J. OPERATOR THEORY
77:1(2017), 19–37
doi: 10.7900/jot.2015jun16.2109

DISCRETIZATION OF C^*-ALGEBRAS

CHRIS HEUNEN and MANUEL L. REYES

Communicated by Marius Dădârlat

ABSTRACT. We investigate how a C^*-algebra could consist of functions on a noncommutative set: a *discretization* of a C^*-algebra A is a $*$-homomorphism $A \to M$ that factors through the canonical inclusion $C(X) \subseteq \ell^\infty(X)$ when restricted to a commutative C^*-subalgebra. Any C^*-algebra admits an injective but nonfunctorial discretization, as well as a possibly noninjective functorial discretization, where M is a C^*-algebra. Any subhomogenous C^*-algebra admits an injective functorial discretization, where M is a W*-algebra. However, any functorial discretization, where M is an AW*-algebra, must trivialize $A = B(H)$ for any infinite-dimensional Hilbert space H.

KEYWORDS: *Noncommutative topology, noncommutative set, function algebra, discrete space, profinite completion, pure state, diffuse measure, spectrum obstruction.*

MSC (2010): Primary: 46L85, 46M15; Secondary: 46L30.

1. INTRODUCTION

In operator algebra it is common practice to regard C^*-algebras as noncommutative analogues of topological spaces, and to regard W*-algebras as noncommutative analogues of measurable spaces. What would it mean to make precise how a C^*-algebra is a "noncommutative ring of continuous functions"? Several natural approaches to this question cannot faithfully represent examples as simple as matrix algebras $\mathbb{M}_n(\mathbb{C})$ [4], [7], [35], [36]. Such obstructions suggest more carefully considering what "noncommutative sets" in the foundations of noncommutative geometry should be, before attempting to topologize them.

This article explores the idea of embedding the C^*-algebra in an appropriate noncommutative algebra of "bounded functions on the noncommutative set underlying its spectrum", just like any topological space embeds in a discrete one. More precisely, consider the case of a commutative C^*-algebra A. A representation of A as operating on a Hilbert space H is equivalent to a $*$-homomorphism $A \to B(H)$. Similarly, representing A as continuous complex-valued functions on a compact Hausdorff space X can equivalently be viewed as a $*$-homomorphism

$A \to \ell^\infty(X)$ to the algebra of bounded functions on the set X. More generally, representating A as (discrete) functions on a set X can equivalently be viewed as a $*$-homomorphism to the algebra $\mathbb{C}^X$ of all functions on X.

In the spirit of noncommutative geometry, we thus seek a category $\mathbf{A}$ of $*$-algebras to play the role of the dual to the category of "noncommutative sets". This category should contain the commutative algebras $\ell^\infty(X)$ (or $\mathbb{C}^X$) as a full subcategory, dual to the category of sets. In keeping with the programme of taking commutative subalgebras seriously [6], [7], [16], [17], [18], [19], [20], [35], [36], we posit that a representation of a C^*-algebra as an algebra of functions on a noncommutative set should be an algebra homomorphism $\phi : A \to M$ for some M in $\mathbf{A}$, whose restriction to every commutative C^*-subalgebra $C \simeq C(X)$ of A factors through the natural inclusion $C(X) \subseteq \ell^\infty(X)$ via a morphism $\ell^\infty(X) \to M$ in $\mathbf{A}$. We call such a map ϕ a *discretization* of A:

$$\begin{array}{ccc} A & \xrightarrow{\quad\phi\quad} & M \\ \uparrow & & \uparrow \\ C(X) & \hookrightarrow & \ell^\infty(X) \end{array}$$

Section 2 makes this definition precise, relative to a parameterizing category $\mathbf{A}$ that can then remain unspecified. This approach to terminology gives most flexibility in investigating the open problem of finding a suitable noncommutative extension of the functor $C(X) \mapsto \ell^\infty(X)$. We show that every C^*-algebra admits a discretization into a C^*-algebra M that is injective but nonfunctorial. We also show that there is a universal candidate for a functorial discretization into the category of C^*-algebras, but it remains open whether this functorial discretization is injective for every C^*-algebra.

In Section 3 we show that a sizeable class of C^*-algebras that are "close to being commutative" does indeed have injective functorial discretizations, namely the *subhomogeneous* algebras: subalgebras of $\mathbb{M}_n(C)$ for some commutative C^*-algebra C. The discretization is achieved by profinite completion, suggesting that profinite completion for subhomogeneous algebras is a noncommutative substitute for the "underlying set functor" that sends a compact Hausdorff space to its underlying discrete space.

On the other hand, in Section 4 we show that no subcategory of W*-algebras, or even AW*-algebras, can be dual to noncommutative sets in the sense of injectively discretizing every C^*-algebra. In particular, every functor from C^*-algebras to AW*-algebras taking each C^*-algebra to a discretization must trivialize $A = B(H)$ for any infinite-dimensional Hilbert space H. A number of related examples and obstructions are discussed, including separable algebras A for which the same trivialization occurs. Viewing $*$-homomorphisms out of a C^*-algebra as representing it by functions on a noncommutative set dates back at least to Akemann [1] and Giles and Kummer [14], who took the representation to be the canonical homomorphism $A \to A^{**}$ into the bidual. They noted ([2], p. 10) that their theory was not functorial. Our obstructions amplify this observation by suggesting that W*-algebras indeed cannot play the role of "noncommutative $\ell^\infty(X)$-algebras" for C^*-algebras as large as $B(H)$.

The article concludes with a discussion in Section 5 of the implications of our obstructions, with an eye toward future work on the problem of finding injective functorial discretizations of all C^*-algebras.

2. DISCRETIZATION

We assume throughout this article that all rings, algebras, and subalgebras are unital, and that all homomorphisms preserve units. Write $\mathrm{Spec}(C)$ for the Gelfand spectrum of a commutative C^*-algebra C. Write **Cstar** for the category of C^*-algebras with $*$-homomorphisms and **Wstar** for the subcategory of W*-algebras with normal $*$-homomorphisms.

Recall that a *pro-C^*-algebra* [31], [32] is a topological $*$-algebra that is a directed (or "inverse") limit in the category of topological $*$-algebras of a system of C^*-algebras. Pro-C^*-algebras with continuous $*$-homomorphisms form a category **proCstar**. The algebra $\mathbb{C}^X$ of all complex-valued functions on a set X equipped with its topology of pointwise convergence is a pro-C^*-algebra, as it is the directed limit of the finite-dimensional C^*-algebras $\mathbb{C}^S$ for all finite subsets $S \subseteq X$.

LEMMA 2.1. *The functors $X \mapsto \ell^\infty(X)$ and $X \mapsto \mathbb{C}^X$ are contravariant equivalences between the category of sets and full subcategories of* **Wstar** *and* **proCstar**.

Proof. The proof for the functor ℓ^∞ can be found in Section 6.1 of [40]. We sketch an argument that covers both functors.

It is rather clear that each of the above assignments forms a contravariant functor into the specified category. It only remains to show that each is naturally bijective on Hom-sets. Fix $x \in X$. Let $\mathrm{ev}_x : \mathbb{C}^X \to \mathbb{C}$ denote the continuous $*$-homomorphism given by evaluation at x, whose restriction to $\ell^\infty(X)$ is normal. The maps $X \to \mathbf{proCstar}(\mathbb{C}^X, \mathbb{C})$ and $X \to \mathbf{Wstar}(\ell^\infty(X), \mathbb{C})$, given in each case by $x \mapsto \mathrm{ev}_x$, are both bijections; this follows by verifying that the kernel of either kind of morphism $\mathbb{C}^X \to \mathbb{C}$ or $\ell^\infty(X) \to \mathbb{C}$ is generated as an ideal by a characteristic function χ_S, which entails that $S = X \setminus \{x\}$ for some $x \in X$.

