J. OPERATOR THEORY
77:1(2017), 149–170
doi: 10.7900/jot.2016mar08.2099

FUNDAMENTAL GROUP OF C^*-ALGEBRAS WITH FINITE DIMENSIONAL TRACE SPACE

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Communicated by Stefaan Vaes

ABSTRACT. We introduce the fundamental group $F(\mathcal{A})$ of a unital C^*-algebra $\mathcal{A}$ with finite dimensional trace space. The elements of the fundamental group are restricted by K-theoretical obstruction and positivity. Moreover we shall show there are uncountably many mutually non-isomorphic simple C^*-algebras such that $F(\mathcal{A}) = \{I_n\}$. Our study extends the results on the fundamental group due to Nawata and Watatani.

KEYWORDS: *Fundamental group, C^*-algebra, Picard group, Hilbert module, finite trace.*

MSC (2010): 46L35, 46L08.

INTRODUCTION

We recall some facts of fundamental groups of operator algebras.

The fundamental group $F(M)$ of a II_1-factor M with a normalized trace τ is defined by Murray and von Neumann in [13]. In the paper, the fact that if M is hyperfinite, then $F(M) = \mathbb{R}_+^\times$ is shown, where $\mathbb{R}_+^\times$ is the set of positive invertible real numbers. It was proved that $F(L(\mathbb{F}_\infty))$ of the group factor of the free group $\mathbb{F}_\infty$ contains the positive rationals by Voiculescu in [25] and it was shown that $F(L(\mathbb{F}_\infty)) = \mathbb{R}_+^\times$ by Radulescu in [20]. The fact that $F(L(G))$ is a countable group if G is an ICC group with property (T) is shown by Connes [4]. That either countable subgroup of $\mathbb{R}_+^\times$ or any uncountable group belonging to a certain "large" class can be realized as the fundamental group of some factor of type II_1 is shown by Popa and Vaes in [18] and in [19].

Nawata and Watatani [16], [17] introduced the fundamental group of simple C^*-algebras with unique trace. Their study is essentially based on the computation of Picard groups by Kodaka [8], [9], [10]. Nawata defined the fundamental group of non-unital C^*-algebras [14] and, by using fundamental group, calculated the Picard group of some projectionless C^*-algebras with strict comparison in [15]. In this paper, we define the of C^*-algebras with finite dimensional trace space. This fundamental group is a "numerical invariant". Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. We define the fundamental group $F(\mathcal{A})$ of $\mathcal{A}$ and the determinant fundamental group $F_{\det}(\mathcal{A})$ by using self-similarity and the extremal points of the tracial state space of $\mathcal{A}$. Then $F(\mathcal{A})$ and $F_{\det}(\mathcal{A})$ are multiplicative subgroups of $GL_n(\mathbb{R})$ and $\mathbb{R}_+^\times$ respectively. We shall show that elements of the fundamental group are restricted by K-theoretical obstruction and positivity. We will have that $A = DU(\sigma)$ for some diagonal matrix D and for some permutation unitary $U(\sigma)$ for any A in $F(\mathcal{A})$. If the unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ with finite dimensional bounded trace space are Morita equivalent, then $F(\mathcal{A}) = (DU(\sigma))^{-1}F(\mathcal{B})(DU(\sigma))$ for some diagonal matrix D and for some permutation unitary $U(\sigma)$. Moreover, we will compute $F(\mathcal{A})$ of several C^*-algebras A. We shall show that given any group G in $\mathrm{GL}_2(\mathbb{R})$ which is isomorphic to $\mathbb{Z}_2$ and whose elements have the form $DU(\sigma)$, there exists a simple AF-algebra $\mathcal{A}$ such that $F(\mathcal{A}) = G$. Furthermore, we shall show that for any $n \in \mathbb{N}$ there exist uncountably many mutually non-isomorphic simple (non)nuclear unital C^*-algebras $\mathcal{A}$ with n-dimensional trace space such that $F(\mathcal{A}) = \{I_n\}$, where I_n is the unit in $M_n(\mathbb{C})$.

We review some of the elementary facts on the trace space. Let $\mathcal{A}$ be a unital C^*-algebra. A *tracial state* φ on $\mathcal{A}$ is a state on $\mathcal{A}$ which satisfies $\varphi(ab) = \varphi(ba)$ for any a, b in $\mathcal{A}$. We denote by $T(\mathcal{A})$ the set of tracial states on $\mathcal{A}$ and denote by $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$ the $\mathbb{C}$-linear span of $T(\mathcal{A})$. Then $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$ is the set of tracial bounded linear functionals. If $T(\mathcal{A}) \neq \{0\}$, then $T(\mathcal{A})$ is a nonempty compact set of $\mathcal{A}^*$ in the weak*-topology so the set of extremal points of $T(\mathcal{A})$, denoted by $\partial_{\mathrm{e}} T(\mathcal{A})$, is nonempty. By 3.1.18 of [24], $T(\mathcal{A})$ is a Choquet simplex. If $\partial_{\mathrm{e}} T(\mathcal{A})$ is a finite set, then it is a canonical basis of $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$. In other words, say $\partial_{\mathrm{e}} T(\mathcal{A}) = \{\varphi_i\}_{i=1}^n$, then $\{\varphi_i\}_{i=1}^n$ is a canonical basis of $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$. The canonical basis is determined up to permutations.

1. HILBERT C^*-MODULES, IMPRIMITIVITY BIMODULES AND PICARD GROUPS

We recall some of the standard facts on Hilbert C^*-modules and imprimitivity bimodules. (See [11], [12], [21].) Let $\mathcal{A}$ be a unital C^*-algebra and let $\mathcal{E}$ be a right Hilbert $\mathcal{A}$-module with a right inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}}$. We denote by $\mathcal{L}(\mathcal{E})$ the set of adjointable operators on $\mathcal{E}$. Then $\mathcal{L}(\mathcal{E})$ is a unital C^*-algebra with a unit I. Let ξ, η be elements of $\mathcal{E}$. We define a rank-one operator $\theta_{\xi,\eta}$ on $\mathcal{E}$ by $\theta_{\xi,\eta}(\zeta) = \xi \langle \eta, \zeta \rangle_{\mathcal{A}}$. Then $\theta_{\xi,\eta} \in \mathcal{L}(\mathcal{E})$. We denote by $\mathcal{K}(\mathcal{E})$ the closed linear span of $\{\theta_{\xi,\eta} : \xi, \eta \in \mathcal{E}\}$ in $\mathcal{L}(\mathcal{E})$. Then $\mathcal{K}(\mathcal{E})$ is a closed ideal of $\mathcal{L}(\mathcal{E})$. If $I \in \mathcal{K}(\mathcal{E})$, then $\mathcal{K}(\mathcal{E}) = \mathcal{L}(\mathcal{E})$. We say that a subset $\{\xi_i\}_{i=1}^n$ of $\mathcal{E}$ is a *finite basis* if $\eta = \sum_{i=1}^n \xi_i \langle \xi_i, \eta \rangle_{\mathcal{A}}$ for any η in $\mathcal{E}$. (See [7], [26].) If $\mathcal{E}$ has a finite basis $\{\xi_i\}_{i=1}^n$, then $I \in \mathcal{K}(\mathcal{E})$ and $\mathcal{K}(\mathcal{E}) = \mathcal{L}(\mathcal{E})$. Put $p = (\langle \xi_i, \xi_j \rangle_{\mathcal{A}})_{ij}$. Then $\mathcal{E}$ is isomorphic to $p\mathcal{A}^n$ as a right Hilbert $\mathcal{A}$-module and $pM_n(\mathcal{A})p = \mathcal{L}(\mathcal{E})$.

Let $\mathcal{B}$ be a unital C^*-algebra. The *dual module* of an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule $\mathcal{E}$, denoted by $\mathcal{E}^*$, is defined to be the set $\{\xi^* : \xi \in \mathcal{E}\}$ with the operations $\xi^* + \eta^* = (\xi + \eta)^*$, $\lambda\xi^* = (\overline{\lambda}\xi)^*$, $b\xi^* a = (a^*\xi b^*)^*$, ${}_{\mathcal{B}}\langle \xi^*, \eta^* \rangle = \langle \xi, \eta \rangle_{\mathcal{B}}$ and $\langle \xi^*, \eta^* \rangle_{\mathcal{A}} = {}_{\mathcal{A}}\langle \xi, \eta \rangle$. Then $\mathcal{E}^*$ is a $\mathcal{B}$-$\mathcal{A}$-imprimitivity bimodule. The C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called *Morita equivalent* if there exists an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. For $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodules $\mathcal{E}_1$ and $\mathcal{E}_2$, $\mathcal{E}_1$ and $\mathcal{E}_2$ are called isomorphic if there exists a linear bijective map Φ from $\mathcal{E}_1$ onto $\mathcal{E}_2$ with the properties $\Phi(a\xi b) = a\Phi(\xi)b$, ${}_{\mathcal{A}}\langle \Phi(\xi), \Phi(\eta) \rangle = {}_{\mathcal{A}}\langle \xi, \eta \rangle$ and $\langle \Phi(\xi), \Phi(\eta) \rangle_{\mathcal{B}} = \langle \xi, \eta \rangle_{\mathcal{B}}$ for any a in $\mathcal{A}$, for any b in $\mathcal{B}$ and for any ξ, η in $\mathcal{E}_1$, where ${}_{\mathcal{A}}\langle \cdot, \cdot \rangle$ and $\langle \cdot, \cdot \rangle_{\mathcal{B}}$ are left and right inner products respectively. We denote by ${}_{\mathcal{A}}E_{\mathcal{B}}$ the set of all isomorphism classes of the $\mathcal{A}$-$\mathcal{B}$ imprimitivity bimodule.

We recall some notations on Picard groups of C^*-algebras introduced by Brown, Green and Rieffel in [3]. The set ${}_{\mathcal{A}}E_{\mathcal{A}}$ forms a group under the product by internal tensor product $\otimes$. The group, denoted by $\mathrm{Pic}(\mathcal{A})$, is called the *Picard group* of $\mathcal{A}$. The identity element of the group $\mathrm{Pic}(\mathcal{A})$ is $[\mathcal{A}]$, where $\mathcal{A}$ is regarded as an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule which has obvious left and right actions and the inner products ${}_{\mathcal{A}}\langle a, b \rangle = ab^*$ and $\langle a, b \rangle_{\mathcal{A}} = a^*b$. Then $[\mathcal{E}^*]$ is the inverse element of $[\mathcal{E}]$ in the Picard group of $\mathcal{A}$. Let α be an automorphism on $\mathcal{A}$. We denote by $\mathcal{E}_\alpha$ the $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule which is the set $\mathcal{A}$ with obvious left actions, obvious left $\mathcal{A}$-valued inner product and with the following right actions and right $\mathcal{A}$-valued inner product: $\xi \cdot a = \xi\alpha(a)$ for any $\xi \in \mathcal{A}$, and $a \in \mathcal{A}$, $\langle \xi, \eta \rangle_{\mathcal{A}} = \alpha^{-1}(\xi^*\eta)$ for any $\xi, \eta \in \mathcal{A}$. For $\alpha, \beta \in \mathrm{Aut}(\mathcal{A})$, $\mathcal{E}_\alpha$ is isomorphic to $\mathcal{E}_\beta$ if and only if there exists a unitary $u \in \mathcal{A}$ such that $\alpha = ad\ u \circ \beta$. Moreover, $\mathcal{E}_\alpha \otimes \mathcal{E}_\beta$ is isomorphic to $\mathcal{E}_{\alpha\circ\beta}$. We denote by $\rho_{\mathcal{A}}$ the injective homomorphism from $\mathrm{Out}(\mathcal{A})$ to $\mathrm{Pic}(\mathcal{A})$ such that $\rho_{\mathcal{A}}(\alpha) = [\mathcal{E}_\alpha]$. We suppose that $\mathcal{A}$ is unital. We say that a projection p is a *full projection* in $M_k(\mathcal{A})$ if the linear span of $\{a^*pb : a, b \in A^k\}$ is dense in $\mathcal{A}$. We say that a projection p is *self-similar* in $M_k(\mathcal{A})$ if there exists an isomorphism from $\mathcal{A}$ onto $pM_k(\mathcal{A})p$. Let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule. Since $\mathcal{E}$ is full as a left Hilbert $\mathcal{A}$-module, there exist some elements ζ_n, η_n in $\mathcal{E}$ such that $\left\| \sum_{i=1}^{n} {}_{\mathcal{A}}\langle \zeta_n, \eta_n \rangle - 1_{\mathcal{A}} \right\| < 1$. Then $\sum_{i=1}^{n} {}_{\mathcal{A}}\langle a\zeta_n, \eta_n \rangle = a\Big(\sum_{i=1}^{n} {}_{\mathcal{A}}\langle \zeta_n, \eta_n \rangle \Big) = 1_{\mathcal{A}}$ for some a in $\mathcal{A}$. For simplicity of notation, we write ζ_n instead of $a\zeta_n$. Then we can assume that $\sum_{i=1}^{n} {}_{\mathcal{A}}\langle \zeta_n, \eta_n \rangle = 1_{\mathcal{A}}$ for some ζ_n, η_n. Put $b = \sum_{i=1}^{n} {}_{\mathcal{A}}\langle \zeta_n + \eta_n, \zeta_n + \eta_n \rangle$, then $b \geqslant 1_{\mathcal{A}}$. Therefore $\sum_{i=1}^{n} {}_{\mathcal{A}}\langle b^{-1/2}(\zeta_n + \eta_n), b^{-1/2}(\zeta_n + \eta_n) \rangle = 1_{\mathcal{A}}$. Put $\xi_i = b^{-1/2}(\zeta_i + \eta_i)$. Then $\sum_{i=1}^{n} {}_{\mathcal{A}}\langle \xi_i, \xi_i \rangle = 1_{\mathcal{A}}$ and $\eta = \sum_{i=1}^{n} \xi_i \langle \xi_i, \eta \rangle_{\mathcal{A}}$ for any η in $\mathcal{E}$. Therefore $\{\xi_i\}_{i=1}^{n}$ is a finite basis of $\mathcal{E}$. Put $p = (\langle \xi_i, \xi_j \rangle_{\mathcal{A}})_{ij} \in M_n(\mathcal{A})$. Then p is a full projection and $\mathcal{E}$ is isomorphic to $p\mathcal{A}^n$ as $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule with an isomorphism of $\mathcal{A}$ to $pM_n(\mathcal{A})p$ as C^*-algebra. Conversely, we suppose that p is a full projection in $M_k(\mathcal{A})$ with an isomorphism $\alpha : \mathcal{A} \to pM_k(\mathcal{A})p$. Then $p\mathcal{A}^n$ is an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule with the operations $a \cdot \xi = \alpha(a)\xi$, $\xi \cdot a = \xi a$.

