

TOPOLOGICAL CONSTRUCTION OF C^* -CORRESPONDENCES FOR GROUPOID C^* -ALGEBRAS

ROHIT DILIP HOLKAR

Communicated by Stefaan Vaes

ABSTRACT. Let (G, α) and (H, β) be locally compact groupoids with Haar systems. We define a topological correspondence from (G, α) to (H, β) to be a G - H -bispaces X on which H acts properly, and X carries a continuous family of measures which is H -invariant and each measure in the family is (G, α) -quasi invariant. We show that a topological correspondence produces a C^* -correspondence from $C^*(G, \alpha)$ to $C^*(H, \beta)$. We give many examples of topological correspondences.

KEYWORDS: *Topological correspondences, morphisms of groupoids, groupoid correspondences.*

MSC (2010): 22D25, 22A22, 46H35, 47L30, 46L08.

INTRODUCTION

A C^* -algebraic correspondence K from a C^* -algebra A to B is an A - B -bimodule which is a Hilbert B -module and A acts on K via the adjointable operators in a non-degenerate fashion. Let $A = C^*(G, \alpha)$ and $B = C^*(H, \beta)$ where the ordered pairs (G, α) and (H, β) consist of a locally compact groupoid and a Haar system for it. Given a G - H -bispaces X carrying an H -invariant family of measures such that each measure in the family is (G, α) -quasi-invariant, we show that if the H -action is proper, then $C_c(X)$ can be completed into a C^* -correspondence from $C^*(G, \alpha)$ to $C^*(H, \beta)$. This work is an extension of some part of my thesis [5], where we worked with Hausdorff topologies with certain countability assumptions. In the present work, we get rid of the Hausdorffness and the countability hypotheses.

Morita equivalence of C^* -algebras is defined by the existence of an imprimitivity bimodule, a special kind of C^* -correspondence. In the well-known result [9] that a Morita equivalence between two locally compact groupoids with Haar systems induces a Morita equivalence between the groupoid C^* -algebras, the imprimitivity bimodule is constructed directly from a bispaces that gives the Morita

equivalence of the two groupoids. The Hausdorff case of Morita equivalence between locally compact groupoids is discussed in [9], and a more general and non-Hausdorff situation is studied in Definition 5.3 of [13]. Which extra structure or conditions are needed for a bispace to give only a C^* -correspondence instead of a Morita equivalence?

In general, we need a family of measures on the bispace as an extra structure to get started. In the Morita equivalence case, a family of measures on the bispace appears automatically (see Example 3.9). The family of measures which we need must be invariant for the right action and each measure of the family must be quasi-invariant for the left action. We also need that the right action be proper.

We use the $*$ -category of a locally compact groupoids introduced in [6], [13], to prove that certain actions and a bilinear form are well-defined (equations (2.3) and (2.4)). The process of constructing a C^* -correspondence from a topological correspondence is divided into two main parts: the first part is to construct the Hilbert module and the second one is to define the representation of the left groupoid C^* -algebra on this Hilbert module. The first part follows from a result of Renault, see Corollaire 5.2 of [13]. Though the statement of Corollaire 5.2 in [13] demands that the action of H is free as well as proper, the remark following the proof of the theorem and just above ([13], Définition 5.3) clarifies that the freeness of the action can be dropped. In the second part, our motivation and techniques are derived from the theory of quasi-invariant measures for locally compact groups ([4], Section 2.6).

As a C^* -correspondence from a C^* -algebra A to B induces a representation of B to that of A , a topological correspondence from a locally compact groupoid with a Haar system (G, α) to (H, β) induces a representation of H to that of G . Renault proves this in [14].

A locally compact, Hausdorff space is a locally compact groupoid with a Haar system, and so is a locally compact group. A well-known fact about groupoid equivalence is that two spaces are equivalent if and only if they are homeomorphic and two groups are equivalent if and only if they are isomorphic. But, since any continuous map between spaces gives a topological correspondence and so does a group homomorphism, see Examples 3.1 and 3.4, respectively, a topological correspondence is far more general than an equivalence.

We give many examples of topological correspondences, most of which are the analogues of the standard examples of C^* -correspondences.

In Example 3.1, we show that a continuous map $f : X \rightarrow Y$ between spaces gives a topological correspondence from Y to X . Example 3.4 shows that a continuous group homomorphism $\phi : G \rightarrow H$ gives a topological correspondences from G to H . Theorem 2.10 gives that the topological correspondences, the one from Y to X and the one from G to H , produce C^* -correspondences from $C_0(Y)$ to $C_0(X)$, and from $C^*(G)$ to $C^*(H)$, respectively. It is easy to see that the C^* -correspondence from $C_0(Y)$ to $C_0(X)$ is exactly the one give by the $*$ -homomorphism $f^* : C_0(Y) \rightarrow \mathcal{M}(C_0(X))$ as in the theory of commutative

C^* -algebras. However, it is not equally easy to see that the C^* -correspondence given by the group homomorphism ϕ agrees with the one which is induced by the $*$ -homomorphism $\phi_* : C^*(G) \rightarrow C^*(H)$.

Example 3.5 shows that if the group homomorphism in Example 3.4 is a proper map, then we get a topological correspondence from H to G .

Let E^0 and E^1 be locally compact, Hausdorff and second countable spaces, and let $s, r : E^1 \rightarrow E^0$ be continuous maps. Let $\lambda = \{\lambda_e\}_{e \in E^0}$ be a continuous family of measures along s . By applying the definition of a topological correspondence, it is straightforward to check that s, r and λ give a topological correspondence from E^0 to itself. Muhly and Tomforde ([10], Definition 3.1) call this correspondence a *topological quiver*. They construct a C^* -correspondence associated to a topological quiver in [10] and the construction in Section 3.1 of [10] is exactly the construction of a C^* -correspondence from a topological correspondence. Muhly and Tomforde define the C^* -algebra associated to a topological quiver ([10], Definition 3.17) which includes a vast class of C^* -algebras: graph C^* -algebras, C^* -algebras of topological graphs, C^* -algebras of branched coverings, C^* -algebras associated with topological relations are all associated to a topological quiver ([10], Section 3.3).

Topological quivers justify our use of families of measures in the definition of a topological correspondence. At a first glance, the families of measures and their quasi-invariance for the left action might look artificial. However, as discussed on page 229, the quasi-invariance of measures is natural to ask for.

We show that the notion of correspondences introduced by Macho Stadler and Ouchi [8], the generalised morphisms introduced by Buneci and Stachura [3] are topological correspondences. See Examples 3.7 and 3.11, respectively.

In [16], Tu defines locally proper generalised morphism for locally compact groupoids with Hausdorff space of units, see Definition 7.3 of [16]. Example 3.7 which shows that a correspondence in the sense of Macho Stadler and Ouchi is a topological correspondence also shows that a locally proper generalised morphism is a topological correspondence. However, Tu proves that a locally proper generalised morphism induces a C^* -correspondence between the reduced C^* -algebras of the groupoids ([16], Theorem 7.8). Whereas, our result involves the full C^* -algebras of the groupoids. We know that the result of Tu holds for topological correspondences when the bispaces involved in the topological correspondence is Hausdorff and second countable, and the right action is amenable ([5], Proposition 2.3.4).

Following is the sectionwise description of the contents.

In Section 1, we mention our conventions, notation, and rewrite some standard definitions and results.

A notion of cohomology for Borel groupoids is introduced in [17] by Westman. In Chapter 1, page 14 of [12], Renault discusses a continuous version of the same cohomology. We need a groupoid equivariant continuous version of this cohomology. For this purpose we define the action of a groupoid on another

groupoid and then define the equivariant cohomology for Borel and continuous groupoids.

Section 2 contains the main construction. Immediately after the definition of topological correspondence (Definition 2.1) we discuss the role of the adjoining function.

Let (X, λ) be a topological correspondence from (G, α) to (H, β) . Then we write the formulae of the actions of $C_c(G)$ and $C_c(H)$ on $C_c(X)$. These actions make $C_c(X)$ into a $C_c(G)$ - $C_c(H)$ -bimodule. We also define the formula of a $C_c(H)$ -valued bilinear map on $C_c(X)$. We complete this setup to get a C^* -correspondence. The process, as mentioned earlier, is divided into two parts: constructing a Hilbert $C^*(H, \beta)$ -module $\mathcal{H}(X)$ and defining a representation of $C^*(G, \alpha)$ on this Hilbert module.