Now the argument that the functors in question are bijective on Hom-sets is purely formal, and can be proved by essentially the same argument as the one given in the algebraic context in Theorem 4.7 of [21]. ∎

The previous lemma leads naturally to the following notion, in keeping with the programme of taking commutative subalgebras seriously. As mentioned in the introduction, the definition is made relative to a category **A** of complex algebras that is a candidate to contain "algebras of bounded functions on noncommutative sets".

DEFINITION 2.2. Let $\mathbf{A}$ denote a category of $\mathbb{C}$-algebras containing the algebras $\ell^\infty(X)$ for any set X with their normal $*$-homomorphisms. Given a C^*-algebra A, a *(bounded)* $\mathbf{A}$*-discretization* is a homomorphism $\phi : A \to M$ whose restriction to each commutative C^*-subalgebra C of A factors through the natural inclusion $C \to \ell^\infty(\mathrm{Spec}(C))$ via a morphism $\phi_C : \ell^\infty(\mathrm{Spec}(C)) \dashrightarrow M$ in $\mathbf{A}$:

$$\begin{array}{ccc} A & \xrightarrow{\quad\phi\quad} & M \\ \uparrow & & \uparrow \phi_C \\ C & \hookrightarrow & \ell^\infty(\mathrm{Spec}(C)) \end{array}$$

We call a discretization ϕ *faithful* when it is injective and all ϕ_C can be chosen injective. We call ϕ *compatible* if the morphisms ϕ_C can be chosen such that ϕ_C factors through ϕ_D via the induced morphism $\ell^\infty(\mathrm{Spec}(C)) \to \ell^\infty(\mathrm{Spec}(D))$ for commutative C^*-subalgebras $C \subseteq D \subseteq A$.

When $\mathbf{A}$ is $\mathbf{Cstar}$ or $\mathbf{Wstar}$ above, we will speak of C^*- or W*-discretizations instead of $\mathbf{A}$-discretizations.

PROPOSITION 2.3. *Every C^*-algebra has a faithful C^*-discretization.*

Proof. Write L for the functor $C \mapsto \ell^\infty(\mathrm{Spec}(C))$. Given a finite family $S = \{C_1, \ldots, C_n\}$ of commutative C^*-subalgebras of A, write A_S for the colimit in $\mathbf{Cstar}$ of the diagram whose objects are A, the C_i, and the $L(C_i)$, along with the inclusions of each C_i into both A and $L(C_i)$. This can be constructed up to isomorphism as an iterated amalgamated free product:

$$A_S \simeq (\cdots((A *_{C_1} L(C_1)) *_{C_2} L(C_2)) \cdots) *_{C_n} L(C_n).$$

Thus the natural maps from A and the $L(C_i)$ into A_S are all embeddings; see Theorem 3.1 of [8] or Theorem 4.2 of [29].

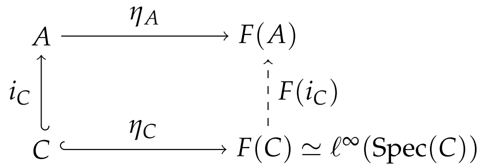
The finite families S of commutative C^*-subalgebras of A form a directed set under inclusion. Consider the directed colimit $M = \operatorname{colim}_S A_S$. By construction the mediating map $\phi : A \to M$ is a C^*-discretization. For finite subfamilies $S \subseteq T$ of commutative C^*-subalgebras of A, the induced map $A_S \to A_T$ is injective because A_T is formed from A_S by iterated pushouts. Thus the natural maps $A_S \to M$ are injective ([37], Theorem 1), from which it follows that ϕ is faithful. ∎

The discretization $\phi : A \to M$ constructed in the proof above is not compatible: for commutative C^*-subalgebras $C \subsetneq D \subseteq A$, the algebra M is obtained by gluing together distinct copies of $L(C)$ and $L(D)$ without regard to the natural inclusion $L(C) \to L(D)$. In Theorem 2.5 below we modify the construction to ensure compatibility, with the caveat that we no longer know that the discretization is even injective. This universally constructed C^*-discretization will in fact satisfy the following natural condition.

DEFINITION 2.4. Let $\mathbf{A}$ be a category as in Definition 2.2. A *functorial* $\mathbf{A}$*-discretization* is a functor $F : \mathbf{Cstar} \to \mathbf{A}$ together with natural homomorphisms

$\eta_A : A \to F(A)$ such that η_C for each commutative C^*-algebra C turns into the natural inclusion $C \to \ell^\infty(\mathrm{Spec}(C))$ by a natural isomorphism $F(C) \simeq \ell^\infty(\mathrm{Spec}(C))$.

A functorial discretization automatically gives compatible discretizations $A \to F(A)$ for every C^*-algebra A: writing $i_C : C \to A$ for the inclusion of a commutative C^*-subalgebra gives the following commutative diagram:



Compatibility follows by applying F to successive inclusions $C \subseteq D \subseteq A$.

Write **cCstar** for the full subcategory of **Cstar** of commutative C^*-algebras. Write $\mathcal{C}(A)$ for the small subcategory of **cCstar** consisting of the commutative C^*-subalgebras of a C^*-algebra A with their inclusion morphisms; we also view this as a partially ordered set.

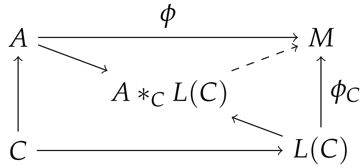
THEOREM 2.5. *The functor* $F : \mathbf{Cstar} \to \mathbf{Cstar}$ *given by*

$$F(A) = \operatorname*{colim}_{C \in \mathcal{C}(A)} A *_C \ell^\infty(\mathrm{Spec}(C))$$

equipped with the naturally induced $*$-homomorphisms $\eta_A : A \to F(A)$ is a functorial C^*-discretization. For each C^*-algebra A, the C^*-discretization $A \to F(A)$ is universal among all compatible C^*-discretizations of A. Thus F is universal among all functorial C^*-discretizations.

Proof. We follow the idea of Theorem 2.15 in [35] but with arrows reversed. Write $L = \ell^\infty \circ \mathrm{Spec} : \mathbf{cCstar} \to \mathbf{Cstar}$. The assignment $A \mapsto \mathcal{C}(A)$ is a functor to the category of small categories. Given a C^*-algebra A, the assignment $C \mapsto A *_C L(C)$ is functorial $\mathcal{C}(A) \to \mathbf{Cstar}$. So $F(A) = \operatorname*{colim}_{C \in \mathcal{C}(A)} A *_C L(C)$ defines a functor $F : \mathbf{Cstar} \to \mathbf{Cstar}$. Moreover, the induced $*$-homomorphisms $\eta_A : A \to F(A)$ are natural by construction. Finally, if A is commutative so that $A \in \mathcal{C}(A)$, then one naturally has an isomorphism $F(A) \simeq \ell^\infty(\mathrm{Spec}(A))$ that turns η_A into the inclusion $A \to \ell^\infty(\mathrm{Spec}(A))$. Thus F is a functorial C^*-discretization.

To verify universality of η_A, fix a compatible C^*-discretization $\phi : A \to M$. Each $C \in \mathcal{C}(A)$ then makes the following outer square commute:



The morphisms ϕ and ϕ_C factor uniquely through the pushout $A *_C L(C)$. Compatibility of the ϕ_C means that these uniquely determined morphisms form a cocone from the diagram of the $A *_C L(C)$ to M. Thus we obtain a $*$-homomorphism $F(A) = \operatorname{colim}_{C \in \mathcal{C}(A)} A *_C L(C) \to M$ through which ϕ factors uniquely, as desired.