2. DEFINITION OF THE FUNDAMENTAL GROUP

We define the fundamental group for unital C^*-algebras with finite dimensional bounded trace space. The definition given in the Section 3 of [16] uses a self-similarity of C^*-algebra. In the same way, we define a fundamental group.

DEFINITION 2.1. Let $\mathcal{A}$ be a unital C^*-algebra. A *self-similar triple* for $\mathcal{A}$, abbreviated s.s.t., is a tuple (k, p, Φ), where k is a natural number, where p is a projection in $M_k(\mathcal{A})$, and where Φ is an isomorphism from $\mathcal{A}$ onto $pM_k(\mathcal{A})p$.

We denote by Tr_k the unnormalized trace on $M_k(\mathbb{C})$. Let $\mathcal{A}$ be a unital C^*-algebra, let $\varphi \in \mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$ and let (k, p, Φ) be a self-similar triple of $\mathcal{A}$. Then $(\mathrm{Tr}_k \otimes \varphi) \circ \Phi$ is in $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$. Therefore we can define a bounded linear map $T_{(k,p,\Phi)}$ on $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$ by $T_{(k,p,\Phi)}(\varphi) = (\mathrm{Tr}_k \otimes \varphi) \circ \Phi$.

We denote by $\mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$ the set of bounded linear maps from $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$ into $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$.

DEFINITION 2.2. Let $\mathcal{A}$ be a unital C^*-algebra. We define the subset $F^{\mathrm{tr}}(\mathcal{A})$ of $\mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$ as follows:

$$F^{\mathrm{tr}}(\mathcal{A}) := \{T_{(k,p,\Phi)} \in \mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})) : (k, p, \Phi) : \text{s.s.t}\}$$

We denote by $\mathcal{GL}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$ the set of invertible elements in $\mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$. It forms a group. We will show $F^{\mathrm{tr}}(\mathcal{A})$ is a subgroup of $\mathcal{GL}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$ by using a Picard group. The following construction generalizes that of the Proposition 2.1 of [16].

PROPOSITION 2.3. *Let $\mathcal{A}$ be a unital C^*-algebra. We define the map $R_{\mathcal{A}} : \mathrm{Pic}(\mathcal{A}) \to \mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$ by $(R_{\mathcal{A}}([\mathcal{E}])(\varphi))(a) = \sum_{i=1}^{k} \varphi(\langle \xi_i, a\xi_i \rangle)$, where $\{\xi_i\}_{i=1}^k$ is a finite basis of $\mathcal{E}$ as a right Hilbert $\mathcal{A}$-module. Then $R_{\mathcal{A}}([\mathcal{E}])$ does not depend on the choice of basis and $R_{\mathcal{A}}$ is well-defined. Moreover $R_{\mathcal{A}}$ is a multiplicative map and $R_{\mathcal{A}}([A]) = \mathrm{id}_{\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})}$.*

Proof. Let φ be a trace on $\mathcal{A}$, a be an element of $\mathcal{A}$, $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule and let $\{\xi_i\}_{i=1}^k$ and $\{\eta_j\}_{j=1}^l$ be finite bases of $\mathcal{E}$. Then

$$\begin{aligned}\sum_{i=1}^{k} \varphi(\langle \xi_i, a\xi_i \rangle_{\mathcal{A}}) &= \sum_{i=1}^{k} \varphi\Big(\Big\langle \xi_i, \sum_{j=1}^{l} \eta_j \langle \eta_j, a\xi_i \rangle_{\mathcal{A}} \Big\rangle_{\mathcal{A}}\Big) = \sum_{i,j=1}^{k,l} \varphi(\langle \xi_i, \eta_j \rangle_{\mathcal{A}} \langle \eta_j, a\xi_i \rangle_{\mathcal{A}}) \\ &= \sum_{i,j=1}^{k,l} \varphi(\langle \eta_j, a\xi_i \rangle_{A} \langle \xi_i, \eta_j \rangle_{\mathcal{A}}) = \sum_{j=1}^{l} \varphi(\langle \eta_j, a\eta_j \rangle_{\mathcal{A}}).\end{aligned}$$

Therefore $R_{\mathcal{A}}([\mathcal{E}])$ is independent on the choice of basis.

Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodules with bases $\{\xi_i\}_{i=1}^k$ and $\{\zeta_j\}_{j=1}^l$. We suppose that there exists an isomorphism Φ of $\mathcal{E}_1$ onto $\mathcal{E}_2$. Then $\{\Phi(\xi_i)\}_{i=1}^k$ is also a basis of $\mathcal{E}_2$. Then

$$\sum_{i=1}^k \varphi(\langle \xi_i, a\xi_i\rangle_{\mathcal{A}}) = \sum_{i=1}^k \varphi(\langle \Phi(\xi_i), a\Phi(\xi_i)\rangle_{\mathcal{A}}) = \sum_{j=1}^l \varphi(\langle \zeta_i, a\zeta_i\rangle_{\mathcal{A}}).$$

Therefore $R_{\mathcal{A}}$ is well-defined.

We shall show that $R_{\mathcal{A}}$ is multiplicative. Let $\mathcal{E}_1$ and $\mathcal{E}_2$ be $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodules with bases $\{\xi_i\}_{i=1}^k$ and $\{\eta_j\}_{j=1}^l$. Then $\{\xi_i \otimes \eta_j\}_{i,j=1}^{k,l}$ is a basis of $\mathcal{E}_1 \otimes \mathcal{E}_2$ and

$$(R_{\mathcal{A}}([\mathcal{E}_1 \otimes \mathcal{E}_2])(\varphi))(a) = \sum_{i,j=1}^{k,l} \varphi(\langle \xi_i \otimes \eta_j, a\xi_i \otimes \eta_j\rangle) = \sum_{i,j=1}^{k,l} \varphi(\langle \eta_j, \langle \xi_i, a\xi_i\rangle_{\mathcal{A}}\eta_j\rangle_{\mathcal{A}}).$$

On the other hand

$$(R_{\mathcal{A}}([\mathcal{E}_1])R_{\mathcal{A}}([\mathcal{E}_2])(\varphi))(a) = \sum_{i=1}^k (R_{\mathcal{A}}([\mathcal{E}_2])(\varphi))(\langle \xi_i, a\xi_i\rangle_{\mathcal{A}}) = \sum_{i,j=1}^{k,l} \varphi(\langle \eta_j, \langle \xi_i, a\xi_i\rangle_{\mathcal{A}}\eta_j\rangle_{\mathcal{A}}).$$

Therefore $R_{\mathcal{A}}$ is multiplicative. ∎

We denote by $\{e_i\}_{i=1}^k$ the canonical basis of $\mathcal{A}^k$.

PROPOSITION 2.4. *Let $\mathcal{A}$ be a unital C^*-algebra. Then $F^{\mathrm{tr}}(\mathcal{A}) = R_{\mathcal{A}}(\mathrm{Pic}(\mathcal{A}))$.*

Proof. Let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule and let $\{\xi_i\}_{i=1}^k$ be a basis of $\mathcal{E}$. Put $p = (\langle \xi_i, \xi_j\rangle)_{ij}$. Then $\mathcal{E}$ is isomorphic to $p\mathcal{A}^k$ as an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule and there exists an $*$-isomorphism $\alpha : \mathcal{A} \to pM_k(\mathcal{A})p$. Then

$$\begin{aligned}(R_{\mathcal{A}}([\mathcal{E}])(\varphi))(a) &= \sum_{i=1}^k \varphi(\langle \xi_i, a\xi_i\rangle) = \sum_{i=1}^k \varphi(\langle pe_i, \alpha(a)pe_i\rangle) \\ &= \sum_{i=1}^k \varphi(e_i^*\alpha(a)e_i) = (\mathrm{Tr}_k \otimes \varphi) \circ (\alpha)(a).\end{aligned}$$

Therefore $F^{\mathrm{tr}}(\mathcal{A}) \supset R_{\mathcal{A}}(\mathrm{Pic}(\mathcal{A}))$. Conversely, we suppose that p is a projection with an $*$-isomorphism $\alpha : \mathcal{A} \to pM_k(\mathcal{A})p$ and that the linear span of $\{a^*pb : a, b \in \mathcal{A}^k\}$ is dense in $\mathcal{A}$. Then, $p\mathcal{A}^k$ is an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule with a basis $\{pe_i\}_{i=1}^k$. Then

$$(\mathrm{Tr}_k \otimes \varphi) \circ (\alpha)(a) = \sum_{i=1}^k \varphi(\langle e_i^*\alpha(a)e_i\rangle) = \sum_{i=1}^k \varphi(\langle pe_i, \alpha(a)pe_i\rangle) = (R_{\mathcal{A}}([p\mathcal{A}^k])(\varphi))(a).$$

Therefore $F^{\mathrm{tr}}(\mathcal{A}) \subset R_{\mathcal{A}}(\mathrm{Pic}(\mathcal{A}))$. Hence $F^{\mathrm{tr}}(\mathcal{A}) = R_{\mathcal{A}}(\mathrm{Pic}(\mathcal{A}))$. ∎

PROPOSITION 2.5. *Let $\mathcal{A}$ be a unital C^*-algebra. Then $F^{\mathrm{tr}}(\mathcal{A})$ is a subgroup of $\mathcal{GL}(\mathrm{lin}_{\mathbb{C}}T(\mathcal{A}))$.*

Proof. By Proposition 2.3, $R_{\mathcal{A}}(\mathrm{Pic}(\mathcal{A}))$ is a subgroup of $\mathcal{GL}(\mathrm{lin}_{\mathbb{C}}T(\mathcal{A}))$. Then, by Proposition 2.4, $F^{\mathrm{tr}}(\mathcal{A})$ is also. ∎

If $\partial_{\mathrm{e}}T(\mathcal{A})$ is a finite set, then $F^{\mathrm{tr}}(\mathcal{A})$ has a matrix representation. We suppose that $\mathcal{A}$ is unital. Then $\mathrm{lin}_{\mathbb{C}}T(\mathcal{A})$ has a canonical basis $\partial_{\mathrm{e}}T(\mathcal{A})$. We define the (determinant) fundamental group $F(\mathcal{A})$ $(F_{\det}(\mathcal{A}))$ using this basis.

DEFINITION 2.6. Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. We define the *fundamental group* $F(\mathcal{A})$ as the matrix representation of $F^{\mathrm{tr}}(\mathcal{A})$ with respect to an indexed set of $\partial_{\mathrm{e}}T(\mathcal{A})$. In other words, say $\partial_{\mathrm{e}}T(\mathcal{A}) = \{\varphi_i\}_{i=1}^n$. We define the representation $S_{\mathcal{A}} : \mathcal{L}(\mathrm{lin}_{\mathbb{C}}T(\mathcal{A})) \to M_n(\mathbb{C})$ by $S_{\mathcal{A}}(T)(e_i) = \sum_{j=1}^{n} a_{ij}e_j$, where $T\varphi_i = \sum_{j=1}^{n} a_{ij}\varphi_j$, where e_i is a canonical basis of $\mathbb{C}^n$. Then $F(\mathcal{A}) = S_{\mathcal{A}}(F^{\mathrm{tr}}(\mathcal{A}))$. The fundamental group is defined up to permutations of the indices. Moreover, we define the determinant fundamental group $F_{\det}(\mathcal{A})$ by $|\det|(F(\mathcal{A}))$ where $|\det|(X) = \{|\det(A)| : A \in X\}$ for some subset X of $M_n(\mathbb{C})$.

