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POISSON WAVE TRACE FORMULA FOR PERTURBED DIRAC OPERATORS

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ABSTRACT. We consider self-adjoint Dirac operators $\mathbb{D} = \mathbb{D}_0 + V(x)$, where $\mathbb{D}_0$ is the free three-dimensional Dirac operator and $V(x)$ is a smooth compactly supported Hermitian matrix. We define resonances of $\mathbb{D}$ as poles of the meromorphic continuation of its cut-off resolvent. An upper bound on the number of resonances in disks, an estimate on the scattering determinant and the Lifshits–Krein trace formula then leads to a global Poisson wave trace formula for resonances of $\mathbb{D}$.

KEYWORDS: *Dirac operator, resonances, Poisson wave trace.*

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1. INTRODUCTION AND MAIN RESULT

For suitable perturbations L of the Laplacian $-\Delta$, on $L^2(\mathbb{R}^n)$ with n odd, the first Poisson wave trace formula (in the sense of distributions) of the form

$$2\,\mathrm{Tr}\,(\cos t\sqrt{L} - \cos t\sqrt{-\Delta}) = \sum \mathrm{e}^{\mathrm{i}t\lambda_j}, \tag{1.1}$$

where the sum extends over all resonances λ_j of L, appeared in Lax and Phillips [11]. They proved the formula for obstacle scattering, but only for $t > 4R$ where the obstacle is located within a ball of radius R. Bardos, Guillot and Ralston [3] investigated the precise distribution of scattering poles associated with perturbations of the wave equation in odd dimensions and one of their main ingredients is an extension of the trace formula (1.1), still based on Lax–Phillips theory [11], but again only valid for comparatively large values of t, namely $t > 2R$. Later the Poisson formula (1.1) was proved for all $t \neq 0$ by Melrose [14] in the case of compactly supported potentials and then further generalized by Sjöstrand and Zworski [23] to more general L.

The main application of the Poisson trace formula has been to obtain lower bounds for the number of resonances in certain regions and to prove the existence of infinitely many resonances, see e.g. [16], [22] and [23].

Zworski [26] obtained a new proof of the trace formula that avoids the use of Lax–Phillips theory and instead is based on an estimate of the scattering determinant in $\mathbb{C}$. This proof allowed Zworski to show the trace formula also in the even-dimensional case, see [27], and his proof also motivated the present work.

Proceeding to the relativistic setting, the only trace formula for Dirac operators involving resonances that we know of is that of Khochman [7], where a local trace formula for resonances in the spirit of Sjöstrand [20] is established.

In the present work we consider Dirac operators $\mathbb{D} = \mathbb{D}_0 + V(x)$ where $\mathbb{D}_0$ is the free Dirac operator in three dimensions (see below) and V is a smooth compactly supported matrix potential. We define resonances as poles of the meromorphic continuation of the cut-off resolvent. Then, following the strategy of Zworski [27] (see also [26]) we establish an upper bound for the resonance counting function, estimate the scattering determinant and apply the Lifshits–Krein trace formula to obtain, in the distributional sense, the following Poisson wave trace formula for the perturbed Dirac operator.

THEOREM 1.1. *Let $\mathcal{R}$ denote the set of resonances of $\mathbb{D}$ away from ± 1 and let m_j be the multiplicity of a resonance λ_j (see Section 2.2 for precise definitions). Then, in the sense of distributions on $\mathbb{R} \setminus \{0\}$,*

$$\begin{aligned} &2\,\mathrm{Tr}\left(\cos(t\mathbb{D}) - \cos(t\mathbb{D}_0)\right) \\ &= \sum_{\lambda_j \in \mathcal{R}\cap\mathbb{C}_+} m_j(\mathrm{e}^{-\mathrm{i}|t|\overline{\lambda}_j} + \mathrm{e}^{\mathrm{i}|t|\lambda_j}) - \sum_{\lambda_j \in \mathcal{R}\cap\mathbb{C}_-} m_j(\mathrm{e}^{-\mathrm{i}|t|\lambda_j} + \mathrm{e}^{\mathrm{i}|t|\overline{\lambda}_j}) + \sum_{\lambda_j \in \mathrm{spec}_{\mathrm{d}}(\mathbb{D})} 2m_j \cos(t\lambda_j). \end{aligned} \tag{1.2}$$

Above $\mathbb{C}_\pm = \{\lambda \in \mathbb{C} : \pm\mathrm{Im}\,\lambda > 0\}$. Existence of resonances of the semiclassical Dirac operator perturbed by smooth, bounded and real-valued scalar potentials V decaying like $\langle x \rangle^{-\delta}$ at infinity for some $\delta > 0$ was recently established by Kungsman and Melgaard [10]. By studying analytic singularities of a certain distribution related to V and by combining two trace formulas, we proved that the perturbed Dirac operators possesses resonances near $\sup V + 1$ and $\inf V - 1$. We also provided a lower bound for the number of resonances near these points expressed in terms of the semiclassical parameter.

Henceforth we denote various positive constants for which the exact numerical values are of no importance by C. These constants may change from line to line without this being indicated.

2. PRELIMINARIES

2.1. THE DIRAC OPERATOR. To discuss perturbed Dirac operators we begin by considering the free, or unperturbed, Dirac operator. The free Dirac operator, describing the motion of a relativistic electron or positron without external forces, is the unique self-adjoint extension of the symmetric operator

$$\mathbb{D}_0 = -\mathrm{i}\sum_{j=1}^{3}\alpha_j\partial_j + \beta = -\mathrm{i}\boldsymbol{\alpha}\cdot\nabla + \beta, \quad \partial_j := \frac{\partial}{\partial x_j}, \ \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3),$$

defined on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ in the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)$. Here the α_j are symmetric 4×4 matrices satisfying the anti-commutation relations

$$\alpha_j\alpha_k + \alpha_k\alpha_j = 2\delta_{jk}I_4, \quad j,k = 1,2,3,$$
$$\alpha_j\beta + \beta\alpha_j = 0, \quad j = 1,2,3,$$

and $\beta^2 = I_4$; the 4×4 identity matrix. The extension, which we also denote by $\mathbb{D}_0$, acts on the Hilbert space $L^2(\mathbb{R}^3;\mathbb{C}^4)$ equipped with the inner product

$$\langle u, v\rangle_{L^2(\mathbb{R}^3;\mathbb{C}^4)} = \sum_{j=1}^{4}\int_{\mathbb{R}^3} u_j(x)\overline{v_j(x)}\,\mathrm{d}x \quad \text{where } u = (u_j)_{1\leqslant j\leqslant 4},\ v = (v_j)_{1\leqslant j\leqslant 4}$$

and it has domain $H^1(\mathbb{R}^3;\mathbb{C}^4)$, the Sobolev space of order one. When there is no risk of confusion we sometimes just write L^2 and H^1, respectively . It is well-known (see, e.g., Thaller [24]) that the spectrum of $\mathbb{D}_0$ is purely absolutely continuous, viz.

