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HYPERCYCLIC BEHAVIOR OF SOME NON-CONVOLUTION OPERATORS ON $H(\mathbb{C}^N)$

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ABSTRACT. We study hypercyclicity properties of a family of non-convolution operators defined on the spaces of entire functions on $\mathbb{C}^N$. These operators are a composition of a differentiation operator and an affine composition operator, and are analogues of operators studied by Aron and Markose on $H(\mathbb{C})$. The hypercyclic behavior is more involved than in the one dimensional case, and depends on several parameters involved.

KEYWORDS: *Non-convolution operators, differentiation operators, composition operators, frequently hypercyclic operators, strongly mixing operators.*

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INTRODUCTION

If T is a continuous linear operator acting on some topological vector space X, the orbit under T of a vector $x \in X$ is the set $\mathrm{Orb}(x,T) := \{x, Tx, T^2x, \dots\}$. The operator T is said to be *hypercyclic* if there exists some vector $x \in X$, called *hypercyclic vector*, whose orbit under T is dense in X. In the Fréchet space setting, an operator T is hypercyclic if and only if it is topologically transitive, that is, if for every pair of non empty open sets U and V, there exists an integer $n_0 \in \mathbb{N}$ such that $T^{n_0}U \cap V \neq \emptyset$. An operator is said to be mixing provided $T^n U \cap V \neq \emptyset$ for all large n, for any pair of non-empty open subsets U and V of X. Recently, some stronger forms of hypercyclicity have gained the attention of researchers, specially the concepts of frequently hypercyclic operators and strongly mixing operators with respect to some invariant probability measure on the space.

The first examples of hypercyclic operators were found by Birkhoff [5] and MacLane [14], whose research was focused on holomorphic functions of one complex variable and not in properties of operators. Birkhoff's result implies that the translation operator $\tau : H(\mathbb{C}) \to H(\mathbb{C})$ defined by $\tau(h)(z) = h(1+z)$ is hypercyclic. Likewise, MacLane's result states that the differentiation operator on $H(\mathbb{C})$ is hypercyclic. In a seminal paper, Godefroy and Shapiro [10] unified and generalized both results, by showing that every continuous linear operator $T : H(\mathbb{C}^N) \to H(\mathbb{C}^N)$ which commutes with translations and which is not a scalar multiple of the identity is hypercyclic. These operators are called non-trivial convolution operators.

Another important class of operators on $H(\mathbb{C}^N)$ are the composition operators C_ϕ, induced by symbols ϕ which are automorphisms of $\mathbb{C}^N$. The hypercyclicity of composition operators induced by affine automorphisms was completely characterized in terms of properties of the symbol by Bernal-González [3].

Besides operators belonging to some of these two classes, there are not many examples of hypercyclic operators on $H(\mathbb{C}^N)$. Motivated by this fact, Aron and Markose [1] studied the hypercyclicity of the following operator on $H(\mathbb{C})$,

$$Tf(z) = f'(\lambda z + b), \quad \text{with } \lambda, b \in \mathbb{C}.$$

The operator T is not a convolution operator unless $\lambda = 1$. They showed that T is hypercyclic for any $|\lambda| \geqslant 1$ (a gap in the proof was corrected in [9]) and that it is not hypercyclic if $|\lambda| < 1$ and $b = 0$. Thus, they gave explicit examples of hypercyclic operators which are neither convolution operators nor composition operators. Recently, these operators were studied in [12], where the authors showed that the operator is frequently hypercyclic when $b = 0$ and $|\lambda| \geqslant 1$, and asked whether it is frequently hypercyclic for any $b \neq 0$. See also [8], [17] for other classes of non-convolution operators. In Section 2, we give a different proof of the result of [1], [9], but for any $\lambda, b \in \mathbb{C}$. We conclude in Proposition 2.3 that T is hypercyclic if and only if $|\lambda| \geqslant 1$, and that in this case, T is even strongly mixing with respect to some Borel probability measure of full support on $H(\mathbb{C})$.

In Sections 3 and 4 we define N-dimensional analogues of the operators considered by Aron and Markose and study the dynamics they induce in $H(\mathbb{C}^N)$. These operators are a composition between a partial differentiation operator and a composition operator induced by some automorphism of $\mathbb{C}^N$. It turns out that their behavior is more complicated than their one variable counterpart. One possible reason is that while the automorphisms of $\mathbb{C}$ have a very simple structure and hypercyclicity inducing properties, the automorphisms of $\mathbb{C}^N$ are much more involved. Even, the characterization of hypercyclic affine automorphisms is non-trivial (see [3]).

In Section 3, we consider the case in which the composition operators are induced by a diagonal operator plus a translation, that is, for $f \in H(\mathbb{C}^N)$ and $z = (z_1, \ldots, z_N) \in \mathbb{C}^N$, we study operators of the form

$$Tf(z) = D^\alpha f((\lambda_1 z_1, \ldots, \lambda_N z_N) + b),$$

where α is a multi-index and b and $\lambda = (\lambda_1, \ldots, \lambda_N)$ are vectors in $\mathbb{C}^N$. In this case we completely characterize the hypercyclicity of these non-convolution operators which, contrary to the one dimensional case studied in [1], does not only depend on the size of λ. In the last section, we study the operators which are a composition of a directional derivative operator with a general affine automorphism of $\mathbb{C}^N$ and determine its hypercyclicity in some cases.

1. PRELIMINARIES

In this section we state some known conditions which ensure that a linear operator is strongly mixing with respect to an invariant Borel probability measure of full support. First we recall the following definitions.

DEFINITION 1.1. A linear operator T on X is called *frequently hypercyclic* if there exists a vector $x \in X$, called a *frequently hypercyclic vector*, whose T-orbit visits each non-empty open set along a set of integers having positive lower density.

DEFINITION 1.2. A Borel probability measure on X is Gaussian if and only if it is the distribution of an almost surely convergent random series of the form $\xi = \sum_0^\infty g_n x_n$, where $(x_n) \subset X$ and (g_n) is a sequence of independent, standard complex Gaussian variables.

DEFINITION 1.3. We say that an operator $T \in \mathcal{L}(X)$ is strongly mixing in the Gaussian sense if there exists some Gaussian T-invariant probability measure μ on X with full support such that any measurable sets A, $B \subset X$ satisfy

$$\lim_{n\to\infty} \mu(A \cap T^{-n}(B)) = \mu(A)\mu(B).$$

We will use the following result, which is a corollary of a theorem due to Bayart and Matheron (see [2]). Essentially this theorem says that a large supply of eigenvectors associated to unimodular eigenvalues that are well distributed along the unit circle implies that the operator is strongly mixing in the Gaussian sense.

THEOREM 1.4 (Bayart, Matheron). *Let X be a complex separable Fréchet space, and let $T \in \mathcal{L}(X)$. Assume that for any set $D \subset \mathbb{T}$ such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the linear span of $\bigcup_{\lambda\in\mathbb{T}\setminus D} \ker(T-\lambda)$ is dense in X. Then T is strongly mixing in the Gaussian sense.*

The following result, proved by Murillo-Arcila and Peris in Theorem 1 of [15], shows that operators defined on Fréchet spaces which satisfy the frequent hypercyclicity criterion are strongly mixing with respect to an invariant Borel measure with full support.

THEOREM 1.5 (Murillo-Arcila, Peris). *Let X be a separable Fréchet space and $T \in \mathcal{L}(X)$. Suppose that there exists a dense subset $X_0 \subset X$ such that $\sum_n T^n x$ is unconditionally convergent for all $x \in X_0$. Suppose further that there exists a sequence of* *maps* $S_k : X_0 \to X$ *such that* $T \circ S_1 = \mathrm{Id}$, $T \circ S_k = S_{k-1}$ *and* $\sum_k S_k(x)$ *is unconditionally convergent for all* $x \in X_0$. *Then there exists a Borel probability measure* μ *in* X, T*-invariant, such that the operator* T *is strongly mixing with respect to* μ.

It can be shown that the hypotheses of Theorem 1.5 imply the corresponding ones of Theorem 1.4. So, in any case, both theorems allow us to conclude the existence of an invariant Gaussian probability measure for linear operators of full support which are strongly mixing. Finally, the next proposition states that the existence of such measures is preserved by linear quasi-conjugation. Its proof is standard.

PROPOSITION 1.6. *Let* X *and* Y *be separable Fréchet spaces and* $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$. *Suppose that* $SJ = JT$ *for some continuous linear mapping* $J : X \to Y$ *of dense range; then, if* T *has an invariant Borel measure then so does* S. *Moreover, if* T *has an invariant Borel measure that is Gaussian, strongly mixing, ergodic or of full support, then so does* S.