REMARK 2.7. This fundamental group measures the enlargement ratio of the self-similarity of a C^*-algebra by its trace space. Suppose $\sharp(\partial_{\mathrm{e}}T(\mathcal{A})) = 2$ and say $\partial_{\mathrm{e}}T(\mathcal{A}) = \{\varphi_1, \varphi_2\}$. Let (k, p, Φ) be a s.s.t. By Proposition 3.6 or Remark 3.7, $T_{(k,p,\Phi)}\varphi_1 = \lambda_1\varphi_1$ and $T_{(k,p,\Phi)}\varphi_2 = \lambda_2\varphi_2$, or $T_{(k,p,\Phi)}\varphi_1 = \lambda_1\varphi_2$ and $T_{(k,p,\Phi)}\varphi_2 = \lambda_2\varphi_1$ for some λ_1, λ_2. Moreover, by substituting $1_{\mathcal{A}}$ in the both sides, $\lambda_1 = \mathrm{Tr}_k \otimes \varphi_1(p)$ and $\lambda_2 = \mathrm{Tr}_k \otimes \varphi_2(p)$. Then $\varphi_1(p)$ and $\varphi_2(p)$ indicate the enlargement ratio of Φ with respect to φ_1 and φ_2.

By using the self-similarity, we shall show the following proposition.

PROPOSITION 2.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras with finite dimensional bounded trace space. We suppose $\sharp(\partial_{\mathrm{e}}T(\mathcal{A})) = n$ and that $\sharp(\partial_{\mathrm{e}}T(\mathcal{B})) = m$. Say $\{\varphi_i\}_{i=1}^n = \partial_{\mathrm{e}}T(\mathcal{A})$ and $\{\psi_j\}_{j=1}^m = \partial_{\mathrm{e}}T(\mathcal{B})$.*

Then $\{\varphi_1\otimes\psi_1, \ldots, \varphi_n\otimes\psi_1, \varphi_2\otimes\psi_1, \ldots, \varphi_n\otimes\psi_m\} = \partial_{\mathrm{e}}T(\mathcal{A}\otimes_{\min}\mathcal{B})$. Moreover

$$F(\mathcal{A}\otimes_{\min}\mathcal{B}) \supset F(\mathcal{A})\otimes F(\mathcal{B}) \quad \text{and} \quad F_{\det}(\mathcal{A}\otimes_{\min}\mathcal{B}) \supset |\det|(F(\mathcal{A})\otimes F(\mathcal{B})),$$

where $A\otimes B = \begin{bmatrix} Ab_{1,1} & \cdots & Ab_{1,m} \\ \vdots & \ddots & \vdots \\ Ab_{m,1} & \cdots & Ab_{m,m} \end{bmatrix}$ *for $A \in M_n(\mathbb{C})$ and $(b_{ij}) = B \in M_m(\mathbb{C})$.*

Proof. Let $A\otimes B$ be an element of $F(\mathcal{A})\otimes F(\mathcal{B})$. Then there exist self-similar triples (k_1, p_0, Φ_1) and (k_2, q_0, Φ_2) such that $\Phi_1 : \mathcal{A} \cong p_0M_{k_1}(\mathcal{A})p_0$, $\Phi_2 : \mathcal{B} \cong q_0M_{k_2}(\mathcal{B})q_0$, and that A and B are representation matrices of $T_{(k_1,p_0,\Phi_1)}$ and $T_{(k_2,q_0,\Phi_2)}$. Put $k = \max\{k_1, k_2\}$, $p = \mathrm{diag}(p_0, 0_{k-k_0})$ and $q = \mathrm{diag}(q_0, 0_{k-k_1})$. Then $p \in M_k(\mathcal{A})$, $q \in M_k(\mathcal{B})$, $\widehat{\Phi}_1 : \mathcal{A} \cong pM_k(\mathcal{A})p$ and $\widehat{\Phi}_2 : \mathcal{B} \cong qM_k(\mathcal{B})q$ with isomorphisms induced by Φ_1 and by Φ_2 respectively. Then $\widehat{\Phi}_1\otimes\widehat{\Phi}_2 : (p\otimes$

$q)(M_k(\mathcal{A}\otimes_{\min}\mathcal{B}))(p\otimes q)\cong\mathcal{A}\otimes_{\min}\mathcal{B}$. Therefore $(k, p\otimes q, \widehat{\Phi}_1\otimes\widehat{\Phi}_2)$ is a self-similar triple of $\mathcal{A}\otimes_{\min}\mathcal{B}$ and $T_{(k,p\otimes q,\widehat{\Phi}_1\otimes\widehat{\Phi}_2)}\varphi_i\otimes\psi_j = T_{(k_1,p_0,\Phi_1)}\varphi_i\otimes T_{(k_2,q_0,\Phi_2)}\psi_j$. Hence $A\otimes B\in F(\mathcal{A}\otimes_{\min}\mathcal{B})$. ∎

EXAMPLE 2.9. Put $\mathcal{A}=M_{2^\infty}$ and $\mathcal{B}=\mathbb{C}\oplus\mathbb{C}$. Then

$$F(\mathcal{A})=\{2^n : n\in\mathbb{Z}\},\quad F(\mathcal{B})=\left\{\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix}\right\},$$
$$F(\mathcal{A}\otimes_{\min}\mathcal{B})=\left\{\begin{bmatrix}2^n&0\\0&2^m\end{bmatrix},\begin{bmatrix}0&2^n\\2^m&0\end{bmatrix} : n,m\in\mathbb{Z}\right\},$$
$$F(\mathcal{A})\otimes F(\mathcal{B})=\left\{\begin{bmatrix}2^n&0\\0&2^n\end{bmatrix},\begin{bmatrix}0&2^n\\2^n&0\end{bmatrix} : n\in\mathbb{Z}\right\},$$
$$F_{\det}(\mathcal{A}\otimes_{\min}\mathcal{B})=\{2^n : n\in\mathbb{Z}\}\quad\text{and}\quad|\det|(F(\mathcal{A})\otimes F(\mathcal{B}))=\{4^n : n\in\mathbb{Z}\}.$$

Therefore $F(\mathcal{A}\otimes_{\min}\mathcal{B})\supsetneq F(\mathcal{A})\otimes F(\mathcal{B})$ and $F_{\det}(\mathcal{A}\otimes_{\min}\mathcal{B})\supsetneq|\det|(F(\mathcal{A})\otimes F(\mathcal{B}))$.

We defined the fundamental group and the determinant fundamental group. Our definitions agree with 3.2 of [16]. This fundamental group is a "numerical invariant". In the case of a simple C^*-algebra with unique trace, the group is a complete numerical invariant. In other words, if $F(\mathcal{A})$ and $F(\mathcal{B})$ are different as a set, then $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic (not Morita equivalent). Put $\mathcal{A}=M_{2^\infty}$ and $\mathcal{B}=M_{3^\infty}$. Then $F(\mathcal{A})=\{2^n : n\in\mathbb{Z}\}$ and $F(\mathcal{B})=\{3^n : n\in\mathbb{Z}\}$. These groups $F(\mathcal{A})$ and $F(\mathcal{B})$ are isomorphic as a group because both of them are isomorphic to $\mathbb{Z}$. However, $F(\mathcal{A})$ and $F(\mathcal{B})$ are not the same as a set, and $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. In the case of the C^*-algebras with finite dimensional trace space, a similar fact will be shown (Proposition 2.11 and Lemma 2.13).

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with finite dimensional trace space. The definition of the fundamental group $F(\mathcal{A})$ depends on the permutation of $\partial_e T(\mathcal{A})$ if $\sharp(\partial_e T(\mathcal{A}))>1$. Moreover the fundamental groups $F(\mathcal{A})$ and $F(\mathcal{B})$ of two C^*-algebras $\mathcal{A},\mathcal{B}$ which are Morita equivalent might be different in our definition (see Example 3.16). In this section, we introduce the concept of being isomorphic and being weightedly isomorphic on the fundamental groups $F(\mathcal{A})$. We define the canonical unitary representation U from a symmetric group S_n into $M_n(\mathbb{C})$ by $U(\sigma)_{ij}=1$ if $j=\sigma(i)$ and $U(\sigma)_{ij}=0$ if $j\neq\sigma(i)$.

DEFINITION 2.10. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras satisfying $\sharp(\partial_e T(\mathcal{A}))=\sharp(\partial_e T(\mathcal{B}))=n$. We say that $F(\mathcal{A})$ is *isomorphic* to $F(\mathcal{B})$ if there exists a permutation σ in S_n such that $F(\mathcal{B})=(U(\sigma))^{-1}F(\mathcal{A})(U(\sigma))$. We say that $F(\mathcal{A})$ is *weightedly isomorphic* to $F(\mathcal{B})$ if there exists an invertible positive diagonal matrix D in $M_n(\mathbb{C})$ and a permutation σ in S_n such that $F(\mathcal{B})=(DU(\sigma))^{-1}F(\mathcal{A})(DU(\sigma))$.

We first consider fundamental groups of C^*-algebras which are isomorphic.

PROPOSITION 2.11. *If two C^*-algebras, which have finite dimensional bounded trace spaces, are isomorphic, then their fundamental groups are the same up to permutation of basis.*

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras which have bounded trace spaces. We suppose $\sharp(\partial_e T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_e T(\mathcal{A})$. If $\mathcal{A}$ and $\mathcal{B}$ are isomorphic, there exists an isomorphism $\alpha : \mathcal{B} \to \mathcal{A}$. Then $\{\varphi_i \circ \alpha\}_{i=1}^n = \partial_e T(\mathcal{B})$. ∎

We shall show that if two unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then their fundamental groups $F(\mathcal{A})$ and $F(\mathcal{B})$ are weightedly isomorphic and $F_{\det}(\mathcal{A}) = F_{\det}(\mathcal{B})$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras which have bounded trace spaces. If $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then there exists an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule $\mathcal{F}$. We define the linear map $R_{\mathcal{A}\mathcal{B}} : {}_{\mathcal{A}}E_{\mathcal{B}} \to \mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{B}), \mathrm{lin}_{\mathbb{C}} T(\mathcal{A}))$ by

$$(R_{\mathcal{A}\mathcal{B}}([\mathcal{F}])(\varphi))(a) = \sum_{i=1}^{k} \varphi(\langle \xi_i, a\xi_i \rangle_{\mathcal{B}}),$$

where $\{\xi_i\}_{i=1}^k$ is a basis of $\mathcal{F}$ as a right Hilbert $\mathcal{B}$-module.

LEMMA 2.12. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be unital C^*-algebras which have bounded trace spaces. We suppose $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are Morita equivalent. Let $\mathcal{E}$ be an imprimitivity $\mathcal{A}$-$\mathcal{B}$-bimodule and let $\mathcal{F}$ be an imprimitivity $\mathcal{B}$-$\mathcal{C}$ bimodule.*

Then $R_{\mathcal{A}\mathcal{B}}([\mathcal{E}])R_{\mathcal{B}\mathcal{C}}([\mathcal{F}]) = R_{\mathcal{A}\mathcal{C}}([\mathcal{E} \otimes \mathcal{F}])$. In particular, $R_{\mathcal{A}\mathcal{B}}([\mathcal{F}])R_{\mathcal{B}\mathcal{A}}([\mathcal{F}^*]) = \mathrm{id}_{\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})}$, $R_{\mathcal{B}\mathcal{A}}([\mathcal{F}^*])R_{\mathcal{A}\mathcal{B}}([\mathcal{F}]) = \mathrm{id}_{\mathrm{lin}_{\mathbb{C}} T(\mathcal{B})}$ and $\sharp(\partial_e T(\mathcal{A})) = \sharp(\partial_e T(\mathcal{B}))$.

Proof. The independence of the choice of the basis of $\mathcal{F}$ and the well-definedness of $R_{\mathcal{A}\mathcal{B}}([\mathcal{F}])$ can be showed similarly with the proof of the definition of $R_{\mathcal{A}}([\mathcal{E}])$ in Proposition 2.3. Moreover, $R_{\mathcal{A}\mathcal{B}}([\mathcal{E}])R_{\mathcal{B}\mathcal{C}}([\mathcal{F}]) = R_{\mathcal{A}\mathcal{C}}([\mathcal{E} \otimes \mathcal{F}])$ can be showed similarly with the proof of the multiplicativity of $R_{\mathcal{A}}([\mathcal{E}])$ in Proposition 2.3. ∎

In particular, $R_{\mathcal{A}\mathcal{A}}([\mathcal{E}])$=$R_{\mathcal{A}}([\mathcal{E}])$ where $\mathcal{E}$ is a $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule.