$$\operatorname{spec}(\mathbb{D}_0) = \operatorname{spec}_{\mathrm{ac}}(\mathbb{D}_0) = (-\infty, -1] \cup [1, \infty).$$

On the resolvent set $\mathbb{C}\setminus\operatorname{spec}(\mathbb{D}_0)$ we denote the free resolvent $(\mathbb{D}_0 - \lambda)^{-1}$ by $R_0(\lambda)$. As usual the Fourier transform is defined by

$$(\mathscr{F}u)(\xi) = \widehat{u}(\xi) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}x\cdot\xi}u(x)\,\mathrm{d}x.$$

The symbol of the free Dirac operator $\mathbb{D}_0$ is given by

$$\boldsymbol{d}_0(\xi) = \mathscr{F}\mathbb{D}_0\mathscr{F}^* = \sum_{j=1}^{3}\alpha_j\xi_j + \beta$$

and it has two doubly degenerate eigenvalues $\pm\sqrt{\xi^2+1} =: \pm\langle\xi\rangle$. Here $\surd$ is the holomorphic square-root on $\mathbb{C}\setminus[0,\infty)$. The corresponding orthogonal projections onto the eigenspaces are given by

$$\Pi_\pm(\xi) = \frac{1}{2}(I_4 \pm \langle\xi\rangle^{-1}\boldsymbol{d}_0(\xi)). \tag{2.1}$$

We are going to consider perturbations of $\mathbb{D}_0$ by smooth compactly supported Hermitian 4×4 matrix potentials $V \in C_0^\infty(\mathbb{R}^3)\otimes M_4(\mathbb{C})$; $M_4(\mathbb{C})$ being the set of 4×4 matrices over $\mathbb{C}$, equipped with the operator norm, designated by $\|\cdot\|_{4\times 4}$. The resulting self-adjoint operator $\mathbb{D} = \mathbb{D}_0 + V$ (defined via Kato–Rellich's theorem) has domain $H^1(\mathbb{R}^3;\mathbb{C}^4)$ and, according to Weyl's theorem,

$\mathrm{spec}_{\mathrm{ess}}(\mathbb{D}) = (-\infty, -1] \cup [1, \infty)$ but, in addition, $\mathbb{D}$ can have finitely many eigenvalues of finite multiplicity in $(-1, 1)$ (see, e.g., Theorem 4.23 of [24]). It is well-known that under our assumptions on V there are no eigenvalues λ with $|\lambda| > 1$ embedded in the continuous spectrum (see, e.g., [1]).

When $A : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4)$ is a compact operator the eigenvalues of $(A^*A)^{1/2}$, indexed in non-increasing order, are called the singular values of A and are denoted by $s_j(A)$. The following inequalities are well-known (see e.g. [19]):

$$s_j(AB) \leqslant \|B\| s_j(A), \quad A \in \mathcal{B}_\infty, B \in \mathcal{B}, \tag{2.2}$$

$$s_{j+k-1}(A+B) \leqslant s_j(A) + s_k(B), \quad A, B \in \mathcal{B}_\infty, \tag{2.3}$$

$$s_{j+k-1}(AB) \leqslant s_j(A) s_k(B), \quad A, B \in \mathcal{B}_\infty, \tag{2.4}$$

where $\mathcal{B}$ and $\mathcal{B}_\infty$ denote the spaces of bounded and compact operators on the space $L^2(\mathbb{R}^3; \mathbb{C}^3)$, respectively. As a consequence of Weyl's inequality we also have

$$|\det(I - A)| \leqslant \prod_{j=1}^{\infty} (1 + s_j(A)) \quad \text{for } A \in \mathcal{B}_1, \tag{2.5}$$

where $\mathcal{B}_1$ is the set of trace class operators.

2.2. RESONANCES. By virtue of (B.3) the free cut-off resolvent $\chi R_0(\lambda)\chi$, $\chi \in C_0^\infty(\mathbb{R}^3)$, is an integral operator with kernel given by

$$\chi(x) R_0(\lambda, x, y) \chi(y) = \chi(x) \Big(\mathrm{i} \frac{\boldsymbol{\alpha} \cdot (x-y)}{|x-y|^2} + \kappa(\lambda) \frac{\boldsymbol{\alpha} \cdot (x-y)}{|x-y|} + \beta + \lambda \Big) \frac{\mathrm{e}^{\mathrm{i}\kappa(\lambda)(x-y)}}{4\pi |x-y|} \chi(y),$$

for $\kappa(\lambda) := \sqrt{\lambda^2 - 1}$, on the branch with $\mathrm{Im}\, \sqrt{\lambda^2 - 1} > 0$. Thus $\chi R_0(\lambda) \chi : L^2(\mathbb{R}^3; \mathbb{C}^4) \to L^2(\mathbb{R}^3; \mathbb{C}^4)$ has a holomorphic extension from

$$\{\mathrm{Re}\, \lambda \leqslant 1, \mathrm{Im}\, \lambda < 0\} \cup \{\mathrm{Re}\, \lambda \geqslant -1, \mathrm{Im}\, \lambda > 0\}$$

across $(-\infty, -1] \cup [1, \infty)$ to the sheet with $\mathrm{Im}\, \sqrt{\lambda^2 - 1} < 0$.