2. NON-CONVOLUTION OPERATORS ON $H(\mathbb{C})$

Let us denote by D and τ_a the derivation and translation operators on $H(\mathbb{C})$, respectively. Namely, for an entire function f, we have

$$D(f)(z) = f'(z) \quad \text{and} \quad \tau_a(f)(z) = f(z+a).$$

MacLane's theorem [14] says that D is a hypercyclic operator, and Birkhoff's theorem [5] states that τ_a is hypercyclic provided that $a \neq 0$. The translation operators are a special class of composition operators on $H(\mathbb{C})$. By a composition operator we mean an operator C_ϕ such that $C_\phi(f) = f \circ \phi$, where ϕ is some automorphism of $\mathbb{C}$. The hypercyclicity of the composition operators on $H(\mathbb{C})$ has been completely characterized in terms of properties of the symbol function ϕ. Precisely, the relevant property of ϕ is the following.

DEFINITION 2.1. A sequence $\{\phi_n\}_{n\in\mathbb{N}}$ of holomorphic maps on $\mathbb{C}$, is called *runaway* if, for each compact set $K \subset \mathbb{C}$, there is an integer $n \in \mathbb{N}$ such that $\phi_n(K) \cap K = \emptyset$. In the case where $\phi_n = \phi^n$ for every $n \in \mathbb{N}$, we will just say that ϕ is runaway.

This definition was first given by Bernal-González and Montes-Rodríguez in [4], where they also proved the following (see also Therorem 4.32 of [11]).

THEOREM 2.2. *Let* ϕ *be an automorphism of* $\mathbb{C}$. *Then* C_ϕ *is hypercyclic if and only if* ϕ *is runaway.*

It is known that the automorphisms of $\mathbb{C}$ are given by $\phi(z) = \lambda z + b$, with $\lambda \neq 0$ and $b \in \mathbb{C}$. In addition, ϕ is runaway if and only if $\lambda = 1$ and $b \neq 0$ (see

Example 4.28 of [11]). This means that the hypercyclic composition operators on $H(\mathbb{C})$ are exactly Birkhoff's translation operators.

Aron and Markose in [1] studied the hypercyclicity of the following operator on $H(\mathbb{C})$,

$$Tf(z) = f'(\lambda z + b),$$

with $\lambda, b \in \mathbb{C}$, which is a composition of MacLane's derivation operator and a composition operator, i.e., $T = C_\phi \circ D$ with $\phi(z) = \lambda z + b$. The main motivation for the study of this operator was the wish to understand the behavior of a concrete operator belonging neither to the class of convolution operators nor to the class of composition operators. As mentioned before, in [1] (see also [9]) the authors proved that T is hypercyclic if $|\lambda| \geqslant 1$, and that it is not hypercyclic if $|\lambda| < 1$ and $b = 0$.

In this section we give a simple proof of the result by Aron and Markose, for the full range on λ, b. This will allow us to illustrate some of the main ideas used in the next section to prove the more involved N-variables case.

Suppose that $\lambda \neq 1$. The key observation is that T is conjugate to an operator of the same type, but with $b = 0$. Indeed, define $T_0 f(z) = f'(\lambda z)$, then we have that the following diagram commutes:

$$\begin{tikzcd} H(\mathbb{C}) \arrow[r, "T"] \arrow[d, "\tau_{[b/(1-\lambda)]}"'] & H(\mathbb{C}) \arrow[d, "\tau_{[b/(1-\lambda)]}"] \\ H(\mathbb{C}) \arrow[r, "T_0"'] & H(\mathbb{C}) \end{tikzcd} .$$

Note that $\frac{b}{1-\lambda}$ is the fixed point of ϕ. This observation will be important later.

PROPOSITION 2.3. *Let T be the operator defined on $H(\mathbb{C})$ by $Tf(z) = f'(\lambda z + b)$. Then T is hypercyclic if and only if $|\lambda| \geqslant 1$. In this case, T is also strongly mixing with respect to some Borel probability measure of full support on $H(\mathbb{C})$.*

Proof. If $\lambda = 1$, then T is a non-trivial convolution operator, thus it is hypercyclic. Moreover, by the Godefroy and Shapiro's theorem and its extensions (see [6], [10], [16]), T is strongly mixing in the Gaussian sense. Hence, by Proposition 1.6, it suffices to prove the case $b = 0$ and $\lambda \neq 1$, i.e. for the operator T_0.

Suppose first that $|\lambda| < 1$ and let $f \in H(\mathbb{C})$. Note that $T_0^n f(z) = \lambda^{n(n-1)/2} \cdot f^{(n)}(\lambda^n z)$. By the Cauchy's estimates we obtain that

$$|T_0^n f(0)| \leqslant |\lambda|^{n(n-1)/2} n! \sup_{\|z\| \leqslant 1} |f(z)| \underset{n\to\infty}{\longrightarrow} 0.$$

Since the evaluation at 0 is continuous, the orbit of f under T_0 can not be dense.

Suppose now that $|\lambda| > 1$. Let us see that we can apply the Murillo-Arcila and Peris criterion, Theorem 1.5. Let X_0 be the set of all polynomials, which is dense in $H(\mathbb{C})$. Then, for each polynomial $f \in X_0$, the series $\sum_n T_0^n f$ is actually a finite sum, thus it is unconditionally convergent.

For $n \in \mathbb{N}$ we define a sequence of linear maps $S_n : X_0 \to X$ as

$$S_n(z^k) = \frac{k!}{(k+n)!} \frac{z^{k+n}}{\lambda^{nk+n(n-1)/2}}.$$

It is easy to see that the sequence (S_n) satisfies the hypotheses of Theorem 1.5:

Case $T_0 \circ S_1 = I$: $T_0 \circ S_1(z^k) = T_0\Big(\frac{1}{k+1}\frac{z^{k+1}}{\lambda^k}\Big) = z^k$.

Case $T_0 \circ S_n = S_{n-1}$:

$$T_0 \circ S_n(z^k) = T_0\Big(\frac{k!}{(k+n)!}\frac{z^{k+n}}{\lambda^{nk+n(n-1)/2}}\Big) = \frac{k!}{(k+n-1)!}\frac{\lambda^{k+n-1}z^{k+n-1}}{\lambda^{nk+n(n-1)/2}}$$
$$= \frac{k!}{(k+n-1)!}\frac{z^{k+n-1}}{\lambda^{(n-1)k+(n-1)(n-2)/2}} = S_{n-1}(z^k).$$

Case the series $\sum_n S_n(f)$ are unconditionally convergent for each $f \in X_0$. If $|z| \leqslant R$, we get that,

$$\sum_n |S_n(z^k)| \leqslant \sum_n \frac{k!}{(k+n)!} R^{k+n} \leqslant k! e^R.$$

Thus, the operator T_0 is strongly mixing in the Gaussian sense. ∎

After the preparation of the manuscript we learned that the hypercyclicity of $C_\phi \circ D$ for $\lambda \geqslant 1$ and $b \neq 0$ was proved independently in [13].

We can summarize the results of this section in the following table. It is worth noticing that neither the hypercyclicity of C_ϕ nor the hypercyclicity of D imply the hypercyclicity of $C_\phi \circ D$.

	$\lambda < 1$	$\lambda = 1$	$\lambda \geqslant 1, \lambda \neq 1$
C_ϕ	Not Hypercyclic	Hypercyclic $\Leftrightarrow b \neq 0$	Not Hypercyclic
D	Hypercyclic	Hypercyclic	Hypercyclic
$C_\phi \circ D$	Not Hypercyclic	Hypercyclic	Hypercyclic

As we shall see in the next section, we may replace $C_\phi \circ D$ by $C_\phi \circ D^r$, for any positive integer r, on the third row of the table above.