We advise the reader to jump to Section 3 after the discussion that follows Definition 2.1 to have a look at some examples.

Section 3 contains examples of topological correspondences.

1. PRELIMINARIES

1.1. GROUPOIDS. The reader should be familiar with the theory of locally compact groupoids ([1], [11], [12], [13], and [16]).

A groupoid G is a small category in which every arrow is invertible. Except a few instances, we denote the space of arrows of groupoid G by the letter G itself, rather than the more precise symbol $G^{(1)}$. We denote the space of units of G by $G^{(0)}$. The source and range maps are denoted by s_G and r_G , respectively. The inverse of an element $\gamma \in G$ is denoted by γ^{-1} . There are instances when we need to write $\gamma \mapsto \gamma^{-1}$ as a function $G \rightarrow G$ and then we denote the function by inv_G .

A pair $(\gamma, \gamma') \in G \times G$ is called composable if $s_G(\gamma) = r_G(\gamma')$. Sometimes we abuse the language by saying “ $\gamma, \gamma' \in G$ are composable” by which we mean that the pair (γ, γ') is composable. For $A, B \subseteq G^{(0)}$ define

$$\begin{aligned} G^A &= \{\gamma \in G : r_G(\gamma) \in A\} = r_G^{-1}(A), \\ G_A &= \{\gamma \in G : s_G(\gamma) \in A\} = s_G^{-1}(A) \quad \text{and} \\ G_B^A &= G^A \cap G_B = \{\gamma \in G : r_G(\gamma) \in A \text{ and } s_G(\gamma) \in B\}. \end{aligned}$$

When $A = \{u\}$ and $B = \{v\}$ are singletons, we write G^u , G_v and G_v^u instead of $G^{\{u\}}$, $G_{\{v\}}$ and $G_{\{v\}}^{\{u\}}$, respectively. For $u \in G^{(0)}$, G_u^u is a group. It is called the *isotropy group* at u .

A topological groupoid and measurable groupoid have their standard meanings [1].

We call a subset $A \subseteq X$ of a topological space X *quasi-compact* if every open cover of A has a finite subcover. And A is called *compact* if it is quasi-compact

and Hausdorff. The space X is called locally compact if every point $x \in X$ has a compact neighbourhood. All the spaces considered in this article are locally compact. For a locally compact space X by $C_c(X)_0$ we denote the set of functions f on X such that $f \in C_c(V)$ where $V \subseteq X$ is open Hausdorff, and f is extended outside V by 0 (see Section 4 of [16]). By $C_c(X)$ we denote the linear span of functions in $C_c(X)_0$. The functions in $C_c(X)$ need not be continuous on X but they are Borel. Furthermore, if μ is a positive σ -finite Radon measure on X , then $C_c(X) \subseteq \mathcal{L}^2(X, \mu)$ is dense.

As in [13], we call a topological groupoid G locally compact if G is a locally compact topological space and $G^{(0)} \subseteq G$ is Hausdorff. In this case, $r_G^{-1}(u) \subseteq G$ (and equivalently, $s_G^{-1}(u) \subseteq G$) are Hausdorff for all $u \in G^{(0)}$.

If X and Y are topological spaces, $X \approx Y$ means X and Y are homeomorphic. If G and H are group(oids)s, then $G \simeq H$ means G and H are isomorphic via a group(oid) homomorphism. All the measures we work with are positive, Radon and σ -finite.

For groupoid actions we do not assume that the momentum maps are open or surjective. However, it is well-known that for a locally compact groupoid with a Haar system the source map (equivalently the range map) is automatically open. We need that each measure in a family of measures along a continuous open map $f : X \rightarrow Y$ is non-zero, but it needs not have full support.

1.2. PROPER ACTIONS AND FAMILIES OF MEASURES. Since we shall not come across any case where there are more than one different left (or right) action of a groupoid G on a space X , we denote the momentum map by r_X (respectively, s_X). When we write “ X is a left (or right) G -space”, where G is a groupoid, without specifying the momentum map, the above convention will be tacitly assumed and then, in such instances, the momentum map is r_X (respectively, s_X).

Let G be a groupoid and $A \subseteq G^{(0)}$. For a left G -space X and a right G -space Y we define X^A, X^u, Y_A and Y_u similar to G^A, G^u, G_A and G_u above.

Let X, Y and Z be spaces, and let $f : X \rightarrow Z$ and $g : Y \rightarrow Z$ be maps. We denote the fibre product of X and Y over Z by $X \times_{f, Z, g} Y$. When there is no confusion about the maps f and g , we simply write $X \times_Z Y$ instead of $X \times_{f, Z, g} Y$.

Let G be a groupoid and X a left G -space. By $G \ltimes X$ we denote the transformation groupoid. Its space of arrows is $G \times_{s_G, G^{(0)}, r_X} X$, which we prefer denoting by $G \times_{G^{(0)}} X$. Recall that a Haar system on G induces a Haar system on $G \ltimes X$.

Let G be a locally compact groupoid with open range map and X a locally compact G -space. Then the quotient map $X \rightarrow X/G$ is open. If the action of G is proper, then X/G is locally compact. If X is Hausdorff (or Hausdorff and second countable) the X/G is also Hausdorff (Hausdorff and second countable, respectively).

Let X be a left G -space. For subsets $K \subseteq G$ and $A \subseteq X$ with $s_G(K) \cap r_X(A) \neq \emptyset$, define $KA = \{\gamma x : \gamma \in K, x \in A \text{ and } (\gamma, x) \in G \times_{G^{(0)}} X\}$. If $s_G(K) \cap r_X(A) = \emptyset$, then we define $KA = \emptyset$. By an abuse of notation, for $x \in X$ we write Kx

instead of $K\{x\}$. The meaning of γA for $\gamma \in G$ is similar. For a right action, we define AK, xK and $A\gamma$ similarly.

Let G and H be groupoids and X a space on which G acts from the left and H from the right. If for all $\gamma \in G, x \in X$ and $\eta \in H$ with $s_G(\gamma) = r_X(x)$ and $s_X(x) = r_H(\eta)$ we have $s_X(\gamma x) = s_X(x), r_X(x\eta) = r_X(x)$, and $(\gamma x)\eta = \gamma(x\eta)$; then we call X a G - H -bispaces.

DEFINITION 1.1. Let H be a locally compact groupoid, and let X and Y be locally compact proper right H -spaces. Let $\pi : X \rightarrow Y$ be an H -equivariant continuous map. An H -invariant continuous family of measures along π is a family of Radon measures $\lambda = \{\lambda_y\}_{y \in Y}$ such that:

- (i) each λ_y is defined on $\pi^{-1}(y)$;
- (ii) (invariance) for all composable pairs $(y, \eta) \in Y \times_{H(0)} H$, the condition $\lambda_{y\eta} = \lambda_{y\eta}$ holds;
- (iii) (continuity condition) for $f \in C_c(X)$ the function $\Lambda(f)(y) := \int_{\pi^{-1}(y)} f d\lambda_y$

on Y is continuous.

We clarify that in the above definition the measure $\lambda_{y\eta}$ is given by

$$\int f d\lambda_{y\eta} = \int f(x\eta) d\lambda_y(x) \text{quadfor } f \in C_c(X).$$

If for each $y \in Y, \text{supp}(\lambda_y) = \pi^{-1}(y)$, we say the family of measures λ has full support. If there is a continuous function f on X with $\Lambda(f) = 1$ on $\pi(X)$, we say that λ is proper. Lemma 1.1.2 in [1] says that, in the continuous case, λ is proper if and only if $\lambda_y \neq 0$ for all $y \in Y$. Thus if λ is continuous and has full support, then λ is proper. In the rest of the article we assume that every family of measures we work with is proper.

Let Pt be the trivial point group(oid). If X and Y are spaces and $\pi : X \rightarrow Y$ is a continuous map, then π is a Pt -equivariant map between Pt -spaces. A continuous Pt -invariant family of measures along π is simply called a continuous family of measures along π .

