the harmonic extension $|\widehat{\varphi}(z)| \geqslant \delta$ for some positive constant δ and for all z in the unit disk, where the harmonic extension $\widehat{\varphi}(z)$ is defined by

$$\widehat{\varphi}(z) = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\mathrm{e}^{\mathrm{i}\theta}) \frac{1 - |z|^2}{|1 - z\mathrm{e}^{-\mathrm{i}\theta}|^2} \mathrm{d}\theta$$

for $z \in \mathbb{D}$. In [3] Douglas posed the following question:

QUESTION 1.1. If φ is in $L^\infty(\partial\mathbb{D})$ and the harmonic extension $|\widehat{\varphi}(z)| \geqslant \delta$ for some positive constant δ and for all z in the unit disk, then is T_φ invertible on the Hardy space H^2?

As mentioned in [4] and [18], Chang and Tolokonnikov obtained a sufficient condition for a Toeplitz operator to be invertible on the Hardy space and showed that if for a constant δ sufficiently close to 1,

$$\delta \leqslant |\widehat{\varphi}(z)| \leqslant 1$$

for all $z \in \mathbb{D}$, then T_φ is invertible. In fact, Tolokonnikov found that $\delta > \frac{45}{46}$ and

$$\|T_\varphi^{-1}\| \leqslant \sqrt{\frac{1}{46\delta - 45}}$$

in [16]. Nikolski [10] proved the invertibility of a Toeplitz operator T_φ (on H^2) with $|\varphi| \leqslant 1$ and estimated

$$\|T_\varphi^{-1}\| \leqslant \sqrt{\frac{1}{24\delta - 23}}$$

under the condition $1 > \delta > \frac{23}{24}$.

Indeed, a slightly better estimation was proved by the reasoning in [10] (see page 374 of [10]):

$$1 > \delta > \sqrt{\frac{4e}{4e+1}} := \Delta$$

is already sufficient for T_φ to be invertible on the Hardy space. It is curious to note that this result of Nikolski follows also from a recent estimate of Hankel operators by Treil (see Theorem 1.1 and its proof in [17]). In a private communication, Nikolski conjectured that the constant Δ defined above is sharp for the invertibility problem of T_φ on the Hardy space H^2 since the methods from [10] and [17] are quite different.

On the other hand, using a suitable martingale, Wolff [18] found an elegant counterexample by constructing a function $\varphi \in L^\infty(\partial\mathbb{D})$ such that *the harmonic function $\widehat{\varphi}$ is bounded below, that is, for some positive δ,*

$$|\widehat{\varphi}(z)| \geqslant \delta$$

for all z in the unit disk, but the corresponding Toeplitz operator T_φ is not invertible on H^2 .

For $\varphi \in L^\infty(\mathbb{D})$, define the Berezin transform $\widetilde{\varphi}$ to be

$$\widetilde{\varphi}(z) = \langle T_\varphi k_z, k_z\rangle = \int_{\mathbb{D}} \varphi(w)|k_z(w)|^2 \mathrm{d}A(w),$$

where k_z is the normalized Bergman reproducing kernel of $L^2_a(\mathbb{D})$ given by

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{1-|z|^2}{(1-\bar{z}w)^2} \quad (z \in \mathbb{D}).$$

By the change of variable formula we have

$$\widetilde{\varphi}(z) = \int_{\mathbb{D}} \varphi(\varphi_z(w))\mathrm{d}A(w),$$

where φ_z is the Möbius map on the unit disk:

$$\varphi_z(w) = \frac{z-w}{1-\bar{z}w} \quad (z, w \in \mathbb{D}).$$

In fact, the Berezin transform of a bounded operator on a reproducing Hilbert space can be defined analogously as above. Thus the harmonic extension $\widehat{\varphi}(z)$ is equal to the Berezin transform of the Toeplitz operator with symbol φ on the Hardy space. These lead to the following natural question on the Bergman space $L^2_a(\mathbb{D})$:

QUESTION 1.2. Is T_φ invertible on the Bergman space if the Berezin transform $|\widetilde{\varphi}(z)| \geqslant \delta$ for some positive constant δ and for all z in the unit disk?

The results [9] on the spectrum of analytic Toeplitz operators on the Bergman space give an affirmative answer to the above question for analytic symbols or co-analytic symbols or the symbols are real harmonic functions. Using Luecking's result [8] on the Toeplitz operators with nonnegative symbols on the Bergman space, we will obtain an affirmative answer to Question 1.2 in the case when symbols are nonnegative functions on the unit disk in Section 3. Moreover, we will show that for a nonnegative function φ on the unit disk, T_φ is invertible on the Bergman space if and only if

$$\inf_{z\in\mathbb{D}} \widetilde{\varphi}(z) > 0.$$

In [20], using the spectral picture theorem (see [11]) and some techniques in [15], the authors proved that if φ is the harmonic function

$$\varphi(z) = c\overline{z} + az + b \quad (a, b, c \in \mathbb{C}),$$

then T_φ is invertible if and only if $|\varphi(z)| \geqslant \delta$ ($\forall z \in \mathbb{D}$) for some constant $\delta > 0$. Noting the harmonic function φ equals its Berezin transform, we obtain also an affirmative answer to Question 1.2 in the case of $\varphi(z) = c\overline{z} + az + b$.

On the other hand, we will show that the answer to Question 1.2 is negative for general functions in $L^\infty(\mathbb{D})$. That is, even if the Berezin transform

$$|\widetilde{\varphi}(z)| \geqslant \delta$$

for all z in $\mathbb{D}$ and a positive constant δ, the Toeplitz operator T_φ may not be invertible on $L^2_a(\mathbb{D})$, see Corollary 3.5 in the third section. However, for harmonic functions on $\mathbb{D}$, the answer to Question 1.2 is still unknown.

For a bounded harmonic function φ on $\mathbb{D}$, we will obtain a sufficient condition for the Toeplitz operator T_φ to be invertible by means of the estimation of the norm of Hankel operators [21]. This condition is analogous to the Chang–Tolokonnikov result in [16] for Hardy–Toeplitz operators. Using the Berezin transform $\widetilde{\varphi}$ and n-th Berezin transform $B_n\varphi$, we also give a sufficient condition for T_φ to be an invertible operator on $L^2_a(\mathbb{D})$ if φ is a bounded function on $\mathbb{D}$. The details are contained in Theorem 4.5. Based on this theorem, one may revise Question 1.2 as follows:

QUESTION 1.3. Is T_φ invertible on the Bergman space if $|\widetilde{\varphi}(z)| \geqslant \delta_1$ and the n-th Berezin transforms $|(B_n\varphi)(z)| \geqslant \delta_2$ for some positive constants δ_1, δ_2 and for all sufficiently large integers n and all z in the unit disk?

We will construct a continuous function on the closed disk which shows that the answer to Question 1.3 is negative in the last section.

2. NOTATIONS AND SOME PRELIMINARIES

In this section, we present some known results that will be needed later on and introduce some notations. First, we introduce the concept of the "n-Berezin transform" of a Toeplitz operator on the Bergman space $L_a^2(\mathbb{D})$, where n is a non-negative integer.

