J. OPERATOR THEORY
75:2(2016), 497–523
doi: 10.7900/jot.2015jul13.2080

A NONCOMMUTATIVE BEURLING THEOREM WITH RESPECT TO UNITARILY INVARIANT NORMS

YANNI CHEN, DON HADWIN and JUNHAO SHEN

Communicated by Hari Bercovici

ABSTRACT. In 1967, Arveson invented a noncommutative generalization of classical H^∞, known as finite maximal subdiagonal subalgebras, for a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. In 2008, Blecher and Labuschagne proved a version of Beurling theorem on H^∞-right invariant subspaces in a noncommutative $L^p(\mathcal{M},\tau)$ space for $1 \leqslant p \leqslant \infty$. In the present paper, we define and study a class of norms $N_c(\mathcal{M},\tau)$ on $\mathcal{M}$, called normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norms, which properly contains the class $\{\|\cdot\|_p : 1 \leqslant p < \infty\}$ and the class of rearrangement invariant quasi Banach function norms studied by Bekjan. For $\alpha \in N_c(\mathcal{M},\tau)$, we define a noncommutative $L^\alpha(\mathcal{M},\tau)$ space and a noncommutative H^α space. Then we obtain a version of the Blecher–Labuschagne–Beurling invariant subspace theorem on H^∞-right invariant subspaces in $L^\alpha(\mathcal{M},\tau)$ spaces and H^α spaces. Key ingredients in the proof of our main result include a characterization theorem of H^α and a density theorem for $L^\alpha(\mathcal{M},\tau)$.

KEYWORDS: *Normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm, maximal subdiagonal algebra, dual space, Beurling theorem, noncommutative Hardy space.*

MSC (2010): Primary 46L52, 30H10; Secondary 47A15.

INTRODUCTION

One of the most celebrated theorems in operator theory is Beurling's invariant subspace theorem, stating that *if $\mathcal{W}$ is a nonzero closed, H^∞-invariant subspace (or, equivalently, $z\mathcal{W} \subseteq \mathcal{W}$) of $H^2(\mathbb{T})$ on the unit circle, then $\mathcal{W} = \psi H^2(\mathbb{T})$ for some $\psi \in H^\infty(\mathbb{T})$ with $|\psi| = 1$ a.e. (μ)* [2]. Later, the Beurling theorem for $H^2(\mathbb{T})$ was generalized to describe closed H^∞-invariant subspaces in the Hardy space $H^p(\mathbb{T})$ with $1 \leqslant p \leqslant \infty$ (see [6], [13], [14], [15], [16], [27] and etc.). Beurling theorem has been extended to many other directions.

In 1967, Arveson [1] invented a noncommutative generalization of classical H^∞, known as finite maximal subdiagonal subalgebras, for a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. Roughly, a subdiagonal algebra $\mathcal{A}$ is a subalgebra of a von Neumann algebra $\mathcal{M}$ which has many of the structural properties of the Hardy space $H^\infty(\mathbb{T})$. Subsequently, several authors studied the invariant subspaces of $\mathcal{A}$ acting on the noncommutative Lebesgue space $L^p(\mathcal{M},\tau)$. In 2008, Blecher and Labuschagne [5] proved a version of Beurling theorem on H^∞-right invariant subspaces in a noncommutative $L^p(\mathcal{M},\tau)$ space for $1 \leqslant p \leqslant \infty$. Very recently, in 2015, T.N. Bekjan [3] obtained the similar Beurling theorem in noncommutative Hardy spaces based on his beautiful study of symmetric Banach spaces.

In the present paper, we set up a Beurling theorem for noncommutative Hardy spaces associated with unitarily invariant norms, which properly contains the class $\{\|\cdot\|_p : 1 \leqslant p < \infty\}$ and the class of rearrangement invariant quasi Banach function norms studied in [3]. It is worth pointing out that many of the classical proofs for the $\|\cdot\|_p$ case use the L^2-result and take cases when $p \leqslant 2$ and $2 < p$ (see Theorem 4.5 in [5] and Theorem 6.5 of [3]). In our general setting, the cases $p \leqslant 2$ and $2 < p$ have no analogue, hence tools available in the setting of L^p-spaces and symmetric Banach spaces are no longer available. In order to achieve this extension, a lot of technology regarding these generalized settings needs to be developed. This is the reason why we proved a new version of Hölder's inequality, a new version of Saito's result [24] and many other results. The approach which we use is not only more elementary, even in the L^p-case, but is much more general.

We now review some of the definitions and notations. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. For each $1 \leqslant p < \infty$, we define a mapping $\|\cdot\|_p : \mathcal{M} \to [0,\infty)$ by $\|x\|_p = (\tau((x^*x)^{p/2}))^{1/p}$ for any $x \in \mathcal{M}$. It is a highly nontrivial fact that $\|\cdot\|_p$ actually defines a norm, an L^p-norm, on $\mathcal{M}$. Thus we let $L^p(\mathcal{M},\tau)$ be the completion of $\mathcal{M}$ under the norm $\|\cdot\|_p$. Moreover, it is not hard to see that there exists an anti-representation ρ of $\mathcal{M}$ on the space $L^p(\mathcal{M},\tau)$ given by $\rho(a)\xi = \xi a$ for $\xi \in L^p(\mathcal{M},\tau)$ and $a \in \mathcal{M}$. Thus we might assume that $\mathcal{M}$ acts naturally on each $L^p(\mathcal{M},\tau)$ space by right multiplication for $1 \leqslant p \leqslant \infty$. We will refer to a wonderful handbook [23] by Pisier and Xu for general knowledge and current development of the theory of noncommutative L^p-spaces.

A (finite maximal) subdiagonal subalgebra of $\mathcal{M}$ is a weak* closed unital subalgebra $\mathcal{A}$ of $\mathcal{M}$ such that if Φ is the unique conditional expectation from $\mathcal{M}$ onto $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$, then

(i) $\mathcal{A} + \mathcal{A}^*$ is weak* dense in $\mathcal{M}$;

(ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{A}$;

(iii) $\tau \circ \Phi = \tau$.

In [10], Exel showed that if $\mathcal{A}$ is weak* closed and τ satisfies (iii), then $\mathcal{A}$ (with respect to Φ) is maximal among those subdiagonal subalgebras (with respect to Φ) satisfying (i), (ii). Such a finite, maximal subdiagonal subalgebra $\mathcal{A}$ of $\mathcal{M}$ is also called an H^∞ space of $\mathcal{M}$. For each $1 \leqslant p < \infty$, the closure of H^∞ in $L^p(\mathcal{M}, \tau)$ is denoted by H^p and the closure of $H_0^\infty = \{x \in H^\infty : \Phi(x) = 0\}$ is denoted by H_0^p.

The concept of unitarily invariant norms was introduced by von Neumann [22] for the purpose of metrizing matrix spaces. These norms have now been generalized and applied in many contexts (for example, see [18], [20], [26] and etc.). Besides all L^p-norms for $1 \leqslant p \leqslant \infty$, there are many other interesting examples of unitarily invariant norms on $\mathcal{M}$ (for example, see [3], [7], [8], [12] and others).

In this paper, we introduce a class $N_c(\mathcal{M}, \tau)$ of normalized, unitarily invariant, $\|\cdot\|_1$-dominating and continuous norms (see Definition 1.2). If $\alpha \in N_c(\mathcal{M}, \tau)$ and H^∞ is a finite, maximal subdiagonal subalgebra of $\mathcal{M}$, then we let $L^\alpha(\mathcal{M}, \tau)$ and H^α be the completion of $\mathcal{M}$, and H^∞ respectively, with respect to the norm α.

In 2008, Fang, Hadwin, Nordgren and Shen set up a generalized noncommutative Lebesgue space associated with unitarily invariant norms. Some classical results in noncommutative L^p-theory (e.g., noncommutative Hölder's inequality, duality and reflexivity of noncommutative L^p-spaces) are obtained for unitarily invariant norms on finite factors.

Motivated by the relation between finite factors and finite von Neumann algebras, in this paper we consider the noncommutative L^p-spaces and the noncommutative H^p-spaces associated with unitarily invariant norms on a finite von Neumann algebra $\mathcal{M}$ and prove a version of Beurling's theorem for H^∞-right invariant subspaces in $L^\alpha(\mathcal{M}, \tau)$, and therefore for H^∞-right invariant subspaces in H^α, when $\alpha \in N_c(\mathcal{M}, \tau)$. More specifically, we are able to obtain the following Beurling theorem for $L^\alpha(\mathcal{M}, \tau)$, built on Blecher and Labuschagne's result in the case of $p = \infty$.

THEOREM 0.1. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$, then $\mathcal{W}H^\infty \subseteq \mathcal{W}$ if and only if*

$$\mathcal{W} = \mathcal{Z} \bigoplus^{\text{col}} (\bigoplus_{i \in \mathcal{I}}^{\text{col}} u_i H^\alpha),$$

where $\mathcal{Z}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{Z} = [\mathcal{Z}H_0^\infty]_\alpha$, and where u_i are partial isometries in $\mathcal{W} \cap \mathcal{M}$ with $u_j^* u_i = 0$ if $i \neq j$, and with $u_i^* u_i \in \mathcal{D}$. Moreover, for each i, $u_i^* \mathcal{Z} = \{0\}$, left multiplication by the $u_i u_i^*$ are contractive projections from $\mathcal{W}$ onto the summands $u_i H^\alpha$, and left multiplication by $1 - \sum_i u_i u_i^*$ is a contractive projection from $\mathcal{W}$ onto $\mathcal{Z}$.

Here $\bigoplus^{\text{col}}$ denotes an internal column sum (see Definition 4.5). Moreover, $\bigoplus_i^{\text{col}} u_i H^\alpha$ and $\mathcal{Z} = [\mathcal{Z} H_0^\infty]_\alpha$ are of type 1, and of type 2 respectively (see [5] for definitions of invariant subspaces of different types).

Many tools used in a noncommutative $L^p(\mathcal{M}, \tau)$ space are no longer available in an arbitrary $L^\alpha(\mathcal{M}, \tau)$ space and new techniques or new proofs need to be invented. Key ingredients in the proof of Theorem 4.7 include a characterization of H^α (see Theorem 3.9), a factorization result in $L^\alpha(\mathcal{M}, \tau)$ (see Proposition 4.2), and a density theorem for $L^\alpha(\mathcal{M}, \tau)$ (see Theorem 4.3), which extend earlier results by Saito in [24].

THEOREM 0.2. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. Then*

$$H^\alpha = H^1 \cap L^\alpha(\mathcal{M}, \tau) = \{x \in L^\alpha(\mathcal{M}, \tau) : \tau(xy) = 0 \textit{ for all } y \in H_0^\infty\}.$$

PROPOSITION 0.3. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $k \in \mathcal{M}$ and $k^{-1} \in L^\alpha(\mathcal{M}, \tau)$, then there are unitary operators $w_1, w_2 \in \mathcal{M}$ and operators $a_1, a_2 \in H^\infty$ such that $k = w_1 a_1 = a_2 w_2$ and $a_1^{-1}, a_2^{-1} \in H^\alpha$.*

THEOREM 0.4. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ and $\mathcal{N}$ is a weak*-closed linear subspace of $\mathcal{M}$ such that $\mathcal{W} H^\infty \subseteq \mathcal{W}$ and $\mathcal{N} H^\infty \subseteq \mathcal{N}$, then*

(i) $\mathcal{N} = [\mathcal{N}]_\alpha \cap \mathcal{M}$;

(ii) $\mathcal{W} \cap \mathcal{M}$ *is weak* closed in* $\mathcal{M}$;

(iii) $\mathcal{W} = [\mathcal{W} \cap \mathcal{M}]_\alpha$;

(iv) *if $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S} H^\infty \subseteq \mathcal{S}$, then*

$$[\mathcal{S}]_\alpha = [\overline{\mathcal{S}}^{w*}]_\alpha,$$

where $\overline{\mathcal{S}}^{w*}$ is the weak closure of $\mathcal{S}$ in $\mathcal{M}$.*

We end the paper with two quick applications of Theorem 4.7, which contain the classical Beurling theorem as a special case by letting $\mathcal{M}$ be $L^\infty(\mathbb{T}, \mu)$.