Finally, if (F', η') is any functorial C^*-discretization, then by the local universality of the previous paragraph the natural morphisms $\eta'_A : A \to F'(A)$ factor uniquely through $\eta_A : A \to F(A)$, from which it readily follows that F' factors through a unique natural transformation $F \Rightarrow F'$ whose composite with η is η'. ∎

Whereas the "incompatible" discretization of Proposition 2.3 is faithful, it is not clear whether the natural C^*-discretizations $A \to F(A)$ of the last theorem are faithful or even injective. Abstract nonsense alone does not answer this question.

QUESTION 2.6. *Is the universal functorial C^*-discretization $\eta_A : A \to F(A)$ of Theorem* 2.5 *injective or faithful for every C^*-algebra A? Equivalently, does every C^*-algebra have an injective or faithful compatible C^*-discretization?*

REMARK 2.7. The definitions and results above carefully used the Gelfand spectrum $\mathrm{Spec}(C)$ of a commutative C^*-algebra C. Henceforth we loosen notation, and write $C = C(X)$ for an arbitrary commutative C^*-algebra, and $C \simeq C(X)$ for an arbitrary commutative C^*-subalgebra of a C^*-algebra A.

Recall from Lemma 2.1 that sets may also be encoded algebraically through the algebra of discrete (possibly unbounded) functions as $X \mapsto \mathbb{C}^X$. The rest of the paper will also discuss "unbounded" discretizations.

DEFINITION 2.8. Let $\mathbf{A}$ denote a category of $\mathbb{C}$-algebras containing the algebras $\mathbb{C}^X$ for any set X with the $*$-homomorphisms that are continuous with respect to the topology of pointwise convergence. Given a C^*-algebra A, an *unbounded* $\mathbf{A}$*-discretization* is a homomorphism $\phi : A \to M$ whose restriction to each commutative C^*-subalgebra $C \simeq C(X)$ of A factors through the inclusion $C(X) \to \mathbb{C}^X$ via a morphism $\phi_C : \mathbb{C}^X \dashrightarrow M$ in $\mathbf{A}$:

$$\begin{array}{ccc} A & \xrightarrow{\quad \phi \quad} & M \\ \uparrow & & \uparrow \phi_C \\ C \simeq C(X) & \hookrightarrow & \mathbb{C}^X \end{array}$$

Define *injective*, *faithful*, and *functorial* unbounded discretizations analogous to the bounded case. For $\mathbf{A} = \mathbf{proCstar}$ we refer to *unbounded pro-C^*-discretizations*.

3. FUNCTORIAL DISCRETIZATIONS THROUGH PROFINITE COMPLETION

For a compact Hausdorff space X, the natural inclusion $C(X) \to \ell^\infty(X)$ is a W*-discretization of the corresponding commutative C^*-algebra. Also, if

A is a finite-dimensional C^*-algebra, then the identity map $A \to A$ is a W*-discretization. This section provides a common generalization of these two examples: Theorems 3.3 and 3.5 below show that the profinite completion of a C^*-algebra is a functorial discretization that is faithful for a large class of algebras.

For a C^*-algebra A, let $\mathcal{F}(A)$ denote the family of closed ideals I of A for which A/I is finite-dimensional. Then $\mathcal{F}(A)$ is closed under finite intersections, as is readily verified by embedding $A/(I \cap J) \to A/I \oplus A/J$ for ideals $I, J \in \mathcal{F}(A)$. Thus the finite-dimensional C^*-algebras A/I for $I \in \mathcal{F}(A)$ form an inversely directed system. We may take the directed limit of this system either in the category **Cstar** to obtain a C^*-algebra, or in the category of topological algebras to obtain a pro-C^*-algebra. We denote these two directed limits by

$$P_{\mathrm{b}}(A) = \lim_{I \in \mathcal{F}(A)} A/I \quad \text{computed in } \mathbf{Cstar},$$

$$P_{\mathrm{u}}(A) = \lim_{I \in \mathcal{F}(A)} A/I \quad \text{computed in } \mathbf{proCstar}.$$

Given a $*$-homomorphism $f : A \to B$ and $J \in \mathcal{F}(B)$, the induced embedding $A/f^{-1}(J) \hookrightarrow B/J$ ensures that $f^{-1}(J) \in \mathcal{F}(A)$. Universality provides a composite $*$-homomorphism

$$P_{\mathrm{b}}(A) = \lim_{I \in \mathcal{F}(A)} A/I \to \lim_{J \in \mathcal{F}(B)} A/f^{-1}(J) \to \lim_{J \in \mathcal{F}(B)} B/J = P_{\mathrm{b}}(B)$$

making the assignments P_{b} and P_{u} functorial.

Notice that the diagram over which the limit $P_{\mathrm{b}}(A)$ is computed consists of W*-algebras with normal $*$-homomorphisms. The subcategory **Wstar** of **Cstar** is closed under limits since the forgetful functor **Wstar** $\to$ **Cstar** is right adjoint to the universal enveloping W*-algebra functor [12]. Thus $P_{\mathrm{b}}(A)$ is a W*-algebra, and for $f : A \to B$ in **Cstar** the induced morphism $P_{\mathrm{b}}(f) : P_{\mathrm{b}}(A) \to P_{\mathrm{b}}(B)$ is a normal $*$-homomorphism. Thus P_{b} is a functor **Cstar** $\to$ **Wstar**.

Each of the two functors P_{b} and P_{u} is a kind of profinite completion [13].

DEFINITION 3.1. We call P_{b} : **Cstar** $\to$ **Wstar** the *bounded profinite completion*, and P_{u} : **Cstar** $\to$ **proCstar** the *unbounded profinite completion*.

Let $b(P) \subseteq P$ denote the set of *bounded* elements of a pro-C^*-algebra P: those elements whose spectrum forms a bounded subset of $\mathbb{C}$. This is a C^*-algebra that lies densely in P ([31], Proposition 1.11).

PROPOSITION 3.2. *If A is a C^*-algebra, then $P_{\mathrm{b}}(A) \simeq b(P_{\mathrm{u}}(A))$: the W*-algebra $P_{\mathrm{b}}(A)$ is $*$-isomorphic to the algebra of bounded elements of the pro-C^*-algebra $P_{\mathrm{u}}(A)$.*

Proof. Suppose that a C^*-algebra B forms a cone over the diagram of finite-dimensional algebras A/I for $I \in \mathcal{F}(A)$. Then B also forms a cone over this diagram in the category **proCstar**, and this cone factors uniquely through a morphism $B \to P_{\mathrm{u}}(A)$. But the image of this morphism lands in the C^*-algebra $b(P_{\mathrm{u}}(A))$ ([31], Corollary 1.13). Thus $b(P_{\mathrm{u}}(A))$ satisfies the universal property of

$\lim_{I \in \mathcal{F}(A)} A/I$ computed in **Cstar**. It follows that the map $P_b(A) \to P_u(A)$ induced by the universal property of the latter is an isomorphism onto $b(P_u(A))$. ∎

Henceforth we identify $P_b(A)$ with the dense subalgebra $b(P_u(A)) \subseteq P_u(A)$. Invoking the universal property of limits once again, for each C^*-algebra A there is a $*$-homomorphism $\eta_A : A \to P_b(A) \subseteq P_u(A)$ that is natural in A. This map makes P_b and P_u into functorial discretizations.

THEOREM 3.3. *Bounded profinite completion is a functorial* W*-*discretization. Unbounded profinite completion is an unbounded functorial pro-*C^**-discretization.*

Proof. For a commutative C^*-algebra $C = C(X)$, each $I \in \mathcal{F}(C)$ is of the form $I = I_S = \{f \in C : f(S) = 0\}$ for some finite subset $S \subseteq X$. The surjection $C \twoheadrightarrow C/I \simeq C(S)$ is Gelfand dual to the inclusion $S \hookrightarrow X$. Thus

$$P_b(C(X)) = \lim_{S \subseteq X} C(S) \simeq \ell^\infty(X), \quad P_u(C(X)) = \lim_{S \subseteq X} C(S) \simeq \mathbb{C}^X,$$

and under these isomorphisms the natural map $\eta_C : C \to P_b(C) \subseteq P_u(C)$ corresponds to the natural inclusion $C(X) \hookrightarrow \ell^\infty(X) \subseteq \mathbb{C}^X$.