Let φ be in $L^\infty(\mathbb{D})$. The n-Berezin transform of a Toeplitz operator T_φ is defined by

$$\begin{aligned}(B_n\varphi)(z) &= (B_nT_\varphi)(z)\\ &= (n+1)(1-|z|^2)^{n+2}\sum_{j=0}^{n}(-1)^j\binom{n}{j}\int\limits_{\mathbb{D}}\frac{\varphi(w)|w|^{2j}}{|1-\bar{z}w|^{2n+4}}\mathrm{d}A(w)\\ &= (n+1)\int\limits_{\mathbb{D}}\varphi(w)(1-|w|^2)^n\frac{(1-|z|^2)^{2+n}}{|1-\bar{z}w|^{2n+4}}\mathrm{d}A(w)\\ &= (n+1)\int\limits_{\mathbb{D}}\varphi(\varphi_z(w))(1-|w|^2)^n\mathrm{d}A(w).\end{aligned}$$

Note that the 0-Berezin transform is the usual Berezin transform which is introduced in Section 1. Since $(n+1)(1-|w|^2)^n\mathrm{d}A(w)$ is a probability measure that tends to concentrate its mass at 0 when $n\to\infty$, $(B_n\varphi)(z)$ is an average of φ satisfying

$$\|B_n\varphi\|_\infty \leqslant \|\varphi\|_\infty$$

for all $\varphi\in L^\infty(\mathbb{D})$. For the n-Berezin transform of a function φ, we have the following lemma, see [13] and [14] for example.

LEMMA 2.1 (Suárez). *Suppose $\varphi\in L^\infty(\mathbb{D})$ and let $B_n(T_\varphi)$ be the n-Berezin transform of the Toeplitz operator T_φ, then:*
(i) $(B_nB_k)(T_\varphi) = (B_kB_n)(T_\varphi)$ *for every* $n,k\in\mathbb{N}$;
(ii) *fix* $k\geqslant 0$, *then* $B_n(B_k(T_\varphi))\to B_k(T_\varphi)$ *uniformly when* $n\to\infty$;
(iii) $T_{B_n\varphi}\to T_\varphi$ *in operator norm when* $n\to\infty$.

To study the invertibility of the Toeplitz operators, we need some basic results of pseudo-hyperbolic metric and Bergman metric, see [21]. For z and w in the open disk $\mathbb{D}$, the pseudo-hyperbolic distance $\rho(z,w)$ between z and w is defined by

$$\rho(z,w) = |\varphi_z(w)|.$$

For $z\in\mathbb{D}$ and $0<r<1$, the pseudo-hyperbolic disk $D(z,r)$ with center z and radius r is defined by

$$D(z,r) = \{w\in\mathbb{D} : \rho(z,w) < r\}.$$

Since $\varphi_z(w)$ is the Möbius map, the pseudo-hyperbolic disk $D(z,r)$ is also a Euclidean disk. More precisely, $D(z,r)$ is a Euclidean disk with center $\mathcal{C}$ and radius

$\mathcal{R}$ given by:

$$\mathcal{C} = \frac{1-r^2}{1-r^2|z|^2}z, \quad \mathcal{R} = \frac{1-|z|^2}{1-r^2|z|^2}r,$$

so the area of $D(z,r)$ is

$$m(D(z,r)) = \frac{(1-|z|^2)^2}{(1-r^2|z|^2)^2}r^2.$$

Furthermore, it is easy to see that there exists a constant $C_r > 0$ (depending only on r) such that

$$m(D(z,r)) \leqslant C_r(1-|z|)^2.$$

Another important metric on the unit disk is the Bergman metric β given by

$$\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|} \quad (z,w \in \mathbb{D}).$$

In particular,

$$\beta(0,z) = \frac{1}{2}\log\frac{1+|z|}{1-|z|} \quad (z \in \mathbb{D}).$$

It is easy to check that the Bergman metric is Möbius invariant:

$$\beta(\varphi_\lambda(z), \varphi_\lambda(w)) = \beta(z,w),$$

where $z, w \in \mathbb{D}$ and φ_λ is a Möbius map which is introduced in Section 1.

Now we introduce the function space BMO. For any $f \in L^2(\mathbb{D}, \mathrm{d}A)$, define

$$\|f\|_{\mathrm{BMO}} = \sup\{[\widetilde{|f|^2}(z) - |\widetilde{f}(z)|^2]^{1/2} : z \in \mathbb{D}\}.$$

Let BMO be the space of functions f with $\|f\|_{\mathrm{BMO}} < +\infty$.

Suppose that φ is a harmonic function on the unit disk, the following two lemmas ([21]) are useful to get our sufficient condition for T_φ to be invertible on the Bergman space.

LEMMA 2.2. *If φ is in* BMO, *then*

$$|\widetilde{\varphi}(z) - \widetilde{\varphi}(w)| \leqslant 2\sqrt{2}\|\varphi\|_{\mathrm{BMO}}\beta(z,w)$$

for all z and w in $\mathbb{D}$.

LEMMA 2.3. *Let $\varphi \in L^\infty(\mathbb{D})$, then there exists a constant $C > 0$ (independent of the function φ) such that*

$$\|H_\varphi\| \leqslant C\|\varphi\|_{\mathrm{BMO}}.$$

As we mentioned in Section 1, Luecking [8] obtained several necessary and sufficient conditions on the invertibility of Toeplitz operators with nonnegative symbols. Now we state his results as the following lemma, which will be used in the next section.

LEMMA 2.4 (Luecking). *Let φ be a bounded nonnegative measurable function on $\mathbb{D}$. Then the following conditions are equivalent:*

(i) *the Toeplitz operator T_φ is invertible on $L^2_a(\mathbb{D})$;*

(ii) *there exists a constant $\eta > 0$ such that*

$$\int_{\mathbb{D}} |\varphi(z)f(z)|^2 \mathrm{d}A(z) \geqslant \eta \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A(z)$$

for all $f \in L^2_a(\mathbb{D})$;

(iii) *there exist $r > 0$, $\delta > 0$ and $0 < \varepsilon < 1$ such that*

$$m(G \cap D(a, \varepsilon)) > \delta m(D(a, \varepsilon))$$

for all $a \in \mathbb{D}$, where $G = \{z \in \mathbb{D} : \varphi(z) > r\}$ and $D(a, r)$ is a pseudo-hyperbolic disk. Here m denotes the area measure on the complex plane.

To end this section, let us recall an important result on the Fredholm theory of the Bergman–Toeplitz operator. The following lemma can be found in [12], which is analogous to Theorem 7.26 in [4].

LEMMA 2.5. *Suppose $\varphi \in C(\overline{\mathbb{D}})$ and T_φ is a Fredholm operator. Then the Fredholm index of T_φ is given by*

$$\operatorname{index}(T_\varphi) = \dim \operatorname{Ker}(T_\varphi) - \dim \operatorname{Ker}(T^*_\varphi) = -\operatorname{wind}(\varphi(\partial\mathbb{D}), 0),$$

where $\operatorname{wind}(\varphi(\partial\mathbb{D}), 0)$ *is the winding number of the closed curve $\varphi(\partial\mathbb{D})$ with respect to the origin, which is defined by*

$$\operatorname{wind}((\varphi(\partial\mathbb{D}), 0) = \frac{1}{2\pi \mathrm{i}} \int_{\varphi(\partial\mathbb{D})} \frac{\mathrm{d}z}{z}.$$

3. TOEPLITZ OPERATORS WITH NONNEGATIVE SYMBOLS VIA BEREZIN TRANSFORM

In this section, we study the invertibility of the Toeplitz operators with nonnegative symbols. First, we use Lemma 2.4 to obtain the following theorem.