COROLLARY 0.5. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{W}\mathcal{M} \subseteq \mathcal{W}$, then there exists a projection e in $\mathcal{M}$ such that $\mathcal{W} = eL^\alpha(\mathcal{M}, \tau)$.*

COROLLARY 0.6. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ such that $H^\infty \cap (H^\infty)^* = \mathbb{C}I$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous* *norm on $\mathcal{M}$. Assume that $\mathcal{W}$ is a closed subspace of $L^{\alpha}(\mathcal{M},\tau)$. If $\mathcal{W}$ is simply H^{∞}-right invariant, i.e. $[\mathcal{W}H^{\infty}]_{\alpha} \subsetneqq \mathcal{W}$, then there exists a unitary $u \in \mathcal{W} \cap \mathcal{M}$ such that $\mathcal{W} = uH^{\alpha}$.*

The organization of the paper is as follows. In Section 1, we introduce a class $N_c(\mathcal{M},\tau)$ of normalized, unitarily invariant, $\|\cdot\|_1$-dominating and continuous norms and study their dual norms on a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ. In Section 2, for each $\alpha \in N_c(\mathcal{M},\tau)$, we show a new version of Hölder's inequality and prove a duality theorem of $L^{\alpha}(\mathcal{M},\tau)$, whose form is different from the usual L^p-spaces for each $1 \leqslant p < \infty$. In Section 3, we define the noncommutative H^{α} spaces and provide a characterization of H^{α}. Finally, in Section 4, based on our density theorem for $L^{\alpha}(\mathcal{M},\tau)$, we obtain the main result of the paper, a version of Beurling theorem for H^{∞}-right invariant subspaces in $L^{\alpha}(\mathcal{M},\tau)$ spaces and in H^{α} spaces.

1. UNITARILY INVARIANT NORMS AND DUAL NORMS ON FINITE VON NEUMANN ALGEBRAS

1.1. UNITARILY INVARIANT NORMS. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. For general knowledge about noncommutative L^p-spaces for $0 < p \leqslant \infty$ associated with a von Neumann algebra $\mathcal{M}$, we will refer to a wonderful handbook [23] by Pisier and Xu. For each $0 < p < \infty$, we let $\|\cdot\|_p$ be the mapping from $\mathcal{M}$ to $[0,\infty)$ (see [23]) as defined by

$$\|x\|_p = (\tau(|x|^p))^{1/p}, \quad \forall\, x \in \mathcal{M}.$$

It is known that $\|\cdot\|_p$ is a norm if $1 \leqslant p < \infty$, and a quasi-norm if $0 < p < 1$. We define $L^p(\mathcal{M},\tau)$, the so called noncommutative L^p-space associated with $(\mathcal{M},\tau)$, to be the completion of $\mathcal{M}$ with respect to $\|\cdot\|_p$ for $0 < p < \infty$.

In the paper, we will mainly focus on the following two classes of unitarily invariant norms of a finite von Neumann algebra.

DEFINITION 1.1. We denote by $N(\mathcal{M},\tau)$ the collection of all norms $\alpha : \mathcal{M} \to [0,\infty)$ satisfying:

(i) $\alpha(I) = 1$, i.e. α is normalized.

(ii) $\alpha(uxv) = \alpha(x)$ for all $x \in \mathcal{M}$ and unitaries u, v in $\mathcal{M}$, i.e. α is unitarily invariant.

(iii) $\|x\|_1 \leqslant \alpha(x)$ for every $x \in \mathcal{M}$, i.e. α is $\|\cdot\|_1$-dominating.

The norm α in $N(\mathcal{M},\tau)$ is called a *normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm* on $\mathcal{M}$.

DEFINITION 1.2. We denote by $N_c(\mathcal{M},\tau)$ the collection of all norms $\alpha : \mathcal{M} \to [0,\infty)$ such that:

(i) $\alpha \in N(\mathcal{M},\tau)$ and

(ii) $\lim_{\tau(e)\to 0} \alpha(e) = 0$ as e ranges over the projections in $\mathcal{M}$ (α is a continuous norm with respect to a trace τ).

The norm α in $N_c(\mathcal{M},\tau)$ is called a *normalized, unitarily invariant,* $\|\cdot\|_1$*-dominating, continuous norm* on $\mathcal{M}$.

EXAMPLE 1.3. Each p-norm, $\|\cdot\|_p$, is in the class $N_c(\mathcal{M},\tau)$ for $1 \leqslant p < \infty$.

EXAMPLE 1.4. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ satisfying the weak Dixmier property (see [12]). Let α be a normalized tracial gauge norm on $\mathcal{M}$. Then Theorem 3.30 in [12] shows that $\alpha \in N(\mathcal{M},\tau)$.

EXAMPLE 1.5. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ and $E(0,1)$ be a rearrangement invariant Banach function space on $(0,1)$. A noncommutative Banach function space $E(\tau)$ together with a norm $\|\cdot\|_{E(\tau)}$, corresponding to $E(0,1)$ and associated with $(\mathcal{M},\tau)$, can be introduced (see [7] or [8]). Moreover $\mathcal{M}$ is a subspace in $E(\tau)$ and the restriction of the norm $\|\cdot\|_{E(\tau)}$ to $\mathcal{M}$ lies in $N(\mathcal{M},\tau)$. If E is also order continuous, then the restriction of the norm $\|\cdot\|_{E(\tau)}$ to $\mathcal{M}$ lies in $N_c(\mathcal{M},\tau)$.

EXAMPLE 1.6. Let $\mathcal{N}$ be a type II_1 factor with a tracial state $\tau_{\mathcal{N}}$. Let $\|\cdot\|_{1,\mathcal{N}}$ and $\|\cdot\|_{2,\mathcal{N}}$ be L^1-norm, and L^2-norm respectively, on $\mathcal{N}$. Let $\mathcal{M} = \mathcal{N} \oplus \mathcal{N}$ be a finite von Neumann algebra with a faithful normal tracial state τ, defined by

$$\tau(x \oplus y) = \frac{\tau_{\mathcal{N}}(x) + \tau_{\mathcal{N}}(y)}{2}, \quad \forall\, x \oplus y \in \mathcal{M}.$$

Let α be a norm of $\mathcal{M}$, defined by

$$\alpha(x \oplus y) = \frac{\|x\|_{1,\mathcal{N}} + \|y\|_{2,\mathcal{N}}}{2}, \quad \forall\, x \oplus y \in \mathcal{M}.$$

Then $\alpha \in N_c(\mathcal{M},\tau)$. But α is neither tracial (see Definition 3.7 in [12]) nor rearrangement invariant (see Definition 2.1 in [9]).

The following lemma is well-known.

LEMMA 1.7. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ and α be a norm on $\mathcal{M}$. If α is unitarily invariant, i.e.*

$$\alpha(uxv) = \alpha(x) \quad \textit{for all } x \in \mathcal{M} \textit{ and unitaries } u, v \textit{ in } \mathcal{M},$$

then

$$\alpha(x_1 y x_2) \leqslant \|x_1\| \cdot \|x_2\| \cdot \alpha(y), \quad \forall\, x_1, x_2, y \in \mathcal{M}.$$

In particular, if α is a normalized unitarily invariant norm on $\mathcal{M}$, then

$$\alpha(x) \leqslant \|x\|, \quad \forall\, x \in \mathcal{M}.$$

Proof. Let $x \in \mathcal{M}$ such that $\|x\| = 1$. Assume that $x = v|x|$ is the polar decomposition of x in $\mathcal{M}$, where v is a unitary in $\mathcal{M}$ and $|x|$ is positive. Then

$u = |x| + \mathrm{i}\sqrt{I - |x|^2}$ is a unitary in $\mathcal{M}$ such that $|x| = (u + u^*)/2$. Thus

$$\alpha(xy) = \alpha(|x|y) = \alpha(\tfrac{uy+u^*y}{2}) \leqslant \frac{\alpha(uy) + \alpha(u^*y)}{2} = \alpha(y).$$

Hence $\alpha(xy) \leqslant \|x\|\alpha(y), \forall\, x, y \in \mathcal{M}$. Similarly, $\alpha(yx) \leqslant \|x\|\alpha(y), \forall\, x, y \in \mathcal{M}$.

Furthermore, if α is a normalized unitarily invariant norm on $\mathcal{M}$, then from the discussion in the preceding paragraph we have that

$$\alpha(x) \leqslant \|x\|\alpha(I) = \|x\|, \quad \forall\, x \in \mathcal{M}. \quad ∎$$

1.2. DUAL NORMS OF UNITARILY INVARIANT NORMS ON $\mathcal{M}$. The concept of dual norm plays an important role in the study of noncommutative L^p-spaces. In this subsection, we will introduce dual norms for unitarily invariant norms on a finite von Neumann algebra.

LEMMA 1.8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$ (see Definition* 1.1*). Define a mapping $\alpha' : \mathcal{M} \to [0, \infty]$ as follows:*

$$\alpha'(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha(y) \leqslant 1\}, \quad \forall\, x \in \mathcal{M}.$$

Then the following statements are true:

(i) $\forall\, x \in \mathcal{M}, \|x\|_1 \leqslant \alpha'(x) \leqslant \|x\|$.

(ii) *α' is a norm on $\mathcal{M}$.*

(iii) *$\alpha' \in N(\mathcal{M}, \tau)$, i.e. α' is a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm.*

(iv) *$|\tau(xy)| \leqslant \alpha(x)\alpha'(y)$ for all x, y in $\mathcal{M}$.*

Proof. (i) Suppose $x \in \mathcal{M}$. If $y \in \mathcal{M}$ with $\alpha(y) \leqslant 1$, then, from the fact that α is $\|\cdot\|_1$-dominating, we have

$$|\tau(xy)| \leqslant \|x\|\|y\|_1 \leqslant \|x\|\alpha(y) \leqslant \|x\|,$$

whence $\alpha'(x) \leqslant \|x\|$. Thus α' is a mapping from $\mathcal{M}$ to $[0, \infty)$.

Now, assume that $x = uh$ is the polar decomposition of x in $\mathcal{M}$, where u is a unitary element in $\mathcal{M}$ and h in $\mathcal{M}$ is positive. Then, from the fact that $\alpha(u^*) = 1$, we have

$$\alpha'(x) \geqslant |\tau(u^*x)| = \tau(h) = \|x\|_1.$$

Therefore $\|x\|_1 \leqslant \alpha'(x)$ for every $x \in \mathcal{M}$. This ends the proof of part (i).

(ii) It is easy to verify that

$$\alpha'(ax) = |a|\alpha'(x), \quad \text{and} \quad \alpha'(x_1 + x_2) \leqslant \alpha'(x_1) + \alpha'(x_2), \quad \forall a \in \mathbb{C}, \forall\, x, x_1, x_2 \in \mathcal{M}.$$

From the result (i), we know that $\alpha'(x) = 0$ implies $x = 0$. Therefore α' is a norm on $\mathcal{M}$.

(iii) It is not hard to verify that α' satisfies conditions (i) and (ii) in the definition of $N(\mathcal{M}, \tau)$. From the result (i), α' also satisfies condition (iii) in the definition of $N(\mathcal{M}, \tau)$. Therefore $\alpha' \in N(\mathcal{M}, \tau)$.

(iv) It follows directly from the definition of α'. ∎

DEFINITION 1.9. The norm α', as defined in Lemma 1.8, is called the *dual norm* of α on $\mathcal{M}$.

Now we are ready to introduce L^α-spaces and $L^{\alpha'}$-spaces for a finite von Neumann algebra $\mathcal{M}$ with respect to the unitarily invariant norms α, and α' respectively, as follows.

DEFINITION 1.10. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$ (see Definition 1.1). Let α' be the dual norm of α on $\mathcal{M}$ (see Definition 1.9). We define $L^\alpha(\mathcal{M},\tau)$ and $L^{\alpha'}(\mathcal{M},\tau)$ to be the completion of $\mathcal{M}$ with respect to α, and α', respectively.

REMARK 1.11. If α is an L^p-norm for some $1 < p < \infty$, then α' is nothing but an L^q-norm where $1/p + 1/q = 1$. Hence $L^\alpha(\mathcal{M},\tau)$, $L^{\alpha'}(\mathcal{M},\tau)$ are the usual $L^p(\mathcal{M},\tau)$, $L^q(\mathcal{M},\tau)$ spaces.

It is known that the dual space of $L^p(\mathcal{M},\tau)$ is $L^q(\mathcal{M},\tau)$ when $1 < p, q < \infty$ and $1/p + 1/q = 1$. However generally, for $\alpha \in N(\mathcal{M},\tau)$, the dual of $L^\alpha(\mathcal{M},\tau)$ might not be $L^{\alpha'}(\mathcal{M},\tau)$.

2. DUAL SPACES OF L^α-SPACES ASSOCIATED WITH FINITE VON NEUMANN ALGEBRAS

In this section we will study the dual spaces of $L^\alpha(\mathcal{M},\tau)$ by investigating some subspaces in $L^1(\mathcal{M},\tau)$.

2.1. DEFINITIONS OF SUBSPACES $L_{\overline{\alpha}}(\mathcal{M},\tau)$ AND $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ OF $L^1(\mathcal{M},\tau)$.