We denote families of measures by small Greek letters. For a given family of measures, the corresponding integration function that appears in the continuity condition in Definition 1.1 will be denoted by the Greek upper case letter used to denote the family of measures. For α, β and μ it will be A, B and M , respectively.

DEFINITION 1.2. (i) Let H be a groupoid and X a left H -space. An H -invariant continuous family of measures along the momentum map r_X is called a left H -invariant continuous family of measures on X . A right H -invariant continuous family of measures on X is defined analogously.

(ii) For a groupoid H , a Haar system on H is a left H -invariant continuous family of measures with full support on H for the left multiplication action of H on itself.

LEMMA 1.3. *Let H be a groupoid, let $\pi : X \rightarrow Y$ be a continuous H -map between the H -spaces X and Y and let λ be a continuous family of measures along π . If λ_y has full support for all $y \in \pi(X)$, then π is an open map onto its image.*

Consider the map $\pi : X \rightarrow \pi(X)$ and then the proof is similar to the one of Proposition 2.2.1 in [11].

COROLLARY 1.4. *If (H, β) is a groupoid with a Haar system, then the range and source maps are open.*

Proof. Lemma 1.3 implies that the range map r_H is open. Since $s_H = r_H \circ \text{inv}_H$ and inv_H is a homeomorphism, s_H is open. ■

We need the following two lemmas from [13].

LEMMA 1.5 ([13], Lemme 1.1). *Let X and Y be spaces, let $\pi : X \rightarrow Y$ be an open surjection and let λ be a family of measures with full support along π . For every open $U \subseteq X$ and for a non-negative function $g \in C_c(\pi(U))$, there is a non-negative function $f \in C_c(U)$ with $\Lambda(f) = g$.*

LEMMA 1.6 ([13], Lemme 1.2). *Let X, Y and Z be spaces, let π and τ be open surjections from X and Y to Z , respectively. Let π_2 denote the projection from the fibre product $X \times_Z Y$ onto the second factor Y . Assume that for each $z \in Z$, there is a measure λ_z on $\pi^{-1}(z)$. For each $y \in Y$ define the measure $\lambda_{2y} = \lambda_{\tau(y)} \times \delta_y$, where δ_y is the point-mass at y . Then λ is continuous if and only if λ_2 is continuous.*

Let H be a locally compact groupoid, and let X and Y be locally compact right H -spaces. For $x \in X$ the equivalence class of x in X/H is denoted by $[x]$. The map π induces a map from X/H to Y/H , which we denote by $[\pi]$. Let $\pi : X \rightarrow Y$ be an H -equivariant map and λ an H -invariant continuous family of measures along π . Then λ induces a continuous family of measures $[\lambda]$ along $[\pi] : X/H \rightarrow Y/H$. The proof is similar to the proof of Proposition 1.3.27 in [5]. For $f \in C_c(X/H)$ and $[y] \in Y/H$ the measure $[\lambda]^{[y]}$ is defined as

$$(1.1) \quad \int_{X/H} f d[\lambda]^{[y]} := \int_{\tilde{X}} f([x]) d\lambda^y(x).$$

EXAMPLE 1.7. Let X be a proper right H -space. Let β be a Haar system on H . Then Lemma 1.6 says that $\beta_1 = \{\delta_x \times \beta_{s_X(x)}\}_{x \in X}$ is a continuous family of measures along the projection map $\pi_1 : X \times_{H(0)} H \rightarrow X$. Take the quotient by the action of H to get a continuous family of measures $[\pi_1]$ along the map $[\pi_1] : (X \times_{H(0)} H)/H \rightarrow X/H$. Identifying $(X \times_{H(0)} H)/H = X$ gives that $[\pi_1]$ is the quotient map, and a small computation gives that for $f \in C_c(X)$ and for $[x] \in X/H$,

$$\int_X f d[\beta_1]^{[x]} = \int_{H^{s_X(x)}} f(x\eta) d\beta^{s_X(x)}(\eta).$$

In the rest of the article we write β_X instead of $[\beta_1]$.

1.3. COHOMOLOGY FOR GROUPOIDS. In this section, the groupoids and maps are assumed to be Borel. The whole discussion goes through when the Borel properties are replaced by the continuous properties.

DEFINITION 1.8. A left action of a groupoid G on another groupoid H is given by maps $r_{H,G} : H \rightarrow G^{(0)}$ and $a : G \times_{s_G, G^{(0)}, r_{H,G}} H \rightarrow H$ which satisfy the following conditions:

(i) if $\eta, \eta' \in H$ are composable, $\gamma \in G$ with $s_G(\gamma) = r_{H,G}(\eta) = r_{H,G}(\eta')$, then $a(\gamma, \eta), a(\gamma, \eta') \in H$ are composable and

$$a(\gamma, \eta)a(\gamma, \eta') = a(\gamma, \eta\eta');$$

(ii) if $u \in G^{(0)}$, then $a(u, \eta) = \eta$ for all $\eta \in r_{H,G}^{-1}(u) \subseteq H$;

(iii) if $\gamma, \gamma' \in G$ are composable, then $(\gamma, a(\gamma', \eta)) \in G \times_{s_G, G^{(0)}, r_{H,G}} H$ and

$$a(\gamma\gamma', \eta) = a(\gamma, a(\gamma', \eta)).$$

To simplify the notation, we write $\gamma \cdot \eta$ or simply $\gamma\eta$ for $a(\gamma, \eta)$, and $G \times_{G^{(0)}} H$ for $G \times_{s_G, G^{(0)}, r_{H,G}} H$. Then (i) and (ii) above read $\gamma \cdot (\eta\eta') = (\gamma \cdot \eta)(\gamma \cdot \eta')$ and $(\gamma\gamma') \cdot \eta = \gamma \cdot (\gamma' \cdot \eta)$, respectively. We call the map $r_{H,G}$ the momentum map for the action and a the action map. When the momentum and action maps are continuous (or Borel) the action is called continuous (or Borel, respectively).

It is not hard to see that Definition 1.8 gives an action of G on H via invertible functors. When G is a group, our definition matches Definition 1.7 in Chapter 1 of [12], which is the action of a group on a groupoid by invertible functors. A proof of this fact is below.

LEMMA 1.9. When G is a group, an action of G on H as in Definition 1.8 above is the same as the action in Definition 1.7, Chapter 1 of [12], that is, there is a homomorphism $\phi : G \rightarrow \text{Aut}(H)$ which gives the action. Here $\text{Aut}(H)$ is the set of all invertible functors from H to itself.

Proof. Definition 1.8 implies Definition 1.7, Chapter 1 of [12]: for $\gamma \in G$ define $\phi(\gamma)(\eta) = \gamma \cdot \eta$. We first prove that each $\phi(\gamma)$ is a functor from H to itself.

Note that an element u in a groupoid is a unit if and only if u is composable with itself and $u^2 = u$. If $u \in G^{(0)}$, then $\phi(\gamma)(u) = \phi(\gamma)(uu) = \phi(\gamma)(u)\phi(\gamma)(u) = (\phi(\gamma)(u))^2$. Hence for each unit $u \in H^{(0)}$, $\phi(\gamma)(u) \in H$ is a unit. Condition (i) of Definition 1.8 gives that for each $\gamma \in G$, $\phi(\gamma)(\eta\eta') = \phi(\gamma)(\eta)\phi(\gamma)(\eta')$. This proves that $\phi(\gamma)$ is functor for each $\gamma \in G$.

Now we show that each of the $\phi(\gamma)$ is invertible. Condition (iii) of Definition 1.8 gives that $\gamma \mapsto \phi(\gamma)$ is a homomorphism. Use (iii) of Definition 1.8 to see that $\phi(\gamma)$ is invertible:

$$\phi(\gamma)\phi(\gamma^{-1})(\eta) = \phi(\gamma\gamma^{-1})(\eta) = \phi(r_G(\gamma))(\eta) = \eta = \text{Id}_H(\eta).$$

Similarly, $\phi(\gamma^{-1})\phi(\gamma) = \text{Id}_H$. Thus $\phi(\gamma)^{-1} = \phi(\gamma^{-1})$. Hence $\phi(\gamma) \in \text{Aut}(H)$.

Definition 1.7, Chapter 1 of [12] implies Definition 1.8. The proof of this part is routine checking of the conditions in Definition 1.8. ■

A continuous (and Borel) version of Lemma 1.9 can be proved along same lines merely by adding continuity (or Borelness) of the action map and the momentum map and the continuity (Borelness) of the group homomorphism ϕ .