THEOREM 3.1. *Let φ be a function in $L^\infty(\mathbb{D})$. If T_φ is invertible, then $\widetilde{|\varphi|}$ is invertible in $L^\infty(\mathbb{D})$. However, this condition is not sufficient.*

Proof. Let φ be in $L^\infty(\mathbb{D})$. If T_φ is invertible, then T_φ is bounded below on $L^2_a(\mathbb{D})$. Thus there exists a constant $\varepsilon > 0$ such that

$$\varepsilon \|f\|_2 \leqslant \|T_\varphi f\|_2 \leqslant \|\varphi f\|_2 = \||\varphi| f\|_2$$

for all f in $L^2_a(\mathbb{D})$. Using (i)⇔(ii) in Lemma 2.4, we get that the positive Toeplitz operator $T_{|\varphi|}$ is invertible. Therefore, there is a constant $\delta > 0$ such that

$$\widetilde{|\varphi|}(z) = \langle T_{|\varphi|} k_z, k_z \rangle \geqslant \delta \langle k_z, k_z \rangle = \delta$$

for all $z \in \mathbb{D}$. This gives that $\widetilde{|\varphi|}$ is invertible.

For the second part of the theorem, we need to construct a function $\varphi \in L^\infty(\mathbb{D})$ such that $\widetilde{|\varphi|}$ is invertible in $L^\infty(\mathbb{D})$ but the Toeplitz operator T_φ is not invertible. Let

$$\varphi(z) = z^2 \quad (z \in \mathbb{D}).$$

Using Proposition 3.4 and the proof of Theorem 2.5 in [19], we obtain

$$\begin{aligned}
\widetilde{|\varphi|}(z) &= \int_{\mathbb{D}} |\varphi(w)| \cdot |k_z(w)|^2 \mathrm{d}A(w) = \int_{\mathbb{D}} |wk_z(w)|^2 \mathrm{d}A(w) \\
&= 2(1-|z|^2)^2 \sum_{n=0}^{\infty} \frac{(n+1)^2}{2n+4} |z|^{2n} \\
&= 2(1-|z|^2)^2 \Big[\frac{1}{4} + \frac{|z|^2}{2(1-|z|^2)^2} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{|z|^{2n}}{n+2}\Big] \\
&= \frac{1}{2} + \frac{1}{2}|z|^4 + (1-|z|^2)^2 \sum_{n=1}^{\infty} \frac{|z|^{2n}}{n+2} \geqslant \frac{1}{2}
\end{aligned}$$

for all $z \in \mathbb{D}$. This gives that $\widetilde{|\varphi|}$ is bounded below on $\mathbb{D}$. However, Lemma 2.5 tells us that T_φ is not invertible since the Fredholm index of T_φ is -2. This completes the proof. ∎

Combining the above theorem and Lemma 2.4 we obtain the following result, which gives a characterization of the invertibility of the Toeplitz operators with nonnegative symbols by their Berezin transforms.

THEOREM 3.2. *Let φ be a nonnegative function in $L^\infty(\mathbb{D})$. Then T_φ is invertible if and only if $\widetilde{\varphi}$ is invertible in $L^\infty(\mathbb{D})$.*

Proof. If T_φ is invertible on the Bergman space and φ is nonnegative, Theorem 3.1 gives that $\widetilde{\varphi} = \widetilde{|\varphi|}$ is invertible in $L^\infty(\mathbb{D})$.

Conversely, suppose $\widetilde{\varphi}$ is invertible in $L^\infty(\mathbb{D})$. Then there exists a constant $\delta > 0$ such that $\widetilde{\varphi}(z) \geqslant \delta$ for all $z \in \mathbb{D}$. Lemma 2.4 implies that we need only to verify condition (iii). To do so, we choose $r = \frac{\delta}{4}$ and define

$$G = \{z \in \mathbb{D} : \varphi(z) > r\}$$

as (iii) in Lemma 2.4.

For each $a \in \mathbb{D}$ and $\varepsilon \in (0,1)$, we observe that

$$\begin{aligned}
\frac{4\|\varphi\|_\infty}{(1-|a|)^2} m(G \cap D(a,\varepsilon)) &\geqslant \int_{G\cap D(a,\varepsilon)} \frac{4\varphi(z)}{(1-|a|)^2} \mathrm{d}A(z) \geqslant \int_{G\cap D(a,\varepsilon)} \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) \\
&= \int_G \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) - \int_{G\setminus D(a,\varepsilon)} \varphi(z)|k_a(z)|^2 \mathrm{d}A(z)
\end{aligned}$$

$$
\begin{aligned}
&\geqslant \int_G \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) - \int_{\mathbb{D}\setminus D(a,\varepsilon)} \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) \\
&\geqslant \int_G \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) - \|\varphi\|_\infty \int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z),
\end{aligned}
$$

where the second inequality comes from

$$
|k_a(z)|^2 \leqslant \frac{4}{(1-|a|)^2}
$$

for each $a \in \mathbb{D}$. Since

$$
\begin{aligned}
\widetilde{\varphi}(a) &= \int_{\mathbb{D}} \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) \\
&= \int_G \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) + \int_{\mathbb{D}\setminus G} \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) \geqslant \delta \quad (a \in \mathbb{D}),
\end{aligned}
$$

we obtain

$$
\begin{aligned}
\frac{4\|\varphi\|_\infty}{(1-|a|)^2} m(G\cap D(a,\varepsilon)) &\geqslant \delta - \int_{\mathbb{D}\setminus G} \varphi(z)|k_a(z)|^2 \mathrm{d}A(z) - \|\varphi\|_\infty \int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z) \\
&\geqslant \delta - r \int_{\mathbb{D}\setminus G} |k_a(z)|^2 \mathrm{d}A(z) - \|\varphi\|_\infty \int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z) \\
&\geqslant \delta - r \int_{\mathbb{D}} |k_a(z)|^2 \mathrm{d}A(z) - \|\varphi\|_\infty \int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z) \\
&= \frac{3\delta}{4} - \|\varphi\|_\infty \int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z),
\end{aligned}
$$

where the second inequality comes from the definition of G. Since

$$
\int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z) = 1 - \int_{D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z) = 1 - \int_{D(0,\varepsilon)} \mathrm{d}A(z) = 1 - \varepsilon^2,
$$

we can choose $\varepsilon \in (0,1)$ such that

$$
\int_{\mathbb{D}\setminus D(a,\varepsilon)} |k_a(z)|^2 \mathrm{d}A(z) < \frac{\delta}{4\|\varphi\|_\infty},
$$

to get

$$
\frac{4\|\varphi\|_\infty}{(1-|a|)^2} m(G\cap D(a,\varepsilon)) \geqslant \frac{3\delta}{4} - \frac{\delta}{4} = \frac{\delta}{2}.
$$