DEFINITION 2.1. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$ (see Definition 1.1). Let α' be the dual norm of α on $\mathcal{M}$ (see Definition 1.9). We define

$$\overline{\alpha} : L^1(\mathcal{M},\tau) \to [0,\infty] \quad \text{and} \quad \overline{\alpha}' : L^1(\mathcal{M},\tau) \to [0,\infty]$$

as follows:

$$\overline{\alpha}(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha'(y) \leqslant 1\}, \quad \forall\, x \in L^1(\mathcal{M},\tau),$$
$$\overline{\alpha}'(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha(y) \leqslant 1\}, \quad \forall\, x \in L^1(\mathcal{M},\tau).$$

We define

$$L_{\overline{\alpha}}(\mathcal{M},\tau) = \{x \in L^1(\mathcal{M},\tau) : \overline{\alpha}(x) < \infty\} \subseteq L^1(\mathcal{M},\tau),$$
$$L_{\overline{\alpha}'}(\mathcal{M},\tau) = \{x \in L^1(\mathcal{M},\tau) : \overline{\alpha}'(x) < \infty\} \subseteq L^1(\mathcal{M},\tau).$$

Thus $\overline{\alpha}$ and $\overline{\alpha}'$, are mappings from $L_{\overline{\alpha}}(\mathcal{M},\tau)$, and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ respectively, into $[0,\infty)$. The next result follows directly from the definitions of $\overline{\alpha}$, $\overline{\alpha}'$, and part (iv) of Lemma 1.8.

LEMMA 2.2. *We have*

$$\overline{\alpha}'(x) = \alpha'(x) \quad \text{and} \quad \overline{\alpha}(x) \leqslant \alpha(x) \quad \text{for every } x \in \mathcal{M}.$$

The following proposition describes properties of $\overline{\alpha}$ and $\overline{\alpha}'$, which imply that $L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ are normed spaces with respect to $\overline{\alpha}$ and $\overline{\alpha}'$, respectively.

PROPOSITION 2.3. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$ (see Definition 1.1). Let α' be the dual norm of α on $\mathcal{M}$ (see Definition 1.9). Let*

$$\overline{\alpha} : L_{\overline{\alpha}}(\mathcal{M},\tau) \to [0,\infty) \quad \text{and} \quad \overline{\alpha}' : L_{\overline{\alpha}'}(\mathcal{M},\tau) \to [0,\infty)$$

be as in Definition 2.1. Then the following statements are true:

(i) $\overline{\alpha}(I) = 1$ *and* $\overline{\alpha}'(I) = 1$.

(ii) *If u, v are unitary elements in $\mathcal{M}$, then*

$$\overline{\alpha}(x) = \overline{\alpha}(uxv), \quad \forall\, x \in L_{\overline{\alpha}}(\mathcal{M},\tau)$$

and

$$\overline{\alpha}'(x) = \overline{\alpha}'(uxv), \quad \forall\, x \in L_{\overline{\alpha}'}(\mathcal{M},\tau).$$

(iii$_1$) *We have*

$$\|x\|_1 \leqslant \overline{\alpha}(x), \quad \forall\, x \in L_{\overline{\alpha}}(\mathcal{M},\tau)$$

and

$$\|x\|_1 \leqslant \overline{\alpha}'(x), \quad \forall\, x \in L_{\overline{\alpha}'}(\mathcal{M},\tau).$$

(iii$_2$) *If x is an element in $\mathcal{M}$, then*

$$\overline{\alpha}(x) \leqslant \|x\| \quad \text{and} \quad \overline{\alpha}'(x) \leqslant \|x\|.$$

(iv) *$\overline{\alpha}$ and $\overline{\alpha}'$ are norms on $L_{\overline{\alpha}}(\mathcal{M},\tau)$, and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$, respectively.*

Proof. (i) Note that $\alpha \in N(\mathcal{M},\tau)$ and $\alpha' \in N(\mathcal{M},\tau)$ from part (iii) of Lemma 1.8. Thus

$$\overline{\alpha}(I) = \sup\{|\tau(y)| : y \in \mathcal{M}, \alpha'(y) \leqslant 1\} = \sup\{||y||_1 : y \in \mathcal{M}, \alpha'(y) \leqslant 1\} = 1.$$

Similarly,

$$\overline{\alpha}'(I) = 1.$$

(ii) If u, v are unitaries in $\mathcal{M}$, then

$$\begin{aligned}
\overline{\alpha}(uxv) &= \sup\{|\tau(uxvy)| : y \in \mathcal{M}, \alpha'(y) \leqslant 1\} \\
&= \sup\{|\tau(xvyu)| : y \in \mathcal{M}, \alpha'(y) \leqslant 1\} \quad \text{(by Definition 2.1)} \\
&= \sup\{|\tau(xy_0)| : y \in \mathcal{M}, \alpha'(y_0) = \alpha'(vyu) = \alpha'(y) \leqslant 1\} \quad (\text{because } \alpha' \in N(\mathcal{M},\tau)) \\
&= \overline{\alpha}(x), \ \forall x \in L_{\overline{\alpha}}(\mathcal{M},\tau).
\end{aligned}$$

Similarly, we have

$$\overline{\alpha}'(x) = \overline{\alpha}'(uxv), \quad \forall\, x \in L_{\overline{\alpha}'}(\mathcal{M},\tau).$$

(iii$_1$) Assume that $x \in L_{\overline{\alpha}}(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$. We let $x = uh$ be the polar decomposition of x in $L^1(\mathcal{M})$, where u is a unitary in $\mathcal{M}$ and $h = |x| \in L^1(\mathcal{M})$. Then, from the result (ii), we obtain that

$$\overline{\alpha}(x) = \overline{\alpha}(uh) = \overline{\alpha}(h) \geqslant |\tau(h)| = \|x\|_1.$$

Similarly, we have

$$\|x\|_1 \leqslant \overline{\alpha}'(x), \quad \forall\, x \in L_{\overline{\alpha}'}(\mathcal{M},\tau).$$

(iii$_2$) Note that $\alpha' \in N(\mathcal{M},\tau)$. Suppose $x \in \mathcal{M}$. If $y \in \mathcal{M}$ with $\alpha'(y) \leqslant 1$. Then

$$|\tau(xy)| \leqslant \|x\|\|y\|_1 \leqslant \|x\|\alpha'(y) \leqslant \|x\|.$$

Now it follows from the definition of $\overline{\alpha}$ that $\overline{\alpha}(x) \leqslant \|x\|$. Similarly, we have $\overline{\alpha}'(x) \leqslant \|x\|, \forall x \in \mathcal{M}$.

(iv) From the definition and the result (iii$_1$), we conclude that $\overline{\alpha}$ and $\overline{\alpha}'$ are norms on $L_{\overline{\alpha}}(\mathcal{M},\tau)$, and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ respectively. ∎

The following lemma is a useful tool for our later results.

LEMMA 2.4. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$ (see Definition* 1.1*). Let α' be the dual norm of α on $\mathcal{M}$ (see Definition* 1.9*). Let $\overline{\alpha}$ and $\overline{\alpha}'$ be as in Definition* 2.1*. Then the following statements are true:*

(i) *For all $x \in L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $a \in \mathcal{M}$ $\overline{\alpha}(xa) \leqslant \overline{\alpha}(x)\|a\|$.*

(ii) *For all $x \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$ and $a \in \mathcal{M}$ $\overline{\alpha}'(xa) \leqslant \overline{\alpha}'(x)\|a\|$.*

Proof. (i) From Proposition 2.3, $\overline{\alpha}$ is a norm on $L_{\overline{\alpha}}(\mathcal{M},\tau)$ satisfying

$$\overline{\alpha}(x) = \overline{\alpha}(uxv), \quad \forall \text{ unitary elements } u,v \in \mathcal{M} \text{ and } x \in L_{\overline{\alpha}}(\mathcal{M},\tau).$$

Now the proof of Lemma 1.7 can also be applied here.

(ii) A similar result holds for $\overline{\alpha}'$. ∎

Our next result shows that $L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ are Banach spaces with respect to $\overline{\alpha}$ and $\overline{\alpha}'$ respectively.

PROPOSITION 2.5. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$ (see Definition* 1.1*). Let α' be the dual norm of α on $\mathcal{M}$ (see Definition* 1.9*). Let $\overline{\alpha}$, $\overline{\alpha}'$, $L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ be as in Definition* 2.1*. Then $L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ are both Banach spaces with respect to norms $\overline{\alpha}$ and $\overline{\alpha}'$, respectively.*

Proof. Since the arguments for $L_{\overline{\alpha}}(\mathcal{M},\tau)$ and for $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ are similar, we will only present the proof that $L_{\overline{\alpha}}(\mathcal{M},\tau)$ is a Banach space here.

From part (iv) of Proposition 2.3, we know that $L_{\overline{\alpha}}(\mathcal{M},\tau)$ is a normed space with respect to $\overline{\alpha}$. To prove the completeness of the space, we suppose $\{x_n\}$ is a Cauchy sequence in $L_{\overline{\alpha}}(\mathcal{M},\tau)$ with respect to $\overline{\alpha}$. Then there is an $M > 0$ such that $\overline{\alpha}(x_n) \leqslant M$ for all n. From part (iii$_1$) of Proposition 2.3, we have that $\|x_m - x_n\|_1 \leqslant \overline{\alpha}(x_m - x_n)$ for $m,n \geqslant 1$. It follows that $\{x_n\}$ is a Cauchy sequence in $L^1(\mathcal{M},\tau)$, which is a complete Banach space. Then there is an $x_0 \in L^1(\mathcal{M}, \tau)$ such that $\|x_n - x_0\|_1 \to 0$.

We claim that $x_0 \in L_{\overline{\alpha}}(\mathcal{M}, \tau)$ and $\overline{\alpha}(x_n - x_0) \to 0$ as n goes to infinity. In fact, we let $y \in \mathcal{M}$ with $\alpha'(y) \leqslant 1$. Since

$$|\tau(x_n y) - \tau(x_0 y)| = |\tau((x_n - x_0)y)| \leqslant \|x_n - x_0\|_1 \|y\| \to 0,$$

we have

$$|\tau(x_0 y)| = \lim_{n\to\infty} |\tau(x_n y)|.$$

By the definition of $\overline{\alpha}$, we have that

$$|\tau(x_0 y)| = \lim_{n\to\infty} |\tau(x_n y)| \leqslant \limsup_{n\to\infty} \overline{\alpha}(x_n)\alpha'(y) \leqslant M,$$

whence $\overline{\alpha}(x_0) \leqslant M$. This implies $x_0 \in L_{\overline{\alpha}}(\mathcal{M}, \tau)$. Furthermore, since $\{x_n\}$ is Cauchy in $L_{\overline{\alpha}}(\mathcal{M}, \tau)$, it follows that, for each $n \geqslant 1$,

$$\begin{aligned}|\tau((x_0 - x_n)y)| &= \lim_{m\to\infty} |\tau((x_m - x_n)y)| \leqslant \limsup_{m\to\infty} \overline{\alpha}(x_m - x_n)\alpha'(y)\\ &\leqslant \limsup_{m\to\infty} \overline{\alpha}(x_m - x_n).\end{aligned}$$

Thus $\overline{\alpha}(x_n - x_0) \leqslant \limsup_{m\to\infty} \overline{\alpha}(x_m - x_n)$ for each $n \geqslant 1$. Again from the fact that $\{x_n\}$ is Cauchy in $L_{\overline{\alpha}}(\mathcal{M}, \tau)$, we conclude that $\overline{\alpha}(x_n - x_0) \to 0$ as n goes to infinity. Therefore $L_{\overline{\alpha}}(\mathcal{M}, \tau)$ is a Banach space with respect to the norm $\overline{\alpha}$. This ends the proof of the whole proposition. ∎

2.2. HÖLDER'S INEQUALITY. In this subsection, we will prove Hölder's inequality for $L^{\alpha}(\mathcal{M}, \tau)$ when α is a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm.

We will need the following result from [29].

LEMMA 2.6 (Corollary III.3.11 in [29]). *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. If ϕ is a bounded linear functional on a von Neumann algebra $\mathcal{M}$, then the following two statements are equivalent:*

(i) *ϕ is normal;*

(ii) *for every orthogonal family $\{e_i\}_{i\in I}$ in $\mathcal{M}$,*

$$\phi\Big(\sum_{i\in I} e_i\Big) = \sum_{i\in I} \phi(e_i).$$

When α is a continuous norm, the following result relates the dual space of $L^{\alpha}(\mathcal{M}, \tau)$ to the space $L_{\overline{\alpha}'}(\mathcal{M}, \tau)$.