It remains to verify that these functors behave as expected on morphisms. Fix a $*$-homomorphism $f : B = C(Y) \to C = C(X)$, which is Gelfand dual to a continuous function $\widehat{f} : X \to Y$. For any finite set $S \subseteq X$, the restriction of $\widehat{f}$ to $S \to \widehat{f}(S)$ is Gelfand dual to $C(\widehat{f}(S)) \simeq B/f^{-1}(I_S) \to C/I_S \simeq C(S)$. Taking the directed limit in **Wstar** over finite subsets $S \subseteq X$, we see that the induced map $P_b(f) : P_b(B) \to P_b(C)$ corresponds to $\ell^\infty(\widehat{f})$ under the isomorphisms $P_b(B) \simeq \ell^\infty(Y)$ and $P_b(C) \simeq \ell^\infty(X)$. This completes the proof for P_b; the analogous argument in **proCstar** also holds for P_u. ∎

EXAMPLE 3.4. Let $A = \mathbb{M}_n(C(X))$ for a compact Hausdorff space X. Then $P_b(A) = \mathbb{M}_n(\ell^\infty(X))$ and $P_u(A) = \mathbb{M}_n(\mathbb{C}^X)$.

Proof. Write $C = C(X)$, and recall that every closed ideal $J \subseteq \mathbb{M}_n(C)$ is of the form $\mathbb{M}_n(I)$ for some closed ideal $I \subseteq C$ ([27], Corollary 17.8). Such an ideal J has finite codimension in A if and only if I has finite codimension in C. Thus

$$P_b(A) = \lim_{J \in \mathcal{F}(A)} A/J = \lim_{I \in \mathcal{F}(C)} \mathbb{M}_n(C)/\mathbb{M}_n(I) \simeq \lim_{I \in \mathcal{F}(C)} \mathbb{M}_n(C/I) \simeq \mathbb{M}_n(\ell^\infty(X))$$

and similarly $P_u(A) \simeq \mathbb{M}_n(\mathbb{C}^X)$. ∎

Let us emphasize that, even though the profinite completion functors yield discretizations of *all* C^*-algebras, there are many C^*-algebras A for which $P_b(A) = P_u(A) = 0$ is trivial. Indeed, if A is any C^*-algebra with no finite-dimensional representations, then by construction of the profinite completions we necessarily have $P_b(A) = P_u(A) = 0$. Example include: the algebra $B(H)$ of bounded operators on an infinite-dimensional Hilbert space H; the CCR algebra [30]; the Calkin algebra $B(H)/K(H)$; and the (separable) Cuntz algebra $\mathcal{O}_n$ generated by

$n \geqslant 2$ isometries [11]. Thus it is interesting to see which algebras have injective or faithful discretizations to their profinite completion. This is addressed in the next theorem.

Recall that a C^*-algebra A is *residually finite-dimensional* when it has a faithful family of finite-dimensional representations. Similarly, A is *subhomogeneous* when there is an integer $n \geqslant 1$ such that every irreducible representation of A has dimension at most n; this is equivalent ([9], Proposition IV.1.4.3) to A being isomorphic to a C^*-subalgebra of $\mathbb{M}_k(C)$ for a commutative C^*-algebra C and an integer $k \geqslant 1$. For a point x in a set X, we let $\delta_x = \chi_{\{x\}} \in \ell^\infty(X) \subseteq \mathbb{C}^X$ denote the indicator function of the singleton $\{x\}$.

THEOREM 3.5. *For a C^*-algebra A, the functorial discretizations P_{b} and P_{u} are:*
(i) *injective if and only if A is residually finite-dimensional;*
(ii) *faithful if A is subhomogeneous.*

Proof. (i) If A is residually finite-dimensional, every nonzero $a \in A$ allows $I_a \in \mathcal{F}(A)$ with $a \notin I_a$ (meaning that a has nonzero image in A/I_a). Thus a is not in the kernel of

$$\eta_A : A \to \lim_{I \in \mathcal{F}(A)} A/I = P_{\mathrm{b}}(A) \subseteq P_{\mathrm{u}}(A).$$

Hence η_A is injective. (See also Lemma 1.10 of [13].) The converse follows directly from the definition.

(ii) Consider a commutative C^*-subalgebra $C(X) \subseteq A$, and $x \in X$. Because the homomorphisms

$$\ell^\infty(X) \simeq P_{\mathrm{b}}(C(X)) \to P_{\mathrm{b}}(A) \quad \text{and} \quad \mathbb{C}^X \simeq P_{\mathrm{u}}(C(X)) \to P_{\mathrm{u}}(A)$$

are respectively normal and continuous, it suffices to show that $\delta_x \in \ell^\infty(X) \subseteq \mathbb{C}^X$ is not in their kernel. Indeed, the kernel I of either morphism is an ideal generated by a characteristic function χ_S for some $S \subseteq X$, so that I contains exactly those δ_x with $x \in S$. Hence if all $\delta_x \notin I$, then $S = \varnothing$ and therefore $I = 0$.

Evaluation at x is a pure state on $C(X)$, which extends ([9], II.6.3.2) to a pure state ρ_x on A. Because A is subhomogeneous, the GNS construction applied to ρ_x yields a finite-dimensional representation $\pi : A \to B(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ for some integer $n \geqslant 1$, with cyclic vector $v_x \in \mathbb{C}^n$. Let $I \in \mathcal{F}(A)$ denote the kernel of π. The induced $*$-homomorphism $\psi : \ell^\infty(X) \to A/I \hookrightarrow \mathbb{M}_n(\mathbb{C})$ has image isomorphic to $C(S)$ for some finite subset $S \subseteq X$; in fact, this set S is characterized as those pure states on $C(X)$ that are induced by vector states of the representation π. Now $\pi(f)v_x = f(x)v_x$ for $f \in C(X)$ by construction of π. Thus $x \in S$, so that δ_x is not in the kernel of ψ. It follows that δ_x has nonzero image in each of the limit algebras $P_{\mathrm{b}}(A)$ and $P_{\mathrm{u}}(A)$, as desired. ∎

REMARK 3.6. For C^*-algebras A that are residually finite-dimensional but not subhomogeneous, the natural map $A \to P_{\mathrm{b}}(A)$ is technically an injective discretization, but it does not satisfy all desiderata for an "algebra of bounded functions on the noncommutative underlying set" of A. Consider the C^*-sum $A =$

$\overline{\bigoplus_{k=1}^{\infty}} \mathbb{M}_k(\mathbb{C})$. Let $I_n \subseteq A$ denote the kernel of the projection $A \twoheadrightarrow \mathbb{M}_1(\mathbb{C}) \oplus \cdots \oplus \mathbb{M}_n(\mathbb{C})$ onto the first n components. By an argument similar to that in Lemma 7.5 of [23], the kernel of any finite-dimensional representation of A must contain some I_n. It follows that the I_n form a cofinal chain in $\mathcal{F}(A)$, so that the profinite completion

$$A \to P_{\mathrm{b}}(A) \simeq \lim_{n\to\infty} A/I_n \simeq A$$

is an isomorphism. But this is far from the behavior one would expect when comparing to the commutative example $C = \overline{\bigoplus_{k=1}^{\infty}} \mathbb{C} \simeq \ell^\infty(\mathbb{N}) \simeq C(\beta\mathbb{N})$; the profinite completion $C \to P_{\mathrm{b}}(C)$ corresponds under this isomorphism to the embedding $C \simeq C(\beta\mathbb{N}) \to \ell^\infty(\beta\mathbb{N})$, indicating that C is "far below" $P_{\mathrm{b}}(C)$ as a subalgebra.