Thus we obtain

$$
m(G\cap D(a,\varepsilon)) \geqslant \frac{\delta(1-|a|)^2}{8\|\varphi\|_\infty} \geqslant \frac{\delta m(D(a,\varepsilon))}{8C_\varepsilon\|\varphi\|_\infty},
$$

where the constant C_ε comes from the remarks in Section 2. This completes the proof. ∎

From the above characterization of the invertibility of Toeplitz operators with nonnegative symbols, one may ask: *Is $\sigma(T_\varphi)$ equal to the essential range $\mathscr{R}(\widetilde{\varphi})$ of the function $\widetilde{\varphi}$ for each nonnegative $\varphi \in L^\infty(\mathbb{D})$?* However, it is easy to see that $\sigma(T_\varphi)$ is a discrete set and $\mathscr{R}(\widetilde{\varphi})$ is not if φ is a continuous *radial function* on $\mathbb{D}$ (i.e., $\varphi(z) = \varphi(|z|)$ for each $z \in \mathbb{D}$). So we have $\mathscr{R}(\widetilde{\varphi}) \nsubseteq \sigma(T_\varphi)$ for general $\varphi \geqslant 0$. But it is not clear that whether $\mathscr{R}(\widetilde{\varphi}) \supset \sigma(T_\varphi)$ for $\varphi \geqslant 0$. These lead us to consider the following question:

QUESTION 3.3. Is $\sigma(T_\varphi)$ contained in $\mathscr{R}(\widetilde{\varphi})$ for every nonnegative function $\varphi \in L^\infty(\mathbb{D})$?

Indeed, we will show the answer to Question 3.3 is negative in the general case by the following example.

PROPOSITION 3.4. *Let $\varphi(z) = |z|^2 + a|z| + b$ $(z \in \overline{\mathbb{D}})$, where a and b are real constants. Then there exist a and b such that $\varphi(z) \geqslant 0$ for all $z \in \overline{\mathbb{D}}$ but $\sigma(T_\varphi) \nsubseteq \mathrm{Ran}(\widetilde{\varphi})$.*

Proof. Denote the eigenvalues of T_φ by λ_n $(n \geqslant 0)$. The standard orthonormal basis $\{\sqrt{n+1}z^n\}_{n=0}^\infty$ of $L^2_a(\mathbb{D})$ diagonalizes both Toeplitz operators $T_{|z|^2}$ and $T_{|z|}$. It follows that the eigenvalues of T_φ are given by

$$\lambda_n = \frac{2n+2}{2n+4} + a\frac{2n+2}{2n+3} + b = a + b + 1 - \Big(\frac{a}{2n+3} + \frac{2}{2n+4}\Big) \quad (n \geqslant 0),$$

see Lemma 3.1 in [19] if needed. Thus $\sigma(T_\varphi) = \{\lambda_n\}_{n=0}^\infty$.

We will show that there exist a and b such that $\varphi \geqslant 0$ and

$$\min\{\lambda_n : n \geqslant 0\} := \lambda_{\min} < \inf_{z\in\mathbb{D}} \widetilde{\varphi}(z). \tag{3.1}$$

If the above holds, then it is easy to see that (3.1) implies $\sigma(T_\varphi) \nsubseteq \mathrm{Ran}(\widetilde{\varphi})$.

We claim that $a = -\frac{3}{2}$ and $b = 1$ satisfy the above conditions. Indeed, it is clear that

$$\varphi(z) = |z|^2 - \frac{3}{2}|z| + 1 = \Big(|z| - \frac{3}{4}\Big)^2 + \frac{7}{16}$$

is positive on $\mathbb{D}$ and

$$\begin{aligned}\lambda_{\min} &= \min\Big\{a + b + 1 - \Big(\frac{a}{2n+3} + \frac{2}{2n+4}\Big) : n \geqslant 0\Big\}\\ &= \min\Big\{\frac{1}{2} - \Big(\frac{-\frac{3}{2}}{2n+3} + \frac{2}{2n+4}\Big) : n \geqslant 0\Big\} = \lambda_2 = \frac{13}{28}.\end{aligned}$$

We will prove the following inequality:

$$\inf_{z\in\mathbb{D}} \widetilde{\varphi}(z) - \lambda_{\min} = \inf_{z\in\mathbb{D}} \widetilde{\varphi}(z) - \lambda_2 = \inf_{z\in\mathbb{D}} \widetilde{\varphi}(z) - \frac{13}{28} > 0.$$

Notice that $(4+a)n+(2a+6) \geqslant 0$ for all $n \geqslant 0$. Let $x = |z|^2 \in [0,1]$ and use the expression of $\widetilde{\varphi}$ (see Lemma 3.3 in [19]), we have

$$\begin{aligned}
\widetilde{\varphi}(z) &= 2(1-x)^2\Big[\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+4}x^n + a\sum_{n=0}^{\infty}\frac{(n+1)^2}{2n+3}x^n\Big] + b \\
&= \Big(\frac{1}{2}+\frac{a}{6}\Big)x^2 + \frac{a}{6}x + \Big(\frac{1}{2}+\frac{2}{3}a+b\Big) + \frac{(1-x)^2}{2}\sum_{n=1}^{\infty}\frac{(4+a)n+(2a+6)}{(n+2)(2n+3)}x^n \\
&\geqslant \frac{1}{4}x^2 - \frac{x}{4} + \frac{1}{2} + \frac{(1-x)^2}{4}\sum_{n=1}^{2}\frac{5n+6}{(n+2)(2n+3)}x^n \quad \Big(\text{since} \quad a=-\frac{3}{2},\ b=1\Big) \\
&= \frac{1}{420}[60x^4 - 43x^3 + 11x^2 - 28x + 210].
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
420\Big[\widetilde{\varphi}(z) - \frac{13}{28}\Big] &\geqslant 60x^4 - 43x^3 + 11x^2 - 28x + 210 - 195 \\
&= 60x^4 - 43x^3 + 11x^2 - 28x + 15 := G(x) \quad (x \in [0,1])
\end{aligned}$$

for all $z \in \overline{\mathbb{D}}$. Taking derivative of $G(x)$ gives

$$G'(x) = 240x^3 - 129x^2 + 22x - 28.$$

Applying Sturm theorem (see Theorem 5.2 in [19]) to the polynomial $G'(x)$, we get that there exists a unique point $x_0 \in [0,1]$ such that $G'(x_0) = 0$. The intermediate value theorem guarantees that $x_0 \in (0.66, 0.67)$. Observe that

$$\min_{x\in[0,1]} G(x) = G(x_0).$$

Let

$$H(t) = -43t^3 + 22t^2 - 84t + 60 \quad (t \in (0.66, 0.67)).$$

It is easy to check that $H'(t) < 0$ for all $t \in (0.66, 0.67)$. Since $4G - G' = H$, we get

$$\min_{x\in[0,1]} G(x) = G(x_0) = \frac{1}{4}H(x_0) \geqslant \frac{1}{4}H(0.67) \geqslant \frac{3}{20},$$

which implies that

$$\inf_{z\in\mathbb{D}} \widetilde{\varphi}(z) - \lambda_{\min} = \inf_{z\in\mathbb{D}} \widetilde{\varphi}(z) - \frac{13}{28} \geqslant \frac{1}{420} \times \min_{x\in[0,1]} G(x) > 0.$$

This completes the proof of Proposition 3.4. ∎

A small modification of the radial function φ in Proposition 3.4 gives the negative answer to Question 1.2.