3. NON-CONVOLUTION OPERATORS ON $H(\mathbb{C}^N)$ — THE DIAGONAL CASE

The operators considered in the previous section were differentiation operators followed by a composition operator. In this section we consider N-dimensional analogues of those operators. First, we will be concerned with symbols $\phi : \mathbb{C}^N \to \mathbb{C}^N$, which are diagonal affine automorphism of the form

$$\phi(z) = \lambda z + b = (\lambda_1 z_1 + b_1, \dots, \lambda_N z_N + b_N),$$

where $\lambda, b \in \mathbb{C}^N$; and the differentiation operator is a partial derivative operator given by a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$,

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2} \cdots \partial z_N^{\alpha_N}}.$$

Thus in this section T will denote the operator on $H(\mathbb{C}^N)$ defined by

$$Tf(z) = C_\phi \circ D^\alpha(f)(z) = D^\alpha f(\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N).$$

Note that, in the definition of T, we allow α to be zero. In this case, the operator is just a composition operator and its hypercyclicity is determined by the symbol ϕ. These symbol functions are special cases of affine automorphisms of $\mathbb{C}^N$. The existence of universal functions for composition operators with affine symbol on $\mathbb{C}^N$ has been completely characterized by Bernal-Gonzalez in [3], where he proved that the hypercyclicity of the composition operator depends on whether or not the symbol is runaway. Recall that an automorphism φ of $\mathbb{C}^N$ is said to be runaway if for any compact subset K there is some $n \geqslant 1$ such that $\varphi^n(K) \cap K = \emptyset$.

THEOREM 3.1 (Bernal-González). *Assume that $\varphi : \mathbb{C}^N \to \mathbb{C}^N$ is an affine automorphism of $\mathbb{C}^N$, say $\varphi(z) = Az + b$. Then, the composition operator C_φ is hypercyclic if and only if φ is a runaway automorphism if and only if the vector b is not in* $\operatorname{ran}(A - I)$ *and* $\det(A) \neq 0$.

The proof of this result is based on the following N-variables generalization of Runge's approximation theorem, which will be useful for us later.

THEOREM 3.2. *If K and L are disjoint convex compact sets in $\mathbb{C}^N$ and f is a holomorphic function on a neighborhood of $K \cup L$, then there is a sequence of polynomials on $\mathbb{C}^N$ that approximate f uniformly on $K \cup L$.*

REMARK 3.3. It is easy to prove that the mapping $\phi(z) = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N)$ is runaway if and only if some coordinate is a translation, that is, for some $i = 1, \ldots, N$ we have, simultaneously, that $\lambda_i = 1$ and $b_i \neq 0$.

If $\lambda_j = 0$ for some j, then we have that the differential $\mathrm{d}(T^n f)(\cdot)(e_j) = 0$, for every $n \in \mathbb{N}$. Since the application $g \mapsto \mathrm{d}(\cdot)(e_j)$ is continuous, we conclude that the orbit of f under T can not be dense.

The next result completely characterizes the hypercyclicity of the operator $T = C_\phi \circ D^\alpha$, with $\lambda \neq 0$ and $\alpha \neq 0$ (the case $\alpha = 0$ is covered in [3], and as mentioned above T is not hypercyclic if $\lambda_j = 0$ for some j). Write $\lambda^\alpha = \prod_{i \leqslant N} \lambda_i^{\alpha_i}$.

THEOREM 3.4. *Let T be the operator on $H(\mathbb{C}^N)$, defined by $Tf(z) = C_\phi \circ D^\alpha f(z)$, where $\alpha \neq 0$, $\phi(z) = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N)$ and $\lambda_i \neq 0$ for all i, $1 \leqslant i \leqslant N$. Then,*

(i) *If $|\lambda^\alpha| \geqslant 1$ then T is strongly mixing in the Gaussian sense.*

(ii) *If for some $i = 1, \ldots, N$ we have that $b_i \neq 0$ and $\lambda_i = 1$, then T is mixing.*

(iii) *In any other case, T is not hypercyclic.*

REMARK 3.5. The item (iii) above includes the following sub-cases:
(a) $|\lambda^\alpha| < 1$ and $b = 0$.
(b) $|\lambda^\alpha| < 1$ and $\lambda_i \neq 1$ for every i, $1 \leqslant i \leqslant N$.
(c) $|\lambda^\alpha| < 1$ and $b_i = 0$ for every i such that $\lambda_i = 1$.

In all three sub-cases we have that the application $\phi(z) = \lambda z + b$ has a fixed point and thus ϕ is not runaway. Also in case (ii) the application ϕ has one coordinate which is a translation, thus it is runaway. So, in particular, Theorem 3.4 implies that $T = C_\phi \circ D^\alpha$ is hypercyclic if and only if either $|\lambda^\alpha| \geqslant 1$ or ϕ is runaway.

We can summarize our main theorem in the following table.

	$\lvert\lambda^\alpha\rvert < 1$ and no coord. of ϕ is a translation	$\lvert\lambda^\alpha\rvert < 1$ and a coord. of ϕ is a translation	$\lvert\lambda^\alpha\rvert \geqslant 1$
C_ϕ	Not Hypercyclic	Hypercyclic	depends on ϕ
D^α	Hypercyclic	Hypercyclic	Hypercyclic
$C_\phi \circ D^\alpha$	Not Hypercyclic	Hypercyclic	Hypercyclic

We will divide the proof of part (i) of Theorem 3.4 in two lemmas. Through a change in the order of the variables, we may suppose that the first j variables, $0 \leqslant j \leqslant N$, correspond to the coordinates in which $\lambda_i = 1$. The operator T is then of the form

$$Tf(z) = D^\alpha f(z_1 + b_1, \ldots, z_j + b_j, \lambda_{j+1} z_{j+1} + b_{j+1}, \ldots, \lambda_N z_N + b_N). \tag{3.1}$$

Moreover, we can assume that $b_i = 0$ for all $i > j$, because T is topologically conjugate to

$$T_0 f(z) = D^\alpha f(z_1 + b_1, \ldots, z_j + b_j, \lambda_{j+1} z_{j+1}, \ldots, \lambda_N z_N) \tag{3.2}$$

through a translation. Indeed, defining $c \in \mathbb{C}^N$ by $c_l = 0$ if $l \leqslant j$, and $c_l = \frac{b_l}{1-\lambda_l}$ if $l > j$, we get that $T_0 \circ \tau_c = \tau_c \circ T$.

We first study the case in which for some i, we have $\lambda_i \neq 1$ and $\alpha_i \neq 0$ (note that if all $\lambda_i = 1$, then T is a convolution operator and it is thus strongly mixing in the Gaussian sense [6], [16]).

LEMMA 3.6. *Let T be as in* (3.1). *Suppose that $|\lambda^\alpha| \geqslant 1$ and $\alpha_i \neq 0$ for some $i > j$. Then T is strongly mixing in the Gaussian sense.*

Proof. By the above comments, we may suppose that $b_i = 0$ for $i > j$, so the operator T is as in (3.2). We apply Theorem 1.5 with

$$X_0 = \operatorname{span}\{e_\gamma z^\beta := \mathrm{e}^{\gamma_1 z_1 + \cdots + \gamma_j z_j} z^\beta \text{ with } \beta_i = 0 \text{ for } i \leqslant j \text{ and } \gamma \in \mathbb{C}^j\}.$$

The set $X_0 \subset H(\mathbb{C}^N)$ is dense. Indeed, since the set $\{e_\gamma : \gamma \in \mathbb{C}^j\}$ generates a dense subspace in $H(\mathbb{C}^j)$ (see for example Proposition 2.4 of [6]), given a monomial $z_1^{\theta_1} \cdots z_j^{\theta_j}$, $\varepsilon > 0$ and $R > 0$, there is $f \in \operatorname{span}\{e_\gamma : \gamma \in \mathbb{C}^j\}$ with

$$\sup_{\|z\| \leqslant R} |f(z_1, \ldots, z_j) - z_1^{\theta_1} \cdots z_j^{\theta_j}| < \varepsilon.$$

We obtain

$$\sup_{\|z\| \leqslant R} |\underbrace{f(z_1, \ldots, z_j) z_{j+1}^{\beta_{j+1}} \cdots z_N^{\beta_N}}_{\in X_0} - z_1^{\theta_1} \cdots z_j^{\theta_j} z_{j+1}^{\beta_{j+1}} \cdots z_N^{\beta_N}| < \varepsilon R^{|\beta|}.$$

Therefore we can approximate any monomial in $H(\mathbb{C}^N)$ by functions of X_0 uniformly on compact sets.