We next consider the full resolvent $R_V(\lambda) := (\mathbb{D} - \lambda)^{-1}$ for $\chi \in C_0^\infty(\mathbb{R}^3)$ such that $\chi V = V$. If we take $|\mathrm{Im}\, \lambda|$ so large that $\|V R_0(\lambda) \chi\| \leqslant C |\mathrm{Im}\, \lambda|^{-1} \leqslant 1/2$ we may write

$$R_V(\lambda) = R_0(\lambda)(I + V R_0(\lambda))^{-1}.$$

Notice that for such λ we have

$$(I - V R_0(\lambda)(1 - \chi))(I + V R_0(\lambda)) = I + V R_0(\lambda) \chi,$$

since $(1 - \chi) V = 0$ and, consequently,

$$R_V(\lambda) = R_0(\lambda)(I + V R_0(\lambda) \chi)^{-1} (I - V R_0(\lambda)(1 - \chi)). \tag{2.6}$$

Since $V R_0(\lambda) \chi : L^2(\mathbb{R}^3; \mathbb{C}^4) \to H^1(\mathrm{supp}\, V; \mathbb{C}^4)$, we infer that $V R_0(\lambda) \chi$ is compact on $L^2(\mathbb{R}^3; \mathbb{C}^4)$ by the Rellich–Kondrachov theorem and, moreover, it depends holomorphically on λ since $R_0(\lambda)$ does. Thus $(I + V R_0(\lambda) \chi)^{-1}$ has a meromorphic extension by the analytic Fredholm theorem. Furthermore, from (2.6) we get for $\chi_0 \in C_0^\infty(\mathbb{R}^3)$ with $\chi_0 V = V$ and $\chi_0\chi = \chi_0$ that

$$\chi_0 R_V(\lambda)\chi_0 = \chi_0 R_0(\lambda)\chi_0(I + VR_0(\lambda)\chi)^{-1} \tag{2.7}$$

since initially $(I + VR_0(\lambda)\chi)^{-1}\chi_0 = \chi_0(I + VR_0(\lambda)\chi)^{-1}$ holds for $|\operatorname{Im}\lambda| \gg 1$ in the physical sheet by considering $(I + VR_0(\lambda)\chi)^{-1}$ as a Neumann series and remains true because both sides have meromorphic extensions. This provides the meromorphic extension of $\chi_0 R_V(\lambda)\chi_0$ for which the poles are those of $(I + VR_0(\lambda)\chi)^{-1}$. These poles will be referred to as resonances of $\mathbb{D}$ and the set of all resonances of $\mathbb{D}$ will be denoted by $\mathcal{R}$.

DEFINITION 2.1. Assume $\lambda_j \in \mathcal{R}$ and let γ_{λ_j} be the circle $\lambda_j + \varepsilon_j e^{i[0,2\pi]}$ where ε_j is chosen sufficiently small so that γ_{λ_j} encircles no other resonances but λ_j. The *multiplicity* of λ_j is then given by

$$m_j = \operatorname{rank} \int_{\gamma_j} R_V(\lambda)\, d\lambda. \tag{2.8}$$

The proof of the following upper bound, which goes back to Melrose [15] in the Schrödinger case, for the number of resonances in disks follows the strategy in Zworski [25] (see also Proposition 6.2 in Kungsman–Melgaard [9] and references therein).

PROPOSITION 2.2. *The following upper bound holds true:*

$$N(r) := \#\{\lambda \in \mathcal{R} : |\lambda| \leqslant r\} \leqslant Cr^3, \tag{2.9}$$

where the number of resonances are counted according to their multiplicities.

Proof. Recall, by (2.6), that the resonances of $\mathbb{D}$ can be characterized as poles of $(I + VR_0(\lambda)\chi)^{-1}$. Since

$$I - (VR_0(\lambda)\chi)^4 = (I + VR_0(\lambda)\chi)(I - VR_0(\lambda)\chi + \cdots - (VR_0(\lambda)\chi)^3)$$

and because (see below) $(VR_0(\lambda)\chi)^4 \in \mathcal{B}_1$, the resonances will appear among the zeros of

$$f(\lambda) = \det(I - (VR_0(\lambda)\chi)^4).$$

We first estimate $|f(\lambda)|$ by using the Weyl inequality (2.5):

$$|f(\lambda)| \leqslant \prod_{j=1}^{\infty}(1 + s_j((VR_0(\lambda)\chi)^4)). \tag{2.10}$$

In view of Ky–Fan's inequality (2.4) and the inequality

$$s_j(VR_0(\lambda)\chi) \leqslant \|V\|_\infty s_j(\chi R_0(\lambda)\chi)$$

it suffices to estimate the singular values of $\chi R_0(\lambda)\chi$. This can be done by comparing them to the singular values of the resolvent of a free Dirac operator on a sufficiently large flat torus $\mathbb{T} = (\mathbb{R}/R\mathbb{Z})^3$ with supp $(\chi) \subset B(0, R)$:

$$s_j(\chi R_0(\lambda)\chi) \leqslant s_j((\mathbb{D}_{\mathbb{T},0} - \mathrm{i})^{-1})\|\chi R_0(\lambda)\chi\|_{L^2\to H^1}.$$

It is well-known (see e.g. [9]) that $s_j((\mathbb{D}_{\mathbb{T},0} - \mathrm{i})^{-1}) \leqslant Cj^{-1/3}$. Moreover, on the branch of the square root where Im $(\kappa(\lambda)) > 0$ we obtain from (B.1) and (B.4) that $\|\chi R_0(\lambda)\chi\|_{L^2\to H^1} \leqslant C\langle\lambda\rangle$.