PROPOSITION 2.13. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras with finite dimensional bounded trace space. If $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, then $F(\mathcal{A})$ is weightedly isomorphic to $F(\mathcal{B})$ and $F_{\det}(\mathcal{A}) = F_{\det}(\mathcal{B})$.*

Proof. Let $\mathcal{F}$ be an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. Then $\mathcal{F}$ induces an isomorphism Ψ of $\mathrm{Pic}(\mathcal{A})$ to $\mathrm{Pic}(\mathcal{B})$ such that $\Psi([\mathcal{E}]) = [\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]$ for $[\mathcal{E}] \in \mathrm{Pic}(\mathcal{A})$. By Lemma 2.12, for $[\mathcal{E}] \in \mathrm{Pic}(\mathcal{A})$

$$\begin{aligned} R_{\mathcal{B}}([\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]) = R_{\mathcal{B}\mathcal{B}}([\mathcal{F}^* \otimes \mathcal{E} \otimes \mathcal{F}]) &= R_{\mathcal{B}\mathcal{A}}([\mathcal{F}^*])R_{\mathcal{A}\mathcal{A}}([\mathcal{E}])R_{\mathcal{A}\mathcal{B}}([\mathcal{F}]) \\ &= R_{\mathcal{B}\mathcal{A}}([\mathcal{F}^*])R_{\mathcal{A}}([\mathcal{E}])R_{\mathcal{A}\mathcal{B}}([\mathcal{F}]). \end{aligned}$$

Put $\sharp(\partial_e T(\mathcal{A})) = \sharp(\partial_e T(\mathcal{B})) = n$. We shall consider the representation of the previous formula into $M_n(\mathbb{C})$ with respect to the basis $\partial_e T(\mathcal{A})$ of $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$ and the basis $\partial_e T(\mathcal{B})$ of $\mathrm{lin}_{\mathbb{C}} T(\mathcal{B})$. Then the representation matrices of $R_{\mathcal{A}}([\mathcal{E}])$ and $R_{\mathcal{B}}([\Psi(\mathcal{E})])$ are $S_{\mathcal{A}}(R_{\mathcal{A}}([\mathcal{E}]))$ and $S_{\mathcal{B}}(R_{\mathcal{B}}([\Psi(\mathcal{E})]))$ respectively. By the definition of $R_{\mathcal{AB}}(\mathcal{F})$ and $R_{\mathcal{BA}}(\mathcal{F}^*)$, they preserve positiveness of traces, so the entries of the matrices which are represented by them are positive. Moreover each of them is the inverse element of the other by Lemma 2.12. Therefore the representation matrix of $R_{\mathcal{AB}}(\mathcal{F})$ satisfies the hypothesis of Lemma 3.5. Hence there exist a positive invertible diagonal matrix D and σ in S_n such that $S_{\mathcal{B}}(R_{\mathcal{B}}([\Psi(\mathcal{E})])) = (DU(\sigma))^{-1}S_{\mathcal{A}}(R_{\mathcal{A}}(\mathcal{F}))(DU(\sigma))$.

Furthermore, since $\det(P^{-1}AP) = \det(A)$ for $P \in GL_n(\mathbb{C})$ and for any $A \in M_n(\mathbb{C})$, $F_{\det}(\mathcal{A}) = F_{\det}(\mathcal{B})$. ∎

3. FORMS OF FUNDAMENTAL GROUPS AND SOME EXAMPLES

We shall show that $F(\mathcal{A})$ is restricted by K-theoretical obstruction and positivity. This computation is motivated by Proposition 3.7 of [16]. We denote by φ^* the map from $K_0(\mathcal{A})$ into $\mathbb{R}$ induced by a bounded trace φ on $\mathcal{A}$. We denote by $H(\mathcal{A})$ the set of isomorphic classes $[\chi]$ of right Hilbert $\mathcal{A}$-modules χ with finite basis $\{\xi_i\}_{i=1}^k$. We define a pairing $\langle\cdot,\cdot\rangle : H(\mathcal{A}) \times \mathrm{lin}_{\mathbb{C}}T(\mathcal{A}) \to \mathbb{C}$ by $\langle[\chi],\varphi\rangle = \sum_{i=1}^{k} \varphi(\langle\xi_i,\xi_i\rangle_{\mathcal{A}})$. As in Proposition 2.3, it will be shown that this pairing is well-defined and does not depend on the chosen basis.

PROPOSITION 3.1. *Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. Then*

$$\langle[\chi\otimes\mathcal{E}],\varphi\rangle = \langle[\chi], R_{\mathcal{A}}([\mathcal{E}])(\varphi)\rangle$$

for any right Hilbert $\mathcal{A}$-modules χ with finite basis, for any $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule $\mathcal{E}$ and for any $\varphi \in \mathrm{lin}_{\mathbb{C}}T(\mathcal{A})$.

Proof. Let $\{\xi_i\}_{i=1}^k$ be a finite basis of χ and let $\{\eta_j\}_{j=1}^l$ be a finite basis of $\mathcal{E}$. Then $\langle[\chi\otimes\mathcal{E}],\varphi\rangle = \sum_{i,j=1}^{k,l} \varphi(\langle\eta_j,\langle\xi_i,\xi_i\rangle_{\mathcal{A}}\eta_j\rangle_{\mathcal{A}}) = \langle[\chi], R_{\mathcal{A}}([\mathcal{E}])(\varphi)\rangle$. ∎

The following computations are based on the pairing.

LEMMA 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. We define the map $\widehat{R}_{\mathcal{A}} : H(\mathcal{A}) \to \mathrm{lin}_{\mathbb{C}}T(\mathcal{A})^*$ by $\widehat{R}_{\mathcal{A}}([\chi])(\varphi) = \sum_{i=1}^{k} \varphi(\langle\xi_i,\xi_i\rangle)$, where $\{\xi_i\}_{i=1}^k$ is a finite basis of $\mathcal{E}$ as a right Hilbert $\mathcal{A}$-module. If $[\mathcal{E}]$ is an element of $\mathrm{Pic}(\mathcal{A})$, then $\widehat{R}_{\mathcal{A}}([\chi\otimes_{\mathcal{A}}\mathcal{E}]) = \widehat{R}_{\mathcal{A}}([\chi])R_{\mathcal{A}}([\mathcal{E}])$.*

Proof. As in Proposition 2.3, $\widehat{R}_{\mathcal{A}}([\chi])$ does not depend on the choice of basis, $\widehat{R}_{\mathcal{A}}$ is well-defined, and $\widehat{R}_{\mathcal{A}}([\chi\otimes_{\mathcal{A}}\mathcal{E}]) = \widehat{R}_{\mathcal{A}}([\chi])R_{\mathcal{A}}([\mathcal{E}])$ for all $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule $\mathcal{E}$. ∎

We denote by $P_k(\mathcal{A})$ the set of projections in $M_k(\mathcal{A})$ and put $P_\infty(\mathcal{A}) = \bigcup_{k=1}^{\infty} P_k(\mathcal{A})$.

LEMMA 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra and let φ be a positive tracial linear functional on $\mathcal{A}$. Then*

$$\begin{aligned}\{\varphi^*([p]_0) : p \in P_\infty(\mathcal{A})\} &= \{\varphi \otimes \mathrm{Tr}_k(p) : p \in M_k(\mathcal{A}), k \in \mathbb{N}\} \\ &= \{\widehat{R}_{\mathcal{A}}([\chi])(\varphi) : [\chi] \in H(\mathcal{A})\}.\end{aligned}$$

Proof. Let χ be a right Hilbert $\mathcal{A}$-module with finite basis $\{\xi_i\}_{i=1}^k$. Put $p = (\langle \xi_i, \xi_j \rangle_{\mathcal{A}})_{ij}$. Then p is a projection in $M_k(\mathcal{A})$. Then $\varphi(p) = \widehat{R}_{\mathcal{A}}([\chi])(\varphi)$. On the other hand, let p be a projection in $M_k(\mathcal{A})$. Then $p\mathcal{A}^k$ is a right Hilbert $\mathcal{A}$-module with finite basis $\{pe_i\}_{i=1}^k$. Therefore $\widehat{R}_{\mathcal{A}}([p\mathcal{A}^k])(\varphi) = \varphi(p)$. ∎

LEMMA 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. We suppose $\sharp(\partial_e T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_e T(\mathcal{A})$. Put $E = \{(\varphi_i^*(g_i))_{i=1,\ldots,n} : g_i \in K_0(\mathcal{A})\}$. Then we can consider E as an additive subgroup of $M_{1,n}(\mathbb{C})$. Then $EB = E$ for all B in $F(\mathcal{A})$. In particular, E is a module over $F(\mathcal{A})$.*

Proof. Let B be an element in $F(\mathcal{A})$. Put $E_0 = \{(\widehat{R}_{\mathcal{A}}([\chi_i])(\varphi_i))_{i=1,\ldots,n} : [\chi_i] \in H(\mathcal{A})\}$. Consider the representation of the equation in Lemma 3.2 with a basis $\partial_e T(\mathcal{A})$, $E_0 B \subset E_0$. Since B is invertible, $E_0 B = E_0$. By Lemma 3.3 and by the fact $K_0(\mathcal{A}) = \{[p]_0 - [q]_0 : p \in P_\infty(\mathcal{A})\}$, $E_0 - E_0 = E$. Hence $EB = E$. ∎

We denote by $M_n(\mathbb{R}^+)$ the set of matrices with nonnegative entries.

LEMMA 3.5. *Let A be an invertible element of $M_n(\mathbb{C})$. If A and A^{-1} are elements of $M_n(\mathbb{R}^+)$, then there exist a permutation σ in S_n and a positive invertible diagonal matrix D, such that $A = DU(\sigma)$.*

Proof. This proof is based on the calculation of matrix elements. It is sufficient to show that there exists only one positive element in each row and in each column. Let $A = \{a_{ij}\}, B = \{b_{ij}\}$ be an element of $M_n(\mathbb{R}^+)$ such that $AB = BA = I_n$. We regard i_0 as fixed and we suppose $a_{i_0 j} = 0$ for all j. Then $\det A = 0$, which contradicts the invertibility of A. Therefore there exists j_0 such that $a_{i_0 j_0} > 0$. We shall show that if $j \neq j_0$, then $a_{i_0 j} = 0$. Since $AB = I_n$, if $j \neq i_0$, $\sum_{k=1}^{n} a_{i_0 k} b_{kj} = 0$. By the positivity of the matrix elements of A and B, if $j \neq i_0$, then $b_{j_0 j} = 0$. Otherwise, we suppose $b_{j_0 i_0} = 0$, then all the elements of the j_0 row are 0, which contradicts the invertibility of B. Therefore $b_{j_0 i_0} > 0$. Since $BA = I_n$, if $j \neq j_0$, then $\sum_{k=1}^{n} b_{j_0 k} a_{kj} = 0$. Because $b_{j_0 i_0} > 0$, if $j \neq j_0$, then $a_{i_0 j} = 0$. Hence there exists only one positive element in each column. By transposing the matrices on both sides of $AB = BA = I_n$, the rest of the proof runs as before. ∎

By the above lemmas, the form of the elements in $F(\mathcal{A})$ is restricted.

PROPOSITION 3.6. *Let A be a unital C^*-algebra with finite dimensional bounded trace space. We suppose $\sharp(\partial_e T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_e T(\mathcal{A})$. For any B in $F(\mathcal{A})$, there exist a positive invertible diagonal matrix D in $M_n(\mathbb{R}^+)$ and $\sigma \in S_n$ such that $B = DU(\sigma)$ and $D_{ii}\varphi_i^*(K_0(\mathcal{A})) = \varphi_{\sigma(i)}^*(K_0(\mathcal{A}))$.*

Proof. We first prove that for any element B in $F(\mathcal{A})$ there exist a σ in S_n and a positive invertible diagonal matrix D such that $B = DU(\sigma)$. Let φ be a positive tracial linear functional on $\mathcal{A}$, let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule, and let $\{\xi_i\}_{i=1}^n$ be a basis of $\mathcal{E}$. Because φ is positive, $R_{\mathcal{A}}([\mathcal{E}])(\varphi)$ is positive. Therefore considering the representation of $R_{\mathcal{A}}([\mathcal{E}])$ and $R_{\mathcal{A}}([\mathcal{E}^*])$ with the basis $\partial_e T(\mathcal{A})$, we see that all entries of the matrices are positive. By Lemma 3.5, the first step is proved and we can put $B = DU(\sigma)$.