EXAMPLE 1.10. Let G be a groupoid and H a space. Then an action of G on H is the same as an action of the groupoid G on the space H . In this case, condition (i) in Definition 1.8 is irrelevant and then the Definition 1.8 matches the usual definition of an action of a groupoid on a space.

EXAMPLE 1.11. Let G and H be groupoids and let X be a G - H -bispaces. Define an action of G on the transformation groupoid $X \rtimes H$ by $\gamma(x, \eta) := (\gamma x, \eta)$. The momentum map for this action is $(x, \eta) \mapsto r_X(x) \in G^{(0)}$. Let $(x\eta, \eta'), (x, \eta) \in X \times_{H^{(0)}} H$ be composable elements then $(\gamma x\eta, \eta')(\gamma x, \eta)$ are composable and

$$\gamma \cdot (x, \eta) \cdot \gamma(x\eta, \eta') = (\gamma x, \eta)(\gamma x\eta, \eta') = (\gamma x, \eta\eta') = \gamma(x, \eta\eta').$$

This verifies (i) of Definition 1.8. The other conditions are easy to check. Thus H acts on the groupoid $G \times X$ in our sense. This is an important example for us.

Let H be a Borel groupoid and assume that H acts on a Borel groupoid G . Let $G^{(0)}$ and $G^{(1)}$ have the usual meaning. For $n = 2, 3, \dots$ define

$$G^{(n)} = \{(\gamma_0, \dots, \gamma_{n-1}) \in \underbrace{G \times G \times \dots \times G}_{n\text{-times}} : s_G(\gamma_i) = r_G(\gamma_{i+1}) \text{ for } 0 \leq i < n-2\}.$$

DEFINITION 1.12. Let G, H be Borel groupoids, let A be an abelian Borel group and let H act on G . The A -valued H -invariant Borel cochain complex $(BC_H^\bullet(G; A), d^\bullet)$ is defined as follows:

(i) The abelian groups BC_H^n are:

(a) $BC_H^0(G; A) := \{f : G^{(0)} \rightarrow A : f \text{ is an } H\text{-invariant Borel map}\},$

(b) for $n > 0$ $BC_H^n(G; A) := \{f : G^{(n)} \rightarrow A : f \text{ is an } H\text{-invariant Borel map and } f(\gamma_0, \dots, \gamma_{n-1}) = 0 \text{ if } \gamma_i \in G^{(0)} \text{ for some } 0 \leq i \leq n-1\};$

(ii) the coboundary map d is:

(a) $d^0 : BC_H^0(G; A) \rightarrow BC_H^1(G; A)$ is $d^0(f)(\gamma) = f(s_G(\gamma)) - f(r_G(\gamma)),$

(b) for $n > 0$, $d^n : BC_H^n(G; A) \rightarrow BC_H^{n+1}(G; A)$ is

$$\begin{aligned} & d^n(f)((\gamma_0, \dots, \gamma_n)) \\ &= f(\gamma_1, \dots, \gamma_n) + \sum_{i=1}^n (-1)^i f(\gamma_0, \dots, \gamma_{i-1}\gamma_i, \dots, \gamma_n) + (-1)^{n+1} f(\gamma_0, \dots, \gamma_{n-1}). \end{aligned}$$

The n -th cohomology group of this complex is the n -th H -invariant Borel cohomology of G for $n \geq 0$, and it is denoted by $H_{\text{Bor}, H}^n(G; A)$. By adding the action of H to all the maps and spaces, one can make sense of the machinery and the results in Sections 1 and 2 of [17].

REMARK 1.13. Any H -invariant Borel function f on $G^{(0)}$ is a 0-cochain. A cochain $f \in BC_H^0(G; A)$ is a cocycle if and only if $d^0(f) = 0$ which is true if and only if f is constant on the orbits of $G^{(0)}$. A cochain $k \in BC_H^1(G; A)$ is a cocycle if and only if $k(\gamma_0) - k(\gamma_0\gamma_1) + k(\gamma_1) = 0$ for all composable γ_0 and γ_1 , which is equivalent to saying that k is an H -invariant Borel groupoid homomorphism.

We drop the suffixes B and Bor and write merely $C_H^0(G; A)$ and $H_H^n(G; A)$.

REMARK 1.14. Let $b, b' \in C_H^0(G; A)$ with $d^0(b) = d^0(b')$. Then $c = b - b'$ is a 0-cocycle since

$$d^0(c) = d^0(b) - d^0(b') = 0.$$

Remark 1.13 now gives that c is constant on the orbits of $G^{(0)}$. Thus c is a function on $G^{(0)}/G$.

1.4. QUASI-INVARIANT MEASURES. Let (G, α) be a locally compact groupoid with a Haar system. Using α we get a right invariant family of measures α^{-1} along the source map. For $f \in C_c(G)$ set $\int f(\gamma) d\alpha_u^{-1}(\gamma) = \int f(\gamma^{-1}) d\alpha^u(\gamma)$. Let X be a left G -space and let μ be a measure on X . We define a measure $\mu \circ \alpha^{-1}$ on the space $G \times_{G^{(0)}} X$ by

$$\int_{G \times_{G^{(0)}} X} f d(\mu \circ \alpha^{-1}) = \int_X \int_{G^{r_X(x)}} f(\gamma^{-1}, x) d\alpha^{r_X(x)}(\gamma) d\mu(x)$$

for $f \in C_c(G \times_{G^{(0)}} X)$. Similarly we define the measure $\mu \circ \alpha$.

DEFINITION 1.15. Let (G, α) be a groupoid with a Haar system and X a G -space. A measure μ on X is called (G, α) -quasi-invariant if $\mu \circ \alpha$ and $(\mu \circ \alpha) \circ \text{inv}_{G \times X}$ are equivalent.

In the above definition, $\text{inv}_{G \times X}$ is the inverse function on the transformation groupoid $G \times X$. Thus for $f \in C_c(G \times_{G^{(0)}} X)$

$$(\mu \circ \alpha) \circ \text{inv}_{G \times X}(f) = \int_X \int_{G^{r_X(x)}} f(\gamma, \gamma^{-1}x) d\alpha^{r_X(x)}(\gamma) d\mu(x).$$

When the groupoid (G, α) with a Haar system in the discussion is fixed and there is no possibility of confusion, we write “ μ is a G -quasi-invariant measure”.

REMARK 1.16. As in [1] or [3], it can be shown that the Radon–Nikodym derivative $d(\mu \circ \alpha)/d(\mu \circ \alpha^{-1})$ is a $\mu \circ \alpha$ -almost everywhere a groupoid homomorphism from the transformation groupoid $G \times X$ to \mathbb{R}_+^* .

REMARK 1.17. A (G, α) -quasi-invariant measure μ is G -invariant if and only if the Radon–Nikodym derivative $d(\mu \circ \alpha)/d(\mu \circ \alpha^{-1}) = 1$ $\mu \circ \alpha$ -almost everywhere on $G \times_{G^{(0)}} X$.

A measured groupoid is a triple (G, α, μ) where (G, α) is a locally compact groupoid with a Haar system and μ is a (G, α) -quasi-invariant measure on $G^{(0)}$.

DEFINITION 1.18. The *modular function* of a measured groupoid (G, α, μ) is the Radon–Nikodym derivative $d(\mu \circ \alpha) / d(\mu \circ \alpha^{-1})$.

The reason to use the article “the” for the modular function is that if Δ and Δ' are two modular, then $\Delta = \Delta' \mu \circ \alpha$ -almost everywhere.

1.5. C^* -CORRESPONDENCES. We shall use the theory of Hilbert modules and assume that the reader is familiar with the basics of the theory (for example, [7]).

DEFINITION 1.19. Let A and B be C^* algebras. A C^* -correspondence from A to B is a Hilbert B -module \mathcal{H} with a non-degenerate $*$ -representation $A \rightarrow \mathbb{B}_B(\mathcal{H})$.