Almost all faithful discretizations of C^*-algebras we know are supplied by Theorem 3.5 above. We conclude this section by describing another significant example of a faithful compatible discretization that is not of this form.

EXAMPLE 3.7. For an infinite-dimensional Hilbert space H, consider the C^*-subalgebra $A = \mathbb{C} \oplus K(H)$ of $B(H)$ generated by the identity and the compact operators. The embedding $A \hookrightarrow B(H)$ is a faithful compatible W*-discretization.

Proof. Any commuting set of self-adjoint compact operators on H has an orthonormal basis of H of simultaneous eigenvectors, so the same remains true for commuting sets of self-adjoint operators in A. Let $C \simeq C(X) \subseteq A$ be a commutative C^*-subalgebra. For $x \in X$ let $p_x \in B(H)$ denote the projection onto the simultaneous eigenspace $\{v \in H : f \cdot v = f(x)v \text{ for all } f \in C\}$. Now each $p_x \neq 0$ and $\sum p_x = 1$ in $B(H)$. It follows that the W*-subalgebra W_C generated by the p_x is isomorphic to $\ell^\infty(X)$, and the fact that $fp_x = p_xf = f(x) \cdot p_x$ for all $f \in C$ guarantees that the natural inclusion $C \subseteq W_C$ corresponds under this isomorphism to the natural inclusion $C(X) \subseteq \ell^\infty(X)$. Thus the discretization is faithful.

Compatibility for commutative C^*-subalgebras $C \subseteq D \subseteq A$ is readily established from the simple observation that a simultaneous eigenspace for D restricts to a simultaneous eigenspace for C. ∎

The example above is a faithful compatible W*-discretization for which we do not know of any extension to an unbounded discretization.

4. OBSTRUCTIONS TO DISCRETIZATIONS WITH MANY PROJECTIONS

Can the bounded faithful functorial W*-discretization for subhomogeneous C^*-algebras of Theorem 3.5 be extended to general C^*-algebras through some method other than profinite completion? Perhaps surprisingly, we prove in this section that the answer is no: any W*-discretization of the algebra $B(H)$ for an infinite-dimensional Hilbert space H is necessarily zero. In fact, the obstruction is even more serious: if we replace the category of W*-algebras ("noncommutative measurable spaces") with the category of AW*-algebras [5], [22] ("noncommutative complete Boolean algebras" [20]), the obstruction persists.

The next definition is crucial to our obstructions, and relies on the following notions from measure theory. An *atom* of a measure space (X, μ) is a measurable subset $U \subseteq X$ with $\mu(U) > 0$, such that $\mu(V) < \mu(U)$ implies $\mu(V) = 0$ for any measurable subset $V \subseteq U$. An atom of a regular Borel measure on a locally compact Hausdorff space is necessarily a singleton ([24], 2.IV). A measure is *diffuse* if it has no atoms. We will say that a positive linear functional $\psi : C(X) \to \mathbb{C}$ of a commutative C^*-algebra, given by $\psi(f) = \int f \, \mathrm{d}\mu$ for a regular Borel measure μ on X, is *diffuse* when μ is diffuse.

DEFINITION 4.1. Let A be a C^*-algebra. A pair of commutative C^*-subalgebras C and D is *relatively diffuse* when every extension of a pure state of D to a state of A restricts to a diffuse state on C.

EXAMPLE 4.2. Consider the separable Hilbert space $H = L^2[0,1]$, and the C^*-algebra $A = B(H)$. Write D for the discrete maximal abelian W*-subalgebra generated by the projections q_n onto the Fourier basis vectors $e_n = \exp(2\pi i n -)$ for $n \in \mathbb{Z}$, and C for the continuous maximal abelian W*-subalgebra $L^\infty[0,1]$. Then C and D are relatively diffuse.

Proof. There is a canonical conditional expectation $E : A \to D$ that sends $f \in A$ to its diagonal part $\sum q_n f q_n$. For $f \in C$ then $E(f) = \int_0^1 f(t) \, \mathrm{d}t$ because

$$\langle f e_n, e_n \rangle = \int_0^1 f(t) \cdot \mathrm{e}^{2\pi i n t} \cdot \overline{\mathrm{e}}^{\,2\pi i n t} \, \mathrm{d}t = \int_0^1 f(t) \, \mathrm{d}t.$$

Because ψ is a pure state of D now $\psi = \psi \circ E$ by the solution of the Kadison–Singer problem [28]. Hence $\psi(f) = \psi(E(f)) = \psi\Big(\int_0^1 f(t) \, \mathrm{d}t\Big) = \int_0^1 f(t) \, \mathrm{d}t$. ∎

EXAMPLE 4.3. For $H = L^2[0,1]$, consider any separable C^*-subalgebra $C \subseteq L^\infty[0,1] \subseteq B(H)$ for which the state $f \mapsto \int_0^1 f(t) \, \mathrm{d}t$ is diffuse (such as $C = C[0,1]$). Then there is a separable C^*-subalgebra $A \subseteq B(H)$ containing C and a commutative C^*-subalgebra D generated by projections, with C and D relatively diffuse.

Proof. Let e_n and E be as in Example 4.2. Because C is separable, we can fix a sequence $\{f_i\}_{i=1}^\infty$ of elements whose linear span is dense in C. For each f_i and for each integer $j \geqslant 1$, the positive solution to the paving conjecture [28] ensures that there is a finite set of projections $p_k = p_k^{(i,j)}$ in the discrete maximal abelian subalgebra of $B(H)$ relative to the Fourier basis e_n with $\sum p_k = 1$ and $\|p_k(f_i - E(f_i))p_k\| \leqslant 1/j$. Let D be the commutative C^*-subalgebra of $B(H)$

generated by the $p_k^{(i,j)}$ for all i, j, and k. Let A be the C^*-subalgebra of $B(H)$ generated by C and D; as both C and D are countably generated, the same is true of A, whence A is separable. An argument familiar in the literature on the Kadison–Singer problem (as in p. 310 of [3]) shows that any extension of a pure state ψ_0 on D to a state ψ on A satisfies $\psi(f) = \psi_0(E(f))$ for all $f \in C$. The same computation as in Example 4.2 shows that $\psi(f) = \int_0^1 f(t)\,\mathrm{d}t$, which is diffuse on C by hypothesis. ∎

REMARK 4.4. It is possible to modify Examples 4.2 and 4.3 so that the conclusions can be reached without using the full force of Kadison–Singer. In either case, identify the algebra $C = C(\mathbb{T})$ of continuous functions on the unit circle with the subalgebra $\{f : f(0) = f(1)\} \subseteq C[0,1] \subseteq B(H)$. The algebra of Fourier polynomials — or more generally, the Wiener algebra $A(\mathbb{T})$ — is a dense subalgebra of C and lies in the algebra $M_0 \subseteq B(H)$ of operators that are l_1-bounded in the sense of Tanbay [38] with respect to the Fourier basis $\{e_n : n \in \mathbb{Z}\}$. Thus C lies in the norm closure M of M_0, and it was shown in [38] (without the full force of Kadison–Singer) that every element of M is compressible (that is, the operator $f - E(f)$ satisfies paving with respect to the basis e_n for any $f \in M$). The computations in either example given above may now proceed in the same manner.

The relatively diffuse subalgebras C and D in the examples above had pure states of D inducing a *unique* diffuse state on C. We thank the referee for the following example which allows for possibly non-unique extensions.

EXAMPLE 4.5. Let A and D be as in Example 4.2, but consider the commutative C^*-subalgebra of A generated by the bilateral shift $e_n \mapsto e_{n+1}$, and let C be its bicommutant. Then C and D are relatively diffuse.