To estimate the singular values of the extended resolvent we use

$$\chi(R_0(\lambda) - \widetilde{R}_0(\lambda))\chi = 2\pi\mathrm{i}E_\chi(\overline{\lambda})^* E_\chi(\lambda),$$

(see (A.5) and (B.5)), where we temporarily denote the extended resolvent by $\widetilde{R}_0(\lambda)$. Therefore, it suffices to estimate

$$s_j(E_\chi(\overline{\lambda})^* E_\chi(\lambda)) \leqslant \|E_\chi(\lambda)\|_{L^2\to L^2(S^2)} s_j(E_\chi(\lambda)).$$

To this end, denote by Δ_ω the Laplace–Beltrami operator on S^2. Then

$$s_j(E_\chi(\lambda)) \leqslant s_j((I - \Delta_\omega)^{-k})\|(I - \Delta_\omega)^k E_\chi(\lambda)\|_{L^2(\mathbb{R}^3)\to L^2(S^2)},$$

where

$$s_j((I - \Delta_\omega)^{-k}) \leqslant C^k j^{-k} \quad \text{and} \quad \|(I - \Delta_\omega)^k E_\chi(\lambda)\|_{L^2(\mathbb{R}^3)\to L^2(S^2)} \leqslant (2k)!\mathrm{e}^{C|\lambda|}$$

are well-known (see, e.g., p. 72 of [21] for details). Hence, for $k = [j^{1/2}/(2C)] + 1$,

$$s_j(E_\chi(\lambda)) \leqslant j^{-k} C^k (2k)!\mathrm{e}^{C|\lambda|} \leqslant C_1 \mathrm{e}^{C_1|\lambda| - j^{1/2}/C_1},$$

by Stirling's formula. Since $\|E_\chi(\lambda)\|_{L^2\to L^2(S^2)} \leqslant C\mathrm{e}^{C|\lambda|}$ we now get

$$s_j((VR_0(\lambda)\chi)^4) \leqslant C_2 \exp(C_2|\lambda| - j^{-1/2}/C_2) + C_2 j^{-4/3},$$

so that, especially,

$$s_j((VR_0(\lambda)\chi)^4) \leqslant \begin{cases} C_3 \exp(C_3|\lambda|) & \text{for } j \leqslant C_3|\lambda|^2, \\ C_3 j^{-4/3} & \text{for } j \geqslant C_3|\lambda|^2. \end{cases}$$

It follows from (2.5) that

$$|f(\lambda)| \leqslant \prod_{j\leqslant C_3|\lambda|^2} (1 + C_3 \exp(C_3|\lambda|)) \Big(\exp\big(\textstyle\sum_{j\geqslant C_3|\lambda|^2} C_3 j^{-4/3}\big)\Big) \leqslant \exp(C_4|\lambda|^3). \tag{2.11}$$

The result now follows by an application of Jensen's formula (see, e.g., p. 63 of [21]). ∎

3. PROOF OF THE POISSON TRACE FORMULA

Let $s(\lambda)$ be the determinant of the scattering matrix, as introduced in Appendix A. Meromorphic extension of the identity $S(\lambda)^{-1} = S(\lambda)^*$ for any $\lambda \in \operatorname{spec}(\mathbb{D}_0)$ implies $S(\lambda)^{-1} = S(\overline{\lambda})^*$ and, therefore, the scattering determinant satisfies

$$\frac{1}{s(\lambda)} = \overline{s(\overline{\lambda})}.$$

It follows that if $z_j \in \mathcal{R}$ then $s(\overline{z_j}) = 0$ and vice versa. From the Weierstrass factorization theorem it follows that

$$s(\lambda) = \mathrm{e}^{g(\lambda)} \frac{P_{\overline{\mathcal{R}}}(\lambda)}{P_{\mathcal{R}}(\lambda)}, \tag{3.1}$$

where $P_{\mathcal{R}}$ is the canonical product

$$P_{\mathcal{R}}(\lambda) = \prod_{z_j \in \mathcal{R}} E_3\Big(\frac{\lambda}{z_j}\Big)^{m_j}, \quad E_p(z) = (1-z)\exp\Big(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\Big) \tag{3.2}$$

and g an entire function. We choose the genus to be $p = 3$ so the infinite product converges (see e.g. [5]). Moreover, it follows from Paper IV, Lemma C.1 of [8] together with Proposition 2.2 that

$$|P_{\mathcal{R}}(\lambda)| \leqslant C\mathrm{e}^{C|\lambda|^4}. \tag{3.3}$$

LEMMA 3.1. *For any constant $C > 0$ there exists a constant C_0 such that the inequality*

$$\prod_{j=1}^{\infty}(1 + C\mathrm{e}^{C|\lambda|^N - \sqrt{j}/C}) \leqslant C_0 \mathrm{e}^{C_0|\lambda|^{3N}}$$

holds for all $\lambda \in \mathbb{R}$.

Proof. Clearly

$$\sum_{j=1}^{\infty} \log(1 + C\mathrm{e}^{C|\lambda|^N - \sqrt{j}/C}) \leqslant \int_0^{\infty} \log(1 + C\mathrm{e}^{C|\lambda|^N - \sqrt{x}/C})\,\mathrm{d}x =: L,$$

and an upper bound of the product is given by e^L. Integrating by parts and denoting $\varphi_0(\lambda) = 2(\log C + C|\lambda|^N)$ we get

$$L = C^2 \int_0^{\infty} x^2 \frac{C\mathrm{e}^{C|\lambda|^N}}{\mathrm{e}^x + C\mathrm{e}^{C|\lambda|^N}}\,\mathrm{d}x \leqslant C^2 \int_0^{\varphi_0(\lambda)} x^2\,\mathrm{d}x + C^2 \int_{\varphi_0(\lambda)}^{\infty} \frac{x^2}{\mathrm{e}^{x/2}+1}\,\mathrm{d}x \leqslant C_1 + C_1|\lambda|^{3N}$$

and the result follows. ∎

Next we estimate the determinant of the scattering matrix. The proof of the following lemma is in the spirit of Zworski's work [26], [27] on Schrödinger operators; see also p. 9 of [17].