Here $\mathbb{B}_B(\mathcal{H})$ denotes the C^* -algebra of adjointable operators on \mathcal{H} . A Hilbert B -module \mathcal{H} is *full* if the linear span of the image of $\mathcal{H} \times \mathcal{H}$ under the inner product map is dense in B . We call a C^* -correspondence \mathcal{H} from A to B *proper* if A acts on \mathcal{H} by compact operators, that is, the action of A is given by a non-degenerate $*$ -representation $A \rightarrow \mathbb{K}_B(\mathcal{H})$.

DEFINITION 1.20. An *imprimitivity bimodule* from A to B is an A - B -bimodule \mathcal{H} such that:

- (i) \mathcal{H} is a full left Hilbert A -module with an inner product ${}_A\langle \cdot, \cdot \rangle$;
- (ii) \mathcal{H} is a full right Hilbert B -module with an inner product $\langle \cdot, \cdot \rangle_B$;
- (iii) $(\mathcal{H}, {}_A\langle \cdot, \cdot \rangle)$ is a correspondence from B to A ;
- (iv) $(\mathcal{H}, \langle \cdot, \cdot \rangle_B)$ is a correspondence from A to B ;
- (v) for $a, b, c \in \mathcal{H}$, $a\langle b, c \rangle_B = {}_A\langle a, b \rangle c$.

Our notion of C^* -correspondence (Definition 1.19) is wider, in the sense that many authors demand that the Hilbert module involved in a C^* -correspondence is *full*, or for some authors a C^* -correspondence is what we call a *proper* correspondence.

In [15], Rieffel shows that an A - B -imprimitivity bimodule induces an isomorphism between the representation categories of B and A . In general, if \mathcal{H} is a C^* -correspondence from A to B , then \mathcal{H} induces a functor from $\text{Rep}(B)$, the representation category of B , to $\text{Rep}(A)$.

2. TOPOLOGICAL CORRESPONDENCES

2.1. TOPOLOGICAL CORRESPONDENCE AND CONSTRUCTION OF THE HILBERT MODULE.

DEFINITION 2.1. A *topological correspondence* from a locally compact groupoid G with a Haar system α to a locally compact groupoid H equipped with a Haar system β is a pair (X, λ) where:

- (i) X is a locally compact G - H -bispaces;
- (ii) the action of H is proper;

(iii) $\lambda = \{\lambda_u\}_{u \in H^{(0)}}$ is an H -invariant proper continuous family of measures along the momentum map $s_X : X \rightarrow H^{(0)}$;

(iv) Δ is a continuous function $\Delta : G \times_{G^{(0)}} X \rightarrow \mathbb{R}^+$ such that for each $u \in H^{(0)}$ and $F \in C_c(G \times_{G^{(0)}} X)$,

$$\int_{X_u} \int_{G^r X(x)} F(\gamma^{-1}, x) d\alpha^{r_X(x)}(\gamma) d\lambda_u(x) = \int_{X_u} \int_{G^r X(x)} F(\gamma, \gamma^{-1}x) \Delta(\gamma, \gamma^{-1}x) d\alpha^{r_X(x)}(\gamma) d\lambda_u(x).$$

If Δ' is another function that satisfies condition (iv) in Definition 2.1, then $\Delta = \Delta' \lambda_u \circ \alpha$ -almost everywhere for each $u \in H^{(0)}$. As both Δ and Δ' are continuous, we get $\Delta = \Delta'$. We call the function Δ the *adjoining function* of the correspondence (X, λ) .

REMARK 2.2. Note that we do not need that the momentum maps s_X and r_X are open surjections. We also do not demand that the family of measures λ has full support, hence the Hilbert module in the resulting C^* -correspondence need not be full. The resulting C^* -correspondence need not be proper, as well.

REMARK 2.3. Referring to Definition 1.15, we can see that condition (iv) in Definition 2.1 says that the measure $\alpha \times \lambda_u$ on $G \times_{G^{(0)}} X_u$ is (G, α) -quasi-invariant for each $u \in H^{(0)}$ where the measure $\alpha \times \lambda_u$ is defined as follows: for $f \in C_c(G \times_{G^{(0)}} X_u)$,

$$\int_{G \times_{G^{(0)}} X_u} f d(\alpha \times \lambda_u) = \int_{X_u} \int_{G^r X(x)} f(\gamma^{-1}, x) d\alpha^{r_X(x)}(\gamma) d\lambda_u(x).$$

In short, “A topological correspondence from (G, α) to (H, β) is a pair (X, λ) where X is a G - H -bispaces with a proper H -action, and λ is an H -invariant family of measures on X along the right momentum map s_X such that each measure in λ is G -quasi-invariant.”

REMARK 2.4. As in [1] or [3], it can be shown that Δ restricted to $G \times_{G^{(0)}} X_u$ is $\alpha \times \lambda_u$ -almost everywhere a groupoid homomorphism for all $u \in H^{(0)}$. So the function Δ in (iv) of Definition 2.1 is a continuous 1-cocycle on the groupoid $G \times X$. We shall use this fact in many computations.

REMARK 2.5. Example 1.11 gives a right action of H on $G \times X$. In [3], Buneci and Stachura use an adjoining function exactly like us. The topological correspondence Buneci and Stachura define is a special case of our construction (Example 3.11). They show that the adjoining function in their case is H -invariant (see Lemma 11 of [3]). In the similar fashion, we may prove that Δ is H -invariant under the right action of H , that is,

$$\Delta(\gamma, x\eta) = \Delta(\gamma, x)$$

for all composable triples $(\gamma, x, \eta) \in G \times_{s_G, G^{(0)}, r_X} X \times_{s_X, H^{(0)}, r_H} H$. Thus, in fact, $\Delta : G \times X/H \rightarrow \mathbb{R}_+^*$. Now Remark 2.4 can be made finer by saying that Δ is an H -invariant continuous 1-cocycle on the groupoid $G \times X$.

2.2. USE OF THE FAMILY OF MEASURES AND THE ADJOINING FUNCTION. In the following discussion, we explain the role of the adjoining function. Let (X, λ) be a topological correspondence from (G, α) to (H, β) with Δ as the adjoining function. We make $C_c(X)$ into a $C_c(H)$ -module using the same formula as in [9] or [8]. To make $C_c(X)$ into a $C^*(H, \beta)$ -pre-Hilbert module, we need to define a $C_c(H)$ -valued inner product on $C_c(X)$. The formula for this inner product cannot be copied directly from either [9] or [8]. This formula has to be modified, and it uses the family of measures λ .

Talking about the left action, for $\phi \in C_c(G)$ and $f \in C_c(X)$, [9] and [8] define $\phi \cdot f \in C_c(X)$ by

$$(2.1) \quad (\phi \cdot f)(x) = \int_G \phi(\gamma) f(\gamma^{-1}x) \, d\alpha^{r_X(x)}(\gamma).$$

For our definition of topological correspondence, the action of $C_c(G)$ on the pre-Hilbert $C^*(H, \beta)$ -module $C_c(X)$ defined by formula (2.1) is not necessarily an action by adjointable operators. For ϕ and f as above we define the left action by

$$(2.2) \quad (\phi \cdot f)(x) := \int_G \phi(\gamma) f(\gamma^{-1}x) \Delta^{1/2}(\gamma, \gamma^{-1}x) \, d\alpha^{r_X(x)}(\gamma).$$

We shall see that the adjoining function gives a nice scaling factor for the action of $C_c(G) \subseteq C^*(G, \alpha)$ and makes this action a $*$ -homomorphism. This is the reason we call Δ the *adjoining* function.

Two examples of topological correspondences are: an equivalence between groupoids (see [9] or Definition 3.10) and the correspondence of Macho Stadler and O’uchi (see [8] or Example 3.7). For equivalences and Macho Stadler–O’uchi correspondences the adjoining function is the constant function 1, and then formulae (2.1) and (2.2) match. To understand the role of Δ the reader may have a look at Lemma 2.9.

To support the necessity of the adjoining function, consider a toy example: let G be a locally compact group and let it act on a locally compact (possibly Hausdorff) space X carrying a measure λ . The left multiplication action of $C_c(G)$ on $C_c(X) \subseteq \mathcal{L}^2(X, \lambda)$ defined by equation (2.1) is not necessarily bounded. To make this action bounded, it is sufficient that λ is G -quasi-invariant, which brings the adjoining function into the picture. Then the left action of $C_c(G)$ given by equation (2.2) becomes a $*$ -representation. This motivated us to introduce condition (iv) in Definition 2.1. Buneci and Stachura [3] also use the adjoining function.