We next prove $D_{ii}\varphi_i^*(K_0(\mathcal{A})) = \varphi_{\sigma(i)}^*(K_0(\mathcal{A}))$. Put $E_i = \varphi_i^*(K_0(\mathcal{A}))$. By Lemma 3.4 and by $B = DU(\sigma)$, we can obtain $D_{ii}E_i \subset E_{\sigma(i)}$. Considering the invertibility of B in $F(\mathcal{A})$, $\frac{1}{D_{ii}}E_{\sigma(i)} \subset E_i$. Hence $D_{ii}E_i = E_{\sigma(i)}$. ∎

REMARK 3.7. Let $\mathcal{A}$ be a unital C^*-algebra. We say that a positive tracial linear functional φ on $\mathcal{A}$ is an extreme ray if for any fixed positive tracial linear functional ψ on $\mathcal{A}$, if $\psi \leqslant \varphi$, then there exists a positive real number λ such that $\lambda\varphi = \psi$. If a positive tracial linear functional φ is an extreme ray, then there exists a positive real number λ and an extremal point ϕ of $T(\mathcal{A})$ such that $\varphi = \lambda\phi$. Let $\mathcal{B}$ be a unital C^*-algebra. We suppose that $\partial_e T(\mathcal{A})$ and $\partial_e T(\mathcal{B})$ are finite sets and that $\sharp(\partial_e T(\mathcal{A})) = \sharp(\partial_e T(\mathcal{B}))$. Let T be an invertible element of $\mathcal{L}(\mathrm{lin}_{\mathbb{C}} T(\mathcal{A}), \mathrm{lin}_{\mathbb{C}} T(\mathcal{B}))$. If T and T^{-1} are positive, then $T\varphi$ is an extreme ray for any extreme ray φ. We can also prove the part of the previous proposition by this fact. Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. We suppose $\sharp(\partial_e T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_e T(\mathcal{A})$. By the fact of the extreme ray, there exist a permutation σ and a positive real number λ_i such that $T_{(k,p,\Phi)}(\varphi_i) = \lambda_i \varphi_{\sigma(i)}$.

COROLLARY 3.8. *Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space. If $\mathcal{A}$ is separable, then $F(\mathcal{A})$ and $F_{\det}(\mathcal{A})$ are countable groups.*

Proof. We suppose $\sharp(\partial_e T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_e T(\mathcal{A})$. Then $\{(\varphi_i^*(g_i))_{i=1,\dots,n} : g_i \in K_0(\mathcal{A})\}$ is a countable additive subgroup of $\mathbb{R}^n$. Let $B \in F(\mathcal{A})$. There exists an invertible positive diagonal matrix D in $M_n(\mathbb{R}^+)$ and $\sigma \in S_n$ such that $B = DU(\sigma)$. Since $1 \in \varphi_i^*(K_0(\mathcal{A}))$ for any i, $D_{ii} \in \varphi_{\sigma(i)}^*(K_0(\mathcal{A}))$ for any k, l. Therefore $F(\mathcal{A})$ and $F_{\det}(\mathcal{A})$ are countable. ∎

So, Proposition 3.6 enables us to calculate $F(\mathcal{A})$ easily. We shall show some examples.

COROLLARY 3.9. *Let $\mathcal{A}$ be a unital C^*-algebra with a unique normalized bounded trace τ, let $\mathcal{B} \equiv \bigoplus_{i=1}^n \mathcal{A}_i$ where $\mathcal{A}_i = \mathcal{A}$. If $F(\mathcal{A}) = (\tau^*(K_0(\mathcal{A})) \setminus \{0\}) \cap (\tau^*(K_0(\mathcal{A})) \setminus$*

$\{0\})^{-1} \cap \mathbb{R}^+$, *then*

$$F(\mathcal{B}) = \{DU(\sigma) : \sigma \in S_n, D \text{ is a diagonal matrix in } M_n(\mathbb{R}^+) \text{ such that } D_{ii} \in F(\mathcal{A})\}.$$

Proof. Let τ_i be the normalized trace of $\mathcal{A}_i$. We define $\alpha(\sigma) : \mathcal{B} \to \mathcal{B}$ by $\alpha(\sigma)(a_1, a_2, \ldots, a_n) = (a_{\sigma^{-1}(1)}, a_{\sigma^{-1}(2)}, \ldots, a_{\sigma^{-1}(n)})$ where $\sigma \in S_n$.

Considering $[\mathcal{E}_{\alpha(\sigma)}]$ in $\mathrm{Pic}(\mathcal{B})$, $F(\mathcal{B})$ includes permutations of $M_n(\mathbb{R})$. Moreover $\{D$: diagonal matrix in $M_n(\mathbb{R}^+)$: $D_{ii} \in F(\mathcal{A})\}$ is a subset(subgroup) of $F(\mathcal{B})$. Hence (RHS)$\subset F(\mathcal{B})$. But, by Proposition 3.6, $F(\mathcal{B}) \subset$(RHS). ∎

COROLLARY 3.10. *Let $\mathcal{A}$ be unital C^*-algebras with finite dimensional bounded trace space. We suppose $\sharp(\partial_e T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_e T(\mathcal{A})$. If there exist i_0 and i_1 such that $d\varphi_{i_0}^*(K_0(\mathcal{A})) \neq \varphi_{i_1}^*(K_0(\mathcal{A}))$ for any $d \in \mathbb{R}^+$, then $B_{i_0 i_1} = B_{i_1 i_0} = 0$ for all element B in $F(\mathcal{A})$.*

Proof. We suppose $B_{i_0 i_1} \neq 0$. Then $B_{i_0 i_1}\varphi_{i_1}^*(K_0(\mathcal{A})) = \varphi_{i_0}^*(K_0(\mathcal{A}))$. This contradicts our assumption. ∎

EXAMPLE 3.11. Put $\mathcal{A} = M_{2^\infty} \oplus M_{2^\infty} \oplus M_{3^\infty}$, then

$$F(\mathcal{A}) = \left\{ \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 2^l & 0 \\ 0 & 0 & 3^m \end{bmatrix}, \begin{bmatrix} 0 & 2^k & 0 \\ 2^l & 0 & 0 \\ 0 & 0 & 3^m \end{bmatrix} : k, l, m \in \mathbb{Z} \right\} \text{ and } F_{\det}(\mathcal{A}) = \{ 2^n 3^m : n, m \in \mathbb{Z} \}.$$

EXAMPLE 3.12. Let $\{ p_n \}_{n=1}^\infty$ be an increasing enumeration of all prime numbers and let $\mathcal{A}_n = M_{\prod_{k=1}^n (4p_k^2)}(\mathbb{C}) \oplus M_{\prod_{k=1}^n (4p_k^2)}(\mathbb{C})$. We define $*$-homomorphisms $\psi_n : A_n \to A_{n+1}$ by $\psi_n((a,b)) = (\mathrm{diag}(a, a, \ldots, a, b, b), \mathrm{diag}(b, b, \ldots, b, a, a))$. Let $\mathcal{A}$ be the inductive limit of the sequence

$$\mathcal{A}_0 \xrightarrow{\psi_0} \mathcal{A}_1 \xrightarrow{\psi_1} \mathcal{A}_2 \longrightarrow \cdots$$

Then $\mathcal{A}$ is simple. Let $\tau_1^{(n)}$ and $\tau_2^{(n)}$ be the normalized traces on $\mathcal{A}_n$ such that

$$\tau_1^{(n)} = \frac{1}{\prod_{k=1}^n (4p_k^2)} \mathrm{Tr}_{\prod_{k=1}^n (4p_k^2)} \oplus 0 \quad \text{and} \quad \tau_2^{(n)} = 0 \oplus \frac{1}{\prod_{k=1}^n (4p_k^2)} \mathrm{Tr}_{\prod_{k=1}^n (4p_k^2)}.$$

Put $\varphi_1^{(n)}, \varphi_2^{(n)}$ as follows:

$$\varphi_1^{(n)} = \Big(\frac{1}{2} + \frac{3\prod_{k=1}^n p_k^2}{(\prod_{k=1}^n (p_k^2 - 1))\pi^2} \Big) \tau_1^{(n)} + \Big(\frac{1}{2} - \frac{3\prod_{k=1}^n p_k^2}{(\prod_{k=1}^n (p_k^2 - 1))\pi^2} \Big) \tau_2^{(n)} \quad \text{and}$$

$$\varphi_2^{(n)} = \Big(\frac{1}{2} - \frac{3\prod_{k=1}^{n+1} p_k^2}{(\prod_{k=1}^n (p_k^2 - 1))\pi^2} \Big) \tau_1^{(n)} + \Big(\frac{1}{2} + \frac{3\prod_{k=1}^n p_k^2}{(\prod_{k=1}^n (p_k^2 - 1))\pi^2} \Big) \tau_2^{(n)}.$$

Then $\varphi_i^{(n)}$ is a tracial state on $\mathcal{A}_n$ and satisfies $\varphi_i^{(n)} = \varphi_i^{n+1} \circ \psi_n$. Therefore there exists a tracial state φ_i on $\mathcal{A}$ such that $\varphi_i|_{\mathcal{A}_n} = \varphi_i^{(n)}$. Since φ_1 and φ_2 are linearly independent, then $\sharp(\partial_e(T(\mathcal{A}))) = 2$. Moreover, $\{ \varphi_1, \varphi_2 \} = \partial_e(T(\mathcal{A}))$.

We compute the fundamental group of $\mathcal{A}$ and we shall show

$$F(\mathcal{A}) = \left\{ \begin{bmatrix} 2^n & 0 \\ 0 & 2^n \end{bmatrix}, \begin{bmatrix} 0 & 2^n \\ 2^n & 0 \end{bmatrix} (n \in \mathbb{Z}) \right\} \quad \text{and} \quad F_{\det}(\mathcal{A}) = \{ 4^n : n \in \mathbb{Z} \}.$$

LEMMA 3.13. *Let $\mathcal{A}$ be the inductive limit of the above sequence. Let φ_1, φ_2 as above. Then both $\varphi_1^*(K_0(\mathcal{A}))$ and $\varphi_2^*(K_0(\mathcal{A}))$ are the same additive group E. If there exists a positive real number λ such that $\lambda E = E$, then $\lambda = 2^n (n \in \mathbb{Z})$.*

Proof. Then both $\varphi_1^*(K_0(\mathcal{A}))$ and $\varphi_2^*(K_0(\mathcal{A}))$ are the same additive group E generated by $\frac{1}{2^{2n+1}\prod_{k=1}^n p_k^2} \pm \frac{3}{4^n(\prod_{k=1}^n (p_k^2-1))\pi^2}$ $(n \in \mathbb{N})$. Since $1 \in E$, there exist rational numbers q_1, q_2, q_3, q_4 such that $\lambda = q_1 + \frac{q_2}{\pi^2}$, $(q_1 + \frac{q_2}{\pi^2})(q_3 + \frac{q_4}{\pi^2}) = 1$. Since π is a transcendental number, $q_2 = q_4 = 0$. Then λ is a rational number. Put $\lambda = 2^a \cdot \frac{l}{m}$, where a is an integer and l, m are non-zero positive odd numbers satisfying $gcd(l, m) = 1$. We will show $l = m = 1$. Indeed, suppose $l \neq 1$. Then there exist a prime number p_{n_0} and an integer l_1 such that $l = p_{n_0} l_1$. Since $gcd(l, m) = 1$, $\frac{1}{2^{2n_0+1}\prod_{k=1}^{n_0} p_k^2} + \frac{3}{4^{n_0}(\prod_{k=1}^{n_0}(p_k^2-1))\pi^2} \notin \lambda E$. This contradicts the fact $\lambda E = E$. Suppose $m \neq 1$. Similarly, we can denote $m = p_{n_1} m_1$, where p_{n_1} is a prime number and m_1 is an integer. Then $\frac{1}{m}\left(\frac{1}{2^{2n_1+1}\prod_{k=1}^{n_1} p_k^2} + \frac{3}{4^{n_1}(\prod_{k=1}^{n_1}(p_k^2-1))\pi^2}\right) \notin E$. We next show $\lambda = 2^m (m \in \mathbb{Z})$, then $\lambda E = E$. It is sufficient to show the case $m > 0$. Let $\alpha = \frac{1}{2^{2n+1}\prod_{k=1}^n p_k^2} \pm \frac{3}{4^n(\prod_{k=1}^n(p_k^2-1))\pi^2}$ be a generator of E. For an integer n_0 more than $\frac{m}{2} + n$,

$$\alpha = 2^m \cdot \Big(\frac{2^{2n_0-m-2n}\prod_{k=n+1}^{n_0} p_k^2}{2^{2n_0+1}\prod_{k=1}^{n_0} p_k^2} \pm \frac{3(2^{2n_0-m-2n}\prod_{k=n+1}^{n_0}(p_k^2-1))}{4^{n_0}(\prod_{k=1}^{n_0}(p_k^2-1))\pi^2} \Big).$$

Therefore $\lambda E \supset E$. Obviously, $\lambda E \subset E$, so $\lambda E = E$. ∎