Proof. Write C_0 for the C^*-subalgebra generated by the shift $u : H \to H$; its Gelfand spectrum is the unit circle $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$ ([15], Problem 84). Let $f_n \in C(\mathbb{T})$ be a decreasing sequence converging to the characteristic function $\delta_\lambda = \chi_{\{\lambda\}}$ of some $\lambda \in \mathbb{T}$. Then, since the bounded sequence (f_n) converges pointwise to δ_λ, the sequence $(f_n(u))$ in C_0 converges strongly to the projection $\delta_\lambda(u)$ in C. But $\lim_n \langle f_n(u)(e_0), e_0\rangle = \langle \delta_\lambda(e_0), e_0\rangle$ vanishes because u has no eigenvectors. Hence $\|E(f_n(u))\| = \|\langle f_n(u)(e_0), e_0\rangle 1_H\| \to 0$. Thus a state ψ of A that is pure on D satisfies $\psi(f_n) = \psi(E(f_n)) \to 0$, and is therefore diffuse on C. ∎

Relatively diffuse pairs of commutative C^*-subalgebras are inherited along $*$-homomorphisms, as follows.

LEMMA 4.6. *Let $\phi : A \to B$ be a morphism in* **Cstar**. *If two commutative C^*-subalgebras $C, D \subseteq A$ are relatively diffuse, then so are $\phi(C), \phi(D) \subseteq B$.*

Proof. Fix a pure state ψ_0 on $\phi(D)$, and let ψ be any extension to a state on B. Then $\psi \circ \phi$ is a state on A that extends $\psi_0 \circ \phi$ from D; observe that the latter is a pure state on D as it is a composition of a $*$-homomorphism with a pure state. By hypothesis, the restriction of $\psi \circ \phi$ to C is diffuse. As the restriction of ϕ to $C \twoheadrightarrow \phi(C)$ is Gelfand dual to the inclusion $\operatorname{Spec}(\phi(C)) \hookrightarrow \operatorname{Spec}(C)$ of a closed subspace, the measure on $\operatorname{Spec}(\phi(C))$ corresponding to $\psi|_{\phi(C)}$ is the restriction of the measure on $\operatorname{Spec}(C)$ corresponding to $\psi_0|_C$, which is diffuse. It follows that the restriction of ψ to C' is diffuse. ∎

The major result below and its many corollaries will refer to commutative diagrams of the following kind, where A is a C^*-algebra with relatively diffuse commutative C^*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$:

$$\begin{array}{ccc} C \simeq C(X) & \hookrightarrow & \ell^\infty(X) \\ \downarrow & & \downarrow \phi_C \\ A & \xrightarrow{\ \phi\ } & M \\ \uparrow & & \uparrow \phi_D \\ D \simeq C(Y) & \hookrightarrow & \ell^\infty(Y) \end{array} \tag{4.1}$$

THEOREM 4.7. *If a C^*-algebra A has relatively diffuse commutative C^*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$, and if there is a C^*-algebra M with $*$-homomorphisms ϕ, ϕ_C and ϕ_D making the diagram* (4.1) *commute, then for any $x \in X$ and $y \in Y$:*

$$\phi_C(\delta_x)\phi_D(\delta_y) = 0.$$

Proof. Let $x \in X$ and $y \in Y$, and write $p = \phi_C(\delta_x)$ and $q = \phi_D(\delta_y)$. Fix any state σ on the C^*-algebra qMq, and let ψ denote the state on A given by $\psi(a) = \sigma(q\phi(a)q)$. For $g \in D$, observe $\psi(g) = \sigma(\phi_D(\delta_y g \delta_y)) = \sigma(\phi_D(g(y)\delta_y)) = g(y)\sigma(q) = g(y)$, so that ψ restricts to a pure state on D. By hypothesis, the restriction of ψ to C is of the form $f \mapsto \int f \, d\mu$ for some diffuse Radon measure μ on X. Thus for each integer $n \geqslant 1$ we may find an open neighborhood U_n of x with $\mu(U_n) \leqslant 1/n$. Urysohn's lemma provides a continuous function $f_n : X \to [0,1]$ that vanishes on $X \setminus U_n$ and satisfies $f_n(x) = 1$. Since $\delta_x \leqslant f_n$ in $\ell^\infty(X)$ we have $p = \phi_C(\delta_x) \leqslant \phi_C(f_n)$. Positivity of $b \mapsto \sigma(qbq)$ yields

$$\sigma(qpq) \leqslant \sigma(q\phi_C(f_n)q) = \psi(f_n) = \int f_n \, d\mu \leqslant \mu(U_n) \leqslant \frac{1}{n}.$$

As $n \to \infty$ we find that $\sigma(pqp) = 0$ for all states σ on qMq, making $qpq = 0$. It follows that $\|qp\|^2 = \|qpq\| = 0$ and thus $pq = (qp)^* = 0$. ∎

Write **AWstar** for the category of AW*-algebras with $*$-homomorphisms whose restriction to the projection lattices preserve arbitrary least upper bounds (see Lemma 2.2 of [20] for further characterizations of these morphisms); **Wstar** is a full subcategory. We call **AWstar**-discretizations *AW*-discretizations*.

COROLLARY 4.8. *If a C^*-algebra A has two relatively diffuse commutative C^*-subalgebras, then any* AW*-*discretization $\phi : A \to M$ satisfies $M = 0$. Consequently, every functorial* AW*-*discretization $F : \mathbf{Cstar} \to \mathbf{AWstar}$ has $F(A) = 0$ for such A.*

Proof. Let $C \simeq C(X)$ and $D \simeq C(Y)$ be the relatively diffuse commutative C^*-subalgebras, and let $\phi_C : \ell^\infty(X) \to M$ and $\phi_D : \ell^\infty(Y) \to M$ be the discretizing morphisms as in Definition 2.2, yielding a commuting diagram (4.1). For $x \in X$ and $y \in Y$, set $p_x = \phi_C(\delta_x)$ and $q_y = \phi_D(\delta_y)$. As $\sum \delta_x = 1_C$ and $\sum \delta_y = 1_D$ (in the sense of least upper bounds of orthogonal projections), and as ϕ_C and ϕ_D are morphisms in **AWstar**, we have $\sum p_x = 1 = \sum q_y$ in M. By Theorem 4.7, each p_x is orthogonal to all of the q_y, so that p_x is orthogonal to $\sum q_y = 1 \in M$. Therefore $p_x = 0$ for all $x \in X$, whence $1 = \sum p_x = 0$ in M and $M = 0$. ∎

EXAMPLE 4.9. If there is a morphism $B(H) \to A$ in **Cstar** for some infinite-dimensional Hilbert space, then A has no nontrivial AW*-discretization.

Proof. First note that H as above is unitarily isomorphic to $L^2[0,1] \otimes H$, so $a \mapsto a \otimes 1$ is a $*$-homomorphism $B(L^2[0,1]) \to B(L^2[0,1]) \otimes B(H) \simeq B(H)$. Example 4.2 along with Lemma 4.6 show that A contains a relatively diffuse commutative C^*-subalgebras, so that Corollary 4.8 applies. ∎

In particular, by the last example the Calkin algebra $A = B(H)/K(H)$ has no nontrivial AW*-discretization for $H = L^2[0,1]$.

Theorem 4.7 has the following consequence for purely ring-theoretic discretizations, with much tamer conclusion than those of Corollaries 4.8 or 4.11.