PROPOSITION 2.7. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$ (see Definition 1.2). Let $L_{\overline{\alpha}'}(\mathcal{M}, \tau)$ be as in Definition 2.1. Then for every bounded linear functional $\phi \in (L^{\alpha}(\mathcal{M}, \tau))^{\sharp}$, there is a $\xi \in L_{\overline{\alpha}'}(\mathcal{M}, \tau)$ such that $\overline{\alpha}'(\xi) = \|\phi\|$ and $\phi(x) = \tau(x\xi)$ for all $x \in \mathcal{M}$.*

Proof. Suppose $\alpha \in N_c(\mathcal{M},\tau)$ and $\phi \in (L^\alpha(\mathcal{M},\tau))^\sharp$. Let $\{e_n\}$ be a family of orthogonal projections in $\mathcal{M}$. It is easily verified that $\sum_{n=N}^{\infty} e_n \to 0$ in the strong operator topology as N approaches infinity. Since τ is normal, by Lemma 2.6, we have that $\lim_{N\to\infty} \tau\Big(\sum_{n=N}^{\infty} e_n\Big) \to 0$. Note that $\alpha \in N_c(\mathcal{M},\tau)$. Then the continuity of α with respect to τ implies that $\lim_{N\to\infty} \alpha\Big(\sum_{n=N}^{\infty} e_n\Big) \to 0$. From the fact that $\phi \in (L^\alpha(\mathcal{M},\tau))^\sharp$, we know that

$$\lim_{N\to\infty} \phi\Big(\sum_{n=1}^{\infty} e_n - \sum_{n=1}^{N-1} e_n\Big) = \lim_{N\to\infty} \phi\Big(\sum_{n=N}^{\infty} e_n\Big) = 0.$$

Now Lemma 2.6 implies that ϕ is a normal functional on $\mathcal{M}$. Hence ϕ is in the predual space of $\mathcal{M}$, i.e. there is a $\xi \in L^1(\mathcal{M},\tau)$ such that $\phi(x) = \tau(x\xi)$ for all $x \in \mathcal{M}$. Furthermore, since $\mathcal{M}$ is dense in $L^\alpha(\mathcal{M},\tau)$, we see that

$$\begin{aligned}\|\phi\| &= \sup\{|\phi(x)| : x \in \mathcal{M}, \alpha(x) \leqslant 1\} \\ &= \sup\{|\tau(x\xi)| : x \in \mathcal{M}, \alpha(x) \leqslant 1\} = \overline{\alpha}'(\xi),\end{aligned}$$

which implies that $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$. This ends the proof of the result. ∎

For a finite von Neumann algebra $\mathcal{M}$ acting on a Hilbert space $\mathcal{H}$, the set of possibly unbounded, closed and densely defined operators on $\mathcal{H}$ which are affiliated to $\mathcal{M}$, forms a topological $*$-algebra where the topology is the noncommutative topology of convergence in measure [21]. We will denote this algebra by $\widetilde{\mathcal{M}}$; it is the closure of $\mathcal{M}$ in the topology just mentioned. We let $\widetilde{\mathcal{M}}_+$ be the set of positive operators in $\widetilde{\mathcal{M}}$. Then the trace

$$\tau : \mathcal{M}_+ \to [0,\infty)$$

can be extended to a generalized trace

$$\widetilde{\tau} : \widetilde{\mathcal{M}}_+ \to [0,\infty].$$

We refer to [21], [25], [30] for more details on the noncommutative integration theory.

We will summarize some properties of the generalized trace on $\widetilde{\mathcal{M}}_+$ as follows.

LEMMA 2.8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ acting on a Hilbert space $\mathcal{H}$. Let $\widetilde{\mathcal{M}}$ be the set of closed and densely-defined operators affiliated to $\mathcal{M}$ and $\widetilde{\mathcal{M}}_+$ be the set of positive operators in $\widetilde{\mathcal{M}}$. If $a \in \widetilde{\mathcal{M}}_+$, there is a family $\{e_\lambda\}_{\lambda>0}$ of projections (spectral resolution of a) in $\mathcal{M}$ such that:*

(i) *$e_\lambda \to I$ increasingly;*

(ii) *$e_\lambda a = a e_\lambda \in \mathcal{M}$ for every $0 < \lambda < \infty$;*

(iii) *$\widetilde{\tau}(a) = \sup_{\lambda>0} \tau(e_\lambda a)$ ($\widetilde{\tau}(a)$ could be infinity);*

(iv) *if* $a \in L^1(\mathcal{M}, \tau)$, *then* $\|e_\lambda a - a\|_1 \to 0$.
Assume that x *is an element in* $\widetilde{\mathcal{M}}$. *Then* $x \in L^1(\mathcal{M}, \tau)$ *if and only if* $\widetilde{\tau}(|x|) < \infty$.

The result is well-known. More details could be found in Section 1.1 of [11] or in [30].

If no confusion arises, we still use τ to denote the generalized trace $\widetilde{\tau}$ on $\widetilde{\mathcal{M}}_+$.

A consequence of the preceding lemma is the following result.

COROLLARY 2.9. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful normal tracial state* τ *acting on a Hilbert space* $\mathcal{H}$. *Let* α *be a normalized, unitarily invariant,* $\|\cdot\|_1$*-dominating, continuous norm on* $\mathcal{M}$ *(see Definition* 1.2*). Let* α' *be the dual norm of* α *on* $\mathcal{M}$ *(see Definition* 1.9*). Let* $\overline{\alpha}$ *and* $\overline{\alpha}'$ *be as defined in Definition* 2.1. *Then*

$$\alpha(x) = \overline{\alpha}(x) \quad \textit{and} \quad \alpha'(x) = \overline{\alpha}'(x) \quad \textit{for all } x \in \mathcal{M}.$$

Proof. It is clear by Lemma 2.2 that $\alpha'(x) = \overline{\alpha}'(x)$ and $\overline{\alpha}(x) \leqslant \alpha(x)$ for all $x \in \mathcal{M}$. We will need only to show that $\overline{\alpha}(x) \geqslant \alpha(x)$ for all $x \in \mathcal{M}$.

Now suppose $x \in \mathcal{M}$ with $\alpha(x) = 1$. By the Hahn–Banach theorem, there is a continuous linear functional $\phi \in (L^\alpha(\mathcal{M}, \tau))^\sharp$ such that $\phi(x) = \alpha(x) = 1$ and $\|\phi\| = 1$. Since $\phi \in (L^\alpha(\mathcal{M}, \tau))^\sharp$, from Proposition 2.7, there is an element $\xi \in L_{\overline{\alpha}'}(\mathcal{M}, \tau)$ such that $\phi(x) = |\tau(x\xi)| = 1$ and $\overline{\alpha}'(\xi) = \|\phi\| = 1$.

Let $\xi = uh$ be the polar decomposition of $\xi \in L_{\overline{\alpha}'}(\mathcal{M}, \tau)$, where $u \in \mathcal{M}$ is a unitary and $h \in L_{\overline{\alpha}'}(\mathcal{M}, \tau) \subseteq L^1(\mathcal{M})$ is positive. Then it follows from Lemma 2.8 that there exists a family $\{e_\lambda\}_{\lambda>0}$ of projections in $\mathcal{M}$ such that

$$\|h - he_\lambda\|_1 \to 0 \tag{2.1}$$

and $e_\lambda h = he_\lambda \in \mathcal{M}$ for every $0 < \lambda < \infty$. Thus $uhe_\lambda \in \mathcal{M}$. It follows from Lemma 2.2 and Lemma 2.4 that

$$\alpha'(uhe_\lambda) = \overline{\alpha}'(uhe_\lambda) \leqslant \overline{\alpha}'(uh)\|e_\lambda\| \leqslant \overline{\alpha}'(uh) = \overline{\alpha}'(\xi) = 1. \tag{2.2}$$

Therefore,

$$\begin{aligned} |\tau(x\xi)| &= |\tau(xuh)| \\ &= \lim_{\lambda\to\infty} |\tau(xuhe_\lambda)| \quad (\text{by (2.1) and } xu \in \mathcal{M}) \\ &\leqslant \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha'(y) \leqslant 1\} \quad \text{by (2.2)}. \end{aligned}$$

Hence, from the definition of $\overline{\alpha}$ we obtain

$$\overline{\alpha}(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha'(y) \leqslant 1\} \geqslant |\tau(x\xi)| = 1 = \alpha(x).$$

This finishes the proof of the result. ∎

A quick corollary of the preceding result is the following conclusion.

PROPOSITION 2.10. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful normal tracial state* τ *acting on a Hilbert space* $\mathcal{H}$. *Let* α *be a normalized, unitarily invariant,* $\|\cdot\|_1$*-dominating, continuous norm on* $\mathcal{M}$ *(see Definition* 1.2*). Let* α' *be the dual norm* *of α on $\mathcal{M}$ (see Definition 1.9). Let $\overline{\alpha}$ and $\overline{\alpha}'$ be as defined in Definition 2.1. There are natural isometric embeddings*

$$L^{\alpha}(\mathcal{M},\tau) \hookrightarrow L_{\overline{\alpha}}(\mathcal{M},\tau) \quad \text{and} \quad L^{\alpha'}(\mathcal{M},\tau) \hookrightarrow L_{\overline{\alpha}'}(\mathcal{M},\tau),$$

such that

$$x \mapsto x \quad \text{and} \quad x \mapsto x, \quad \forall\, x \in \mathcal{M}.$$

Thus $L^{\alpha}(\mathcal{M},\tau)$ and $L^{\alpha'}(\mathcal{M},\tau)$ are Banach subspaces of $L_{\overline{\alpha}}(\mathcal{M},\tau)$, and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$, respectively.

The following theorem is a generalization of Hölder's inequality in non-commutative L^p-spaces.

THEOREM 2.11. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ acting on a Hilbert space $\mathcal{H}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$ (see Definition 1.2). Let α' be the dual norm of α on $\mathcal{M}$ (see Definition 1.9). Let $L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ be as defined in Definition 2.1. If $x \in L_{\overline{\alpha}}(\mathcal{M},\tau)$ and $y \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$, then $xy \in L^1(\mathcal{M},\tau)$ and $\|xy\|_1 \leqslant \overline{\alpha}(x)\overline{\alpha}'(y)$.*

In particular, if $x \in L^{\alpha}(\mathcal{M},\tau)$ and $y \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$, then $xy \in L^1(\mathcal{M},\tau)$ and $\|xy\|_1 \leqslant \alpha(x)\overline{\alpha}'(y)$.

Proof. Suppose $x \in L_{\overline{\alpha}}(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$ and $y \in L_{\overline{\alpha}'}(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$. Then $xy \in \widetilde{\mathcal{M}}$, where $\widetilde{\mathcal{M}}$ is the set of closed and densely defined operators affiliated with $\mathcal{M}$. Let $xy = uh$ be the polar decomposition of xy in $\widetilde{\mathcal{M}}$, where $u \in \mathcal{M}$ is a unitary and $h = |xy| \in \widetilde{\mathcal{M}}_+$. From Lemma 2.8, there exists an increasing family $\{e_\lambda\}_{\lambda>0}$ of projections in $\mathcal{M}$, such that $e_\lambda h = he_\lambda \in \mathcal{M}$ for each $\lambda > 0$ and such that $\tau(h) = \sup_{\lambda>0} \tau(e_\lambda h)$. We will show that $\tau(h) \leqslant \overline{\alpha}(x)\overline{\alpha}'(y)$.