The series $\sum_n T^n(e_\gamma z^\beta)$ is unconditionally convergent because the operator T differentiates in some variable z_i with $i > j$, and so it is a finite sum. On the other hand, if we denote by $\alpha_{(1)} := (\alpha_1, \ldots, \alpha_j)$ and $\alpha_{(2)} := (\alpha_{j+1}, \ldots, \alpha_N) \neq 0$, we obtain

$$T^n(e_\gamma z^\beta) = \gamma^{n\alpha_{(1)}} \mathrm{e}^{n\langle \gamma, b\rangle} \lambda^{n\beta - n(n+1)/2\alpha_{(2)}} \frac{\beta!}{(\beta - n\alpha_{(2)})!} e_\gamma z^{\beta - n\alpha_{(2)}}.$$

Now, we define a sequence of maps $S_n : X_0 \to X_0$. First, we do that on the set $\{e_\gamma z^\beta\}$ and then extending them by linearity

$$S_n(e_\gamma z^\beta) = \frac{\beta!}{\gamma^{n\alpha_{(1)}} \mathrm{e}^{n\langle \gamma, b\rangle} \lambda^{n\beta + n(n-1)/2\alpha_{(2)}} (\beta + n\alpha_{(2)})!} e_\gamma z^{\beta + n\alpha_{(2)}}.$$

The following assertions hold:

Case $T \circ S_1 = I$:

$$\begin{aligned} T \circ S_1(e_\gamma z^\beta) &= \frac{1}{\gamma^{\alpha_{(1)}} \mathrm{e}^{\langle \gamma, b\rangle} \lambda^\beta} \frac{\beta!}{(\beta + \alpha_{(2)})!} T(e_\gamma z^{\beta + \alpha_{(2)}}) \\ &= \frac{1}{\gamma^{\alpha_{(1)}} \mathrm{e}^{\langle \gamma, b\rangle} \lambda^\beta} \frac{\beta!}{(\beta + \alpha_{(2)})!} \gamma^{\alpha_{(1)}} \mathrm{e}^{\langle \gamma, b\rangle} e_\gamma \frac{(\beta + \alpha_{(2)})!}{\beta!} z^\beta \lambda^\beta = e_\gamma z^\beta. \end{aligned}$$

Case $T \circ S_n = S_{n-1}$:

$$\begin{aligned} &T \circ S_n(e_\gamma z^\beta) \\ &= \frac{1}{\gamma^{n\alpha_{(1)}} \mathrm{e}^{n\langle \gamma, b\rangle} \lambda^{n\beta + n(n-1)/2\alpha_{(2)}}} \frac{\beta!}{(\beta + n\alpha_{(2)})!} T(e_\gamma z^{\beta + n\alpha_{(2)}}) \\ &= \frac{\beta! \gamma^{\alpha_{(1)}} \mathrm{e}^{\langle \gamma, b\rangle} \lambda^{\beta + (n-1)\alpha_{(2)}} (\beta + n\alpha_{(2)})!}{\gamma^{n\alpha_{(1)}} \mathrm{e}^{n\langle \gamma, b\rangle} \lambda^{n\beta + n(n-1)/2\alpha_{(2)}} (\beta + n\alpha_{(2)})! (\beta + (n-1)\alpha_{(2)})!} e_\gamma z^{\beta + (n-1)\alpha_{(2)}} \\ &= \frac{\beta!}{\gamma^{(n-1)\alpha_{(1)}} \mathrm{e}^{(n-1)\langle \gamma, b\rangle} \lambda^{(n-1)\beta + (n-1)(n-2)/2\alpha_{(2)}} (\beta + (n-1)\alpha_{(2)})!} e_\gamma z^{\beta + (n-1)\alpha_{(2)}} \end{aligned}$$

$$= S_{n-1}(e_\gamma z^\beta).$$

Case given $R > 0$, let $|z| \leqslant R$ and denote $C = \left|\frac{R^{\alpha(2)}}{\lambda^\beta \gamma^{\alpha(1)} e^{\langle \gamma, b\rangle}}\right|$. We have $|S_n(e_\gamma z^\beta)| \leqslant M \frac{C^n}{(\beta + n\alpha_{(2)})!}$ for some constant $M > 0$ not depending on n. Since, $\alpha_{(2)} \neq 0$, we get that for each $\gamma \in \mathbb{C}^j$ and $\beta \in \mathbb{C}^N$ with $\beta_i = 0$ for $i \leqslant j$, $\sum_n |S_n(e_\gamma z^\beta)|$ is uniformly convergent on compact sets.

We have thus shown that the hypotheses of Theorem 1.5 are fulfilled. Hence T is strongly mixing in the Gaussian sense, as we wanted to prove. ∎

The other case we need to prove is when T does not differentiate in the variables z_i with $i > j$. This means that $\alpha_i = 0$ for all $i > j$. To prove this case we will use Theorem 1.4.

LEMMA 3.7. *Let T be as in* (3.1). *Suppose that $|\lambda^\alpha| \geqslant 1$ and $\alpha_i = 0$ for every $i > j$. Then T is strongly mixing in the Gaussian sense.*

Proof. We may suppose that $b_i = 0$ for $i > j$, so the operator T is as in (3.2). The functions $e_\gamma z^\beta$, with $\gamma_i = 0$ for all $i > j$ and $\beta_i = 0$ for every $i \leqslant j$, are eigenfunctions of T. Indeed,

$$T(e_\gamma z^\beta) = \gamma^{\alpha(1)} e^{\sum \gamma_i (z_i + b_i)} (\lambda z)^\beta = \gamma^{\alpha(1)} \lambda^\beta e^{\langle \gamma, b\rangle} e_\gamma z^\beta,$$

where, as in the proof of the last lemma, $\alpha_{(1)} = (\alpha_1, \ldots, \alpha_j) \neq 0$ (note that in this case we have $\alpha_{(2)} = (\alpha_{j+1}, \ldots, \alpha_N) = 0$).

By Theorem 1.4 it is enough to show that for every set $D \subset \mathbb{T}$ such that $\mathbb{T} \setminus D$ is dense in $\mathbb{T}$, the set

$$\{e_\gamma z^\beta : \beta \in \mathbb{C}^N \text{ with } \beta_i = 0 \text{ for } i \leqslant j \text{ and } \gamma_i = 0 \text{ for } i > j, \tag{3.3}$$
$$\text{such that } \gamma^\alpha \lambda^\beta e^{\langle \gamma, b\rangle} \in \mathbb{T} \setminus D\},$$

spans a dense subspace on $H(\mathbb{C}^N)$.

Fix $\beta \in \mathbb{C}^N$ with $\beta_i = 0$ for every $i \leqslant j$ and consider the map

$$f_\beta : \mathbb{C}^j \to \mathbb{C}$$
$$\gamma \mapsto \gamma^\alpha \lambda^\beta e^{\langle \gamma, b\rangle}.$$

The application f_β is holomorphic and non constant. So there exists $\gamma_0 \in \mathbb{C}^j$ such that $|\gamma_0{}^\alpha \lambda^\beta e^{\langle \gamma_0, b\rangle}| = 1$. Since, $\mathbb{T} \setminus D$ is a dense set in $\mathbb{T}$, the vector γ_0 is an accumulation point of $\mathbb{T} \setminus D$. Thus, by Proposition 2.4 of [6], we get that the set

$$\{e_\gamma : \text{with } \gamma \text{ such that } \gamma^\alpha \lambda^\beta e^{\langle \gamma, b\rangle} \in \mathbb{T} \setminus D\}$$

spans a dense subspace in $H(\mathbb{C}^j)$. It is then easy to see that the set defined in (3.3) spans a dense subspace in $H(\mathbb{C}^N)$. In particular, we have shown that the set of eigenvectors of T associated to eigenvalues belonging to $\mathbb{T} \setminus D$ span a dense subspace in $H(\mathbb{C}^N)$. So, the hypotheses of Theorem 1.4 are satisfied and hence T is strongly mixing in the Gaussian sense. ∎

The following remark will be useful for the next proof and in the rest of the article.

REMARK 3.8. Recall Cauchy's formula for holomorphic functions in $\mathbb{C}^N$,

$$D^\alpha f(z_1,\ldots,z_N) = \frac{\alpha!}{(2\pi \mathrm{i})^N} \int\limits_{|w_1-z_1|=r_1} \cdots \int\limits_{|w_N-z_N|=r_N} \frac{f(w_1,\ldots,w_N)}{\prod_{i=1}^N (w_i-z_i)^{\alpha_i+1}}\, \mathrm{d}w_1\cdots \mathrm{d}w_N.$$

Therefore, we can estimate the supremum of $D^\alpha f$ over a set of the form $B(z_1,r_1)\times \cdots \times B(z_N,r_N)$, where $B(z_j,r_j)$ denotes the closed disk of center $z_j\in\mathbb{C}$ and radius r_j. Fix positive real numbers $\varepsilon_1,\ldots,\varepsilon_N$, then

$$\|D^\alpha f\|_{\infty,B(z_1,r_1)\times\cdots\times B(z_N,r_N)} \leqslant \frac{\alpha!}{(2\pi)^N} \frac{\|f\|_{\infty,B(z_1,r_1+\varepsilon_1)\times\cdots\times B(z_N,r_N+\varepsilon_N)}}{\varepsilon_1^{\alpha_1+1}\cdots\varepsilon_N^{\alpha_N+1}}. \tag{3.4}$$

Proof of Theorem 3.4. Part (i) is proved by Lemmas 3.6 and 3.7.