COROLLARY 3.5. *Let $\psi(z) = \varphi(z) - \lambda_2$, where φ and λ_n are given in the above proof. Then the Berezin transform $\widetilde{\psi}$ is invertible in $L^\infty(\mathbb{D})$, but the corresponding Toeplitz operator T_ψ is not invertible on the Bergman space $L^2_a(\mathbb{D})$.*

Proof. From the proof of the above proposition, we have

$$\psi(z) = |z|^2 - \frac{3}{2}|z| + \frac{15}{28}.$$

Thus $\inf_{z\in\mathbb{D}} \widetilde{\psi}(z) > 0$ and the eigenvalues μ_n of T_ψ are given by $\mu_n = \lambda_n - \lambda_2$, so that $\mu_2 = 0$ and T_ψ is not invertible. This completes the proof. ∎

4. INVERTIBILITY OF TOEPLITZ OPERATORS WITH HARMONIC SYMBOLS

In this section we deal with the Toeplitz operators with harmonic symbols. To prove our main theorem, we need an estimation on the norm of the Hankel operator. In fact, the following result tells us that the positive constant C in Lemma 2.3 can be estimated easily if the function φ is harmonic on $\mathbb{D}$.

LEMMA 4.1. *Suppose that φ is a bounded harmonic function on $\mathbb{D}$. Then we have*

$$\|H_\varphi\| \leqslant 27\|\varphi\|_{\mathrm{BMO}}.$$

Proof. To estimate the norm of the Hankel operator H_φ, we use the technique in Lemma 7.3.2 of [21]. By the definition of Hankel operator and Lemma 2.2, we have

$$\begin{aligned}|H_\varphi f(z)| = |H_{\widetilde{\varphi}} f(z)| &\leqslant \int_{\mathbb{D}} |\widetilde{\varphi}(z) - \widetilde{\varphi}(w)| \cdot |K_z(w)| \cdot |f(w)| \mathrm{d}A(w) \\ &\leqslant 2\sqrt{2}\|\varphi\|_{\mathrm{BMO}} \int_{\mathbb{D}} \beta(z,w) \cdot |K_z(w)| \cdot |f(w)| \mathrm{d}A(w)\end{aligned}$$

for all $f \in L_a^2(\mathbb{D})$. Now we consider the following linear operator T on $L^2(\mathbb{D})$:

$$Tf(z) = \int_{\mathbb{D}} |K_z(w)| \beta(z,w) f(w) \mathrm{d}A(w),$$

then

$$\|H_\varphi f\|_2 \leqslant 2\sqrt{2}\|\varphi\|_{\mathrm{BMO}}\|T(|f|)\|_2 \leqslant 2\sqrt{2}\|\varphi\|_{\mathrm{BMO}}\|T\| \cdot \|f\|_2$$

for all $f \in L_a^2(\mathbb{D})$. Thus we need to use Schur's test to estimate the norm of the operator T. By the properties of the Bergman metric and Lemma 7.3.2 in [21], it suffices to determine the maximum value of the function

$$\beta(0,w)(1 - |w|^2)^{1/4}$$

and estimate the value of the integral

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^{-3/4}}{|1 - z\overline{w}|} \mathrm{d}A(w).$$

First we consider the above function. By the definition of Bergman metric, we get

$$\beta(0,w)(1-|w|^2)^{1/4} = \frac{1}{2}(1-|w|^2)^{1/4}\log\frac{1+|w|}{1-|w|}.$$

Let

$$F(x) = (1-x^2)^{1/4}\log\frac{1+x}{1-x} \quad (x \in [0,1]).$$

Now we are going to estimate the maximum value of $F(x)$ on $[0,1]$. Taking derivative of $F(x)$ gives

$$F'(x) = \frac{x}{2(1-x^2)^{3/4}}\Big[\frac{4}{x} - \log\frac{1+x}{1-x}\Big] := \frac{x}{2(1-x^2)^{3/4}}G(x).$$

Simple calculations give that $G(x)$ is decreasing on $(0,1)$ and

$$G(\tfrac{121}{125}) > 0 \quad \text{and} \quad \lim_{x\to 1^-} G(x) < 0.$$

Thus there exists a unique point $x_0 \in (\frac{121}{125},1)$ such that $G(x_0) = 0$. By the definition of $G(x)$ and $F(x) \geqslant 0$ for all $x \in [0,1]$, we obtain that x_0 is the unique point where $F(x)$ reaches its maximum value and x_0 satisfies that

$$\log\frac{1+x_0}{1-x_0} = \frac{4}{x_0}.$$

Therefore,

$$\sup_{x\in[0,1]} F(x) = F(x_0) = \frac{4(1-x_0^2)^{1/4}}{x_0} \leqslant \frac{3\sqrt{2}}{2} \quad (\text{since } x_0 > \tfrac{121}{125}).$$

Now we turn to estimate the above integral. We use the gamma function to get

$$\begin{aligned}\int_{\mathbb{D}} \frac{(1-|w|^2)^{-3/4}}{|1-z\overline{w}|}\mathrm{d}A(w) &= \frac{\Gamma(\frac{1}{4})}{[\Gamma(\frac{1}{2})]^2}\sum_{n=0}^{+\infty}\frac{[\Gamma(n+\frac{1}{2})]^2}{n!\Gamma(n+\frac{5}{4})}|z|^{2n} \leqslant \frac{\Gamma(\frac{1}{4})}{[\Gamma(\frac{1}{2})]^2}\sum_{n=0}^{+\infty}\frac{[\Gamma(n+\frac{1}{2})]^2}{n!\Gamma(n+\frac{5}{4})}\\ &= \frac{\Gamma(\frac{1}{4})}{\pi}\sum_{n=0}^{10}\frac{[\Gamma(n+\frac{1}{2})]^2}{n!\Gamma(n+\frac{5}{4})} + \frac{\Gamma(\frac{1}{4})}{\pi}\sum_{n=11}^{+\infty}\frac{[\Gamma(n+\frac{1}{2})]^2}{n!\Gamma(n+\frac{5}{4})}\\ &= \mathrm{I}+\mathrm{II}.\end{aligned}$$