LEMMA 3.2. *For any $\varepsilon, \delta > 0$ we have*

$$|s(\lambda)| \leqslant C e^{|\lambda|^{9+3\varepsilon}} \quad \text{for } \lambda \notin \bigcup_{z_j \in \mathcal{R}} D(z_j, \langle z_j \rangle^{-3-\delta}).$$

Proof. Introduce (see also (A.4))

$$A(\lambda) := 2\pi i E_\chi(\lambda) V (I + \chi(\mathbb{D}_0 - \lambda)^{-1} V)^{-1} E_\chi(\lambda)^*,$$

with

$$(E_\chi(\lambda) f)(\omega) = (2\pi)^{-3/2} (\lambda^2(\lambda^2 - 1))^{1/4} \Pi_\pm(\kappa(\lambda)\omega) \int_{\mathbb{R}^3} e^{-i\kappa(\lambda)\omega\cdot x} \chi(x) f(x)\, dx.$$

Next we estimate

$$|s(\lambda)| \leqslant \prod_{j=1}^{\infty} (1 + s_j(A(\lambda))), \tag{3.4}$$

where

$$s_j(A(\lambda)) \leqslant C \|V(I + \chi(\mathbb{D}_0 - \lambda)^{-1} V)^{-1}\| \|E_\chi(\lambda)\| s_j(E_\chi(\lambda)). \tag{3.5}$$

From Theorem 5.1 of [6] we have

$$\|V(I + \chi(\mathbb{D}_0 - \lambda)^{-1} V)^{-1}\| \leqslant \frac{\det(I + |\chi(\mathbb{D}_0 - \lambda)^{-1} V|^4)}{|\det(I + (\chi(\mathbb{D}_0 - \lambda)^{-1} V)^4)|}. \tag{3.6}$$

For the numerator it follows from (2.11) that we have the upper bound

$$\det(I + |\chi(\mathbb{D}_0 - \lambda)^{-1} V|^4) \leqslant C e^{C|\lambda|^3}. \tag{3.7}$$

Then Cartan's minimum modulus principle for entire functions (see, e.g., Chapter I of [12]) gives, for any $\varepsilon, \delta > 0$,

$$|\det(I + (\chi(\mathbb{D}_0 - \lambda)^{-1} V)^4)| \geqslant C e^{-C|\lambda|^{3+\varepsilon}} \quad \text{for } \lambda \notin \bigcup_{z_j \in \mathcal{R}} D(z_j, \langle z_j \rangle^{-3-\delta}). \tag{3.8}$$

From (3.6), (3.7) and (3.8) it then follows that

$$\|V(I + \chi(\mathbb{D}_0 - \lambda)^{-1} V)^{-1}\| \leqslant C e^{C|\lambda|^{3+\varepsilon}} \quad \text{for } \lambda \notin \bigcup_{z_j \in \mathcal{R}} D(z_j, \langle z_j \rangle^{-3-\delta}).$$

Combined with (3.4) and (3.5) this results in the upper bound

$$|s(\lambda)| \leqslant \prod (1 + C e^{C|\lambda|^{3+\varepsilon}} j^{-k}) \leqslant \prod (1 + C e^{C|\lambda|^{3+\varepsilon} - \sqrt{j}/C}) \leqslant C e^{C|\lambda|^{9+3\varepsilon}},$$

where we have taken $k = [j^{1/2}] + 1$ and the last step uses Lemma 3.1. ∎

We are now ready to show one of the main ingredients of the proof of Theorem 1.1.

LEMMA 3.3. *The entire function g in* (3.1) *is a polynomial of degree $\leqslant 9$.*

Proof. Applying the maximum principle to the holomorphic function f given by $f(\lambda) = s(\lambda)P_{\mathcal{R}}(\lambda) = \mathrm{e}^{g(\lambda)}P_{\overline{\mathcal{R}}}(\lambda)$, we get from Lemma 3.2 and (3.3) that

$$|f(\lambda)| = |\mathrm{e}^{g(\lambda)}||P_{\overline{\mathcal{R}}}(\lambda)| \leqslant C\mathrm{e}^{C|\lambda|^{9+3\varepsilon}}.$$

Away from $\mathcal{R}$ we have (see Chapter XI, Section 3, Lemma 3.1 of [5])

$$\frac{\mathrm{d}^N}{\mathrm{d}\lambda^N}\Big(\frac{f'(\lambda)}{f(\lambda)}\Big) = -N!\sum(\overline{z_j}-\lambda)^{-N+1}, \quad \text{for } N \geqslant 9.$$

From

$$\frac{f'(\lambda)}{f(\lambda)} = g'(\lambda) - \frac{P'_{\overline{\mathcal{R}}}(\lambda)}{P_{\overline{\mathcal{R}}}(\lambda)}$$

we therefore obtain

$$-N!\sum(\overline{z_j}-\lambda)^{-N+1} = g^{(N+1)}(\lambda) - \frac{\mathrm{d}^N}{\mathrm{d}\lambda^N}\Big(\frac{P'_{\overline{\mathcal{R}}}(\lambda)}{P_{\overline{\mathcal{R}}}(\lambda)}\Big).$$

A direct calculation of the second term on the right hand side gives $g^{(N+1)}(\lambda) = 0$ for $N \geqslant 9$. ∎

The proof of Lemma 3.3 is inspired by Zworski [26], [27] (Schrödinger operator case). With these preparations we are now ready to prove our main result.

Proof of Theorem 1.1. Let $u(t) = 2\,\mathrm{Tr}\,(\cos(t\mathbb{D}) - \cos(t\mathbb{D}_0))$ and $\varphi \in C_0^\infty(\mathbb{R} \setminus \{0\})$. Then we can write $\varphi = \varphi_- + \varphi_+$ where $\varphi_\pm$ are the restrictions of φ to $\mathbb{R}_\pm$. We obtain

$$\begin{aligned}\langle u, \varphi\rangle_{\mathscr{D}',\mathscr{D}} &= \sum_{\pm} \mathrm{Tr}\,(\widehat{\varphi}_\pm(\mathbb{D}) + \widehat{\varphi}_\pm(-\mathbb{D}) - \widehat{\varphi}_\pm(\mathbb{D}_0) - \widehat{\varphi}_\pm(-\mathbb{D}_0))\\ &= \sum_{\pm} \mathrm{Tr}\,(f_\pm(\mathbb{D}) - f_\pm(\mathbb{D}_0)),\end{aligned}$$

where we have defined $f_\pm(\lambda) = \widehat{\varphi}_\pm(\lambda) + \widehat{\varphi}_\pm(-\lambda)$. Using (A.6) we obtain