(ii) Suppose that $b_l\neq 0$ for some l such that $\lambda_l=1$. We will prove that T is a mixing operator, i.e., that for every pair U and V of non empty open sets for the local uniform topology of $H(\mathbb{C}^N)$, there exists $n_0\in\mathbb{N}$ such that $T^n(U)\cap V\neq\emptyset$ for all $n\geqslant n_0$. Let f and g be two holomorphic functions on $H(\mathbb{C}^N)$, L be a compact set of $\mathbb{C}^N$ and θ a positive real number. We can assume that

$$U=\{h\in H(\mathbb{C}^N): \|f-h\|_{\infty,L}<\theta\} \quad \text{and} \quad V=\{h\in H(\mathbb{C}^N): \|g-h\|_{\infty,L}<\theta\},$$

and that g is a polynomial and that L is a closed ball of $(\mathbb{C}^N,\|\cdot\|_\infty)$. We do so because we can define a right inverse map over the set of polynomials. Since $T=C_\phi\circ D^\alpha$, we can define

$$I^\alpha(z^\beta)=\frac{\beta!}{(\alpha+\beta)!}z^{\alpha+\beta}.$$

Thus, $S=I^\alpha\circ C_{\phi^{-1}}$ is a right inverse for T when restricted to polynomials. Hence, we assume that $L=B(0,r)\times B(0,r)\times\cdots\times B(0,r)$, for some $r>0$ and denote $\phi_i(z)=\lambda_i z+b_i$, for $z\in\mathbb{C}$. We get that $\phi(z_1,\ldots,z_N)=(\phi_1(z_1),\ldots,\phi_N(z_N))$ and $\phi_i(B(z_i,r_i))=B(\phi_i(z_i),|\lambda_i|r_i)$.

Now, suppose that P is a polynomial in $\mathbb{C}^N$. Applying the inequality (3.4) several times, in which each time we use it we divide each ε_i by 2, we get that

$$\begin{aligned}
&\|g-T^nP\|_{\infty,L}\\
&\quad=\|C_\phi\circ D^\alpha(Sg-T^{n-1}P)\|_{\infty,L}=\|D^\alpha(Sg-T^{n-1}P)\|_{\infty,\phi(L)}\\
&\quad=\|D^\alpha(Sg-T^{n-1}P)\|_{\infty,\prod B(b_i,|\lambda_i|r)}\\
&\quad\leqslant\frac{\alpha!}{(2\pi)^N\varepsilon_1^{\alpha_1+1}\cdots\varepsilon_N^{\alpha_N+1}}\|Sg-T^{n-1}P\|_{\infty,\prod B(b_i,|\lambda_i|r+\varepsilon_i)}\\
&\quad\leqslant\frac{\alpha!}{(2\pi)^N\varepsilon_1^{\alpha_1+1}\cdots\varepsilon_N^{\alpha_N+1}}\|C_\phi\circ D^\alpha(S^2g-T^{n-2}P)\|_{\infty,\prod B(b_i,|\lambda_i|r+\varepsilon_i)}
\end{aligned}$$

$$\leqslant \frac{\alpha!}{(2\pi)^N \varepsilon_1^{\alpha_1+1}\cdots\varepsilon_N^{\alpha_N+1}} \|D^\alpha(S^2 g - T^{n-2}P)\|_{\infty,\prod B((\lambda_i+1)b_i,|\lambda_i|(|\lambda_i|r+\varepsilon_i))}$$
$$\leqslant \frac{2^{|\alpha|+N}\alpha!^2}{(2\pi)^{2N}\varepsilon_1^{2(\alpha_1+1)}\cdots\varepsilon_N^{2(\alpha_N+1)}} \|S^2 g - T^{n-2}P\|_{\infty,\prod B((\lambda_i+1)b_i,|\lambda_i|(|\lambda_i|r+\varepsilon_i)+\varepsilon_i/2)}.$$

Continuing this process we obtain

$$\|g - T^n P\|_{\infty,L} \leqslant \frac{2^{(n(n+1)/2)(|\alpha|+N)}\alpha!^n}{(2\pi)^{nN}\varepsilon_1^{n(\alpha_1+1)}\cdots\varepsilon_N^{n(\alpha_N+1)}} \cdot \|S^n g - P\|_{\infty,\prod B(\phi_i^n(0),|\lambda_i|^n r+\varepsilon_i\sum_{k=0}^{n-1}|\lambda_i|^k/2^{n-k-1})}.$$

Let us denote by l, the coordinate of ϕ that is a translation in $\mathbb{C}$. Thus, we have that $\lambda_l = 1$ and $b_l \neq 0$. This implies that

$$B\Big(\phi_l^n(0), |\lambda_l|^n r + \varepsilon_l \sum_{k=0}^{n-1} \frac{|\lambda_l|^k}{2^{n-k-1}}\Big) = B\Big(nb_l, r + \varepsilon_l \sum_{k=0}^{n-1} \frac{1}{2^k}\Big) \subset B(nb_l, r+2\varepsilon_l).$$

Fix $n_0 \in \mathbb{N}$, such that $B(0,r) \cap B(nb_l, r+2\varepsilon_l) = \emptyset$ for all $n \geqslant n_0$. Now, take $\delta_n > 0$ and Λ_n a ball of $(\mathbb{C}^N, \|\cdot\|_\infty)$, such that $[L+\delta_n] \cap [\Lambda_n + \delta_n] = \emptyset$ for all $n \geqslant n_0$ and

$$\prod_{i=1}^N B\Big(\phi_l^n(0), |\lambda_l|^n r + \varepsilon_l \sum_{k=0}^{n-1} \frac{|\lambda_l|^k}{2^{n-k-1}}\Big) \subset \Lambda_n.$$

Also, denote $K_n = \frac{2^{(n(n+1)/2)(|\alpha|+N)}\alpha!^n}{(2\pi)^{nN}\varepsilon_1^{n(\alpha_1+1)}\cdots\varepsilon_N^{n(\alpha_N+1)}}$. Then, use Theorem 3.2 with $h_n = \chi_{L+\delta_n} f + \chi_{\Lambda_n+\delta_n} S^n g$. We get a polynomial P_n such that

$$\|f - P_n\|_L < \theta \quad \text{and} \quad \|S^n g - P_n\|_{\Lambda_n} < \frac{\theta}{K_n}.$$

Hence,

$$\|f - P_n\|_L < \theta \quad \text{and} \quad \|g - T^n P_n\|_L < \theta.$$

Thus, $P_n \in U \cap T^{-n}V$ for all $n \geqslant n_0$ and T is a mixing operator as we wanted to prove.

(iii) Let $\frac{b}{1-\lambda} = (\frac{b_1}{1-\lambda_1}, \ldots, \frac{b_N}{1-\lambda_N})$ where, if $b_j = 0$ and $\lambda_j = 1$ for some $j = 1, \ldots, N$, we will understand that $\frac{b_j}{1-\lambda_j} = 0$. Then $\frac{b}{1-\lambda}$ is a fixed point of ϕ, and thus

$$T^n f\Big(\frac{b}{1-\lambda}\Big) = \lambda^{n(n-1)/2\alpha} D^{n\alpha} f\Big(\frac{b}{1-\lambda}\Big).$$

Applying the Cauchy estimates we obtain

$$\Big|T^n f\Big(\frac{b}{1-\lambda}\Big)\Big| \leqslant |\lambda^\alpha|^{n(n-1)/2} \Big|D^{n\alpha} f\Big(\frac{b}{1-\lambda}\Big)\Big| \leqslant \frac{|\lambda^\alpha|^{n(n-1)/2}(n\alpha)!}{r^{n|\alpha|}} \sup_{\|z\|\leqslant r} |f(z)| \underset{n\to\infty}{\longrightarrow} 0.$$

Since the evaluation at the vector $\frac{b}{1-\lambda}$ is a continuous functional, this implies that the orbit of f under T is not dense. ∎

Notice that in case (ii) of Theorem 3.4 we do not know if the operator $C_\phi \circ D^\alpha$ is strongly mixing in the Gaussian sense or even frequently hypercyclic. If $|\lambda_i| \leqslant 1$ for $1 \leqslant i \leqslant N$, we are able to show that the operator is frequently hypercyclic. To achieve this we prove that $C_\phi \circ D^\alpha$ is Runge transitive.