PROPOSITION 3.14. *Let $\mathcal{A}$ be the inductive limit of the sequence in Example* 3.12. *Then $F(\mathcal{A}) = \{ \left[\begin{smallmatrix} 2^n & 0 \\ 0 & 2^n \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 2^n \\ 2^n & 0 \end{smallmatrix}\right] (n \in \mathbb{Z}) \}$ and $F_{\det}(\mathcal{A}) = \{ 4^n : n \in \mathbb{Z} \}$.*

Proof. We first show $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in F(\mathcal{A})$. Let α_n be an automorphism from $\mathcal{A}_n$ onto $\mathcal{A}_n$ such that $\alpha_n((a,b)) = (b,a)$. Since $\psi_n \circ \alpha_n = \alpha_{n+1} \circ \psi_n$ for any $n \in \mathbb{N}$, there exists an automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ such that $\alpha|_{\mathcal{A}_n} = \alpha_n$. Therefore $\varphi_2 = \varphi_1 \circ \alpha$ and $\varphi_1 = \varphi_2 \circ \alpha$. On the other hand,

$$F(\mathcal{A}) \subset \left\{ \begin{bmatrix} 2^n & 0 \\ 0 & 2^m \end{bmatrix}, \begin{bmatrix} 0 & 2^n \\ 2^m & 0 \end{bmatrix} (n, m \in \mathbb{Z}) \right\}$$

by Definition 3.6 and Lemma 3.13. We next show that if $\left[\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right] \in F(\mathcal{A})$, then $a = b$. Let $\mathcal{E}$ be an $\mathcal{A}$-$\mathcal{A}$-imprimitivity bimodule such that $R_{\mathcal{A}}([\mathcal{E}])\varphi_1 = a\varphi_1$ and $R_{\mathcal{A}}([\mathcal{E}])\varphi_2 = b\varphi_2$. Then $\sum_{i=1}^n \varphi_1(\langle \xi_i, \xi_i \rangle_{\mathcal{A}}) = a$ and $\sum_{i=1}^n \varphi_2(\langle \xi_i, \xi_i \rangle_{\mathcal{A}}) = b$, where $\{\xi_i\}_{i=1}^n$ is a basis of $\mathcal{E}$. Put a projection $p = (\langle \xi_i, \xi_j \rangle_{\mathcal{A}})_{ij}$ on $M_n(\mathcal{A})$. Then $\mathrm{Tr}_n \otimes \varphi_1(p) = a$ and $\mathrm{Tr}_n \otimes \varphi_2(p) = b$. Since $M_n(\mathcal{A}) = \lim_{i\to\infty} M_n(\mathcal{A}_i)$, there exist a projection p_0 in $M_n(\mathcal{A}_i)$ such that $\mathrm{Tr}_n \otimes \varphi_1|_{\mathcal{A}_i}(p_0) = a$ and $\mathrm{Tr}_n \otimes \varphi_2|_{\mathcal{A}_i}(p_0) = b$. Put $p_0 = p_0^1 \oplus p_0^2$, where p_0^1 and p_0^2 are projections of $M_{n\prod_{k=1}^i 4p_k^2}(\mathbb{C})$. Then

$$\begin{aligned} a = {} & \mathrm{Tr}_n \otimes \Big(\frac{1}{2} + \frac{3\prod_{k=1}^{i+1} p_k^2}{(\prod_{k=1}^{i+1}(p_k^2-1))\pi^2}\Big) \cdot \frac{1}{\prod_{k=1}^{i}(4p_k^2)} \mathrm{Tr}_{\prod_{k=1}^{i}(4p_k^2)}(p_1) \\ & + \mathrm{Tr}_n \otimes \Big(\frac{1}{2} - \frac{3\prod_{k=1}^{i+1} p_k^2}{(\prod_{k=1}^{i}(p_k^2-1))\pi^2}\Big) \cdot \frac{1}{\prod_{k=1}^{i}(4p_k^2)} \mathrm{Tr}_{\prod_{k=1}^{i}(4p_k^2)}(p_2) \quad \text{and} \\ b = {} & \mathrm{Tr}_n \otimes \Big(\frac{1}{2} - \frac{3\prod_{k=1}^{i+1} p_k^2}{(\prod_{k=1}^{i}(p_k^2-1))\pi^2}\Big) \cdot \frac{1}{\prod_{k=1}^{i}(4p_k^2)} \mathrm{Tr}_{\prod_{k=1}^{i}(4p_k^2)}(p_1) \\ & + \mathrm{Tr}_n \otimes \Big(\frac{1}{2} + \frac{3\prod_{k=1}^{i+1} p_k^2}{(\prod_{k=1}^{i}(p_k^2-1))\pi^2}\Big) \cdot \frac{1}{\prod_{k=1}^{i}(4p_k^2)} \mathrm{Tr}_{\prod_{k=1}^{i}(4p_k^2)}(p_2). \end{aligned}$$

Let q_1, q_2 be rational numbers. Since a, b are rational numbers, if

$$\begin{aligned} &\Big(\frac{1}{2} + \frac{3\prod_{k=1}^{n} p_k^2}{(\prod_{k=1}^{n}(p_k^2-1))\pi^2}\Big)q_1 + \Big(\frac{1}{2} - \frac{3\prod_{k=1}^{n} p_k^2}{(\prod_{k=1}^{n}(p_k^2-1))\pi^2}\Big)q_2 = a \quad \text{and} \\ &\Big(\frac{1}{2} - \frac{3\prod_{k=1}^{n} p_k^2}{(\prod_{k=1}^{n}(p_k^2-1))\pi^2}\Big)q_1 + \Big(\frac{1}{2} - \frac{3\prod_{k=1}^{n} p_k^2}{(\prod_{k=1}^{n}(p_k^2-1))\pi^2}\Big)q_2 = b, \end{aligned}$$

then $a = b$. Therefore $a = b$. Finally, we show $\left[\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right] \in F(\mathcal{A})$. Let $\mathcal{B}_n = M_{\prod_{k=1}^n (2p_k^2)} \oplus M_{\prod_{k=1}^n (2p_k^2)}$. We define $*$-homomorphisms $\phi_n : \mathcal{A}_n \to \mathcal{A}_{n+1}$ by

$$\phi_n((a,b)) = (\mathrm{diag}(a, a, \ldots, a, b), \mathrm{diag}(b, b, \ldots, b, a)).$$

Let $\mathcal{B}$ be the inductive limit of the sequence

$$\mathcal{B}_1 \xrightarrow{\phi_1} \mathcal{B}_2 \xrightarrow{\phi_2} \mathcal{B}_3 \longrightarrow \cdots$$

Then $\mathcal{A}$ is isomorphic to $M_{2^\infty} \otimes B$. Therefore $\mathcal{A}$ is isomorphic to $M_2(\mathcal{A})$. ∎

EXAMPLE 3.15. Let $p > 2$ be a prime number. Put $\mathcal{B}_n = M_{\prod_{k=1}^n p^{k-1}}(\mathbb{C}) \oplus M_{\prod_{k=1}^n p^{k-1}}(\mathbb{C})$. We define $*$-homomorphisms $\psi_n : B_n \to B_{n+1}$ by $\psi_n((a,b)) = (\mathrm{diag}(a, \ldots, a, b, \ldots, b), \mathrm{diag}(b, \ldots, b, a, \ldots, a))$, where the multiplicities of a are $p^n - 2^{n-1}$ and 2^{n-1} respectively and the one of b are 2^{n-1} and $p^n - 2^{n-1}$ respectively. Let $\mathcal{B}$ be the inductive limit of the sequence

$$\mathcal{B}_1 \xrightarrow{\psi_1} \mathcal{B}_2 \xrightarrow{\psi_2} \mathcal{B}_3 \longrightarrow \cdots$$

Put $\mathcal{A} = M_{p^\infty} \otimes \mathcal{B}$. Then $\mathcal{A}$ is simple and $\dim \mathrm{lin}_{\mathbb{C}} \mathrm{T}(\mathcal{A}) = 2$. Since $\frac{2}{p}$ is an algebraic number, $\prod_{n=1}^{\infty}(1 - (\frac{2}{p})^n)$ is a transcendental number. As in the same proof of the previous example, $F(\mathcal{A}) = \left\{ \begin{bmatrix} p^n & 0 \\ 0 & p^n \end{bmatrix}, \begin{bmatrix} 0 & p^n \\ p^n & 0 \end{bmatrix} (n \in \mathbb{Z}) \right\}$ and $F_{\det}(\mathcal{A}) = \{ p^{2n} : n \in \mathbb{Z} \}$.

EXAMPLE 3.16. Let $\{n_i\}_{i=1}^m$ be an indexed finite subset of $\mathbb{N}$ such that $1 \leqslant n_1$ and $n_i \leqslant n_{i+1}$ for any i. Put $\mathcal{B} = \bigoplus_{i=1}^m M_{n_i}(\mathbb{C})$. We denote by φ_i the normalized trace on $M_{n(i)}(\mathbb{C})$. We show

$$F(\mathcal{B}) = \Big\{ DU(\sigma) : D_{ii} = \frac{n(i)}{n(\sigma(i))}\ \sigma \in S_N \Big\}. \tag{3.1}$$

Put $\mathcal{A} = \mathbb{C}^m$. The C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent and $\bigoplus_{i=1}^m M_{1,n_i}(\mathbb{C})$, denoted by $\mathcal{F}$, is an $\mathcal{A}$-$\mathcal{B}$-imprimitivity bimodule. Then the representation matrix P of $R_{\mathcal{A}\mathcal{B}}(\mathcal{F})$ is a diagonal matrix and $P_{ii} = n(i)$. By Proposition 2.13, $F(\mathcal{B}) = R_{\mathcal{B}\mathcal{A}}(\mathcal{F}^*)\{U(\sigma) : \sigma \in S_N\}R_{\mathcal{A}\mathcal{B}}(\mathcal{F})$. Especially, if $\mathcal{A} = \mathbb{C}^2$ and $\mathcal{B} = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$, then

$$\begin{aligned} F(\mathcal{A}) &= \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, \\ F(\mathcal{B}) &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} F(\mathcal{A}) \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{3}{2} \\ \frac{2}{3} & 0 \end{bmatrix} \right\} \end{aligned}$$

and $F_{\det}(\mathcal{A}) = F_{\det}(\mathcal{B}) = \{1\}$.

EXAMPLE 3.17. Let θ be an irrational number. The irrational rotation algebra $\mathcal{A}_\theta$ has a unique normalized trace φ_θ. The irrational rotation algebras $\mathcal{A}_\theta, \mathcal{A}_\eta$ are Morita equivalent if and only if $\eta = \frac{a\theta+b}{c\theta+d}$, $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{GL}_2(\mathbb{Z})$ where $\mathrm{GL}_2(\mathbb{Z})$ is the set of integer-valued 2×2 matrices A which satisfy $\det A = 1$ or -1. If X is an $\mathcal{A}_\eta$-$\mathcal{A}_\theta$-imprimitivity bimodule with $\mathcal{A}_\eta$-valued left inner product ${}_{\mathcal{A}_\eta}\langle \cdot, \cdot \rangle$ and with $\mathcal{A}_\theta$-valued right inner product $\langle \cdot, \cdot \rangle_{\mathcal{A}_\theta}$, we have the following equation: $\varphi_\theta(\langle \xi, \zeta \rangle_{\mathcal{A}_\theta}) = |c\theta + d|\varphi_\eta({}_{\mathcal{A}_\eta}\langle \zeta, \xi \rangle)$. These facts can be found in [22] and [23]. Put $\mathcal{B} = \mathcal{A}_\theta \oplus \mathcal{A}_\eta$ and $\mathcal{C} = \mathcal{A}_\theta \oplus \mathcal{A}_\theta$. Since $\mathcal{A}_\theta \oplus X$ is a $\mathcal{B}$-$\mathcal{C}$-imprimitivity bimodule, $F(\mathcal{B}) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & |c\theta+d| \end{smallmatrix}\right] F(\mathcal{C}) \left[\begin{smallmatrix} 1 & 0 \\ 0 & \frac{1}{|c\theta+d|} \end{smallmatrix}\right]$ and $F_{\det}(\mathcal{B}) = F_{\det}(\mathcal{C})$. From Corollary 3.9 and Corollary 3.18 of [16], we can calculate $F(\mathcal{B})$. In particular let $\theta = \sqrt{5}$ and $\eta = \frac{1}{\sqrt{5}}$, then

$$F(\mathcal{B}) = \left\{ \begin{bmatrix} (\sqrt{5}+2)^n & 0 \\ 0 & (\sqrt{5}+2)^m \end{bmatrix}, \begin{bmatrix} 0 & \frac{1}{\sqrt{5}}(\sqrt{5}+2)^n \\ \sqrt{5}(\sqrt{5}+2)^m & 0 \end{bmatrix} : n, m \in \mathbb{Z} \right\}$$

and $F_{\det}(\mathcal{B}) = \{\, (\sqrt{5}+2)^n : n \in \mathbb{Z} \,\}$. Let θ be a non-quadratic number and $\eta = \frac{a\theta+b}{c\theta+d}$, then $F(\mathcal{B}) = \left\{ \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & \frac{1}{|c\theta+d|} \\ |c\theta+d| & 0 \end{smallmatrix}\right] \right\}$ and $F_{\det}(\mathcal{B}) = \{1\}$.