$$\langle u, \varphi\rangle_{\mathscr{D}',\mathscr{D}} = -\frac{1}{2\pi \mathrm{i}}\sum_{\pm}\Big(\int_{\mathbb{R}} \widehat{\varphi}_\pm(\pm\lambda)\partial_\lambda(\log s(\lambda))\,\mathrm{d}\lambda + \sum_{\lambda_j\in \mathrm{spec}_{\,\mathrm{d}}(\mathbb{D})} f_\pm(\lambda_j)\Big) \tag{3.9}$$

with all four sign combinations in the integral. Now define $h_\pm(\lambda) = \mathscr{F}[\varphi_\pm/t^9](\lambda)$ so that $\partial_\lambda^9(h_\pm)(\lambda) = \widehat{\varphi}_\pm(\lambda)$, integrate by parts and use the factorization (3.1) to obtain

$$\begin{aligned}-\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} \widehat{\varphi}_+(\lambda)\partial_\lambda(\log s(\lambda))\,\mathrm{d}\lambda &= -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} h_+(\lambda)\partial_\lambda^{10}(\log s(\lambda))\,\mathrm{d}\lambda\\ &= -\frac{1}{2\pi\mathrm{i}}\sum_{\lambda_j\in\mathcal{R}} m_j\int_{\mathbb{R}} h_+(\lambda)\Big(\frac{9!}{(\lambda-\overline{\lambda}_j)^{10}} - \frac{9!}{(\lambda-\lambda_j)^{10}}\Big)\,\mathrm{d}\lambda\\ &= \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_+} m_j\widehat{\varphi}_+(\overline{\lambda}_j) - \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_-} m_j\widehat{\varphi}_+(\lambda_j),\end{aligned}$$

where we have used the fact that $|h_+(\lambda)| = \mathcal{O}(\langle\lambda\rangle^{-\infty})$ for Im $\lambda \leqslant 0$ to deform the integral over $\mathbb{R}$ into the lower half-plane. We treat the remaining terms in (3.9) similarly to obtain

$$\begin{aligned}
&\langle u, \varphi\rangle_{\mathscr{D}',\mathscr{D}} \\
&= \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_+} m_j\widehat{\varphi}_+(\overline{\lambda}_j) - \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_-} m_j\widehat{\varphi}_+(\lambda_j) - \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_-} m_j\widehat{\varphi}_+(-\overline{\lambda}_j) + \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_+} m_j\widehat{\varphi}_+(-\lambda_j) \\
&\quad - \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_-} m_j\widehat{\varphi}_-(\overline{\lambda}_j) + \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_+} m_j\widehat{\varphi}_-(\lambda_j) + \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_+} m_j\widehat{\varphi}_-(-\overline{\lambda}_j) \\
&\quad - \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_-} m_j\widehat{\varphi}_-(-\lambda_j) + \sum_{\lambda_j\in\operatorname{spec}_{\mathrm{d}}(\mathbb{D})} (f_-(\lambda_j) + f_+(\lambda_j)) \\
&= \Big\langle \varphi, \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_+} m_j(\mathrm{e}^{-\mathrm{i}|t|\overline{\lambda}_j} + \mathrm{e}^{\mathrm{i}|t|\lambda_j}) - \sum_{\lambda_j\in\mathcal{R}\cap\mathbb{C}_-} (\mathrm{e}^{-\mathrm{i}|t|\lambda_j} + \mathrm{e}^{\mathrm{i}|t|\overline{\lambda}_j}) + \sum_{\lambda_j\in\operatorname{spec}_{\mathrm{d}}(\mathbb{D})} 2m_j\cos(t\lambda_j)\Big\rangle
\end{aligned}$$

which proves (1.1). ∎

We believe that it is possible to avoid the assumption $\pm 1 \notin \mathcal{R}$ in Theorem 1.1 but it requires a substantial analysis of the threshold behaviour of $\mathbb{D}$ which is outside the scope of the present paper. We intend to address this matter in a future work.

Appendix A. SCATTERING THEORY

It is well-known (see e.g. [24]) that under the assumption that $V \in C_0^\infty(\mathbb{R}^3)$ the wave operators

$$W_\pm = \operatorname*{s-lim}_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}t\mathbb{D}}\mathrm{e}^{-\mathrm{i}t\mathbb{D}_0}$$

exist, are asymptotically complete and fulfill the intertwining relation $\mathbb{D}W_\pm = W_\pm\mathbb{D}_0$. The scattering operator $\mathbb{S} = W_+^*W_-$ for the pair $(\mathbb{D}, \mathbb{D}_0)$ is unitary on $L^2(\mathbb{R}^3)$ and commutes with $\mathbb{D}_0$ and consequently it can be represented as multiplication by the so called scattering matrix $S(\lambda)$.

To obtain such stationary representations of $S(\lambda)$ we need to discuss spectral representations of the Dirac operator. To this end we introduce $E_0(\lambda) : L^2(\mathbb{R}^3;\mathbb{C}^4) \to L^2(S^2;\mathbb{C}^4)$ by

$$(E_0(\lambda)f)(\omega) = (2\pi)^{-3/2}(\lambda^2(\lambda^2-1))^{1/4}\Pi_\pm(\kappa(\lambda)\omega)\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\kappa(\lambda)\omega\cdot x} f(x)\,\mathrm{d}x, \tag{A.1}$$

for $\pm\lambda > 1$, with $\Pi_\pm$ as in (2.1) and $\kappa(\lambda) = \sqrt{\lambda^2-1}$. The adjoint operator $E_0(\lambda)^* : L^2(S^2;\mathbb{C}^4) \to L^2(\mathbb{R}^3;\mathbb{C}^4)$ is then given by

$$(E_0(\lambda)^*f)(x) = (2\pi)^{-3/2}(\lambda^2(\lambda^2-1))^{1/4}\int_{S^2} \mathrm{e}^{\mathrm{i}\kappa(\lambda)\omega\cdot x}\Pi_\pm(\kappa(\lambda)\omega)f(\omega)\,\mathrm{d}\omega.$$

Then

$$\begin{aligned}[E_0(\lambda)\mathbb{D}_0 f](\omega) &= (\lambda^2(\lambda^2-1))^{1/4}\Pi_\pm(\kappa(\lambda)\omega)\mathscr{F}(\mathbb{D}_0 f)(\kappa(\lambda)\omega)\\ &= (\lambda^2(\lambda^2-1))^{1/4}\Pi_\pm(\kappa(\lambda)\omega)\boldsymbol{d}_0(\kappa(\lambda)\omega)\mathscr{F}(f)(\kappa(\lambda)\omega)\\ &= \lambda[E_0(\lambda)f](\omega),\end{aligned}$$

which is to say that $E_0(\lambda)\mathbb{D}_0 E_0(\lambda)^{-1} = \lambda$ where the right-hand side denotes multiplication by λ.