Direct calculation gives that

$$\mathrm{I} \leqslant 6 + \frac{21}{100}.$$

For the second term, recall that the Gautschi inequality gives

$$n^{1-\alpha} \leqslant \frac{n!}{\Gamma(n+\alpha)} \leqslant (n+1)^{1-\alpha} \quad (0 \leqslant \alpha \leqslant 1)$$

for all $n \geqslant 1$. Thus we obtain

$$\begin{aligned}
\mathrm{II} &\leqslant \frac{\Gamma(\frac{1}{4})}{\pi} \sum_{n=11}^{+\infty} \frac{(n+1)^{3/4}}{n(n+\frac{1}{4})} \\
&\leqslant \frac{\Gamma(\frac{1}{4})}{\pi} \sum_{n=11}^{+\infty} \frac{(\frac{12}{11}n)^{3/4}}{n^2} \quad (\text{since } n \geqslant 11) \\
&= \frac{\Gamma(\frac{1}{4})}{\pi} \cdot \Big(\frac{12}{11}\Big)^{3/4} \sum_{n=11}^{+\infty} \frac{1}{n^{5/4}} \leqslant \frac{\Gamma(\frac{1}{4})}{\pi} \Big(\frac{12}{11}\Big)^{3/4} \int_{10}^{+\infty} x^{-5/4} \mathrm{d}x \\
&= \frac{\Gamma(\frac{1}{4})}{\pi} \cdot \Big(\frac{12}{11}\Big)^{3/4} \cdot \frac{4}{10^{1/4}} \leqslant 2 + \frac{78}{100}.
\end{aligned}$$

Combining I, II and the maximum value of $F(x)$ we obtain

$$\|T\| \leqslant \frac{3\sqrt{2}}{4} \times 9.$$

Thus we have

$$\|H_\varphi\| \leqslant 2\sqrt{2} \times \frac{3\sqrt{2}}{4} \times 9\|\varphi\|_{\mathrm{BMO}} = 27\|\varphi\|_{\mathrm{BMO}}.$$

This completes the proof of Lemma 4.1. ∎

Using Lemma 4.1 we will establish the following theorem, which is analogous to the result of the invertibility of Hardy–Toeplitz operators, see [16].

THEOREM 4.2. *Suppose that φ is a bounded harmonic function on $\mathbb{D}$. If there exists a $\delta \in (0,1)$ such that*

$$|\varphi(z)| \geqslant \delta\|\varphi\|_\infty > \frac{27}{\sqrt{730}}\|\varphi\|_\infty$$

for all $z \in \mathbb{D}$, then T_φ is invertible on $L^2_a(\mathbb{D})$ and

$$\|T_\varphi^{-1}\| \leqslant \frac{1}{\sqrt{\delta^2 - 27^2(1-\delta^2)}\|\varphi\|_\infty}.$$

Proof. Recall that $L^\infty(\mathbb{D}) \subset \mathrm{BMO}$ and

$$\|\varphi\|^2_{\mathrm{BMO}} = \sup_{z\in\mathbb{D}}[\widetilde{|\varphi|}^2(z) - |\widetilde{\varphi}(z)|^2].$$

Thus we have

$$\begin{aligned}
\|T_\varphi f\|^2 = \|\varphi f\|_2^2 - \|H_\varphi f\|^2 &\geqslant \delta^2\|\varphi\|^2_\infty\|f\|^2 - \|H_\varphi\|^2\|f\|^2 \\
&\geqslant \delta^2\|\varphi\|^2_\infty\|f\|^2 - 27^2\sup_{z\in\mathbb{D}}[\widetilde{|\varphi|}^2(z) - |\widetilde{\varphi}(z)|^2]\|f\|^2 \quad (\text{by Lemma 4.1}) \\
&\geqslant \delta^2\|\varphi\|^2_\infty\|f\|^2 - 27^2(\|\varphi\|^2_\infty - \delta^2\|\varphi\|^2_\infty)\|f\|^2 \quad (\text{since } \varphi \text{ is harmonic}) \\
&= [\delta^2 - 27^2(1-\delta^2)]\|\varphi\|^2_\infty\|f\|^2
\end{aligned}$$

for each $f \in L_a^2(\mathbb{D})$. Since $\delta > \frac{27}{\sqrt{730}}$, we see that T_φ is bounded below and

$$\|T_\varphi f\| \geqslant \sqrt{\delta^2 - 27^2(1-\delta^2)}\|\varphi\|_\infty \|f\|$$

for all $f \in L_a^2(\mathbb{D})$. Using that $T_{\overline{\varphi}} = T_\varphi^*$, we also get

$$\|T_\varphi^* f\| \geqslant \sqrt{\delta^2 - 27^2(1-\delta^2)}\|\varphi\|_\infty \|f\|$$

for all $f \in L_a^2(\mathbb{D})$. Thus T_φ is invertible. This finishes the proof of the theorem. ∎

The above theorem gives a sufficient condition for T_φ to be invertible when the symbol φ is a harmonic function. However, if φ is real and harmonic on $\mathbb{D}$, we can characterize the invertibility of T_φ easily because the spectrum of T_φ has been computed explicitly by McDonald and Sundberg in Proposition 12 of [9].

THEOREM 4.3. *Let φ be a real harmonic function in $L^\infty(\mathbb{D})$, then T_φ is invertible on $L_a^2(\mathbb{D})$ if and only if the symbol φ is invertible in $L^\infty(\mathbb{D})$.*

Note that the above result tells us that if for a harmonic function φ the Toeplitz operator T_φ is self-adjoint, then $\frac{1}{\varphi} \in L^\infty(\mathbb{D})$ if and only if T_φ is invertible. Furthermore, the above theorem also holds if φ is a harmonic function such that T_φ is normal.

THEOREM 4.4. *Let φ be a harmonic function in $L^\infty(\mathbb{D})$. If T_φ is a normal operator, then T_φ is invertible on $L_a^2(\mathbb{D})$ if and only if φ is invertible in $L^\infty(\mathbb{D})$.*

Proof. By Corollary 17 of [1], $\varphi(\mathbb{D})$ lies on some line in $\mathbb{C}$. Then there exist constants a, b and a real valued function ψ such that

$$\varphi = a\psi + b.$$

We need only to consider $a \neq 0$. Note that

$$0 \in \sigma(T_\varphi) = \sigma(aT_\psi + b)$$

if and only if

$$-\frac{b}{a} \in \sigma(T_\psi) = \operatorname{clos}[\psi(\mathbb{D})],$$

which is equivalent to $0 \in \operatorname{clos}[\varphi(\mathbb{D})]$. This completes the proof. ∎

In the rest of this section, we will use the n-Berezin transform to study the invertibility of Bergman–Toeplitz operators with bounded symbols. The following theorem gives us a sufficient condition for T_φ to be invertible on the Bergman space.

THEOREM 4.5. *Let $\varphi \in L^\infty(\mathbb{D})$ and C be the constant in Lemma 2.3. Then there exists some integer $N = N(\varphi)$ (depending only on φ) such that the inequalities*

$$\inf_{z\in\mathbb{D}} |\widetilde{\varphi}(z)| > \delta\|\varphi\|_\infty \quad \text{and} \quad |(B_{N_0}\varphi)(z)| \geqslant \varepsilon\|\varphi\|_\infty$$

hold for all $z \in \mathbb{D}$, where $1 > \delta > 0$, $1 > \varepsilon > C\sqrt{1-\delta^2}$ are constants and $N_0 \geqslant N(\varphi)$ imply that the Toeplitz operator T_φ is invertible on $L_a^2(\mathbb{D})$.