COROLLARY 4.10. *If a C^*-algebra A has relatively diffuse C^*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$, and if there is a commutative diagram of the form* (4.1) *where M is a ring and ϕ, ϕ_C, ϕ_D are ring homomorphisms, then for every $x \in X$ and $y \in Y$:*

$$\phi_C(\delta_x)\phi_D(\delta_y) = \phi_D(\delta_y)\phi_C(\delta_x) = 0.$$

Proof. Invoking Theorem 4.7 in the case where

$$M_1 = (A *_{C(X)} \ell^\infty(X)) *_{C(Y)} \ell^\infty(Y)$$

is the colimit in **Cstar** of the diagram (4.1) with M deleted, we obtain that the images of δ_x and δ_y are orthogonal in M_1. Now let $R \circledast_S T$ denote the amalgamated free product of rings (which coincides with the amalgamated free product of $\mathbb{C}$-algebras when S is a unital subalgebra of algebras R and T), and let

$$M_0 = (A \circledast_{C(X)} \ell^\infty(X)) \circledast_{C(Y)} \ell^\infty(Y)$$

be the colimit in the category of rings of the diagram (4.1) with M deleted. There is a natural map $M_0 \to M_1$ induced by the universal property of M_0. It is a folk result that this is an embedding [10], [34]. Thus the images of δ_x and δ_y in M_0 are already orthogonal. But the morphisms ϕ, ϕ_C, and ϕ_D of (4.1) factor universally through M_0, so the images of δ_x and δ_y in M are orthogonal. ∎

We conclude this section with an obstruction for unbounded discretizations into topological algebras. Write **TAlg** for the category of Hausdorff topological $\mathbb{C}$-algebras with continuous homomorphisms. Recall ([39], Chapter 10) that a family $(a_i)_{i\in I}$ of elements in a Hausdorff topological ring R is *summable* if the net (a_J) indexed by finite subsets $J \subseteq I$ converges, where $a_J = \sum_{j\in J} a_j$; in that case we write $\sum a_i$ for the limit.

COROLLARY 4.11. *Let A be a C^*-algebra with relatively diffuse C^*-subalgebras $C \simeq C(X)$ and $D \simeq C(Y)$. Then every unbounded* **TAlg***-discretization of A is zero. More precisely, if there is a commutative diagram*

$$\begin{array}{ccc} C \simeq C(X) & \hookrightarrow & \mathbb{C}^X \\ \downarrow & & \downarrow \phi_C \\ A & \xrightarrow{\phi} & M \\ \uparrow & & \uparrow \phi_D \\ D \simeq C(Y) & \hookrightarrow & \mathbb{C}^Y \end{array}$$

where M is a Hausdorff topological ring, ϕ_C and ϕ_D are continuous homomorphisms, and ϕ is a homomorphism, then $M = 0$.

Proof. It suffices to prove the second, more general claim. Because the natural embedding $C(X) \hookrightarrow \mathbb{C}^X$ has image in the subring $\ell^\infty(X) \subseteq \mathbb{C}^X$ and similarly for $C(Y)$, we may apply Corollary 4.10 to conclude that the idempotents $p_x = \phi_C(\delta_x)$ and $q_y = \phi_D(\delta_y)$ satisfy $p_x q_y = 0$ for all $x \in X$ and $y \in Y$.

The orthogonal set of idempotents $\{\delta_x : x \in X\}$ is summable with $\sum \delta_x = 1$ in $\mathbb{C}^X$, so the family of images $(p_x)_{x\in X}$ under the continuous homomorphism ϕ_C is also summable in M with $\sum p_x = 1$. Similarly, we have $(q_y)_{y\in Y}$ summable in M with $\sum q_y = 1$.

Now consider the net $(p_I q_J)$ indexed by the directed set of all "rectangular" subsets $I \times J \subseteq X \times Y$ with both $I \subseteq X$ and $J \subseteq Y$ finite. As both (p_I) and (q_J) converge to 1, we have $p_I q_J \to 1^2 = 1$. But each $p_I q_J = \sum_I \sum_J p_x q_y = 0$, so we have $1 = \lim p_I q_J = 0$. Thus $M = 0$. ∎

Just as in Example 4.9, if there is a morphism $B(H) \to A$ in **Cstar** with H an infinite-dimensional Hilbert space, then every unbounded **TAlg**-discretization of A is trivial.

REMARK 4.12. Similar to the C^*-discretization in Proposition 2.3, one could construct a pro-C^*-discretization by replacing the pushouts $A *_C \ell^\infty(\mathrm{Spec}(C))$ in **Cstar** with the pushouts $A *_C \mathbb{C}^{\mathrm{Spec}(C)}$ in **proCstar**. However, the previous corollary shows that this construction must trivialize for algebras A that have relatively diffuse commutative C^*-subalgebras.

We close with one further example of a separable algebra having no injective W*-discretizations. We only sketch its proof, as the complete argument would require us to modify several results above to account for possibly nonunital commutative subalgebras, a technicality that we have avoided for the sake of readability.

EXAMPLE 4.13. Let $H = L^2[0,1]$ and $C = C[0,1] \subseteq L^\infty[0,1] \subseteq B(H)$. Then $A = C + K(H)$ is a separable C^*-algebra of type I for which every AW*-discretization and every unbounded **TAlg**-discretizations has nonzero kernel. (It does, however, have nonzero non-injective such discretizations that factor through the commutative C^*-algebra $A/K(H)$.)

Proof. Let e_n, and q_n be as in Example 4.2. Within $B(H)$, write $C_0(\mathbb{Z}) \simeq D \subseteq K(H)$ for the nonunital commutative C^*-subalgebra generated by the q_n. If one alters Definition 4.1 to allow for possibly nonunital C^*-subalgebras, then C and D are relatively diffuse. Indeed, each pure state ψ_0 on D is supported on some projection $p = q_n$, and every extension of ψ_0 to a state ψ on A satisfies $\psi(f) = \psi(pfp) = \left(\int_0^1 f\,\mathrm{d}t\right)\psi(p) = \int_0^1 f\,\mathrm{d}t$ for all $f \in C[0,1]$. A suitable modification of Theorem 4.7 holds for such C and D, with hardly a change to the proof.

Now if $\phi : A \to M$ is an AW*-discretization or an unbounded **TAlg**-discretization, then we claim that $K(H) \subseteq \ker(\phi)$. Indeed, the same method of proof of Corollaries 4.8 and 4.11 shows that D is contained in $\ker(\phi)$ (noting that C is still a unital subalgebra), and $K(H)$ is the ideal generated by D. ∎

5. CONCLUSION

In contrast to the obstructions [35], [7], [4], based on the Kochen–Specker theorem [25] from quantum physics, the fact that profinite completion faithfully discretizes all finite-dimensional C^*-algebras shows that the results in Section 4 are truly infinite-dimensional obstructions and are therefore independent of the Kochen–Specker theorem.

From the perspective of discretization as discussed in this paper, the search for a suitable candidate **A** for a category of algebras dual to “noncommutative sets” remains open. Having ruled out various candidates, we now briefly discuss the implications, including possible avenues to avoid these obstructions.

Within the category **Cstar**, there remains the interesting open Question 2.6 of whether every C^*-algebra has a functorial (or equivalently, compatible) C^*-discretization that is injective or faithful. This question is addressed in recent work of Kornell [26] that takes a radically different approach: passing to a model of set theory in which every subset of $\mathbb{R}$ is measurable, so that the Axiom of Choice fails.

A positive answer to Question 2.6 would still not entail a candidate category of algebras dual to “noncommutative sets”. That would require isolating a suitable subcategory **A** of **Cstar** containing the algebras $\ell^\infty(X)$ and their normal $*$-homomorphisms as a full subcategory (dual to "classical" sets). One of the most notable feature of the algebras $\ell^\infty(X)$ and $\mathbb{C}^X$ is their abundance of projections. But using this structure as a guide makes Corollaries 4.8 and 4.11 particularly troubling. Suppose that A, $C(X)$, and $C(Y)$ are as in Theorem 4.7. Let $\phi : A \to M$ be the discretization of Proposition 2.3. On the one hand, that proposition demonstrates that $\ell^\infty(X)$ and $\ell^\infty(Y)$ embed faithfully into M. On the other hand, for all $x \in X$ and $y \in Y$, Theorem 4.7 implies that the images of $\delta_x \in \ell^\infty(X)$ and $\delta_y \in \ell^\infty(Y)$ are orthogonal in M. So it is not contradictory to faithfully embed both $\ell^\infty(X)$ and $\ell^\infty(Y)$ into a common discretization making all $\delta_x\delta_y$ vanish.