Put $G_j = \left\{ \frac{m}{\prod_{i=1}^k p_i^j} : m \in \mathbb{Z},\ k \in \mathbb{N} \right\}$. Then G_j is an additive dense subgroup of $\mathbb{R}$.

LEMMA 3.18. *Let l, m be different natural numbers. If $G_l = \lambda G_l$ for any positive real number λ, then $\lambda = 1$. Moreover, there is no positive real number λ such that $G_l = \lambda G_m$.*

Proof. We suppose $G_l = \lambda G_l$. Since G_l is a subset of $\mathbb{Q}^+$, λ is a positive rational number. Put $\lambda = \frac{p}{q}$ where p and q are mutually prime. Suppose $p \neq 1$. Then there exists a prime factor p_0 of p. While $\frac{q}{p_0^l} \in qG_l$, $\frac{q}{p_0^l} \notin pG_l$. Therefore $pG_l \neq qG_l$. This contradicts $G_l = \lambda G_l$. Then $p = 1$. Similarly, $q = 1$. Therefore $\lambda = 1$. We suppose $l > m$ and $G_l = \lambda G_m$. Then λ is a positive rational number. Obviously $\lambda \neq 1$. Put $\lambda = \frac{p}{q}$ where p and q are natural numbers. Let r be a prime number which is neither a prime factor of p nor that of q. Then $\frac{q}{r^l} \in qG_l$ and $\frac{q}{r^l} \notin pG_m$. Therefore $pG_l \neq qG_l$. This contradicts $G_l = \lambda G_l$. Hence there is no positive real number λ such that $G_l = \lambda G_m$. ∎

Using Lemma 3.18, the following examples can be shown.

EXAMPLE 3.19. Let n be a natural number. Put $G = \prod_{l=1}^{n} G_l$ and $G^+ = \{ (g_1, g_2, \ldots, g_n) : g_i > 0 \} \cup 0$, where 0 is the additive unit of $\mathbb{R}^n$. Then (G, G^+) is unperforated and has the Riesz interpolation property. We denote by $\mathcal{A}_{\text{unit}}$ the unital simple AF-algebra the triples of which are isomorphic to $(G, G^+, (1, 1, \ldots, 1))$. Then $F(\mathcal{A}_{\text{unit}}) = \{I_n\}$ and $F_{\det}(\mathcal{A}_{\text{unit}}) = \{ 1 \}$, where I_n is a unit of $M_n(\mathbb{C})$.

EXAMPLE 3.20. Let n be a natural number. Put $G = \prod_{l=1}^{n} G_1$ and $G^+ = \{ (g_1, g_2, \ldots, g_n) : g_i > 0 \} \cup 0$, where 0 is the additive unit of $\mathbb{R}^n$. Then (G, G^+) is unperforated and has the Riesz interpolation property. We denote by $\mathcal{A}_{S_n}$ the unital simple AF-algebra the triples of which are isomorphic to $(G, G^+, (1, 1, \ldots, 1))$. Then $F(\mathcal{A}_{S_n}) = U(S_n)$.

EXAMPLE 3.21. Let θ be a non-quadratic number. Put $G_\theta = (G_1 + \theta G_1) \oplus (G_1 + \theta G_1)$ and $G_\theta^+ = \{ (g, h) : g > 0, h > 0 \} \cup 0$. Then (G_θ, G_θ^+) is unperforated and has the Riesz interpolation property. Moreover, $(1, 1)$ and $(1, \theta)$ are order units of (G_θ, G_θ^+) . We denote by $\mathcal{A}_{(1,1)}$ and by $\mathcal{A}_{(1,\theta)}$ the unital simple AF-algebras the triples of which are isomorphic to $(G_\theta, G_\theta^+, (1, 1))$ and $(G_\theta, G_\theta^+, (1, \theta))$ respectively. Then $\mathcal{A}_{(1,1)}$ and $\mathcal{A}_{(1,\theta)}$ are Morita equivalent. Moreover,

$$F(\mathcal{A}_{(1,1)}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, \quad F(\mathcal{A}_{(1,\theta)}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \theta \\ \frac{1}{\theta} & 0 \end{bmatrix} \right\} \quad \text{and}$$
$$F_{\det}(\mathcal{A}_{(1,1)}) = F_{\det}(\mathcal{A}_{(1,\theta)}) = \{ 1 \}.$$

EXAMPLE 3.22. Let θ be a quadratic number. Put $\overline{G}_\theta = (G_1 + \theta G_1 + \pi G_1) \oplus (G_1 + \theta G_1 + \pi G_1)$ and $\overline{G}_\theta^+ = \{ (g, h) : g > 0, h > 0 \} \cup 0$. Then $(\overline{G}_\theta, \overline{G}_\theta^+)$ is unperforated and has the Riesz interpolation property. Moreover, $(1, 1)$ and $(1, \theta)$ are order units of $(\overline{G}_\theta, \overline{G}_\theta^+)$. We denote by $\mathcal{B}_{(1,1)}$ and by $\mathcal{B}_{(1,\theta)}$ the unital simple AF-algebras the triples of which are isomorphic to $(\overline{G}_\theta, \overline{G}_\theta^+, (1,1))$ and $(\overline{G}_\theta, \overline{G}_\theta^+, (1,\theta))$ respectively. Then $\mathcal{B}_{(1,1)}$ and $\mathcal{B}_{(1,\theta)}$ are Morita equivalent. Moreover,

$$F(\mathcal{B}_{(1,1)}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}, \quad F(\mathcal{B}_{(1,\theta)}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \theta \\ \frac{1}{\theta} & 0 \end{bmatrix} \right\} \quad \text{and}$$
$$F_{\det}(\mathcal{A}_{(1,1)}) = F_{\det}(\mathcal{A}_{(1,\theta)}) = \{ 1 \}$$

Hence the next theorem follows.

THEOREM 3.23. *Let G be a subgroup of $GL_2(\mathbb{R})$ whose elements are represented as $DU(\sigma)$, where D is a positive diagonal matrix and $\sigma \in S_n$. If G is isomorphic to $\mathbb{Z}_2$ as a group, then there exists a simple* AF*-algebra $\mathcal{A}$ such that $F(\mathcal{A}) = G$.*

EXAMPLE 3.24. Let n be a natural number. Let $\theta_1, \dots, \theta_n$ be non-quadratic numbers which satisfy $\theta_i \neq \frac{a+b\theta_j}{c+d\theta_j}$ for any a, b, c, d in $\mathbb{Z}$ and for any i, j. Put

$$G_{\{\theta_1,\dots,\theta_n\}} = \bigoplus_{i=1}^{n} (\mathbb{Z} + \theta_i \mathbb{Z}) \quad \text{and} \quad G^+_{\{\theta_1,\dots,\theta_n\}} = \{ (g_1, \dots, g_n) : g_i > 0 \} \cup \{0\}.$$

Then $(G_{\{\theta_1,\dots,\theta_n\}}, G^+_{\{\theta_1,\dots,\theta_n\}})$ is unperforated and has the Riesz interpolation property. Moreover, $(1, \dots, 1)$ is an order unit of $(G_{\{\theta_1,\dots,\theta_n\}}, G^+_{\{\theta_1,\dots,\theta_n\}})$.

We denote by $\mathcal{A}_{\{\theta_1,\dots,\theta_n\}}$ the unital simple AF-algebra the triple of which is isomorphic to $(G_{\{\theta_1,\dots,\theta_n\}}, G^+_{\{\theta_1,\dots,\theta_n\}}, (1, \dots, 1))$. Then $F(\mathcal{A}_{\{\theta_1,\dots,\theta_n\}}) = \{I_n\}$.

By using Example 3.24, we shall show the following theorem.

THEOREM 3.25. *For any natural number n, there exist uncountably many mutually non-isomorphic simple (non)nuclear unital C^*-algebras $\mathcal{A}$ with n-dimensional trace space such that $F(\mathcal{A}) = \{I_n\}$.*

Proof. We show the case $n = 2$. If $\mathcal{A}_{\{\theta_1,\theta_2\}}$ is isomorphic to $\mathcal{A}_{\{\theta'_1,\theta'_2\}}$, then there exist $m_1, m_2 \in \mathbb{Z}$ such that $\{\theta'_1, \theta'_2\} = \{\theta_1 + m_1, \theta_2 + m_2\}$. Since there exist uncountably many sets $\{\theta_1, \theta_2\}$ each pair of which do not have that relation, there exist uncountably many mutually non-isomorphic simple unital AF-algebras. In the case of nonnuclear, all one have to do is to consider $\mathcal{A}_{\{\theta_1,\theta_2\}} \otimes C^*_r(\mathbb{F}_2)$. ∎

PROPOSITION 3.26. *Let G be a countable subgroup of $\mathbb{R}^\times_+$ and let n be a natural number. Then there exist uncountably many mutually non-isomorphic separable simple nonnuclear unital C^*-algebras $\mathcal{A}$ with n-dimensional trace space such that $F_{\det}(\mathcal{A}) \supset G$.*

Proof. Let $r \notin G$ be a real number of $\mathbb{R}^\times_+$. We denote G_r the subgroup of $\mathbb{R}^\times_+$ generated by r and G. Then there exist a separable non-nuclear unital C^*-algebra $\mathcal{B}_r$ with unique trace such that $F(\mathcal{B}_r) = \{ g^{1/n} : g \in G_r \}$. Then $\mathcal{B}_r \otimes \mathcal{A}_{S_n}$ is a separable non-nuclear unital C^*-algebra with n-dimensional trace space such that $F_{\det}(\mathcal{B}_r \otimes \mathcal{A}_{S_n}) \supseteq G_r \supsetneq G$. Since r is arbitrarily chosen and $F_{\det}(\mathcal{B}_r \otimes \mathcal{A}_{S_n})$ is countable, there exist uncountably many mutually non-isomorphic separable simple nonnuclear unital C^*-algebras. ∎

COROLLARY 3.27. *For any natural number n, there exist uncountably many mutually non-isomorphic separable simple nonnuclear unital C^*-algebras $\mathcal{A}$ with n-dimensional trace space whose fundamental groups $F(\mathcal{A})$ are all different.*

4. EXACT SEQUENCE OF PICARD GROUPS AND FUNDAMENTAL GROUPS

In this section, we shall show the diagram with respect to fundamental groups $F(\mathcal{A})$ and Picard groups of C^*-algebra $\mathcal{A}$. This construction generalizes the Proposition 3.26 of [16]. We denote by $\mathrm{Int}(\mathcal{A})$ the set of inner automorphisms of $\mathcal{A}$. We say that an automorphism α is *trace invariant* if $\varphi \circ \alpha = \varphi$ for any φ in $T(\mathcal{A})$.

DEFINITION 4.1. Let $\mathcal{A}$ be a unital C^*-algebra. We denote by $\mathrm{Aut}_{T(\mathcal{A})}(\mathcal{A})$ the set of automorphisms which are trace invariant. Then $\mathrm{Int}(\mathcal{A})$ is a normal subgroup of $\mathrm{Aut}_{T(\mathcal{A})}(\mathcal{A})$. We denote by $\mathrm{Out}_{T(\mathcal{A})}(\mathcal{A})$ the quotient group $\mathrm{Aut}_{T(\mathcal{A})}(\mathcal{A})/\mathrm{Int}(\mathcal{A})$.

Because $\mathrm{Aut}_{T(\mathcal{A})}(\mathcal{A})$ is normal subgroup of $\mathrm{Aut}(\mathcal{A})$, $\mathrm{Out}_{T(\mathcal{A})}(\mathcal{A})$ is a normal subgroup of $\mathrm{Out}(\mathcal{A})$. We denote by $\rho_{\mathcal{A}}|$ the restriction of $\rho_{\mathcal{A}}$ to $\mathrm{Aut}_{T(\mathcal{A})}(\mathcal{A})$. We say that $T(\mathcal{A})$ separates equivalence classes of projections if for any fixed natural number n and for any projections $p, q \in M_n(\mathcal{A})$, if $\mathrm{Tr}_n \otimes \varphi(p) = \mathrm{Tr}_n \otimes \varphi(q)$ for all φ in $T(\mathcal{A})$, then p and q are Murray–von Neumann equivalent.