A representation of the scattering matrix is shown in [2] but they use a different spectral representation that we now briefly discuss. We begin by introducing the so called Foldy–Wouthuysen (F-W) transform which diagonalizes $\mathbb{D}_0$ as in [4] (see also [2] and [24]). In the ξ-representation it is given by the unitary 4×4 matrix defined by $\widehat{G}(\xi) = \exp(\beta(\boldsymbol{\alpha}\cdot\xi)\theta(|\xi|))$ where $\theta(t) = (2t)^{-1}\arctan t$ for $t>0$. A direct calculation gives

$$\widehat{G}(\xi)\boldsymbol{d}_0(\xi)\widehat{G}(\xi)^{-1} = (\xi^2+1)^{1/2}\beta. \tag{A.2}$$

We then define the F-W transform as the unitary operator G on $L^2(\mathbb{R}^3;\mathbb{C}^4)$ defined by $G = \mathscr{F}^{-1}\widehat{G}(\xi)\mathscr{F}$ and it transforms $\mathbb{D}_0$ into

$$\widetilde{\mathbb{D}}_0 := G\mathbb{D}_0 G^{-1} = (-\Delta+1)^{1/2}\beta.$$

We now define the restrictions of the so called free trace operator (see [2]) $T_0^\pm(\lambda) : L^2(\mathbb{R}^3;\mathbb{C}^4) \to L^2(S^2;\mathbb{C}^4)$ by

$$(T_0^\pm(\lambda)f)(\omega) = (2\pi)^{-3/2}(\lambda^2(\lambda^2-1))^{1/4}\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\kappa(\lambda)\omega\cdot x}P_\pm G f(x)\,\mathrm{d}x$$

where $P_\pm = 2^{-1}(I_4\pm\beta)$ and take the free trace operator to be $T_0(\lambda) = T_0^\pm(\lambda)$ depending on whether $\pm\lambda > 1$. Similarly to above it is easy to see that $T_0(\lambda)^{-1}\mathbb{D}_0 T_0(\lambda)$ is also multiplication by λ.

In [2] it is shown that the scattering matrix has the stationary representation

$$\widetilde{S}(\lambda) = I - 2\pi\mathrm{i}T_0(\lambda)(V - VR_V(\lambda+\mathrm{i}0)V)T_0(\lambda) \quad \text{for } |\lambda|>1. \tag{A.3}$$

We can relate this representation to the one given by $E(\lambda)$ in (A.1) by noting that by (A.2)

$$\begin{aligned}&[\widehat{G}(\kappa(\lambda)\omega)E_0(\lambda)f](x)\\ &\quad= (\lambda^2(\lambda^2-1))^{1/4}\widehat{G}(\kappa(\lambda)\omega)\frac{1}{2}(I_4+\lambda^{-1}\boldsymbol{d}_0(\kappa(\lambda)\omega))\mathscr{F}(f)(\kappa(\lambda)\omega)\\ &\quad= (\lambda^2(\lambda^2-1))^{1/4}P_\pm\widehat{G}(\kappa(\lambda)\omega)\mathscr{F}(f)(\kappa(\lambda)\omega)\\ &\quad= (\lambda^2(\lambda^2-1))^{1/4}P_\pm(\mathscr{F}Gf)(\kappa(\lambda)\omega) = [T_0 f](\omega).\end{aligned}$$

This together with (A.3) results in the representation

$$S(\lambda) := \widehat{G}(\kappa(\lambda)\omega)^{-1}\widetilde{S}(\lambda)\widehat{G}(\kappa(\lambda)\omega) = I - 2\pi\mathrm{i}E_0(\lambda)(V - VR_V(\lambda+\mathrm{i}0)V)E_0(\lambda)^*$$

for $|\lambda| > 1$. The scattering matrix is unitary for $|\lambda| > 1$ with

$$S(\lambda)^{-1} = I + 2\pi \mathrm{i} E_0(\lambda)(V - VR_V(\lambda - \mathrm{i}0)V)E_0(\lambda)^*.$$

By taking $\chi \in C_0^\infty(B(0, R_0))$ with $\chi V = V$ we use the identity

$$V(I - R_V(\lambda)V) = V(I + \chi(\mathbb{D}_0 - \lambda)^{-1}V)^{-1}$$

to rewrite the extended scattering matrix as

$$S(\lambda) = I - A(\lambda) := I - 2\pi\mathrm{i}E_\chi(\lambda)V(I + \chi(\mathbb{D}_0 - \lambda)^{-1}V)^{-1}E_\chi(\lambda)^*. \tag{A.4}$$

where

$$\begin{aligned}(E_\chi(\lambda)f)(\omega) \\ = (2\pi)^{-3/2}(\lambda^2(\lambda^2-1))^{1/4}\Pi_\pm(\kappa(\lambda)\omega)\int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}\kappa(\lambda)\omega\cdot x}\chi(x)f(x)\,\mathrm{d}x.\end{aligned} \tag{A.5}$$