Assume, to the contrary, that

$$\tau(h) = \sup_{\lambda>0} \tau(e_\lambda h) > \overline{\alpha}(x)\overline{\alpha}'(y).$$

Then there is a projection $e \in \mathcal{M}$ and $\varepsilon > 0$ such that $eh \in \mathcal{M}$ and

$$\tau(eh) > \overline{\alpha}(x)\overline{\alpha}'(y) + \varepsilon.$$

Note that $eh = eu^*xy$. Let $eu^*x = h_2u_2$, where $u_2^*h_2$ is the polar decomposition of x^*ue in $\widetilde{\mathcal{M}}$. It is clear that $u_2 \in \mathcal{M}$ is a unitary and $h_2 \in \widetilde{\mathcal{M}}_+$. Again from Lemma 2.8, we may choose $\{f_\lambda\}_{\lambda>0}$ to be an increasing family of projections in $\mathcal{M}$ such that (i) $f_\lambda \to I$ increasingly in the strong operator topology, (ii) $f_\lambda h_2 = h_2 f_\lambda \in \mathcal{M}$, and (iii) $\tau(eu^*xu_2^*) = \tau(h_2) = \sup_{\lambda} \tau(f_\lambda h_2)$. From (ii), we have $f_\lambda h_2 u_2 \in \mathcal{M}$ for each $\lambda > 0$. It follows that, for each $\lambda > 0$,

$$\begin{aligned} |\tau(f_\lambda eh)| &= |\tau(f_\lambda eu^*xy)| = |\tau(f_\lambda h_2u_2y)| \\ &\leqslant \alpha(f_\lambda h_2u_2)\overline{\alpha}'(y) \quad \text{(by definition of } \overline{\alpha}'\text{)} \\ &= \overline{\alpha}(f_\lambda h_2u_2)\overline{\alpha}'(y) \quad \text{(by Corollary 2.9)} \end{aligned}$$

$$\begin{aligned}
&\leqslant \|f_\lambda\|\overline{\alpha}(h_2u_2)\overline{\alpha}'(y) \quad \text{(by Lemma 2.4)}\\
&\leqslant \overline{\alpha}(h_2)\overline{\alpha}'(y) \quad \text{(by properties of } \overline{\alpha})\\
&= \overline{\alpha}(eu^*xu_2^*)\overline{\alpha}'(y)\\
&\leqslant \|e\|\overline{\alpha}(u^*xu_2^*)\overline{\alpha}'(y) \quad \text{(by Lemma 2.4)}\\
&\leqslant \overline{\alpha}(x)\overline{\alpha}'(y) \quad \text{(by properties of } \overline{\alpha}).
\end{aligned}$$

Moreover, since $f_\lambda \to I$ increasingly in the strong operator topology and $eh \in \mathcal{M}$, we have $f_\lambda eh \to eh$ in the strong operator topology. Since τ is normal, τ is continuous on bounded subsets of $\mathcal{M}$ in the strong operator topology. Therefore, we have

$$\tau(eh) = |\tau(eh)| = \lim_\lambda |\tau(f_\lambda eh)| \leqslant \overline{\alpha}(x)\overline{\alpha}'(y),$$

which is a contradiction. Therefore

$$\|xy\|_1 = \tau(|xy|) = \tau(h) \leqslant \overline{\alpha}(x)\overline{\alpha}'(y),$$

and $xy \in L^1(\mathcal{M})$. If $x \in L^\alpha(\mathcal{M},\tau)$ and $y \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$, then, from Proposition 2.10, $\alpha(x) = \overline{\alpha}(x)$. Hence, $\|xy\|_1 \leqslant \alpha(x)\overline{\alpha}'(y)$. ∎

2.3. DUAL SPACE OF $L^\alpha(\mathcal{M},\tau)$. Now we are ready to describe the dual space of $L^\alpha(\mathcal{M},\tau)$, when α is a normalized, unitarily invariant, $\|\cdot\|_1$-dominating and continuous norm on $\mathcal{M}$.

THEOREM 2.12. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$ (see Definition* 1.2*). Let $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ be as defined in Definition* 2.1*. Then*

$$(L^\alpha(\mathcal{M},\tau))^\sharp = L_{\overline{\alpha}'}(\mathcal{M},\tau),$$

i.e.,

(i) *for every $\phi \in (L^\alpha(\mathcal{M},\tau))^\sharp$, there is a $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$ such that $\overline{\alpha}'(\xi) = \|\phi\|$ and $\phi(x) = \tau(x\xi)$ for all $x \in L^\alpha(\mathcal{M},\tau)$.*

(ii) *for every $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$, the mapping $\phi : L^\alpha(\mathcal{M},\tau) \to \mathbb{C}$, defined by $\phi(x) = \tau(x\xi)$ for all x in $L^\alpha(\mathcal{M},\tau)$, is in $(L^\alpha(\mathcal{M},\tau))^\sharp$. Moreover, $\|\phi\| = \overline{\alpha}'(\xi)$.*

Proof. (i) Assume that $\phi \in (L^\alpha(\mathcal{M},\tau))^\sharp$. From Proposition 2.7, there exists a $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$ such that $\overline{\alpha}'(\xi) = \|\phi\|$ and $\phi(y) = \tau(y\xi)$ for all $y \in \mathcal{M}$. Thus we need only to show that $\phi(x) = \tau(x\xi)$ for all $x \in L^\alpha(\mathcal{M},\tau)$.

Suppose $x \in L^\alpha(\mathcal{M},\tau)$. Then there is a sequence $\{x_n\}$ in $\mathcal{M}$ such that $\alpha(x_n - x) \to 0$. Note that $\phi \in (L^\alpha(\mathcal{M},\tau))^\sharp$. Then $\phi(x_n - x) \to 0$. By the generalized Hölder's inequality (Theorem 2.11), we have

$$|\tau(x_n\xi) - \tau(x\xi)| = |\tau((x_n - x)\xi)| \leqslant \alpha(x_n - x)\overline{\alpha}'(\xi) \to 0.$$

Thus $\tau(x\xi) = \lim_{n\to\infty} \tau(x_n\xi) = \lim_{n\to\infty} \phi(x_n) = \phi(x)$.

(ii) It follows directly from the definition of $\overline{\alpha}'$ in Definition 2.1 and the fact that $\mathcal{M}$ is dense in $L^{\alpha}(\mathcal{M},\tau)$, that

$$\begin{aligned}\|\phi\| &= \sup\{|\phi(x)| : x \in \mathcal{M}, \alpha(x) \leqslant 1\}\\ &= \sup\{|\tau(x\xi)| : x \in \mathcal{M}, \alpha(x) \leqslant 1\} = \overline{\alpha}'(\xi) < \infty,\end{aligned}$$

and thus $\phi \in (L^{\alpha}(\mathcal{M},\tau))^{\sharp}$. ∎

3. NONCOMMUTATIVE HARDY SPACES H^{α}

Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ. Given a von Neumann subalgebra $\mathcal{D}$ of $\mathcal{M}$, a conditional expectation $\Phi : \mathcal{M} \to \mathcal{D}$ is defined to be a positive linear map which preserves the identity and satisfies $\Phi(x_1 y x_2) = x_1 \Phi(y) x_2$ for all $x_1, x_2 \in \mathcal{D}$ and $y \in \mathcal{M}$. For a finite von Neumann algebra $\mathcal{M}$ with a faithful normal tracial state τ and a von Neumann subalgebra $\mathcal{D}$, it is a well-known fact that there exists a unique, faithful, normal, conditional expectation Φ from $\mathcal{M}$ onto $\mathcal{D}$ such that $\tau(\Phi(y)) = \tau(y)$, for all $y \in \mathcal{M}$. Furthermore it is known that such $\Phi : \mathcal{M} \to \mathcal{D}$ can be extended to a contractive linear mapping $\Phi : L^1(\mathcal{M},\tau) \to L^1(\mathcal{D},\tau)$ satisfying $\tau(y) = \tau(\Phi(y))$ for all $y \in L^1(\mathcal{M},\tau)$ (for example, see Proposition 3.9 in [19].)

3.1. ARVESON'S NONCOMMUTATIVE HARDY SPACES.

We now recall the noncommutative analogue of the classical Hardy space $H^{\infty}(\mathbb{T})$ by Arveson in [1] (also see [10]).

DEFINITION 3.1. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal tracial state τ. Let $\mathcal{A}$ be a weak* closed unital subalgebra of $\mathcal{M}$, and let Φ be a faithful, normal conditional expectation from $\mathcal{M}$ onto the diagonal von Neumann algebra $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$. Then $\mathcal{A}$ is called a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ with respect to Φ if

(i) $\mathcal{A} + \mathcal{A}^*$ is weak* dense in $\mathcal{M}$;
(ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for all $x, y \in \mathcal{A}$;
(iii) $\tau \circ \Phi = \tau$.

Such a finite, maximal subdiagonal subalgebra $\mathcal{A}$ of $\mathcal{M}$ is also called an H^{∞} space of $\mathcal{M}$.

EXAMPLE 3.2. Let $\mathcal{M} = M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices with complex entries equipped with a trace τ. Let $\mathcal{A}$ be the subalgebra of upper triangular matrices. Now $\mathcal{D}$ is the diagonal matrices and Φ is the natural projection onto the diagonal. Then $\mathcal{A}$ is a finite maximal subdiagonal algebra of $\mathcal{M}$.

EXAMPLE 3.3. Let $\mathcal{M} = L^{\infty}(X,\mu)$, where (X,μ) is a probability space. Let $\tau(f) = \int f \mathrm{d}\mu$ for all f in $L^{\infty}(X,\mu)$. Let $\mathcal{A}$ be a weak* closed subalgebra of $L^{\infty}(X,\mu)$ such that $I \in \mathcal{A}$, $\mathcal{A} + \mathcal{A}^*$ is weak* dense in $L^{\infty}(X,\mu)$, and such that

$\int fg\mathrm{d}\mu = (\int f\mathrm{d}\mu)(\int g\mathrm{d}\mu)$ for all $f,g \in \mathcal{A}$. Let $\Phi(f) = (\int f\mathrm{d}\mu)I$ for all f in $L^\infty(X,\mu)$. Then $\mathcal{A}$ is a finite, maximal subdiagonal algebra in $L^\infty(X,\mu)$. These examples are the weak* Dirichlet algebras of Srinivasan and Wang [28].

3.2. NONCOMMUTATIVE H^α SPACES. Let H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. We let

$$H_0^\infty = \{x \in H^\infty : \Phi(x) = 0\}.$$

For $\mathcal{S} \subseteq L^p(\mathcal{M},\tau), 0 < p < \infty$, let $[\mathcal{S}]_p$ denote the closure of $\mathcal{S}$ in $L^p(\mathcal{M},\tau)$ with respect to $\|\cdot\|_p$. Let

$$H^p = [H^\infty]_p \quad \text{and} \quad H_0^p = [H_0^\infty]_p.$$

For $\mathcal{S} \subseteq \mathcal{M}$, let $\overline{\mathcal{S}}^{w*}$ denote the weak* closure of $\mathcal{S}$ in $\mathcal{M}$.

The following characterization of noncommutative H^p spaces for $1 \leqslant p \leqslant \infty$ was proved by Saito in [24].

PROPOSITION 3.4 (from [24]). *Let* $1 \leqslant p \leqslant \infty$. *Then*
(i) $H^1 \cap L^p(\mathcal{M},\tau) = H^p$ *and* $H_0^1 \cap L^p(\mathcal{M},\tau) = H_0^p$.
(ii) $H^p = \{x \in L^p(\mathcal{M},\tau) : \tau(xy) = 0$ *for all* $y \in H_0^\infty\}$.
(iii) $H_0^p = \{x \in L^p(\mathcal{M},\tau) : \tau(xy) = 0$ *for all* $y \in H^\infty\} = \{x \in H^p : \Phi(x) = 0\}$.

Similarly, we have the following definition in $L^\alpha(\mathcal{M},\tau)$ spaces.

DEFINITION 3.5. Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal tracial state τ. Let H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Suppose α is a normalized, unitarily invariant, continuous, $\|\cdot\|_1$-dominating norm on $\mathcal{M}$. For $\mathcal{S} \subseteq L^\alpha(\mathcal{M},\tau)$, let $[\mathcal{S}]_\alpha$ denote the closure of $\mathcal{S}$ in $L^\alpha(\mathcal{M},\tau)$ with respect to the norm α. In particular, We define H^α to be the α-closure of H^∞, i.e.,

$$H^\alpha = [H^\infty]_\alpha.$$

3.3. CHARACTERIZATIONS OF H^α SPACES. In this section, our object is to provide an analogue of Saito's result stated in Proposition 3.4 in the new setting H^α, where α is a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$.

It is proved in [4] that the multiplicativity of the conditional expectation Φ on H^∞ surprisingly extends to multiplicativity on H^p for all $0 < p < \infty$.

LEMMA 3.6 (from [4]). *The conditional expectation* Φ *is multiplicative on Hardy spaces. More precisely,* $\Phi(ab) = \Phi(a)\Phi(b)$ *for all* $a \in H^p$ *and* $b \in H^q$ *with* $0 < p,q \leqslant \infty$.

Next we will prove two lemmas before we state the main result of the section.

LEMMA 3.7. *Let* $\mathcal{M}$ *be a finite von Neumann algebra with a faithful normal tracial state* τ, *and* H^∞ *be a finite, maximal subdiagonal subalgebra of* $\mathcal{M}$. *Let* α *be a normalized, unitarily invariant,* $\|\cdot\|_1$*-dominating, continuous norm on* $\mathcal{M}$ *(see Definition* 1.2*). Let*

$L_{\overline{\alpha}'}(\mathcal{M},\tau)$ *be as defined in Definition* 2.1.*Then*

$$H^{\alpha} = \{x \in L^{\alpha}(\mathcal{M},\tau) : \tau(xy) = 0 \textit{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)\}.$$

Proof. Let

$$X = \{x \in L^{\alpha}(\mathcal{M},\tau) : \tau(xy) = 0 \text{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)\}.$$

Suppose $x \in H^{\infty}$. If $y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau) \subseteq H_0^1$, then it follows from part (iii) of Proposition 3.4 that $\tau(xy) = 0$, which implies $x \in X$, and so $H^{\infty} \subseteq X$.