Proof. Let $\psi_n = B_n\varphi$, $n \geqslant 1$. By Lemma 2.1, we obtain that

$$\lim_{n\to\infty} \|T_{\psi_n} - T_\varphi\| = 0.$$

Thus there exists an integer $N_1 = N_1(\varphi) \geqslant 1$ such that

$$\|T_{\psi_n} - T_\varphi\| < \frac{\alpha}{2}$$

for all $n \geqslant N_1$, where $\alpha = \sqrt{\varepsilon^2 - C^2(1-\delta^2)}\|\varphi\|_\infty$ is a fixed positive constant. Thus, if $n \geqslant N_1$, then

$$\|T_{\psi_n} f - T_\varphi f\| \leqslant \frac{\alpha}{2}\|f\|$$

for each $f \in L_a^2(\mathbb{D})$. Therefore, we obtain

$$\|T_\varphi f\| \geqslant \|T_{\psi_n} f\| - \frac{\alpha}{2}\|f\| \tag{4.1}$$

for all $n \geqslant N_1$ and $f \in L_a^2(\mathbb{D})$.

By Lemma 2.1 again, we have

$$\widetilde{\psi}_n = B_0(B_n\varphi) = B_n(B_0\varphi) \to B_0\varphi$$

uniformly as $n \to \infty$. Since

$$\inf_{z\in\mathbb{D}} |(B_0\varphi)(z)| = \inf_{z\in\mathbb{D}} |\widetilde{\varphi}(z)| > \delta\|\varphi\|_\infty,$$

there exists $N_2 = N_2(\varphi) \geqslant 1$ such that

$$|\widetilde{\psi}_n(z)| \geqslant |\widetilde{\varphi}(z)| - \Big(\inf_{z\in\mathbb{D}}|\widetilde{\varphi}(z)| - \delta\|\varphi\|_\infty\Big) \geqslant \inf_{z\in\mathbb{D}}|\widetilde{\varphi}(z)| - \Big(\inf_{z\in\mathbb{D}}|\widetilde{\varphi}(z)| - \delta\|\varphi\|_\infty\Big) = \delta\|\varphi\|_\infty$$

for all $n \geqslant N_2$ and $z \in \mathbb{D}$. By the definition of Berezin transform, we get

$$\begin{aligned}
\widetilde{|\psi_n|^2}(z) &= \int_{\mathbb{D}} |\psi_n(w)|^2 |k_z(w)|^2 \mathrm{d}A(w) \leqslant \|\psi_n\|_\infty^2 \\
&= \|B_n\varphi\|_\infty^2 \quad \text{(by the definition of } \psi_n) \\
&\leqslant \|\varphi\|_\infty^2
\end{aligned}$$

for all $n \geqslant 1$ and $z \in \mathbb{D}$. If $n \geqslant N_2$, we obtain

$$\begin{aligned}
\|T_{\psi_n} f\|^2 = \|\psi_n f\|^2 - \|H_{\psi_n} f\|^2 &\geqslant \|\psi_n f\|^2 - \|H_{\psi_n}\|^2\|f\|^2 \\
&\geqslant \|\psi_n f\|^2 - C^2 \sup_{z\in\mathbb{D}} [\widetilde{|\psi_n|^2}(z) - |\widetilde{\psi}_n(z)|^2] \|f\|_2^2 \\
&\geqslant \|\psi_n f\|^2 - C^2(1-\delta^2)\|\varphi\|_\infty^2 \|f\|^2
\end{aligned}$$

for all $f \in L_a^2(\mathbb{D})$. The second "$\geqslant$" comes from Lemma 2.3.

Suppose that $N_0 \geqslant \max\{N_1, N_2\} := N(\varphi)$. If φ satisfies the following condition

$$|\psi_{N_0}(z)| = |(B_{N_0}\varphi)(z)| \geqslant \varepsilon\|\varphi\|_\infty$$

for all $z \in \mathbb{D}$, then we have

$$\begin{aligned}\|T_{\psi_{N_0}} f\|^2 &\geqslant \|\psi_{N_0} f\|^2 - C^2(1-\delta^2)\|\varphi\|_\infty^2 \|f\|^2 \\ &\geqslant [\varepsilon^2 - C^2(1-\delta^2)]\|\varphi\|_\infty^2 \|f\|^2 = \alpha^2 \|f\|^2\end{aligned}$$

for each $f \in L_a^2(\mathbb{D})$. Since $N_0 \geqslant N_1$, by (4.1) we get

$$\|T_\varphi f\| \geqslant \|T_{\psi_{N_0}} f\| - \frac{\alpha}{2}\|f\| \geqslant \alpha\|f\| - \frac{\alpha}{2}\|f\| = \frac{\alpha}{2}\|f\|$$

for $f \in L_a^2(\mathbb{D})$. This gives that T_φ is bounded below, and so is T_φ^*. This completes the proof. ∎

As mentioned in the introduction, we will show that there exists a function $\varphi \in C(\overline{\mathbb{D}})$ such that $\widetilde{\varphi}$ is invertible and $|B_n(\varphi)|$ are bounded below for all sufficiently large n, but the corresponding Bergman–Toeplitz operator is not invertible. Before doing this, we recall some important results on the n-th Berezin transform, see Theorem 6.19 in [22].

LEMMA 4.6. *If φ is a function in $L^1(\mathbb{D}, \mathrm{d}A)$, then*

$$\lim_{n\to\infty} \|B_n(\varphi) - \varphi\|_{L^1} = 0.$$

If $\varphi \in C(\overline{\mathbb{D}})$, then we have

$$\lim_{n\to\infty} \|B_n(\varphi) - \varphi\|_\infty = 0.$$

Based on the above lemma, we see that if $\varphi \in C(\overline{\mathbb{D}})$ and $|\varphi|$ is bounded below by some positive constant δ, then the n-th Berezin transforms $|B_n(\varphi)|$ are bounded below for all sufficiently large n. In view of this observation we need only to construct a continuous function φ with the following properties:

(i) φ is invertible in $L^\infty(\mathbb{D})$;
(ii) the Berezin transform $\widetilde{\varphi}$ is also invertible;
(iii) T_φ is not invertible on $L_a^2(\mathbb{D})$.

Note that for the real valued continuous function φ, if φ is invertible in $L^\infty(\mathbb{D})$, then T_φ is invertible on $L_a^2(\mathbb{D})$, which can be proved easily using the same idea in Proposition 7.18 of [4]. Consequently, we need to find a complex valued continuous function that satisfies the above three conditions.

THEOREM 4.7. *Suppose that $\varphi(z) = |z|^2 + a|z| + b$, where a, b are constants. Then there exist $a, b \in \mathbb{C}\backslash\mathbb{R}$ and $\delta_1, \delta_2 > 0$ such that*

$$|\varphi(z)| \geqslant \delta_1 \quad \text{and} \quad |\widetilde{\varphi}(z)| \geqslant \delta_2$$

for all $z \in \mathbb{D}$, but the Toeplitz operator T_φ is not invertible on $L_a^2(\mathbb{D})$.