DEFINITION 3.9. An operator T on a Fréchet space X is called *Runge transitive* if there is an increasing sequence (p_n) of seminorms defining the topology of X and numbers $N_m \in \mathbb{N}$, $C_{m,n} > 0$ for $m, n \in \mathbb{N}$ such that:

(i) for all $m, n \in \mathbb{N}$ and $x \in X$,

$$p_m(T^n x) \leqslant C_{m,n} p_{n+N_m}(x);$$

(ii) for all $m, n \in \mathbb{N}$, $x, y \in X$ and $\varepsilon > 0$ there is some $z \in X$ such that

$$p_n(z - x) < \varepsilon \quad \text{and} \quad p_m(T^{n+N_m} z - y) < \varepsilon.$$

The concept of Runge transitivity was introduced by Bonilla and Grosse-Erdmann in [7]. They proved in Theorem 3.3 of [7] that every Runge transitive operator on a Fréchet space is frequently hypercyclic. They also showed that every translation operator on $H(\mathbb{C})$ is Runge transitive. However, the differentiation operator on $H(\mathbb{C})$ is not Runge transitive, even though we know that it is strongly mixing in the Gaussian sense. Now, we prove that some of the operators which are included in the case (ii) are frequently hypercyclic.

PROPOSITION 3.10. *Let T be the operator on $H(\mathbb{C}^N)$, defined by $Tf(z) = C_\phi \circ D^\alpha f(z)$, with $\alpha \neq 0$, $\phi(z) = (\lambda_1 z_1 + b_1, \ldots, \lambda_N z_N + b_N)$ and $\lambda_i \neq 0$ for all i, $1 \leqslant i \leqslant N$. Then, if $|\lambda_i| \leqslant 1$ for every i, $1 \leqslant i \leqslant N$ and we have that $b_j \neq 0$ and $\lambda_j = 1$ for some j, $1 \leqslant j \leqslant N$, then T is Runge transitive.*

Proof. Define the increasing sequence of seminorms

$$p_m(f) = \sup_{\prod_{i=1}^N B(0, r_i(m))} |f(z)|,$$

where the radiuses $r_i(m)$ are defined as follows:

$$r_i(m) = \begin{cases} |b_i| m & \text{if } b_i \neq 0, \\ m & \text{if } b_i = 0. \end{cases}$$

We will prove that both conditions of the Definition 3.9 are satisfied with $N_m = m+1$. For the first condition, we proceed as in the proof of part (iii) of Theorem 3.4. We will apply several times the Cauchy inequalities (3.4) with ε_i defined as

$$\varepsilon_i = \begin{cases} \frac{|b_i|}{2} & \text{if } b_i \neq 0, \\ \frac{1}{2} & \text{if } b_i = 0, \end{cases}$$

and in each step we divide it by 2. So, we get that

$$p_m(T^n f) \leqslant \frac{2^{(n(n+1)/2)(|\alpha|+N)} \alpha!^n}{(2\pi)^{nN} \varepsilon_1^{n(\alpha_1+1)} \cdots \varepsilon_N^{n(\alpha_N+1)}} \sup_{\Lambda} |f(z)|,$$

where $\Lambda = \prod B\Big(\phi_i^n(0), |\lambda_i|^n r_i(m) + \varepsilon_i \sum_{k=0}^{n-1} \frac{|\lambda_i|^k}{2^{n-k-1}}\Big)$.

Since $|\lambda_i| \leqslant 1$ for every i, $1 \leqslant i \leqslant N$, we obtain that

$$|\phi_i^n(0)| = \Big|b_i \sum_{k=0}^{n-1} \lambda_i^k\Big| \leqslant |b_i| n,$$

and that

$$|\lambda_i|^n r_i(m) + \varepsilon_i \sum_{k=0}^{n-1} \frac{|\lambda_i|^k}{2^{n-k-1}} \leqslant r_i(m) + 2\varepsilon_i.$$

From here it is easy to prove that $\Lambda \subseteq \prod B(0, r_i(n+m+1))$. Thus, if we denote

$$C_{m,n} = \frac{2^{(n(n+1)/2)(|\alpha|+N)} \alpha!^n}{(2\pi)^{nN} \varepsilon_1^{n(\alpha_1+1)} \cdots \varepsilon_N^{n(\alpha_N+1)}},$$

we get that

$$p_m(T^n f) \leqslant C_{m,n} p_{n+m+1}(f).$$

Suppose that ε is a positive number, n and m are two integer numbers and that f, g are two holomorphic functions on $H(\mathbb{C}^N)$, we want to prove that there exists some function $h \in H(\mathbb{C}^N)$ such that

$$p_n(f-h) < \varepsilon \quad \text{and} \quad p_m(T^{n+m+1}h - g) < \varepsilon.$$

Similarly, for the second condition we can estimate $p_m(T^{n+m+1}h - g)$ in the same way we did previously by making use of the right inverse for T. We get that

$$p_m(T^{n+m+1}h - g) \leqslant C \sup_{\Gamma} |S^{n+m+1}g - h|$$

where C is some positive constant and

$$\Gamma = \prod B\Big(\phi_i^{n+m}(0), |\lambda_i|^{n+m+1} r_i(m) + \varepsilon_i \sum_{k=0}^{n+m} \frac{|\lambda_i|^k}{2^{n-k-1}}\Big).$$

To assure the existence of such function h, by Runge's Theorem 3.2, it is enough to prove that $\Gamma \cap \prod B(0, r_i(n)) = \emptyset$. We study this set in the j-th coordinate. We get that

$$\Gamma_j = B(b_j(n+m), r_j(m) + 2\varepsilon_j) = B(b_j(n+m), |b_j|(m+1)),$$

which is disjoint from $B(0, |b_j| n)$. Then, we have proved that the operator T is Runge transitive, hence it is frequently hypercyclic. ∎

4. THE NON-DIAGONAL CASE

We are now interested in the case in which the automorphism $\phi(z) = Az + b$, is given by any invertible matrix $A \in \mathbb{C}^{N\times N}$. Let $v \neq 0$ be any vector in $\mathbb{C}^N$ and let T be the operator on $H(\mathbb{C}^N)$ defined by

$$Tf(z) = C_\phi \circ D_v f(z) = D_v f(Az + b),$$

where $D_v f$ is the differential operator in the direction of v,

$$D_v f(z_0) = \lim_{s\to 0} \frac{f(z_0 + sv) - f(z_0)}{s} = \nabla f(z_0)\cdot v = \mathrm{d}f(z_0)(v).$$

The next two remarks show that we may consider a simplified version of the operator T.

REMARK 4.1. We can assume that the matrix A is given in its Jordan form. Indeed, let Q be an invertible matrix in $\mathbb{C}^{N\times N}$ such that $A = QJQ^{-1}$, where J is the Jordan form of A. Also let $c = Q^{-1}b$ and denote $Q^*(f)(z) = f(Qz)$ for $f \in H(\mathbb{C}^N)$. Thus, we have that

$$Q^*(C_\phi \circ D_v f)(z) = \nabla f(AQz + b)\cdot v.$$

If we denote $\psi(z) = Jz + c$ and $w = Q^{-1}v$ then,

$$(C_\psi \circ D_w)Q^*(f)(z) = \nabla f(Q(Jz + c))\cdot Qw = \nabla f(AQz + b)\cdot v.$$

We have proved that the following diagram commutes:

$$\begin{tikzcd} H(\mathbb{C}^N) \arrow[r, "C_\phi\circ D_v"] \arrow[d, "Q^*"'] & H(\mathbb{C}^N) \arrow[d, "Q^*"] \\ H(\mathbb{C}^N) \arrow[r, "C_\psi\circ D_w"'] & H(\mathbb{C}^N) \end{tikzcd}$$

This shows that $C_\phi \circ D_v$ is linearly conjugate to $C_\psi \circ D_w$.