We claim that X is α-closed in $L^{\alpha}(\mathcal{M},\tau)$. In fact, suppose $\{x_n\}$ is a sequence in X and $x \in L^{\alpha}(\mathcal{M},\tau)$ such that $\alpha(x_n - x) \to 0$. If $y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)$, then by the generalized Hölder's inequality (Theorem 2.11), we have

$$|\tau(xy) - \tau(x_n y)| = |\tau((x - x_n)y)| \leqslant \alpha(x - x_n)\overline{\alpha}'(y) \to 0.$$

Since $x_n \in X$ for all $n \in \mathbb{N}$, it follows that $\tau(xy) = \lim_{n\to\infty} \tau(x_n y) = 0$ for all $y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)$. By the definition of X, we know that $x \in X$. Hence X is closed in $L^{\alpha}(\mathcal{M},\tau)$. Therefore

$$H^{\alpha} = [H^{\infty}]_{\alpha} \subseteq X.$$

Next, we show that $H^{\alpha} = X$. Assume, via contradiction, that $H^{\alpha} \subsetneq X \subseteq L^{\alpha}(\mathcal{M},\tau)$. By the Hahn–Banach theorem, there is a $\phi \in (L^{\alpha}(\mathcal{M},\tau))^{\sharp}$ and $x \in X$ such that

(i) $\phi(x) \neq 0$, and

(ii) $\phi(y) = 0$ for all $y \in H^{\alpha}$.

Since α is a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$, it follows from Proposition 2.7 that there exists a $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau)$ such that

(iii) $\phi(z) = \tau(z\xi)$ for all $z \in L^{\alpha}(\mathcal{M},\tau)$.

Hence from (ii) and (iii) we can conclude that

(iv) $\tau(y\xi) = \phi(y) = 0$ for every $y \in H^{\infty} \subseteq H^{\alpha} \subseteq L^{\alpha}(\mathcal{M},\tau)$.

Since $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$, it follows from part (iii) of Proposition 3.4 and (iv) as above that $\xi \in H_0^1$, which means $\xi \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)$. Combining this with the fact that $x \in X = \{x \in L^{\alpha}(\mathcal{M},\tau) : \tau(xy) = 0 \text{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)\}$, we obtain that $\tau(x\xi) = 0$. Note, again, that $x \in X \subseteq L^{\alpha}(\mathcal{M},\tau)$. From (i) and (iii), it follows that $\tau(x\xi) = \phi(x) \neq 0$. This is a contradiction. Therefore

$$H^{\alpha} = X = \{x \in L^{\alpha}(\mathcal{M},\tau) : \tau(xy) = 0 \text{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)\}. \quad \blacksquare$$

LEMMA 3.8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^{∞} be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$ (see Definition* 1.2*). Let $L_{\overline{\alpha}'}(\mathcal{M},\tau)$ be as defined in Definition* 2.1. *Then*

$$H^1 \cap L^{\alpha}(\mathcal{M},\tau) = \{x \in L^{\alpha}(\mathcal{M},\tau) : \tau(xy) = 0 \textit{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M},\tau)\}.$$

Proof. Let

$$X = \{x \in L^\alpha(\mathcal{M}, \tau) : \tau(xy) = 0 \text{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M}, \tau)\}.$$

It is clear that $X \subseteq L^\alpha(\mathcal{M}, \tau)$.

Now we suppose $x \in X$, that is $x \in L^\alpha(\mathcal{M}, \tau)$ such that $\tau(xy) = 0$ for all $y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M}, \tau)$. Since $H_0^\infty \subseteq H^\infty \subseteq \mathcal{M} \subseteq L_{\overline{\alpha}'}(\mathcal{M}, \tau)$ and $H_0^\infty \subseteq H_0^1$, it follows that $\tau(xy) = 0$ for all $y \in H_0^\infty$. Then by part (ii) of Proposition 3.4, $x \in H^1$, which implies $X \subseteq H^1 \cap L^\alpha(\mathcal{M}, \tau)$.

To prove $H^1 \cap L^\alpha(\mathcal{M}, \tau) \subseteq X$, suppose $x \in H^1 \cap L^\alpha(\mathcal{M}, \tau)$. Then $x \in L^\alpha(\mathcal{M}, \tau)$. Assume that $y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M}, \tau)$. So $\Phi(y) = 0$. Note that $xy \in H^1 H_0^1 \subseteq H^{1/2}$. From Lemma 3.6, we know that $\Phi(xy)$ is in $L^{1/2}(\mathcal{D}, \tau)$ (see Theorem 2.1 in [4]) and $\Phi(xy) = \Phi(x)\Phi(y) = 0$. Moreover, since $x \in L^\alpha(\mathcal{M}, \tau)$ and $y \in L_{\overline{\alpha}'}(\mathcal{M}, \tau)$, it follows from Theorem 2.11 that $xy \in L^1(\mathcal{M}, \tau)$, whence $\Phi(xy)$ is also in $L^1(\mathcal{M}, \tau)$. Thus $\tau(xy)$ is well defined and $\tau(xy) = \tau(\Phi(xy)) = 0$. By the definition of X, we conclude that $x \in X$. Therefore $H^1 \cap L^\alpha(\mathcal{M}, \tau) \subseteq X$. Now we can obtain that

$$H^1 \cap L^\alpha(\mathcal{M}, \tau) = \{x \in L^\alpha(\mathcal{M}, \tau) : \tau(xy) = 0 \text{ for all } y \in H_0^1 \cap L_{\overline{\alpha}'}(\mathcal{M}, \tau)\}. \quad \blacksquare$$

The following theorem gives a characterization of H^α.

THEOREM 3.9. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. Then*

$$H^\alpha = H^1 \cap L^\alpha(\mathcal{M}, \tau) = \{x \in L^\alpha(\mathcal{M}, \tau) : \tau(xy) = 0 \textit{ for all } y \in H_0^\infty\}.$$

The result follows directly from Lemma 3.7, Lemma 3.8 and Proposition 3.4.

4. BEURLING INVARIANT SUBSPACE THEOREM

In this section, we extend the classical Beurling theorem to Arveson's noncommutative Hardy spaces associated with unitarily invariant norms.

4.1. A FACTORIZATION RESULT.

In [24], Saito proved the following useful factorization theorem.

LEMMA 4.1 (from [24]). *Suppose $\mathcal{M}$ is a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. If $k \in \mathcal{M}$ and $k^{-1} \in L^2(\mathcal{M}, \tau)$, then there are unitary operators $u_1, u_2 \in \mathcal{M}$ and operators $a_1, a_2 \in H^\infty$ such that $k = u_1 a_1 = a_2 u_2$ and $a_1^{-1}, a_2^{-1} \in H^2$.*

We shall show that in fact it is possible to choose a_1 and a_2 with their inverses in H^α.

PROPOSITION 4.2. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a* *normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $k \in \mathcal{M}$ and $k^{-1} \in L^\alpha(\mathcal{M},\tau)$, then there are unitary operators $w_1, w_2 \in \mathcal{M}$ and operators $a_1, a_2 \in H^\infty$ such that $k = w_1 a_1 = a_2 w_2$ and $a_1^{-1}, a_2^{-1} \in H^\alpha$.*

Proof. Suppose $k \in \mathcal{M}$ with $k^{-1} \in L^\alpha(\mathcal{M},\tau)$. Assume that $k = vh$ is the polar decomposition of k in $\mathcal{M}$, where v is a unitary operator in $\mathcal{M}$ and h in $\mathcal{M}$ is positive. Then from the assumption that $k^{-1} = h^{-1}v^* \in L^\alpha(\mathcal{M},\tau)$, we see $h^{-1} \in L^\alpha(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$. Since h in $\mathcal{M}$ is positive, we can conclude that $h^{-1/2} \in L^2(\mathcal{M},\tau)$. Note that $h^{1/2} \in \mathcal{M}$. It follows from Lemma 4.1 that there exist a unitary operator $u_1 \in \mathcal{M}$ and $h_1 \in H^\infty$ such that $h^{1/2} = u_1 h_1$ and $h_1^{-1} \in H^2$.

Now $h = h^{1/2} \cdot h^{1/2} = u_1(h_1 u_1)h_1$. Since $h_1 u_1$ is in $\mathcal{M}$ and $(h_1 u_1)^{-1} = u_1^* h_1^{-1} \in L^2(\mathcal{M},\tau)$, by Lemma 4.1 there exist a unitary operator $u_2 \in \mathcal{M}$ and $h_2 \in H^\infty$ such that $h_1 u_1 = u_2 h_2$ and $h_2^{-1} \in H^2$. Thus

$$k = vh = vu_1h_1u_1h_1 = vu_1u_2h_2h_1 = w_1a_1,$$

where $w_1 = vu_1u_2$ is a unitary operator in $\mathcal{M}$ and $a_1 = h_2h_1 \in H^\infty$ with

$$a_1^{-1} = (h_2h_1)^{-1} = h_1^{-1}h_2^{-1} \in H^2 \cdot H^2 \subseteq H^1.$$

Since $k^{-1} = (w_1a_1)^{-1} = a_1^{-1}w_1^* \in L^\alpha(\mathcal{M},\tau)$, we obtain that $a_1^{-1} \in L^\alpha(\mathcal{M},\tau)$. Then by Theorem 3.9, we have

$$a_1^{-1} \in H^1 \cap L^\alpha(\mathcal{M}) = H^\alpha.$$

Hence w_1 is a unitary in $\mathcal{M}$ and a_1 is in H^∞ such that $k = w_1a_1$ and $a_1^{-1} \in H^\alpha$.

Similarly, there exist a unitary operator $w_2 \in \mathcal{M}$ and $a_2 \in H^\infty$ such that $k = a_2w_2$ and $a_2^{-1} \in H^\alpha$. ∎

4.2. DENSE SUBSPACES. The following theorem plays an important role in the proof of our main result of the paper.

THEOREM 4.3. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful normal tracial state τ, and H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ and $\mathcal{N}$ is a weak* closed linear subspace of $\mathcal{M}$ such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$ and $\mathcal{N}H^\infty \subseteq \mathcal{N}$, then*

(i) $\mathcal{N} = [\mathcal{N}]_\alpha \cap \mathcal{M}$;

(ii) *$\mathcal{W} \cap \mathcal{M}$ is weak* closed in $\mathcal{M}$;*

(iii) $\mathcal{W} = [\mathcal{W} \cap \mathcal{M}]_\alpha$;

(iv) *if $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^\infty \subseteq \mathcal{S}$, then*

$$[\mathcal{S}]_\alpha = [\overline{\mathcal{S}}^{w*}]_\alpha,$$

where $\overline{\mathcal{S}}^{w*}$ is the weak closure of $\mathcal{S}$ in $\mathcal{M}$.*

Proof. (i) It is clear that $\mathcal{N} \subseteq [\mathcal{N}]_\alpha \cap \mathcal{M}$. Assume, via contradiction, that $\mathcal{N} \subsetneqq [\mathcal{N}]_\alpha \cap \mathcal{M}$. Note that $\mathcal{N}$ is a weak* closed linear subspace of $\mathcal{M}$ and

$L^1(\mathcal{M},\tau)$ is the predual space of $\mathcal{M}$. It follows from the Hahn–Banach theorem that there exist a $\xi \in L^1(\mathcal{M},\tau)$ and an $x \in [\mathcal{N}]_\alpha \cap \mathcal{M}$ such that

(a) $\tau(\xi x) \neq 0$, but
(b) $\tau(\xi y) = 0$ for all $y \in \mathcal{N}$.

We claim that there exists a $z \in \mathcal{M}$ such that

(a′) $\tau(zx) \neq 0$, but
(b′) $\tau(zy) = 0$ for all $y \in \mathcal{N}$.

Actually assume that $\xi = |\xi^*|v$ is the polar decomposition of ξ in $L^1(\mathcal{M},\tau)$, where v is a unitary element in $\mathcal{M}$ and $|\xi^*|$ in $L^1(\mathcal{M},\tau)$ is positive. Let f be a function on $[0,\infty)$ defined by the formula $f(t) = 1$ for $0 \leqslant t \leqslant 1$ and $f(t) = 1/t$ for $t > 1$. We define $k = f(|\xi^*|)$ by the functional calculus. Then by the construction of f, we know that $k \in \mathcal{M}$ and $k^{-1} = f^{-1}(|\xi^*|) \in L^1(\mathcal{M},\tau)$. It follows from Theorem 4.2 that there exist a unitary $u \in \mathcal{M}$ and $a \in H^\infty$ such that $k = ua$ and $a^{-1} \in H^1$. Therefore, we can further assume that $\{a_n\}_{n=1}^\infty$ is a sequence of elements in H^∞ such that $\|a^{-1} - a_n\|_1 \to 0$. Observe that

(1) since a, a_n are in H^∞, for each $y \in \mathcal{N}$ we have that $ya_na \in \mathcal{N}H^\infty \subseteq \mathcal{N}$ and
$$\tau(a_na\xi y) = \tau(\xi y a_n a) = 0;$$
(2) we have $a\xi = (u^*u)a(|\xi^*|v) = u^*(k|\xi^*|)v \in \mathcal{M}$, by the choice of a and u;
(3) from (a) and (ii), we have
$$0 \neq \tau(\xi x) = \tau(a^{-1}a\xi x) = \lim_{n\to\infty} \tau(a_na\xi x).$$

Combining (1), (2) and (3), we are able to find an $N \in \mathbb{N}$ such that $z = a_Na\xi \in \mathcal{M}$ satisfying

(a′) $\tau(zx) \neq 0$, but
(b′) $\tau(zy) = 0$ for all $y \in \mathcal{N}$.