From (2.7) we see that the resonances will appear as poles of the extended $S(\lambda)$. It also follows that resonances appear as poles of the scattering determinant

$$s(\lambda) = \det(S(\lambda)) = \det(I + A(\lambda)).$$

A.0.1. THE LIFSHITS–KREIN TRACE FORMULA. The so called spectral shift function $\xi \in \mathscr{D}'(\mathbb{R})$ is a generalization of the eigenvalue counting function that makes sense also on the absolutely continuous spectrum $(-\infty, -1] \cup [1, \infty)$ where it is smooth. It is well-known that the Lifshits–Krein trace formula [13]

$$\mathrm{tr}\,(f(\mathbb{D}) - f(\mathbb{D}_0)) = \int_{\mathbb{R}} f(\lambda)\xi'(\lambda)\,\mathrm{d}\lambda,$$

holds for any $f \in \mathscr{S}(\mathbb{R})$. Also, by the Birman–Krein formula we have

$$s(\lambda) = \mathrm{e}^{-2\pi\mathrm{i}\xi(\lambda)}$$

for almost every $\lambda \in (-\infty, -1] \cup [1, \infty)$. Therefore we can choose a branch of the logarithm such that

$$\xi(\lambda) = -\frac{1}{2\pi\mathrm{i}}\log s(\lambda), \quad \text{for a.e. } \pm\lambda > 1,$$

and obtain

$$\mathrm{tr}\,(f(\mathbb{D}) - f(\mathbb{D}_0)) = -\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{R}} f(\lambda)\partial_\lambda(\log s(\lambda))\,\mathrm{d}\lambda + \sum_{\lambda_j \in \mathrm{spec}_{\mathrm{d}}(\mathbb{D})} m_j f(\lambda_j), \tag{A.6}$$

under the assumption that $\pm 1 \notin \mathcal{R}$.

Appendix B. RESOLVENT OF FREE DIRAC OPERATOR

From the identity $\mathbb{D}_0^2 = -\Delta + 1$ it follows that

$$R_0(\lambda) = (\mathbb{D}_0 + \lambda) R_{00}(\sqrt{\lambda^2 - 1}), \tag{B.1}$$

where $R_{00}(z) = (-\Delta - z^2)^{-1}$. It is well-known that $R_{00}(z)$ is a convolution operator (see e.g. [16] and [21]) and that its kernel is given by $(4\pi)^{-1}|x|^{-1}\mathrm{e}^{\mathrm{i}z|x|}$ where $\operatorname{Im} z > 0$. Consequently, for $\lambda \in \mathbb{C} \setminus \operatorname{spec}(\mathbb{D}_0)$, we have

$$[(\mathbb{D}_0 - \lambda)^{-1}u](x) = (\boldsymbol{\alpha} \cdot \nabla + \beta + \lambda)\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda^2-1}|x-y|}}{|x-y|} u(y)\,\mathrm{d}y, \tag{B.2}$$

for $u \in L^2(\mathbb{R}^3;\mathbb{C}^4)$ on the branch where $\operatorname{Im}(\sqrt{\lambda^2-1}) > 0$. It is not difficult to show that the resolvent kernel of $\mathbb{D}_0$ on $C_0^\infty(\mathbb{R}^3;\mathbb{C}^4)$ is given by

$$R_0(\lambda, x) = \Big(\mathrm{i}\frac{\boldsymbol{\alpha}\cdot x}{|x|^2} + \sqrt{\lambda^2-1}\frac{\boldsymbol{\alpha}\cdot x}{|x|} + \beta + \lambda\Big)\frac{\mathrm{e}^{\mathrm{i}\sqrt{\lambda^2-1}x}}{4\pi|x|}. \tag{B.3}$$

It is also well-known (see e.g. [21]) that for $\chi \in C_0^\infty(\mathbb{R}^3)$ the cut-off resolvent $\chi R_{00}(z)\chi$ can be extended holomorphically to all of $\mathbb{C}$ and that it admits the following upper bounds:

$$\|\chi R_{00}(z)\chi\|_{L^2\to H^j} \leqslant C(|z|^{j-1}\mathrm{e}^{C(\operatorname{Im} z)_-}), \quad \text{for } j = 0,1,2 \tag{B.4}$$

where the constant C depends only on the support of χ.

If temporarily we denote the extended resolvent by $\widetilde{R}_{00}(z)$ it can be related to the standard resolvent by

$$\chi R_{00}(z)\chi - \chi\widetilde{R}_{00}(-z)\chi = T_\chi(z), \quad \operatorname{Im} z > 0,$$

where $T(z)$ is the convolution operator with kernel

$$T(x,y,z) = \chi(x)\frac{\mathrm{i}}{2}\frac{z}{(2\pi)^2}\int_{S^2} \mathrm{e}^{\mathrm{i}z\omega\cdot(x-y)}\,\mathrm{d}\omega\,\chi(y).$$

It follows that if $\chi\widetilde{R}_0(\lambda)\chi$ denotes the resolvent extended to $\operatorname{Im}\kappa(\lambda) < 0$ we have

$$\begin{aligned}
&[(\chi R_0(\lambda)\chi - \chi\widetilde{R}_0(\lambda)\chi)f](x)\\
&\quad = \chi(x)[(\mathbb{D}_0+\lambda)(R_{00}(\kappa(\lambda)) - \widetilde{R}_{00}(-\kappa(\lambda)))(\chi f)](x)\\
&\quad = \frac{\mathrm{i}}{2}\frac{\kappa(\lambda)}{(2\pi)^2}\chi(x)(\mathbb{D}_0+\lambda)\int_{S^2}\mathrm{e}^{\mathrm{i}\kappa(\lambda)\omega\cdot x}\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\kappa(\lambda)\omega\cdot y}\chi(y)f(y)\,\mathrm{d}y
\end{aligned}$$

$$= \frac{\mathrm{i}}{2}\frac{\kappa(\lambda)}{(2\pi)^2}\chi(x)\int\limits_{S^2}(\boldsymbol{d}_0(\kappa(\lambda)\omega)+\lambda)\mathrm{e}^{\mathrm{i}\kappa(\lambda)\omega\cdot x}\int\limits_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\kappa(\lambda)\omega\cdot y}\chi(y)f(y)\,\mathrm{d}y$$

$$= \frac{\mathrm{i}\lambda\kappa(\lambda)}{(2\pi)^2}\chi(x)\int\limits_{S^2}\frac{1}{2}(I_4\pm\lambda^{-1}\boldsymbol{d}_0(\kappa(\lambda)\omega))\mathrm{e}^{\mathrm{i}\kappa(\lambda)\omega\cdot x}\int\limits_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}\kappa(\lambda)\omega\cdot y}\chi(y)f(y)\,\mathrm{d}y$$

$$= [2\pi\mathrm{i}E_\chi(\overline{\lambda})^*E_\chi(\lambda)f](x). \tag{B.5}$$

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