For the left multiplication action of G on G/K , where K is a closed subgroup of G , the space G/K always carries a G -quasi-invariant measure λ . Hence there

is a representation of G on $\mathcal{L}^2(G/K, \lambda)$. Quasi-invariant measures and the corresponding *adjoining functions* are studied very well in the group case, for example, see Section 2.6 of [4].

At this point, readers may peep into Section 3 to see some examples of adjoining functions.

We start with the main construction now. For $\phi \in C_c(G)$, $f \in C_c(X)$ and $\psi \in C_c(H)$ define functions $\phi \cdot f$ and $f \cdot \psi$ on X as follows:

$$(2.3) \quad \begin{cases} (\phi \cdot f)(x) := \int_{G^r x(x)} \phi(\gamma) f(\gamma^{-1}x) \Delta^{1/2}(\gamma, \gamma^{-1}x) \, d\alpha^{r x(x)}(\gamma), \\ (f \cdot \psi)(x) := \int_{H^s x(x)} f(x\eta) \psi(\eta^{-1}) \, d\beta^{s x(x)}(\eta). \end{cases}$$

For $f, g \in C_c(X)$ define the function $\langle f, g \rangle$ on H by

$$(2.4) \quad \langle f, g \rangle(\eta) := \int_{X_{r_H(\eta)}} \overline{f(x)} g(x\eta) \, d\lambda_{r_H(\eta)}(x).$$

Most of the times we write ϕf and $f\psi$ instead of $\phi \cdot f$ and $f \cdot \psi$.

LEMMA 2.6. *Among the functions ϕf , $f\psi$ and $\langle f, g \rangle$ defined above, the first two are in $C_c(X)$ and the last one is in $C_c(H)$.*

Proof. The proof follows from Lemme 3.1 of [13] and the $*$ -category $C_c(H)$ used in [13] or [6]. Detailed computations can be found in Section 3.3.1 of [5] in which Tables 3.1 and 3.2 list the equivalence of the operations in Definitions 2.3 and 2.4 and those in the $*$ -category $C_c(H)$.

We prove that $\langle f, g \rangle \in C_c(H)$ and the remaining claims can be proved similarly. Let $\mathfrak{C}_c(H)$ denote the $*$ -category for H . Then (H, β) and (X, λ) are objects in $\mathfrak{C}_c(H)$. We identify X with $(X \times_{H^{(0)}} H)/H$ and then think of $f, g \in C_c((X \times_{H^{(0)}} H)/H)$ as arrows from (X, λ) to (H, β) . Then $\langle f, g \rangle = f^* *_{\lambda} g$, where the latter is in $C_c((H \times_{r_H, H^{(0)}, r_H} H)/H)$. Now identify $C_c((H \times_{r_H, H^{(0)}, r_H} H)/H)$ with $C_c(H)$ to conclude the proof. ■

Both $C_c(G)$ and $C_c(H)$ are $*$ -algebras. Denote the convolution product on them by $*$.

LEMMA 2.7. *Let $\phi, \phi' \in C_c(G)$, $\psi, \psi' \in C_c(H)$ and $f, g, g' \in C_c(X)$. Then*

$$(2.5) \quad (\phi * \phi')f = \phi(\phi'f),$$

$$(2.6) \quad f(\psi * \psi') = (f\psi)\psi',$$

$$(2.7) \quad (\phi f)\psi = \phi(f\psi),$$

$$(2.8) \quad \langle f, g + g' \rangle = \langle f, g \rangle + \langle f, g' \rangle,$$

$$(2.9) \quad \langle f, g \rangle^* = \langle g, f \rangle,$$

$$(2.10) \quad \langle f, g \rangle * \psi = \langle f, g\psi \rangle,$$

$$(2.11) \quad \langle \phi f, g \rangle = \langle f, \phi^* g \rangle.$$

Proof. Checking most of the equalities above are straightforward computations involving obvious change of variables, using the invariances of the families of measures, Fubini's theorem and some properties of the adjoining function. We write two detailed computations and hints for computing the others. Let $\gamma \in G$, $x \in X$ and $\eta \in H$.

Equation (2.5):

$$\begin{aligned} & ((\phi * \phi')f)(x) \\ &= \int_{G^{r_X(x)}} (\phi * \phi')(\gamma) f(\gamma^{-1}x) \Delta(\gamma, \gamma^{-1}x)^{1/2} d\alpha^{r_X(x)}(\gamma) \\ &= \int_{G^{r_X(x)}} \int_{G^{r_G(\gamma)}} \phi(\zeta) \phi'(\zeta^{-1}\gamma) f(\gamma^{-1}x) \Delta(\gamma, \gamma^{-1}x)^{1/2} d\alpha^{r_G(\gamma)}(\zeta) d\alpha^{r_X(x)}(\gamma). \end{aligned}$$

First apply Fubini's theorem and then change the variable $\gamma \mapsto \zeta\gamma$ and use the invariance of α to see that the last term equals

$$\int_{G^{r_G(\gamma)}} \int_{G^{r_X(x)}} \phi(\zeta) \phi'(\gamma) f(\gamma^{-1}\zeta^{-1}x) \Delta(\zeta\gamma, \gamma^{-1}\zeta^{-1}x)^{1/2} d\alpha^{r_X(x)}(\gamma) d\alpha^{r_G(\gamma)}(\zeta).$$

We observe that $(\zeta\gamma, \gamma^{-1}\zeta^{-1}x) = (\zeta, \zeta^{-1}x)(\gamma, \gamma^{-1}\zeta^{-1}x)$ in the transformation groupoid $G \times X$. This relation, Remark 2.4 and the associativity of the left action together allow us to write the previous term as

$$\begin{aligned} & \int_{G^{r_G(\gamma)}} \int_{G^{r_X(x)}} \phi(\zeta) \phi'(\gamma) f(\gamma^{-1}\zeta^{-1}x) \Delta(\zeta, \zeta^{-1}x)^{1/2} \Delta(\gamma, \gamma^{-1}\zeta^{-1}x)^{1/2} d\alpha^{r_X(x)}(\gamma) d\alpha^{r_G(\gamma)}(\zeta) \\ &= \int_{G^{r_G(\gamma)}} \phi(\zeta) \left(\int_{G^{r_X(x)}} \phi'(\gamma) f(\gamma^{-1}\zeta^{-1}x) \Delta(\gamma, \gamma^{-1}\zeta^{-1}x)^{1/2} d\alpha^{r_X(x)}(\gamma) \right) \Delta(\zeta, \zeta^{-1}x)^{1/2} d\alpha^{r_G(\gamma)}(\zeta) \\ &= \int_{G^{r_G(\gamma)}} \phi(\zeta) (\phi'f)(\zeta^{-1}x) \Delta(\zeta, \zeta^{-1}x)^{1/2} d\alpha^{r_G(\gamma)}(\zeta) = (\phi(\phi'f))(x). \end{aligned}$$

Equation (2.6): this computation is similar to the above computation for equation (2.5) except that there is no adjoining function here.

Equation (2.7): this uses Fubini's theorem and the H -invariance of Δ .

Equation (2.8): this is a direct computation.

Equation (2.9): this involves a change of variable and the right invariance of the family of measures λ .

Equation (2.10): this uses a change of variable, the left invariance of β and Fubini's theorem.