Recall that $x \in [\mathcal{N}]_\alpha$. Then there is a sequence $\{x_n\}$ in $\mathcal{N}$ such that $\alpha(x - x_n) \to 0$. We have
$$|\tau(zx_n) - \tau(zx)| = |\tau(z(x - x_n))| \leqslant \|x - x_n\|_1\|z\| \leqslant \alpha(x - x_n)\|z\| \to 0.$$
Combining with (b′) we conclude that $\tau(zx) = \lim_{n\to\infty} \tau(zx_n) = 0$. This contradicts the result (a′). Therefore $\mathcal{N} = [\mathcal{N}]_\alpha \cap \mathcal{M}$.

(ii) Let $\overline{\mathcal{W} \cap \mathcal{M}}^{w*}$ be the weak* closure of $\mathcal{W} \cap \mathcal{M}$ in $\mathcal{M}$. In order to show that $\mathcal{W} \cap \mathcal{M} = \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$, it suffices to show that $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{W}$. Assume, to the contrary, that $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \nsubseteq \mathcal{W}$. Thus there exists an element x in $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{M} \subseteq L^\alpha(\mathcal{M},\tau)$, but $x \notin \mathcal{W}$. Since $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$, by the Hahn–Banach theorem and Theorem 2.12, there exists a $\xi \in L_{\overline{\alpha}'}(\mathcal{M},\tau) \subseteq L^1(\mathcal{M},\tau)$ such that $\tau(\xi x) \neq 0$ and $\tau(\xi y) = 0$ for all $y \in \mathcal{W}$. Since $\xi \in L^1(\mathcal{M},\tau)$, the linear mapping $\tau_\xi : \mathcal{M} \to \mathbb{C}$, defined by $\tau_\xi(a) = \tau(\xi a)$ for all $a \in \mathcal{M}$, is weak* continuous. Note that $x \in \overline{\mathcal{W} \cap \mathcal{M}}^{w*}$ and $\tau(\xi y) = 0$ for all $y \in \mathcal{W}$. But then we know that $\tau(\xi x) = 0$, which contradicts the assumption that $\tau(\xi x) \neq 0$. Hence $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} \subseteq \mathcal{W}$, whence $\overline{\mathcal{W} \cap \mathcal{M}}^{w*} = \mathcal{W} \cap \mathcal{M}$.

(iii) Since $\mathcal{W}$ is α-closed, it is easy to see $[\mathcal{W} \cap \mathcal{M}]_\alpha \subseteq \mathcal{W}$. Now we assume $[\mathcal{W} \cap \mathcal{M}]_\alpha \subsetneqq \mathcal{W} \subseteq L^\alpha(\mathcal{M}, \tau)$. By the Hahn–Banach theorem and Theorem 2.12 there exist an $x \in \mathcal{W}$ and $\xi \in L_{\overline{\alpha}'}(\mathcal{M}, \tau)$ such that $\tau(\xi x) \neq 0$ and $\tau(\xi y) = 0$ for all $y \in [\mathcal{W} \cap \mathcal{M}]_\alpha$. Let $x = v|x|$ be the polar decomposition of x in $L^\alpha(\mathcal{M}, \tau)$, where v is a unitary element in $\mathcal{M}$. Let f be a function on $[0, \infty)$ defined by the formula $f(t) = 1$ for $0 \leqslant t \leqslant 1$ and $f(t) = 1/t$ for $t > 1$. We define $k = f(|x|)$ through the functional calculus. Then we see $k \in \mathcal{M}$ and $k^{-1} = f^{-1}(|x|) \in L^\alpha(\mathcal{M}, \tau)$. It follows from Theorem 4.2 that there exist a unitary $u \in \mathcal{M}$ and $a \in H^\infty$ such that $k = au$ and $a^{-1} \in H^\alpha$. A little computation shows that $|x|k \in \mathcal{M}$, which implies that

$$xa = xauu^* = xku^* = v(|x|k)u^* \in \mathcal{M}.$$

Since $a \in H^\infty$, we know $xa \in \mathcal{W}H^\infty \subseteq \mathcal{W}$, and thus $xa \in \mathcal{W} \cap \mathcal{M}$. Furthermore, note that $(\mathcal{W} \cap \mathcal{M})H^\infty \subseteq \mathcal{W} \cap \mathcal{M}$. Thus, if $b \in H^\infty$, we see $xab \in \mathcal{W} \cap \mathcal{M}$, and so $\tau(\xi xab) = 0$. Since H^∞ is dense in H^α and ξ is in $L_{\overline{\alpha}'}(\mathcal{M}, \tau)$, it follows from Theorem 2.11 that $\tau(\xi xab) = 0$ for all $b \in H^\alpha$. Since $a^{-1} \in H^\alpha$, we see $\tau(\xi x) = \tau(\xi xaa^{-1}) = 0$. This contradicts the assumption that $\tau(\xi x) \neq 0$. Therefore $\mathcal{W} = [\mathcal{W} \cap \mathcal{M}]_\alpha$.

(iv) Assume that $\mathcal{S}$ is a subspace of $\mathcal{M}$ such that $\mathcal{S}H^\infty \subseteq \mathcal{S}$ and $\overline{\mathcal{S}}^{w*}$ is the weak* closure of $\mathcal{S}$ in $\mathcal{M}$. Then $[\mathcal{S}]_\alpha H^\infty \subseteq [\mathcal{S}]_\alpha$. Note that $\mathcal{S} \subseteq [\mathcal{S}]_\alpha \cap \mathcal{M}$. From (ii), we know that $[\mathcal{S}]_\alpha \cap \mathcal{M}$ is weak*-closed. Therefore $\overline{\mathcal{S}}^{w*} \subseteq [\mathcal{S}]_\alpha \cap \mathcal{M}$. Hence $[\overline{\mathcal{S}}^{w*}]_\alpha \subseteq [\mathcal{S}]_\alpha$, whence $[\overline{\mathcal{S}}^{w*}]_\alpha = [\mathcal{S}]_\alpha$. ∎

4.3. MAIN RESULT. Before we state our main result in this section, we will need the following definition from [17].

DEFINITION 4.4. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, tracial, normal state τ. Let X be a weak* closed subspace of $\mathcal{M}$. Then X is called an *internal column sum* of a family of weak* closed subspaces $\{X_i\}_{i \in \mathcal{I}}$ of $\mathcal{M}$, denoted by

$$X = \bigoplus_{i \in \mathcal{I}}^{\mathrm{col}} X_i$$

if

(i) $X_j^* X_i = \{0\}$ for all distinct $i, j \in \mathcal{I}$; and

(ii) the linear span of $\{X_i : i \in \mathcal{I}\}$ is weak* dense in X, i.e.,

$$X = \overline{\mathrm{span}\{X_i : i \in \mathcal{I}\}}^{w*}.$$

Similarly, we introduce a concept of internal column sum of subspaces in $L^\alpha(\mathcal{M}, \tau)$ as follows.

DEFINITION 4.5. Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, tracial, normal state τ, α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating and continuous norm on $\mathcal{M}$. Let X be a closed subspace of $L^\alpha(\mathcal{M},\tau)$. Then X is called an *internal column sum* of a family of closed subspaces $\{X_i\}_{i\in\mathcal{I}}$ of $L^\alpha(\mathcal{M},\tau)$, denoted by

$$X = \bigoplus_{i\in\mathcal{I}}^{\text{col}} X_i$$

if

(i) $X_j^* X_i = \{0\}$ for all distinct $i, j \in \mathcal{I}$; and

(ii) the linear span of $\{X_i : i \in \mathcal{I}\}$ is dense in X, i.e.,

$$X = [\text{span}\{X_i : i \in \mathcal{I}\}]_\alpha.$$

In [5], David P. Blecher and Louis E. Labuschagne proved a version of Beurling theorem for $L^p(\mathcal{M},\tau)$ spaces when $1 \leqslant p \leqslant \infty$.

LEMMA 4.6 (from [5]). *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, tracial, normal state τ, and H^∞ be a maximal subdiagonal subalgebra of $\mathcal{M}$ with $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Suppose that $\mathcal{K}$ is a closed H^∞-right-invariant subspace of $L^p(\mathcal{M},\tau)$, for some $1 \leqslant p \leqslant \infty$. (For $p = \infty$ we assume that $\mathcal{K}$ is weak* closed.) Then $\mathcal{K}$ may be written as a column L^p-sum $\mathcal{K} = \mathcal{Z} \bigoplus^{\text{col}} (\bigoplus_i^{\text{col}} u_i H^p)$, where $\mathcal{Z}$ is a closed (indeed weak* closed if $p = \infty$) subspace of $L^p(\mathcal{M},\tau)$ such that $\mathcal{Z} = [\mathcal{Z}H_0^\infty]_p$, and where u_i are partial isometries in $\mathcal{M} \cap \mathcal{K}$ with $u_j^* u_i = 0$ if $i \neq j$, and with $u_i^* u_i \in \mathcal{D}$. Moreover, for each i, $u_i^* \mathcal{Z} = \{0\}$, left multiplication by the $u_i u_i^*$ are contractive projections from $\mathcal{K}$ onto the summands $u_i H^p$, and left multiplication by $I - \sum\limits_i u_i u_i^*$ is a contractive projection from $\mathcal{K}$ onto $\mathcal{Z}$.*

Now we are ready to prove the main result of the paper, a generalized version of the classical theorem of Beurling [2] in a noncommutative $L^\alpha(\mathcal{M},\tau)$ space for a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm α.

THEOREM 4.7. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ and $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$, then $\mathcal{W}H^\infty \subseteq \mathcal{W}$ if and only if*

$$\mathcal{W} = \mathcal{Z} \bigoplus^{\text{col}} (\bigoplus_{i\in\mathcal{I}}^{\text{col}} u_i H^\alpha),$$

where $\mathcal{Z}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{Z} = [\mathcal{Z}H_0^\infty]_\alpha$, and where u_i are partial isometries in $\mathcal{W} \cap \mathcal{M}$ with $u_j^* u_i = 0$ if $i \neq j$, and with $u_i^* u_i \in \mathcal{D}$. Moreover, for each i, $u_i^* \mathcal{Z} = \{0\}$, left multiplication by the $u_i u_i^*$ are contractive projections from $\mathcal{W}$ onto the summands $u_i H^\alpha$, and left multiplication by $I - \sum\limits_i u_i u_i^*$ is a contractive projection from $\mathcal{W}$ onto $\mathcal{Z}$.

Proof. The if part is obvious. Suppose $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{W}H^\infty \subseteq \mathcal{W}$. Then it follows from part (ii) of Theorem 4.3 that $\mathcal{W} \cap \mathcal{M}$

is weak* closed in $\mathcal{M}$. It follows from Lemma 4.6, in the case $p=\infty$, that

$$\mathcal{W}\cap\mathcal{M}=\mathcal{Z}_1\bigoplus^{\text{col}}(\bigoplus^{\text{col}}_{i\in\mathcal{I}}u_iH^\infty),$$

where $\mathcal{Z}_1$ is a weak* closed subspace in $\mathcal{M}$ such that $\mathcal{Z}_1=\overline{\mathcal{Z}_1H_0^\infty}^{w*}$, and u_i are partial isometries in $\mathcal{W}\cap\mathcal{M}$ with $u_j^*u_i=0$ if $i\neq j$, and with $u_i^*u_i\in\mathcal{D}$. Moreover, for each i, $u_i^*\mathcal{Z}_1=\{0\}$, left multiplication by the $u_iu_i^*$ are contractive projections from $\mathcal{W}\cap\mathcal{M}$ onto the summands u_iH^∞, and left multiplication by $I-\sum_i u_iu_i^*$ is a contractive projection from $\mathcal{W}\cap\mathcal{M}$ onto $\mathcal{Z}_1$.