THEOREM 4.2. *Let A be a unital C^*-algebra with n dimensional bounded trace space. If $T(\mathcal{A})$ separates equivalence classes of projections, then we have the following commutative diagram whose horizontal lines are exact:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Out}_{T(\mathcal{A})}(\mathcal{A}) & \xrightarrow{\rho_{\mathcal{A}}|} & \mathrm{Pic}(\mathcal{A}) & \xrightarrow{S_{\mathcal{A}}R_{\mathcal{A}}} & F(\mathcal{A}) & \longrightarrow & 1 \\
 & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \rho_{\mathcal{A}}}\uparrow & & {\scriptstyle i_F}\uparrow & & \\
1 & \longrightarrow & \mathrm{Out}_{T(\mathcal{A})}(\mathcal{A}) & \xrightarrow{i_{\mathrm{out}}} & \mathrm{Out}(\mathcal{A}) & \xrightarrow{\rho_{\mathcal{A}}S_{\mathcal{A}}R_{\mathcal{A}}} & U(S_n) \cap F(\mathcal{A}) & \longrightarrow & 1
\end{array}.$$

In this diagram, i_{out} and i_F are the inclusion maps from $\mathrm{Out}_{T(\mathcal{A})}(\mathcal{A})$ into $\mathrm{Out}(\mathcal{A})$ and from $U(S_n) \cap F(\mathcal{A})$ into $F(\mathcal{A})$ respectively.

Proof. It is sufficient to show horizontal lines are exact. We show the first line is exact. The map $\rho_{\mathcal{A}}$ is one-to-one and $\mathrm{Im}\rho_{\mathcal{A}} \subset \mathrm{Ker}(S_{\mathcal{A}}R_{\mathcal{A}})$ by the Chapter 2 and $S_A R_A$ is onto by definition. We shall show that $\mathrm{Ker}(S_A R_A) \subset \mathrm{Im}\rho_{\mathcal{A}}$. Let $[\mathcal{E}]$ be in $\mathrm{Ker}(S_{\mathcal{A}}R_{\mathcal{A}})$, let $\{\xi_i\}_{i=1}^k$ be the basis of $\mathcal{E}$, let $p = (\langle \xi_i, \xi_j \rangle_{\mathcal{A}})_{ij}$ in $M_k(\mathcal{A})$, and let Φ be the isomorphism from $\mathcal{A}$ to $pM_k(\mathcal{A})p$. Because $S_{\mathcal{A}}R_{\mathcal{A}}([\mathcal{E}]) = \mathrm{id}_{T(\mathcal{A})}$,

$\sum_{i=1}^{k} \varphi_j(\langle \xi_i, a\xi_i \rangle_{\mathcal{A}}) = \varphi_j(a)$ for all φ_j in $\partial_e T(\mathcal{A})$ and for all a in A. Substituting $a = 1_{\mathcal{A}}$ into the previous formula, we can obtain the formula $\mathrm{Tr}_k \otimes \varphi(p) = \mathrm{Tr}_k \otimes \varphi(1_{\mathcal{A}} \otimes e_{11}) = 1$. By assumption, there exists a partial isometry w such that $p = w^*w$ and $1 \otimes e_{11} = ww^*$ in $M_k(\mathcal{A})$. Then there exists an automorphism α of $\mathcal{A}$ such that $w\Phi(a)w^* = \alpha(a) \otimes e_{11}$. Hence $[\mathcal{E}] = [\mathcal{E}_\alpha]$. Since $[\mathcal{E}]$ is in $\mathrm{Ker}(S_{\mathcal{A}}R_{\mathcal{A}})$, α is an element of $\mathrm{Out}_{T(\mathcal{A})}(\mathcal{A})$. We next show that $\rho_{\mathcal{A}}S_{\mathcal{A}}R_{\mathcal{A}}$ is onto. Since $S_{\mathcal{A}}R_{\mathcal{A}}$ is onto, for all $U(\sigma) \in U(S_n) \cap F(\mathcal{A})$ there exists an imprimitivity bimodule $\mathcal{E}$ such that $S_{\mathcal{A}}R_{\mathcal{A}}([\mathcal{E}]) = U(\sigma)$. Let $\{\xi_i\}_{i=1}^{l}$ be a basis of $\mathcal{E}$. Then $\sum_{i=1}^{l} \varphi_j(\langle \xi_i, a\xi_i \rangle_{\mathcal{A}}) = \varphi_{\sigma(j)}(a)$. As in the same proof in Theorem 4.2, there exists an automorphism α such that $[\mathcal{E}] = [\mathcal{E}_\alpha]$. ∎

By using the above diagram, we have the following corollary.

COROLLARY 4.3. *Let A be a unital C^*-algebra with finite dimensional bounded trace space. Put $\sharp(\partial_e(T(\mathcal{A}))) = n$. Under the same assumption as in Theorem* 4.2, *if $F(\mathcal{A}) \subset U(S_n)$, then* $\mathrm{Out}(\mathcal{A})$ *is isomorphic to* $\mathrm{Pic}(\mathcal{A})$.

We will show the relation between the scaling group and the fundamental group.

DEFINITION 4.4. Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional bounded trace space and let $\{\varphi_i\}_{i=1}^{n}$ be a basis of $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$. A dual system of $\{\varphi_i\}_{i=1}^{n}$ in $\mathcal{A}$ is a subset $\{ u_i \}_{i=1}^{n}$ of $\mathcal{A}$ satisfying $\varphi_i(u_j) = \delta_{ij}$, where δ_{ij} is Kronecker's delta.

LEMMA 4.5. *Let $\mathcal{A}$ be a unital C^*-algebra with n dimensional bounded trace space. There exists a dual system $\{ u_i \}_{i=1}^{n}$ for any basis* $\mathrm{lin}_{\mathbb{C}} T(\mathcal{A})$.

Proof. We define a linear map $\Phi : \mathcal{A} \to \mathbb{C}^n$ by $\Phi(a) = (\varphi_1(a), \varphi_2(a), \ldots, \varphi_n(a))$. It is sufficient to show $\dim \mathrm{Im}(\Phi) = n$. Obviously, $\dim \mathrm{Im}(\Phi) \leqslant n$. We will show $\dim \mathrm{Im}(\Phi) = \dim(\mathcal{A}/\mathrm{Ker}(\Phi)) \geqslant n$. The linear map $\iota : \mathrm{lin}_{\mathbb{C}} T(\mathcal{A}) \to (\mathcal{A}/\mathrm{Ker}(\Phi))^*$ given by $\iota(\varphi)([a]) = \varphi(a)$ is well-defined, where $[a]$ is an equivalent class of a in $\mathcal{A}$. Suppose $\iota(\varphi_1) = \iota(\varphi_2)$. Then $\varphi_1(a) = \varphi_2(a)$ for any a in $\mathcal{A}$. Therefore ι is injective. Hence $\dim(\mathcal{A}/\mathrm{Ker}(\Phi)) \geqslant n$. ∎

REMARK 4.6. Let $\mathcal{A}$ be a simple C^*-algebra and let τ be a non-zero bounded trace on $\mathcal{A}$. Since τ is a trace, $I = \{ a \in \mathcal{A} : \tau(a^*a) = 0 \}$ is a closed two-sided ideal. Therefore $I = \{0\}$ because $\mathcal{A}$ is simple. We suppose $\sharp(\partial_e T(\mathcal{A})) \geqslant 2$. Then no elements of a dual system of $\partial_e T(\mathcal{A})$ is positive because $\tau(a) > 0$ for any τ in $T(\mathcal{A})$ and for any positive element a in $\mathcal{A}$.

Using this dual basis, we can see

$$F(\mathcal{A}) = \{ ((\mathrm{Tr}_k \otimes \varphi_i) \circ \Phi(u_j))_{ij} : (k, p, \Phi) : \text{s.s.p.} \}.$$

We denote by Tr the canonical trace on $\mathbb{K}$.

Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional trace space and with no unbounded trace. We suppose $\sharp(\partial_{\mathrm{e}}(T(\mathcal{A})) = n$. Say $\{\varphi_i\}_{i=1}^n = \partial_{\mathrm{e}}(T(\mathcal{A}))$. Let $\{ u_j \}_{j=1}^n$ be a dual basis of $\{\varphi_i\}_{i=1}^n$. If $\alpha \in \mathrm{Aut}(\mathcal{A} \otimes \mathbb{K})$, then $(\tau \otimes \mathrm{Tr}) \circ \alpha$ is a densely defined, lower semicontinuous trace for any $\tau \in T(\mathcal{A})$. Since $\mathcal{A}$ has no unbounded trace, we can write $(\tau \otimes \mathrm{Tr}) \circ \alpha$ as a linear combination of $\{ \varphi_i \otimes \mathrm{Tr} \}_{i=1}^n$. Put

$$\mathcal{S}(\mathcal{A}) = \{ ((\varphi_i \otimes \mathrm{Tr}) \circ \alpha(u_j \otimes e_{11}))_{ij} : \alpha \in \mathrm{Aut}(\mathcal{A} \otimes \mathbb{K}) \} .$$

Even if $\mathcal{A} \otimes \mathbb{K}$ is isomorphic to $\mathcal{B} \otimes \mathbb{K}$, $S(\mathcal{A})$ might not be $S(\mathcal{B})$ because $S(\mathcal{A})$ depends on $\partial_{\mathrm{e}}(T(\mathcal{A}))$ (see Example 4.8). We shall show that this set is equal to $F(\mathcal{A})$. We denote by $\mathrm{M}(\mathcal{B})$ the multiplier algebra of the C^*-algebra $\mathcal{B}$.

PROPOSITION 4.7. *Let $\mathcal{A}$ be a unital C^*-algebra with finite dimensional trace space and with no unbounded trace. Then $\mathcal{S}(\mathcal{A}) = F(\mathcal{A})$.*

Proof. Let p be a self-similar full projection and let Φ be an isomorphism from $\mathcal{A} \to pM_n(\mathcal{A})p$. Then $\Phi \otimes \mathrm{id}_{\mathbb{K}} : \mathcal{A} \otimes \mathbb{K} \to pM_n(\mathcal{A})p \otimes \mathbb{K} = (p \otimes I)M_n(\mathcal{A}) \otimes \mathbb{K}(p \otimes I)$ is an isomorphism. Since p is full, there exists a partial isometry $v \in \mathrm{M}(M_n(\mathcal{A}) \otimes \mathbb{K})$ such that $v^*v = p \otimes I$ and that $vv^* = 1$ by Lemma 2.5 on [2]. Put $\beta_v(x) = vxv^*$ for $x \in \mathrm{M}(M_n(\mathcal{A}) \otimes \mathbb{K})$ and $\alpha = (\psi_n \otimes \mathrm{id}_{\mathcal{A}}) \circ \beta_v \circ (\Phi \otimes \mathrm{id}_{\mathbb{K}})$, where ψ_n is an isomorphism from $M_n(\mathbb{C}) \otimes \mathbb{K}$ onto $\mathbb{K}$ and $\mathrm{id}_{\mathcal{A}}$ is an identity map on $\mathcal{A}$. Then α is an automorphism on $\mathcal{A} \otimes \mathbb{K}$ and $(\tau \otimes \mathrm{Tr}) \circ \alpha(a) = \tau \otimes \mathrm{Tr}_n \circ \Phi(a)$ for any a in $\mathcal{A}$. Therefore $\mathcal{S}(\mathcal{A}) \supset F(\mathcal{A})$. Conversely, let α be an automorphism on $\mathcal{A} \otimes \mathbb{K}$. Put $p = \alpha(1 \otimes e_{11})$. Then there exists a projection q in $M_n(\mathcal{A})$ such that p and q are Murray–von Neumann equivalent. We define an isomorphism $\Psi : p(\mathcal{A} \otimes \mathbb{K})p \to q(M_n(\mathcal{A}))q$ by $\Psi(a) = vav^*$, where v is a partial isometry satisfying $v^*v = p$ and $vv^* = q$. Since $\Psi \circ \alpha$ is the isomorphism of the map from $\mathcal{A}$ onto $q(M_n(\mathcal{A}))q$, if Φ is the induced map by this, $(\tau \otimes \mathrm{Tr}) \circ \alpha(a) = \tau \otimes \mathrm{Tr}_n \circ \Phi(a)$ for any a in $\mathcal{A}$. Because $1 \otimes e_{11}$ is full, q is a full projection of $M_n(\mathcal{A})$. Therefore $\mathcal{S}(\mathcal{A}) \subset F(\mathcal{A})$. Hence $\mathcal{S}(\mathcal{A}) = F(\mathcal{A})$. ∎

EXAMPLE 4.8. Put $\mathcal{A} = M_2(\mathbb{C}) \oplus M_3(\mathbb{C})$ and $\mathcal{B} = \mathbb{C} \oplus \mathbb{C}$. Then $S(\mathcal{A}) = F(\mathcal{A}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & \frac{3}{2} \\ \frac{2}{3} & 0 \end{bmatrix} \right\}$ and $S(\mathcal{B}) = F(\mathcal{B}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}$ by Example 3.16. Although $\mathcal{A}$ and $\mathcal{B}$ are Morita equivalent, $F(\mathcal{A}) \neq F(\mathcal{B})$.

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Received March 8, 2016; revised May 14, 2016 and June 12, 2016.