Thus Corollaries 4.8 and 4.11 merely indicate that globally "gluing" projections via the structure of an AW*-algebra or via convergence of nets of finite sums is inadequate for functorial discretization. This suggests exploring new structures imposing a suitable "global coherence" on projections in noncommutative $*$-algebras beyond AW*-algebras or topological algebras. To speculate only about a single possibility: the notion of contramodule [33] formalizes "infinite summation" operations that cannot be interpreted as convergence of sums in any topology.

Acknowledgements. C. Heunen was supported by EPSRC Fellowship EP/L002388/1. M. Reyes was supported by NSF grant DMS-1407152. We thank the anonymous referee for an insightful report that sharpened the statement of Theorem 4.7, informed us of Example 4.5, and inspired us to significantly expand and rewrite this paper.

REFERENCES

[1] C.A. AKEMANN, The general Stone–Weierstrass problem, *J. Funct. Anal.* **4**(1969), 277–294.

[2] C.A. AKEMANN, A Gelfand representation theory for C^*-algebras, *Pacific J. Math.* **39**(1971), 1–11.

[3] J. ANDERSON, Extensions, restrictions, and representations of states on C^*-algebras, *Trans. Amer. Math. Soc.* **249**(1979), 303–329.

[4] M. BEN-ZVI, A. MA, M.L. REYES, A Kochen–Specker theorem for integer matrices and noncommutative spectrum functors, arXiv:1509.03618.

[5] S.K. BERBERIAN, *Baer $*$-Rings*, Springer-Verlag, New York-Berlin 1972; second edition, AMS Chelsea Publ., Providence, RI 2011.

[6] B. VAN DEN BERG, C. HEUNEN, Noncommutativity as a colimit, *Appl. Categ. Structures* **20**(2012), 393–414.

[7] B. VAN DEN BERG, C. HEUNEN, Extending obstructions to noncommutative functorial spectra, *Theory Appl. Categ.* **29**(2014), 457–474.

[8] B.E. BLACKADAR, Weak expectations and nuclear C^*-algebras, *Indiana Univ. Math. J.* **27**(1978), 1021–1026.

[9] B.E. BLACKADAR, *Operator Algebras: Theory of C^*-Algebras and von Neumann Algebras*, Encyclopaedia Math. Sci., vol. 122, Springer-Verlag, Berlin 2006.

[10] D.P. BLECHER, V.I. PAULSEN, Explicit construction of universal operator algebras and applications to polynomial factorization, *Proc. Amer. Math. Soc.* **112**(1991), 839–850.

[11] J. CUNTZ, Simple C^*-algebras generated by isometries, *Comm. Math. Phys.* **57**(1977), 173–185.

[12] J. DAUNS, Categorical W*-tensor product, *Trans. Amer. Math. Soc.* **166**(1972), 439–456.

[13] R. EL HARTI, N.C. PHILLIPS, P.R. PINTO, Profinite pro-C^*-algebras and pro-C^*-algebras of profinite groups, *Houston J. Math.* **40**(2014), 791–816.

[14] R. GILES, H. KUMMER, A non-commutative generalization of topology, *Indiana Univ. Math. J.* **21**(1971), 91–102.

[15] P. HALMOS, *A Hilbert Space Problem Book*, 2nd ed., Graduate Texts in Math., vol. 19; Encyclopedia Math. Appl., vol. 17, Springer-Verlag, New York-Berlin 1982.

[16] J. HAMHALTER, Isomorphisms of ordered structures of abelian C*-algebras, *J. Math. Anal. Appl.* **383**(2011), 391–399.

[17] J. HAMHALTER, Dye's theorem and Gleason's theorem for AW*-algebras, *J. Math. Anal. Appl.* **422**(2015), 1103–1115.

[18] C. HEUNEN, The many classical faces of quantum structures, arXiv:1412.2177[quant-ph]

[19] C. HEUNEN, N.P. LANDSMAN, B. SPITTERS, A topos for algebraic quantum theory, *Comm. Math. Phys.* **291**(2009), 63–110.

[20] C. HEUNEN, M.L. REYES, Active lattices determine AW*-algebras, *J. Math. Anal. Appl.* **416**(2014), 289–313.

[21] M.C. IOVANOV, Z. MESYAN, M.L. REYES, Infinite-dimensional diagonalization and semisimplicity, *Israel J. Math.* **215**(2016), 801–855.

[22] I. KAPLANSKY, Projections in Banach algebras, *Ann. of Math. (2)* **53**(1951), 235–249.

[23] I. KAPLANSKY, The structure of certain operator algebras, *Trans. Amer. Math. Soc.* **70**(1951), 219–255.

[24] J.D. KNOWLES, On the existence of non-atomic measures, *Mathematika* **14**(1967), 62–67.

[25] S. KOCHEN, E.P. SPECKER, The problem of hidden variables in quantum mechanics, *J. Math. Mech.* **17**(1967), 59–87.

[26] A. KORNELL, V^*-algebras, arXiv:1502.01516.

[27] T.-Y. LAM, *Lectures on Modules and Rings*, Grad. Texts in Math., vol. 189, Springer-Verlag, New York 1999.

[28] A. MARCUS, D.A. SPIELMAN, N. SRIVASTAVA, Interlacing families II: Mixed characteristic polynomials and the Kadison–Singer problem, *Ann. Math.* **182**(2015), 327–350.

[29] G.K. PEDERSEN, Pullback and pushout constructions in C^*-algebra theory, *J. Funct. Anal.* **167**(1999), 243–344.

[30] D. PETZ, An invitation to the algebra of canonical commutation relations, Leuven Univ. Press, Leuven 1990.

[31] N.C. PHILLIPS, Inverse limits of C^*-algebras, *J. Operator Theory* **19**(1988), 159–195.

[32] N.C. PHILLIPS, Inverse limits of C^*-algebras and applications, in *Operator Algebras and Applications*, Vol. 1, London Math. Soc. Lecture Note Ser., vol. 135, Cambridge Univ. Press, Cambridge 1988, pp. 127–185.

[33] L. POSITSELSKI, Contramodules, arXiv:1503.00991.

[34] C. RAMSEY, M.L. REYES, Amalgamated free products of C^*-algebras are nondegenerate, preprint 2016.

[35] M.L. REYES, Obstructing extensions of the functor Spec to noncommutative rings, *Israel J. Math.* **192**(2012), 667–698.

[36] M.L. REYES, Sheaves that fail to represent matrix rings, in *Ring Theory and its Applications*, Contemp. Math., vol. 609, Amer. Math. Soc., Providence, RI 2014, pp. 285–297.

[37] Z. TAKEDA, Inductive limit and infinite direct product of operator algebras, *Tôhoku Math. J.* (2) **7**(1955), 67–86.

[38] B. TANBAY, Pure state extensions and compressibility of the l_1-algebra, *Proc. Amer. Math. Soc.* **113**(1991), 707–713.

[39] S. WARNER, *Topological Rings*, North-Holland Math. Stud., vol. 178, North-Holland Publ. Co., Amsterdam 1993.

[40] N. WEAVER, *Mathematical Quantization*, Stud. Adv. Math., Chapman and Hall/CRC, Boca Raton, FL 2001.

CHRIS HEUNEN, SCHOOL OF INFORMATICS, UNIVERSITY OF EDINBURGH, EDINBURGH EH8 9AB, U.K.
E-mail address: chris.heunen@ed.ac.uk

MANUEL L. REYES, DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE, BRUNSWICK, ME 04011–8486, U.S.A.
E-mail address: reyes@bowdoin.edu

Received June 16, 2015; revised July 12, 2016.