REMARK 4.2. We can assume that $b = 0$ if the affine linear map ϕ has a fixed point $z_0 = \phi(z_0)$. Indeed, if we denote $\varphi(z) = Az$, then

$$\tau_{z_0}(C_\phi \circ D_v)(f)(z) = D_v(f)(A(z + z_0) + b) = \tau_{z_0} D_v(f)(Az) = (C_\varphi \circ D_v)\tau_{z_0}(f)(z).$$

We have that the following diagram commutes:

$$\begin{tikzcd} H(\mathbb{C}^N) \arrow[r, "C_\phi\circ D_v"] \arrow[d, "\tau_{z_0}"'] & H(\mathbb{C}^N) \arrow[d, "\tau_{z_0}"] \\ H(\mathbb{C}^N) \arrow[r, "C_\varphi\circ D_v"'] & H(\mathbb{C}^N) \end{tikzcd}$$

We conclude that $C_\phi \circ D_v$ is linearly conjugate to $C_\varphi \circ D_v$.

The first two results of this section deal with affine transformations that have fixed points.

PROPOSITION 4.3. *Let $A \in \mathbb{C}^{N\times N}$ be an invertible matrix and let v be a nonzero vector in $\mathbb{C}^N$. Suppose that the affine linear map $\phi(z) = Az + b$ has a fixed point and that*

$$\lim_{k\to\infty} k! \prod_{i=0}^{k-1} \|A^i v\| < +\infty.$$

Then the operator $C_\phi \circ D_v$ acting on $H(\mathbb{C}^N)$ is not hypercyclic.

Consequently, $C_\phi \circ D_v$ is not hypercyclic if v belongs to an invariant subspace M of A such that the spectral radius of the restriction, $r(A|_M)$, is less than 1. This happens in particular if $r(A) < 1$ or if v is an eigenvector of A associated to an eigenvalue of modulus strictly less than 1.

Proof. We denote by $\mathrm{d}^k f(z)$ to the k-th differential of a function f at z, which is a k-homogenous polynomial, and we denote by $(\mathrm{d}^k f)^\vee(z)$ to the associated symmetric k-linear form.

It is not difficult to see that the orbits of the operator $C_\phi \circ D_v$ are determined by

$$(C_\phi \circ D_v)^k f(z) = (\mathrm{d}^k f)^\vee(\phi^k z)(v, Av, \dots, A^{k-1}v).$$

Assume that z_0 is a fixed point of ϕ, then applying Cauchy's inequalities we get

$$\begin{aligned}|(C_\phi \circ D_v)^k f(z_0)| &= |(\mathrm{d}^k f)^\vee(\phi^k z_0)(v, Av, \dots, A^{k-1}v)| \\ &= |(\mathrm{d}^k f)^\vee(z_0)(v, Av, \dots, A^{k-1}v)| \leqslant k! \prod_{i=0}^{k-1} \|A^i v\| \sup_{|z-z_0|<1} |f(z)|.\end{aligned}$$

Therefore $\{(C_\phi \circ D_v)^k f(z_0)\}$ is a bounded set of $\mathbb{C}$. Since the evaluation at z_0 is continuous, $C_\phi \circ D_v$ cannot have dense orbits.

For the last assertion, first note that if $J = Q^{-1}AQ$ is the Jordan form of A, we have that $w = Q^{-1}v$ belongs to the invariant subspace $Q^{-1}M$ of J and that $r := r(J|_{Q^{-1}M}) < 1$. By Remarks 4.1 and 4.2 it suffices to prove that $C_J \circ D_w$ is not hypercyclic.

It is not difficult to show that for every $i \geqslant N$,

$$\|J^i w\| \leqslant c r^{i-N} i^N \|w\|,$$

where c is a constant that depends only on r and N. Therefore,

$$\begin{aligned}k! \prod_{i=0}^{k-1} \|J^i w\| &\leqslant k! \prod_{i=0}^{N-1} \|J^i w\| \prod_{i=N}^{k-1} c r^{i-N} i^N \|w\| \\ &\leqslant (k!)^{N+1} \|J\|^{(N+1)N/2} c^{k-N} \|w\|^k r^{(k-N)(k-N-1)/2} \to 0,\end{aligned}$$

which implies that $C_J \circ D_w$ is not hypercyclic by the first part of the proposition. ∎

In contrast with the previous result, if the matrix A is expansive when restricted to an invariant subspace then the operator is strongly mixing in the Gaussian sense. This assumption is analogous to the hypothesis of the results in the previous sections. Indeed, in the one dimensional case we have that $\phi(z) = \lambda z + b$ and if $|\lambda| \geqslant 1$, then the operator $C_\phi \circ D$ is strongly mixing in the Gaussian sense. Here, the linear part of the composition operator is expansive. This situation still holds in the diagonal case in $H(\mathbb{C}^N)$. In this last case, we have that $\phi(z_1, \dots, z_N) = (\lambda_1 z_1 + b_1, \dots, \lambda_N z_N + b_N)$. Suppose that α is a multi-index of modulus one, i.e. that D^α is a partial derivative, then the hypothesis $|\lambda^\alpha| \geqslant 1$ turns out to be exactly the same as imposing that the linear part of ϕ is expansive on the subspace spanned by α. The precise result may be stated as follows.

PROPOSITION 4.4. *Let $A \in \mathbb{C}^{N\times N}$ be an invertible matrix and let $v \neq 0$ be a vector in $\mathbb{C}^N$. Suppose that the affine linear map $\phi(z) = Az + b$ has a fixed point and that v belongs to a subspace M that reduces A and such that $\|(A|_M)^{-1}\| < 1$. Then the operator $C_\phi \circ D_v$ acting on $H(\mathbb{C}^N)$ is strongly mixing in the Gaussian sense.*

Proof. We will show that the hypotheses of the Theorem 1.5 are fulfilled, taking as dense sets the polynomials in N complex variables. It is clear that $\sum_n T^n f$ converges unconditionally for every polynomial f. Now we will define a right inverse for $C_\phi \circ D_v$, but first we set some notation. Let us denote the fixed point of ϕ by z_0. Let us denote by π_1 the orthogonal projection over M, $\pi_2 = I - \pi_1$ the orthogonal projection over $M^\perp$. Set $\mu(z) = \frac{\langle z,v\rangle}{\|v\|^2}$. We have that $z \mapsto \mu(z)v$ is the orthogonal projection over $\operatorname{span}\{v\}$, and we denote $\widetilde{\pi} = \pi_1 - \mu(z)v$. Finally, set $\phi_i(z) = Az + \pi_i(b)$, for $i = 1, 2$. Since, M reduces A, we have that ϕ_i is invertible and that $\pi_i(z_0)$ is a fixed point of ϕ_i, for $i = 1, 2$.

We define now for each $g \in H(\mathbb{C}^N)$,

$$Rg(z) = \int_{\mu(z_0)}^{\mu(z)} g(\phi_1^{-1}(tv + \widetilde{\pi}(z)) + \pi_2(z))\mathrm{d}t,$$

and $C(g)(z) = g(\pi_1(z) + \phi_2^{-1}(\pi_2(z)))$. Note that $R \circ C = C \circ R$. Finally, let $S = C \circ R$. Observe that, $Sg(z) = \int_{\mu(z_0)}^{\mu(z)} g(\phi^{-1}(tv + \widetilde{\pi}(z) + \pi_2(z)))\mathrm{d}t$. We have that

$$\begin{aligned} D_v Sg(z) &= \lim_{s\to 0} \frac{Sg(z+sv) - Sg(z)}{s} \\ &= \lim_{s\to 0} \frac{1}{s}\Big[\int_{\mu(z_0)}^{\mu(z+sv)} g(\phi^{-1}(tv+\widetilde{\pi}(z)+\pi_2(z)))\mathrm{d}t - \int_{\mu(z_0)}^{\mu(z)} g(\phi^{-1}(tv+\widetilde{\pi}(z)+\pi_2(z)))\mathrm{d}t \Big] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{s\to 0} \frac{1}{s} \int_{\mu(z)}^{\mu(z)+s} g(\phi^{-1}(tv + \widetilde{\pi}(z) + \pi_2(z)))\mathrm{d}t \\
&= g(\phi^{-1}(\mu(z)v + \widetilde{\pi}(z) + \pi_2(z))) = g(\phi^{-1}z).
\end{aligned}
$$

Thus, $[C_\phi \circ D_v] \circ Sg = g$ for every $g \in H(\mathbb{C}^N)$. To conclude the proof we need to show that $\sum_n S^n g$ converges unconditionally for every polynomial g.