Equation (2.11):

$$\begin{aligned}
 (2.12) \quad \langle \phi f, g \rangle(\eta) &= \int_X \overline{(\phi f)(x)} g(x\eta) d\lambda_{r_H(\eta)}(x) \\
 &= \int_X \int_G \overline{\phi(\gamma) f(\gamma^{-1}x)} g(x\eta) \Delta^{1/2}(\gamma, \gamma^{-1}x) d\alpha^{r_X(x)}(\gamma) d\lambda_{r_H(\eta)}(x) \\
 &= \int_X \int_G \overline{f(\gamma^{-1}x) \phi(\gamma)} g(x\eta) \Delta^{1/2}(\gamma, \gamma^{-1}x) d\alpha^{r_X(x)}(\gamma) d\lambda_{r_H(\eta)}(x).
 \end{aligned}$$

Make a change of variables $(\gamma, \gamma^{-1}x) \mapsto (\gamma^{-1}, x)$. Then we use the fact that Δ is an almost everywhere groupoid homomorphism (see Remark 2.4). Due to Remark 2.5, we know that Δ is H -invariant. Taking into account these facts we see that

$$\begin{aligned}
 \langle \phi f, g \rangle(\eta) &= \int_X \int_G \overline{f(x)} \overline{\phi(\gamma^{-1})} g(\gamma^{-1}x\eta) \Delta^{1/2}(\gamma, \gamma^{-1}x\eta) d\alpha^{r(x)}(\gamma) d\lambda_{r_H(\eta)}(x) \\
 &= \int_X \int_G \overline{f(x)} (\phi^* g)(x\eta) d\lambda_{r_H(\eta)}(x) = \langle f, \phi^* g \rangle(\eta). \quad \blacksquare
 \end{aligned}$$

It can be checked that the left and the right actions and the map $\langle \cdot, \cdot \rangle$ are continuous in the inductive limit topology.

Equations (2.5), (2.6) and (2.7) show that $C_c(X)$ is a $C_c(G)$ - $C_c(H)$ -bimodule. Equations (2.8), (2.9) and (2.10) show that the map $\langle \cdot, \cdot \rangle : C_c(X) \times C_c(X) \rightarrow C_c(H)$ is a $C_c(H)$ -conjugate bilinear map. And equation (2.11) says that $C_c(G)$ acts on $C_c(X)$ by $C_c(H)$ -adjointable operators.

Recall from page 4 of [7] that a pre- C^* -algebra satisfies all the conditions to be a C^* -algebra except that it need not be complete. Let (H, β) be a locally compact groupoid with a Haar system. Then, very often, especially in the theorem that follows, we consider the $*$ -algebra $C_c(H)$ (equipped with an appropriate norm) a pre- C^* -algebra that completes to the C^* -algebra $C^*(H, \beta)$. Now we are ready to state the following well-known result of Renault from [13].

THEOREM 2.8 (cf. Corollaire 5.2 of [13]). *Let (H, β) be a locally compact groupoid with a Haar system and let (X, λ) be a proper right H -space with an H -invariant continuous family of measures. Then the inner product $C_c(H)$ -module $C_c(X)$, which is defined by formulae (2.3) and (2.4), can be completed into a Hilbert $C^*(H, \beta)$ -module $\mathcal{H}(X)$.*

Theorem 2.8 is a particular case of Corollaire 5.2 in [13] which is stated for a C^* -algebraic twisted groupoid dynamical system. In contrast to Corollaire 5.2 of [13], Theorem 2.8 involves merely a groupoid. Note that in Corollaire 5.2 of [13] the space X is a free and proper H -space; however, it is remarked there that the corollaire holds for a proper H -space. The key idea of the proof is to show that the representations of the $*$ -algebra $C_c(H)$ extend to that of a $*$ -category constructed

from the measured proper H -spaces. This $*$ -category is discussed in [6], and Theorem 2.2 of [6] extends the representations of $C_c(H)$ to that of the $*$ -category. The $*$ -category in [6] is a special case of the one constructed in Section 3, pages 74 and 75 of [13]. The latter is used to prove Corollaire 5.2 of [13]. Again, both the $*$ -categories differ by *freeness* of the H -action which can be, as mentioned earlier, overlooked.

In [13], Renault uses Corollaire 5.2 of [13] to prove that an equivalence between two C^* -algebraic twisted groupoid dynamical system produces an equivalence bimodule between the C^* -algebras of the dynamical systems. As mentioned in the Introduction, this result is the motivation of the present work. A result similar to Theorem 2.8 concerning the reduced C^* -algebras is proved in Theorem 1.4 of [8] with a different proof.

2.3. THE LEFT ACTION AND CONSTRUCTION OF THE C^* -CORRESPONDENCE. We now turn our attention to the left action. We wish to extend the action of $C_c(G)$ on $C_c(X)$ to an action of $C^*(G)$ on $\mathcal{H}(X)$. For a groupoid equivalence the adjoining function vanishes (see Example 3.9 and the discussion about the adjoining function in Example 3.7), that is, it becomes the constant function 1, and the formulae for the left actions in equation (2.3) and page 11 of [9] match. Our proof runs along the same lines as in [9].

LEMMA 2.9. *The action of $C_c(G)$ on $C_c(X)$ defined by equation (2.3) extends to a non-degenerate $*$ -homomorphism from $C^*(G, \alpha)$ to $\mathbb{B}_{C^*(H)}(\mathcal{H}(X))$.*

Proof. Equation 2.11 in Lemma 2.7 shows that $C_c(G)$ acts on $C_c(X)$ by adjointable operators. We need to prove that this representation is non-degenerate. For this purpose we show that there is a continuous family of measures with full support $\tilde{\alpha} = \{\tilde{\alpha}_x\}_{x \in X}$ along the projection map $\pi_2 : G \times X \rightarrow X$. Then Lemma 1.5 gives that $\tilde{A} : C_c(G \times_{G(0)} X) \rightarrow C_c(X)$ is a surjection. Since, due to the theorem of Stone–Weierstaß the set $\{g \otimes h : g \in C_c(G), h \in C_c(X)\} \subseteq C_c(G \times_{G(0)} X)$ is dense, the claim of the current lemma will be proved.

Let $f \in C_c(G \times_{G(0)} X)$ be given. For $x \in X$ define the measure $\tilde{\alpha}_x$ on $\pi_2^{-1}(x) = G^{rx(x)} \times \{x\}$ by

$$\int_{\pi_2^{-1}(x)} f \, d\tilde{\alpha}^x = \int_{G^{rx(x)}} f(\eta^{-1}, x) \Delta^{1/2}(\eta^{-1}, x) \, d\alpha^{rx(x)}(\gamma).$$

Since $\alpha^{rx(x)}$ has full support and Δ is non-zero, $\tilde{\alpha}^{S_{G \times_{G(0)} X}}$ has full support. Using an argument similar to Lemma 1.6 we may infer that $\tilde{\alpha} := \{\tilde{\alpha}_x\}_{x \in X}$ is continuous.

Finally, we check that the action is bounded. Once we prove this, then the action extends to $C^*(G, \alpha)$. Let ϵ be a state on $C^*(H, \beta)$. Then $\epsilon(\langle \cdot, \cdot \rangle)$ makes $\mathcal{H}(X)$ into a Hilbert space, say $\mathcal{H}(X)_\epsilon$. Let $V_\epsilon \subseteq \mathcal{H}(X)_\epsilon$ be the dense subspace generated by $\{\zeta f : \zeta \in C_c(G), f \in C_c(X)\}$. Define a representation L of $C_c(G)$ on V_ϵ by $L(\zeta)f = \zeta f$.

(i) The representation L is a non-degenerate representation of $C_c(G)$ on V_ϵ . Non-degenerate means that the set $\{\zeta f : \zeta \in C_c(G), f \in C_c(X)\}$ is dense in V_ϵ .

(ii) The continuity of the operations in Lemma 2.7 in the inductive limit topology implies that L is continuous: for $f, g \in C_c(X)$, $L_{f,g}(\zeta) = \langle f, L(\zeta)g \rangle$ is a continuous functional on $C_c(G)$ when $C_c(G)$ is given by the inductive limit topology.

(iii) L preserves the involution, that is, $\langle \zeta f, g \rangle = \langle f, \zeta^* g \rangle$. This is proved in equation (2.11) in Lemma 2.7.

Proposition 4.2 of [13] says that L is a representation of G on V_ϵ . Hence L is bounded with respect to the norm on $C^*(G)$. Thus

$$\epsilon(\langle \zeta f, \zeta f \rangle) = \epsilon(\langle L(\zeta)f, L(\zeta)f \rangle) \leq \| \zeta \|_{C^*(G)} \epsilon(\langle f, f \rangle)$$

for all $f \in C_c(X)$ and $\zeta \in C_c(G)$. The state ϵ was arbitrary. Hence for all $f \in C_c(X)$ and $\zeta \in C_c(G)$ we get

$$\langle \zeta f, \zeta f \rangle \leq \| \zeta \|_{C^*(G)} \langle f, f \rangle.$$

This shows that the action of $C_c(G)$ on $C_c(X)$ is bounded in the topology induced by the norm of the inner product $\langle \cdot, \cdot \rangle$. Hence the action can be extended to $C^*(G)$. ■

Now we are ready to state the main theorem.