Proof. From the proof of Proposition 3.4, we have that the eigenvalues of T_φ are given by

$$\lambda_n = \frac{2n+2}{2n+4} + a\frac{2n+2}{2n+3} + b \quad (n \geqslant 0).$$

On the other hand, the Berezin transform of φ is given by the following formula (see Lemma 3.3 in [19]):

$$\widetilde{\varphi}(z) = \Big[2 - \frac{1}{|z|^2} - \frac{(1-|z|^2)^2}{|z|^4}\log(1-|z|^2)\Big] + \frac{a}{2}\Big[3 - \frac{1}{|z|^2} + \frac{(1-|z|^2)^2}{2|z|^3}\log\frac{1+|z|}{1-|z|}\Big] + b \quad (z \in \mathbb{D}).$$

Now we take $a = 2(1+\mathrm{i})$ and $b = -\frac{34}{15} - \frac{8}{5}\mathrm{i}$ to show that $0 = \lambda_1 \in \sigma(T_\varphi)$, and the continuous functions φ and $\widetilde{\varphi}$ are both invertible in $L^\infty(\mathbb{D})$. Indeed,

$$\lambda_1 = \frac{4}{6} + \frac{4}{5}a + b = 0.$$

This implies that T_φ is not invertible.

To prove that φ is invertible in $L^\infty(\mathbb{D})$, we need to show that φ has no zeros in $\overline{\mathbb{D}}$. Since

$$\varphi(z) = \Big(|z|^2 + 2|z| - \frac{34}{15}\Big) + 2\mathrm{i}\Big(|z| - \frac{4}{5}\Big),$$

we have that $\varphi(z) = 0$ if and only if $|z| - \frac{4}{5} = 0$ and

$$|z|^2 + 2|z| - \frac{34}{15} = 0.$$

It is clear that the above equations have no solution. Thus φ is invertible.

The difficult part is to prove that the Berezin transform $\widetilde{\varphi}$ is invertible. Under the above assumption, we have

$$\widetilde{\varphi}(z) = P(z) + \mathrm{i}Q(z),$$

where

$$P(z) = \frac{41}{15} - \frac{2}{|z|^2} + \frac{(1-|z|^2)^2}{2|z|^3}\Big[\log\frac{1+|z|}{1-|z|} - \frac{2}{|z|}\log(1-|z|^2)\Big] \quad \text{and}$$

$$Q(z) = \frac{7}{5} - \frac{1}{|z|^2} + \frac{(1-|z|^2)^2}{2|z|^3}\log\frac{1+|z|}{1-|z|}.$$

Thus $\widetilde{\varphi}(z) = 0$ if and only if $P(z) = 0$ and $Q(z) = 0$. Letting $t = |z| \in [0,1]$, we consider the following two functions:

$$F(t) = |z|^4 P(z) = \frac{41}{15}t^4 - 2t^2 + \frac{(1-t^2)^2}{2}\Big[t\log\frac{1+t}{1-t} - 2\log(1-t^2)\Big] \quad (t \in [0,1])$$

and

$$G(t) = |z|^3 Q(z) = \frac{7}{5}t^3 - t + \frac{(1-t^2)^2}{2}\log\frac{1+t}{1-t} \quad (t \in [0,1]).$$

Observe that $F(0) = G(0) = 0$ but

$$\widetilde{\varphi}(0) = \frac{1}{2} + \frac{2}{3}a + b = -\frac{13}{30} - \frac{4}{15}\mathrm{i} \neq 0,$$

so we need only to show that the equations $F(t) = 0$ and $G(t) = 0$ do not have any nonzero solutions.

To do so, our idea is as follows: first we prove that $G(t) = 0$ has only one nonzero root t_1 in $(0, 1)$, next we show that t_1 is not a root of $F(t) = 0$. We consider the monotonicity of the function G. Taking derivative gives

$$G'(t) = 2t(1-t^2)\Big[\frac{8t}{5(1-t^2)} - \log\frac{1+t}{1-t}\Big] := 2t(1-t^2)H(t) \quad (t \in (0,1)).$$

On the other hand, we have

$$H'(t) = \frac{2(9t^2-1)}{5(1-t^2)^2} \quad (t \in (0,1)).$$

From the above computations, we have that $H(t)$ is increasing in $(\frac{1}{3}, 1)$ and $H(t)$ is decreasing in $(0, \frac{1}{3})$. Note that $H(0) = 0$, $H(\frac{1}{3}) = \frac{3}{5} - \log 2 < 0$ and $\lim_{t \to 1^-} H(t) = +\infty$, we obtain that $H(t) = 0$ has only one root t_0 in $(0, 1)$. This implies that: if $t_0 < t < 1$, then $H(t) > 0$ and so $G'(t) > 0$; if $0 < t < t_0$, then $H(t) < 0$, so we have $G'(t) < 0$.

From the arguments above, we get that $G(t)$ is increasing if $t \in (t_0, 1)$ and decreasing if $t \in (0, t_0)$. Observe that $G(0) = 0$ and $G(1) = \frac{2}{5}$, we have that $G(t) = 0$ has exactly one solution t_1 in $(0, 1)$. To approximate t_1, we evaluate values of $G(t)$ for some points in $(0, 1)$ to get that $G(\frac{17}{25}) < 0$ and $G(\frac{7}{10}) > 0$. Thus the intermediate value theorem tells us that $t_1 \in \left(\frac{17}{25}, \frac{7}{10}\right)$.

Now we are going to show that $F(t_1) \neq 0$. If this is not true, we have that $F(t_1) = 0$. We will derive a contradiction. To do so, we need the following function:

$$l(t) = F(t) - tG(t) = t^2\Big(\frac{4}{3}t^2 - 1\Big) - (1-t^2)^2\log(1-t^2) \quad (t \in (0,1)).$$

Using $F(t_1) = 0$ we have

$$l(t_1) = F(t_1) - t_1 G(t_1) = 0.$$

Let $1 - t^2 = x \in (0, 1)$, the function l becomes the following

$$L(x) = \frac{(1-x)(1-4x)}{3} - x^2 \log x \quad (x \in (0,1)).$$

One has $L(x_1) = 0$, where

$$x_1 = 1 - t_1^2 \in \Big(0, 1 - \Big(\frac{17}{25}\Big)^2\Big] \subset (0, 0.54].$$

Simple calculations give us

$$L'(x) = -2x\Big[\frac{5(1-x)}{6x} + \log x\Big] := -2xR(x).$$

For the function $R(x)$, we have

$$R'(x) = \frac{1}{x^2}\Big(x - \frac{5}{6}\Big) \quad (x \in (0,1)).$$

Therefore R decreases on $(0, 0.54]$. Hence $R(x) \geqslant R(0.54) > 0$ on this interval, and one gets that L decreases on $(0, 0.54]$. Since $L(0.54) > 0$, L is positive on $(0, 0.54]$, which contradicts the fact that $x_1 \in (0, 0.54]$ is a root of $L(x) = 0$. The contradiction implies that t_1 is not a zero of F. Thus we have $\widetilde{\varphi}(z) \neq 0$ for any $z \in \overline{\mathbb{D}}$. This implies that the Berezin transform $\widetilde{\varphi}$ is also invertible in $L^\infty(\mathbb{D})$, to complete the proof of the theorem. ∎

Acknowledgements. We thank the referee for providing constructive comments and help in improving the contents of this paper. We are grateful to Professor Nikolai Nikolski for giving many useful remarks and comments on the invertibility problem of Hardy–Toeplitz operators. This work was partially supported by a NSFC grant (11271387) and Chongqing Natural Sience Foundation (cstc 2013jjB0050). The second author was partially supported by Simons Foundation grant #196300.

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Received July 7, 2015.