First we will bound the supremum of $|Rg|$ on $B(\pi_1 z_0, r) \times B(\pi_2 z_0, s)$, for a fixed polynomial g. Suppose that $z \in B(\pi_1 z_0, r) \times B(\pi_2 z_0, s)$ and that $t \in [\mu(z_0), \mu(z)]$ i.e. t lives in the complex segment from $\mu(z_0)$ to $\mu(z)$. Then we have that

$$
\begin{aligned}
\|tv+\widetilde{\pi}(z)-\pi_1 z_0\|^2 &= \|(t-\mu(z_0))v+\widetilde{\pi}(z-z_0)\|^2 = |t-\mu(z_0)|^2\|v\|^2+\|\widetilde{\pi}(z-z_0)\|^2 \\
&\leqslant |\mu(z)-\mu(z_0)|^2\|v\|^2 + \|\widetilde{\pi}(z-z_0)\|^2 = \|\pi_1(z-z_0)\|^2 < r^2.
\end{aligned}
$$

Also, suppose that $\sigma := \|(A|_M)^{-1}\| < 1$. We get that

$$
\begin{aligned}
\|\phi_1^{-1}(\pi_1(z)) - \pi_1(z_0)\| &= \|\phi_1^{-1}(\pi_1(z)) - \phi_1^{-1}(\pi_1(z_0))\| \\
&= \|A^{-1}(\pi_1(z) - \pi_1(b)) - A^{-1}(\pi_1(z_0) - \pi_1(b))\| \\
&\leqslant \|(A|_M)^{-1}\|\|\pi_1(z) - \pi_1(z_0)\| = \sigma r.
\end{aligned}
$$

Gathering the previous statements we get that

$$
\begin{aligned}
|Rg(z)| &\leqslant |\mu(z)-\mu(z_0)| \sup_{t\in[\mu(z_0),\mu(z)]} |g(\phi_1^{-1}(tv+\widetilde{\pi}(z)) + \pi_2(z))| \\
&\leqslant \frac{r}{\|v\|^2} \sup_{w\in B(\pi_1 z_0, r)\times B(\pi_2 z_0, s)} |g(\phi_1^{-1}(\pi_1(w)) + \pi_2(w))| \\
&\leqslant \frac{r}{\|v\|^2} \sup_{w\in B(\pi_1 z_0, \sigma r)\times B(\pi_2 z_0, s)} |g(w)|.
\end{aligned}
$$

Thus, we have proved that

$$
\sup_{B(\pi_1 z_0, r)\times B(\pi_2 z_0, s)} |Rg| \leqslant \frac{r}{\|v\|^2} \sup_{B(\pi_1 z_0, \sigma r)\times B(\pi_2 z_0, s)} |g|.
$$

Following by induction we obtain that

$$
\begin{aligned}
\sup_{B(\pi_1 z_0, r)\times B(\pi_2 z_0, s)} |R^n g| &\leqslant \frac{r}{\|v\|^2} \sup_{B(\pi_1 z_0, \sigma r)\times B(\pi_2 z_0, s)} |R^{n-1} g| \\
&\leqslant \frac{r^n}{\|v\|^{2n}} \sigma^{n(n-1)/2} \sup_{B(\pi_1 z_0, \sigma^n r)\times B(\pi_2 z_0, s)} |g|.
\end{aligned}
$$

Finally, to conclude the proof we compute

$$
\sup_{B(\pi_1 z_0, r)\times B(\pi_2 z_0, s)} |S^n g(z)| = \sup_{B(\pi_1 z_0, r)\times B(\pi_2 z_0, s)} |R^n C^n g(z)|
$$

$$\leqslant \frac{r^n}{\|v\|^{2n}}\sigma^{n(n-1)/2} \sup_{B(\pi_1 z_0,\sigma^n r)\times B(\pi_2 z_0,s)} |C^n g(z)|$$
$$\leqslant \frac{r^n}{\|v\|^{2n}}\sigma^{n(n-1)/2} \sup_{B(\pi_1 z_0,\sigma^n r)\times \phi_2^{-n}(B(\pi_2 z_0,s))} |g(z)|.$$

Since $\sigma < 1$, we have proved that $\sum_n S^n g$ converges unconditionally for every polynomial g. Hence the operator $C_\phi \circ D_v$ is strongly mixing in the Gaussian sense, as we wanted to prove. ∎

We turn now our discussion to the case in which the affine linear map $\phi(z) = Az + b$ does not have a fixed point. This is equivalent to say that $b \notin \mathrm{Ran}(I - A)$. Thus, 1 belongs to the spectrum of A. Then the Jordan form of A, which we denote by J, has a sub-block with ones in the principal diagonal and the first sub-diagonal and zeros elsewhere. It is easy to see that there exists some $k \in \mathbb{N}$, $k \leqslant N$ such that the canonical vector e_k does not belong to $\mathrm{Ran}(I - J)$ and such that $b_k \neq 0$. This argument will be the key to show that ϕ is a runaway map, hence the operator $C_\phi \circ D_v$ is topologically transitive. The proof of this result is in the spirit of part (ii) of Theorem 3.4.

PROPOSITION 4.5. *Let $A \in \mathbb{C}^{N\times N}$ be an invertible matrix and let $v \neq 0$ be a vector in $\mathbb{C}^N$. Suppose that the affine linear map $\phi(z) = Az + b$ does not have a fixed point. Then the operator $C_\phi \circ D_v$ acting on $H(\mathbb{C}^N)$ is mixing.*

Proof. Due to the previous observations it is enough to prove that $C_\psi \circ D_w$ is topologically transitive if $\psi(z) = Jz + b$ with $b \notin \mathrm{Ran}(I - J)$ and $w \in \mathbb{C}^N$, $w \neq 0$. We will denote $T = C_\psi \circ D_w$.

Given K_U, K_V two compact sets of $\mathbb{C}^N$, h_U, h_V two holomorphic functions in $H(\mathbb{C}^N)$ and θ a positive real number, we want to prove that there exists $k \in \mathbb{N}$ and $g \in H(\mathbb{C}^N)$ such that

$$\|g - h_U\|_{K_U} < \theta \quad \text{and} \quad \|(C_\psi \circ D_w)^k g - h_V\|_{K_V} < \theta. \tag{4.1}$$

We will use Runge's theorem to show the existence of such function g. As before, we denote by S the right inverse of D_w. We have that

$$\begin{aligned}\sup_{K_V} |C_\psi \circ D_w g(z) - h_V(z)| &= \sup_{K_V} |C_\psi (D_w g(z) - C_{\psi^{-1}} h_V(z))| \\ &= \sup_{C_\psi(K_V)} |D_w (g(z) - C_{\psi^{-1}} h_V(z))| \\ &= \sup_{J(K_V)+b} |D_w (g(z) - S \circ C_{\psi^{-1}} h_V(z))| \\ &\leqslant \frac{\|w\| N}{\varepsilon_1^N} \sup_{J(K_V)+B_{\varepsilon_1}(b)} |g(z) - S \circ C_{\psi^{-1}} h_V(z)|.\end{aligned}$$

Following in this way inductively, we will get an estimate of $\|(C_\psi \circ D_w)^k g - h_V\|_{K_V}$,

$$\sup_{K_V} |(C_\psi \circ D_w)^l g(z) - h_V(z)| \leqslant \alpha(l) \sup_{A_l} |g(z) - (S \circ C_{\psi^{-1}})^l h_V(z)|,$$

with $\alpha(l) > 0$ and $A_l = J^l(K_V) + \sum_{i=1}^{l} J^i(B(0, \varepsilon_i)) + \sum_{i=1}^{l} J^i(b)$.

It is enough to find some $l \in \mathbb{N}$ such that $K_U \cap A_l = \emptyset$. Without loss of generality we can assume that $e_1 \notin \mathrm{Ran}(J - I)$ and $b_1 \neq 0$ (see the comments before the proposition). This means that J acts like the identity in the first coordinate.

Suppose that $K_V \subset \prod_{i=1}^{N} B(0, r_i)$, then if we project in the first coordinate and choose proper $\varepsilon_i > 0$ we obtain

$$\begin{aligned}[A_l]_1 &= [J^l(K_V)]_1 + \sum_{i=1}^{l} [J^i(B(0, \varepsilon_i))]_1 + \sum_{i=1}^{l} [J^i(b)]_1 \\ &\subset B(0, r_1) + B\left(0, \textstyle\sum_{i=1}^{l} \varepsilon_i\right) + l b_1 \subset B(0, R) + l b_1.\end{aligned}$$

Thus, we will be able to find $l_0 \in \mathbb{N}$ such that $[K_U]_1 \cap [A_l]_1 = \emptyset$ for all $l \geqslant l_0$. Therefore, by Runge's theorem, there exists some $g_l \in H(\mathbb{C}^N)$ such that (4.1) is satisfied for all $l \geqslant l_0$. We have proved that the operator $C_\psi \circ D_w$ is mixing, as we wanted to prove. ∎

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