Let $\mathcal{Z}=[\mathcal{Z}_1]_\alpha$. It is not hard to verify that for each i, $u_i^*\mathcal{Z}=\{0\}$. We also claim that $[u_iH^\infty]_\alpha=u_iH^\alpha$. In fact it is obvious that $[u_iH^\infty]_\alpha\supseteq u_iH^\alpha$. We will need only to show that $[u_iH^\infty]_\alpha\subseteq u_iH^\alpha$. Let $\{a_n\}\subseteq H^\infty$ and $a\in[u_iH^\infty]_\alpha$ be such that $\alpha(u_ia_n-a)\to 0$. By the choice of u_i, we know that $u_i^*u_i\in\mathcal{D}\subseteq H^\infty$, whence $u_i^*u_ia_n\in H^\infty$ for each $n\geqslant 1$. Combining with the fact that $\alpha(u_i^*u_ia_n-u_i^*a)\leqslant\alpha(u_ia_n-a)\to 0$, we obtain that $u_i^*a\in H^\alpha$. Again from the choice of u_i, we know that $u_iu_i^*u_ia_n=u_ia_n$ for each $n\geqslant 1$. This implies that $a=u_i(u_i^*a)\in u_iH^\alpha$. Thus we conclude that $[u_iH^\infty]_\alpha\subseteq u_iH^\alpha$, whence $[u_iH^\infty]_\alpha=u_iH^\alpha$. Now from parts (iii) and (iv) of Theorem 4.3 and from the definition of internal column sum, it follows that

$$\begin{aligned}\mathcal{W}=[\mathcal{W}\cap\mathcal{M}]_\alpha&=[\overline{\text{span}\{\mathcal{Z}_1,u_iH^\infty:i\in\mathcal{I}\}}^{w*}]_\alpha=[\text{span}\{\mathcal{Z}_1,u_iH^\infty:i\in\mathcal{I}\}]_\alpha\\&=[\text{span}\{\mathcal{Z},u_iH^\alpha:i\in\mathcal{I}\}]_\alpha=\mathcal{Z}\bigoplus^{\text{col}}(\bigoplus^{\text{col}}_i u_iH^\alpha).\end{aligned}$$

Next, we will verify that $\mathcal{Z}=[\mathcal{Z}H_0^\infty]_\alpha$. Recall that $\mathcal{Z}=[\mathcal{Z}_1]_\alpha$. It follows from part (i) of Theorem 4.3 we have that

$$[\mathcal{Z}_1H_0^\infty]_\alpha\cap\mathcal{M}=\overline{\mathcal{Z}_1H_0^\infty}^{w*}=\mathcal{Z}_1.$$

Hence from part (iii) of Theorem 4.3 we have that

$$\mathcal{Z}\supseteq[\mathcal{Z}H_0^\infty]_\alpha\supseteq[\mathcal{Z}_1H_0^\infty]_\alpha=[[\mathcal{Z}_1H_0^\infty]_\alpha\cap\mathcal{M}]_\alpha=[\mathcal{Z}_1]_\alpha=\mathcal{Z}.$$

Thus $\mathcal{Z}=[\mathcal{Z}H_0^\infty]_\alpha$.

Moreover, it is not hard to verify that for each i, left multiplication by the $u_iu_i^*$ are contractive projections from $\mathcal{W}$ onto the summands u_iH^α, and left multiplication by $I-\sum_i u_iu_i^*$ is a contractive projection from $\mathcal{W}$ onto $\mathcal{Z}$. Now the proof is completed. ∎

A quick application of Theorem 4.7 yields the following corollary on doubly invariant subspaces in $L^\alpha(\mathcal{M},\tau)$.

COROLLARY 4.8. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. If $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $\mathcal{W}\mathcal{M}\subseteq\mathcal{W}$, then there exists a projection e in $\mathcal{M}$ such that $\mathcal{W}=eL^\alpha(\mathcal{M},\tau)$.*

Proof. Note that $\mathcal{M}$ itself is a finite, maximal subdiagonal subalgebra of $\mathcal{M}$. Let $H^\infty = \mathcal{M}$. Then $\mathcal{D} = \mathcal{M}$ and Φ is the identity map from $\mathcal{M}$ to $\mathcal{M}$. Hence $H_0^\infty = \{0\}$ and $H^\alpha = L^\alpha(\mathcal{M}, \tau)$.

Assume that $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{W}\mathcal{M} \subseteq \mathcal{W}$. From Theorem 4.7,

$$\mathcal{W} = \mathcal{Z} \bigoplus^{\text{col}} (\bigoplus_{i \in \mathcal{I}}^{\text{col}} u_i H^\alpha),$$

where $\mathcal{Z}$ and the $u_i's$ satisfy the hypothesis of Theorem 4.7. From the fact that $H_0^\infty = \{0\}$, we know that $\mathcal{Z} = \{0\}$. Since $\mathcal{D} = \mathcal{M}$, we know that

$$u_i H^\alpha = u_i L^\alpha(\mathcal{M}, \tau) \supseteq u_i u_i^* L^\alpha(\mathcal{M}, \tau) \supseteq u_i u_i^* u_i L^\alpha(\mathcal{M}, \tau) = u_i L^\alpha(\mathcal{M}, \tau) = u_i H^\alpha.$$

So $u_i H^\alpha = u_i u_i^* L^\alpha(\mathcal{M}, \tau)$ and

$$\begin{aligned} \mathcal{W} &= \mathcal{Z} \bigoplus^{\text{col}} (\bigoplus_{i \in \mathcal{I}}^{\text{col}} u_i H^\alpha) = \bigoplus_{i \in \mathcal{I}}^{\text{col}} u_i u_i^* L^\alpha(\mathcal{M}, \tau) \\ &= \Big(\sum_i u_i u_i^* \Big) L^\alpha(\mathcal{M}, \tau) = e L^\alpha(\mathcal{M}, \tau), \end{aligned}$$

where $e = \sum_i u_i u_i^*$ is a projection in $\mathcal{M}$. ∎

The next result is another application of Theorem 4.7 on simply invariant subspaces in weak* Dirichlet algebras.

COROLLARY 4.9. *Let $\mathcal{M}$ be a finite von Neumann algebra with a faithful, normal, tracial state τ. Let H^∞ be a finite, maximal subdiagonal subalgebra of $\mathcal{M}$ such that $H^\infty \cap (H^\infty)^* = \mathbb{C}I$. Let α be a normalized, unitarily invariant, $\|\cdot\|_1$-dominating, continuous norm on $\mathcal{M}$. Assume that $\mathcal{W}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$. Then*

(i) *if $\mathcal{W}$ is simply H^∞-right invariant, i.e. $[\mathcal{W}H^\infty]_\alpha \subsetneqq \mathcal{W}$, then $\mathcal{W} = uH^\alpha$ for some unitary $u \in \mathcal{W} \cap \mathcal{M}$.*

(ii) *if $\mathcal{W}$ is simply H^∞-right invariant in H^α, i.e. $[\mathcal{W}H^\infty]_\alpha \subsetneqq \mathcal{W}$, then $\mathcal{W} = uH^\alpha$ with u an inner element (i.e., u is unitary and $u \in H^\infty$).*

Proof. It is not hard to see that part (ii) follows directly from part (i). We will only need to prove (i). From Theorem 4.7, $\mathcal{W} = \mathcal{Z} \oplus^{\text{col}} (\oplus_{i \in \mathcal{I}}^{\text{col}} u_i H^\alpha)$, where $\mathcal{Z}$ and the $u_i's$ satisfy the hypothesis of Theorem 4.7.

Since $[\mathcal{W}H^\infty]_\alpha \subsetneqq \mathcal{W}$, $\bigoplus_{i \in \mathcal{I}}^{\text{col}} i \in \mathcal{I}$ such that $u_i \neq 0$. Then $u_i^* u_i$ is a nonzero projection in $H^\infty \cap (H^\infty)^* = \mathbb{C}I$, or $u_i^* u_i = I$. This implies that u_i is a unitary element in $\mathcal{W} \cap \mathcal{M}$. From the choice of $\{u_i\}_{i \in \mathcal{I}}$, we further conclude that $\mathcal{W} = u_i H^\alpha$. ∎

Acknowledgements. The authors wish to thank the referee for carefully reading the paper, and many useful suggestions, including bringing the work of T.N. Bekjan [3] to our attention. We are also grateful to E. Nordgren for several useful discussions. The research was supported by the Fundamental Research Funds for the central universities (Grant No. GK201603009) and a Collaboration Grants from the Simons Foundation.

REFERENCES

[1] W.B. ARVESON, Analyticity in operator algebras, *Amer. J. Math.* **89**(1967), 578–642.

[2] A. BEURLING, On two problems concerning linear transformations in Hilbert space, *Acta Math.* **81**(1949), 239–255.

[3] T.N. BEKJAN, Noncommutative symmetric Hardy spaces, *Integral Equations Operator Theory* **81**(2015), 191–212.

[4] T.N. BEKJAN, Q. XU, Riesz and Szegö type factorizations for noncommutative Hardy spaces, *J. Operator Theory* **62**(2009), 215–231.

[5] D. BLECHER, L.E. LABUSCHAGNE, A Beurling theorem for noncommutative L^p, *J. Operator Theory* **59**(2008), 29–51.

[6] S. BOCHNER, Generalized conjugate and analytic functions without expansions, *Proc. Nat. Acad. Sci. U.S.A.* **45**(1959), 855–857.

[7] P. DODDS, T. DODDS, *Some Properties of Symmetric Operator Spaces*, Proc. Centre Math. Appl. Austral. Nat. Univ., vol. 29, Austral. Nat. Univ., Canberra 1992.

[8] P. DODDS, T. DODDS, B. PAGTER, Noncommutative Banach function spaces, *Math. Z.* **201**(1989), 583–597.

[9] P. DODDS, T. DODDS, B. PAGTER, Noncommutative Köthe duality, *Trans. Amer. Math. Soc.* **339**(1993), 717–750.

[10] R. EXEL, Maximal subdiagonal algebras, *Amer. J. Math.* **110**(1988), 775–782.

[11] T. FACK, H. KOSAKI, Generalized s-numbers of -measurable operators, *Pacific J. Math.* **2**(1986), 269–300.

[12] J. FANG, D. HADWIN, E. NORDGREN, J. SHEN, Tracial gauge norms on finite von Neumann algebras satisfying the weak Dixmier property, *J. Funct. Anal.* **255**(2008), 142–183.

[13] P. HALMOS, Shifts on Hilbert spaces, *J. Reine Angew. Math.* **208**(1961), 102–112.

[14] H. HELSON, *Lectures on Invariant Subspaces*, Academic Press, New York-London 1964.

[15] H. HELSON, D. LOWDENSLAGER, Prediction theory and Fourier series in several variables, *Acta Math.* **99**(1958), 165–202.

[16] K. HOFFMAN, Analytic functions and logmodular Banach algebras, *Acta Math.* **108**(1962), 271–317.

[17] M. JUNGE, D. SHERMAN, Noncommutative L^p-modules, *J. Operator Theory* **53**(2005), 3–34.

[18] R.A. KUNZE, L^p-Fourier transforms on locally compact unimodular groups, *Trans. Amer. Math. Soc.* **89**(1958), 519–540.

[19] M. MARSALLI, G. WEST, Noncommutative H^p spaces, *J. Operator Theory* **40**(1998), 339–355.

[20] C.A. MCCARTHY, C_p, *Israel J. Math.* **5**(1967), 249–271.

[21] E. NELSON, Notes on noncommutative integration, *J. Funct. Anal.* **15**(1974), 103–116.

[22] J. VON NEUMANN, Some matrix-inequalities and metrization of matric-space, *Tomsk Univ. Rev.* **1**(1937), 286–300.

[23] G. PISIER, Q. XU, *Noncommutative L^p-Spaces*, Vol. 2, Handbook on Banach Spaces, North-Holland, Amsterdam 2003.

[24] K.S. SAITO, A note on invariant subspaces for finite maximal subdiagonal algebras, *Proc. Amer. Math. Soc.* **77**(1979), 348–352.

[25] I. SEGAL, A noncommutative extension of abstract integration, *Ann. Math.* **57**(1952), 401–457.

[26] B. SIMON, *Trace Ideals and their Applications*, London Math. Soc. Lecture Note Ser., vol. 35, Cambridge Univ. Press, Cambridge-New York 1979.

[27] T.P. SRINIVASAN, Simply invariant subspaces, *Bull. Amer. Math. Soc.* **69**(1963), 706–709.

[28] T. SRINIVASAN, J.K. WANG, Weak*-Dirichlet algebras, in *Proceedings of the International Symposium on Function Algebras, Tulane University, 1965 (Chicago)*, Scott-Foresman, town 1966, pp. 216–249.

[29] M. TAKESAKI, *Theory of Operator Algebras*. I, Springer, New York-Heidelberg 1979.

[30] F. YEADON, Noncommutative L^p-spaces, *Math. Proc. Cambridge Philos. Soc.* **77**(1975), 91–102.

YANNI CHEN, SCHOOL OF MATHEMATICS AND INFORMATION SCIENCE, SHAANXI NORMAL UNIVERSITY, XI'AN, 710119, CHINA
E-mail address: yanni.chen@snnu.edu.cn

DON HADWIN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NH 03824, U.S.A.
E-mail address: don@unh.edu

JUNHAO SHEN, DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, NH 03824, U.S.A.
E-mail address: junhao.shen@unh.edu

Received July 13, 2015.