THEOREM 2.10. *Let (G, α) and (H, β) be locally compact groupoids with Haar systems. Then a topological correspondence (X, λ) from (G, α) to (H, β) produces a C^* -correspondence $\mathcal{H}(X)$ from $C^*(G, \alpha)$ to $C^*(H, \beta)$.*

The proof follows by putting Theorem 2.8 and Lemma 2.9 together.

3. EXAMPLES

EXAMPLE 3.1. Let X and Y be locally compact, Hausdorff spaces, and let $f : X \rightarrow Y$ be a continuous function. We view X and Y as groupoids with Haar systems consisting of Dirac measures on X and Y , $\delta_X = \{\delta_x\}_{x \in X}$ and $\delta_Y = \{\delta_y\}_{y \in Y}$, respectively. We write X' for the space X . We use this notation to avoid confusing the space and the groupoid structures.

The function f is the momentum map for the trivial left action of Y on X' , that is, for $(f(x), x) \in Y \times_{\text{Id}_Y, Y, f} X'$, $f(x) \cdot x = x$. In fact, this is the only possible action of Y on X' . In a similar way, X acts on itself trivially via the momentum map Id_X . The family of Dirac measures δ_X mentioned above is an X -invariant family of measures on X' . Both the actions are proper. If $h \in C_c(Y \times_{\text{Id}_Y, Y, f} X')$, then

$$\int_{X'Y} h(y, x) d(\delta_Y)^{f(a)}(y) d(\delta_X)^a(x) = h(f(a), a) = \int_{X'Y} h(y^{-1}, yx) d(\delta_Y)^{f(a)}(y) d(\delta_X)^a(x).$$

Therefore δ_X is Y -invariant. Thus (X', δ_X) is a topological correspondence from Y to X with the constant function 1 as the adjoining function. The action of $C_c(X)$ on $C_c(X')$ as well as the $C_c(X)$ -valued inner product on $C_c(X')$ are the pointwise multiplication of two functions. For $h \in C_c(Y)$, $k \in C_c(X')$, $(h \cdot k)(x) = h(f(x))k(x)$.

The C^* -correspondence $\mathcal{H}(X') : C_0(Y) \rightarrow C_0(X)$ is the C^* -correspondence associated with the $*$ -homomorphism $f^* : C_0(Y) \rightarrow \mathcal{M}(C_0(X))$ produced by the Gelfand transform.

EXAMPLE 3.2. Let X, Y, X' and f be as in Example 3.1. Let $\lambda = \{\lambda_y\}_{y \in Y}$ be a continuous family of measures along f . It follows from the discussion in Example 3.1 that X is a proper X - Y -bispaces. For $h \in C_c(X \times_{\text{Id}_X, X, \text{Id}_X} X')$,

$$\int_{X' \times X} h(x^{-1}, xz) d(\delta_X)^x(z) d\lambda_y(x) = \int_{X' \times X} h(x, z) d(\delta_X)^x(z) d\lambda_y(x) = \int_{X'} h(f(x), x) d\lambda_y(x).$$

The first equality above is due to the triviality of the action and the second one follows from the definition of the measures $\delta_X \times \lambda_y$ as in Remark 2.3. Thus λ is X -invariant and the modular function is the constant function 1. Hence (X', λ) is a correspondence from X to Y .

EXAMPLE 3.3. Let X and Y be as in Example 3.1. Let $f, b : X \rightarrow Y$ be continuous maps and let λ be a continuous family of measures along f . Make X a Y - Y -bispaces using actions similar to those in Example 3.1. For $f : X \rightarrow Y$ use the family of measures λ and the formulae in Example 3.2 to define a right action of $C_c(Y)$ on $C_c(X)$. It is straightforward to check that (X, λ) is a topological correspondence from Y to itself. When the spaces are second countable, the quintuple (Y, X, s, r, λ) is called a *topological quiver* [10].

In general, assume that X, Y and Z are locally compact Hausdorff spaces, $f : X \rightarrow Y$ and $b : X \rightarrow Z$ are maps. Let λ be a continuous family of measures along f . Then (X, λ) is a topological correspondence from Z to Y .

EXAMPLE 3.4. Let G and H be locally compact groups, $\phi : H \rightarrow G$ a continuous group homomorphism, and α and β the Haar measures on G and H , respectively. The right multiplication action is a proper action of G on itself. The measure α^{-1} is invariant under this action. Using ϕ define an action of H on G as $\eta\gamma = \phi(\eta)\gamma$ for $(\eta, \gamma) \in H \times G$. We claim that α^{-1} is H -quasi-invariant for this H -action. Let δ_G and δ_H be the modular functions of G and H , respectively. The modular functions allow to switch between the left and right invariant Haar measures α and α^{-1} , and β and β^{-1} . The relations are $\alpha^{-1} = \delta_G^{-1}\alpha$ and $\beta^{-1} = \delta_H^{-1}\beta$. If $R_\gamma : G \rightarrow G$ is the right multiplication operator, then

$$\int_G R_\gamma f \, d\alpha = \delta_G(\gamma)^{-1} \int_G f \, d\alpha$$

for $f \in C_c(G)$. A similar equality holds for $g \in C_c(H)$. Let $f \in C_c(H \times G)$,

$$\begin{aligned} & \int_G \int_H f(\eta, \phi(\eta)^{-1}\gamma) \frac{\delta_H(\eta)}{\delta_G(\phi(\eta))} d\beta(\eta) d\alpha^{-1}(\gamma) \\ &= \int_G \int_H f(\eta^{-1}, \phi(\eta)\gamma) \frac{1}{\delta_G(\phi(\eta))} d\beta(\eta) d\alpha^{-1}(\gamma) \quad (\text{by sending } \eta \text{ to } \eta^{-1} \text{ in } H) \\ &= \int_G \int_H f(\eta^{-1}, \gamma) d\beta(\eta) d\alpha^{-1}(\gamma) \quad (\text{by removing } \phi(\eta)^{-1} \text{ in } G). \end{aligned}$$

If one compares the first term of the above computation with the equation in (iv) of Definition 2.1, and uses the fact that the adjoining function is a groupoid homomorphism, then it can be seen that $\Delta(\eta, \eta^{-1}\gamma) = \delta_H(\eta) / (\delta_G \circ \phi(\eta))$. Hence $\Delta(\eta^{-1}, \gamma) = \Delta(\eta, \eta^{-1}\gamma)^{-1} = (\delta_G \circ \phi(\eta)) / \delta_H(\eta)$. Thus a group homomorphism $\phi : H \rightarrow G$ gives a topological correspondence (G, α^{-1}) from (H, β) to (G, α) and $(\delta_G \circ \phi) / \delta_H$ is the adjoining function.

EXAMPLE 3.5. Let $G, H, \alpha, \beta, \delta_H$ and ϕ be as in Example 3.4. Additionally, assume that $\phi : H \rightarrow G$ is a proper function. For the time being, assume that the right action of H on G given by $\gamma\eta := \gamma\phi(\eta)$ for $(\gamma, \eta) \in G \times H$ is proper, which is a fact and we prove it towards the end of this example. With this action of H and the left multiplication action of G on itself, G is a proper G - H -bispaces. α^{-1} is an H -invariant measure. The adjoining function of this action is the constant function 1. To see this, let $f \in C_c(G \times G)$, then

$$\begin{aligned} & \iint f(\gamma^{-1}, \eta) d\alpha(\gamma) d\alpha^{-1}(\eta) \\ &= \iint f(\gamma, \eta) \delta_G(\gamma)^{-1} d\alpha(\gamma) d\alpha^{-1}(\eta) \quad (\text{because } \alpha^{-1} = \delta^{-1}\alpha) \\ &= \iint f(\gamma, \gamma^{-1}\eta) d\alpha(\gamma) d\alpha^{-1}(\eta) \quad (\text{because } L_\gamma\alpha^{-1} = \delta(\gamma)\alpha^{-1}). \end{aligned}$$

Now we prove that the right action of H on G is proper, that is, the map $\Psi : G \times H \rightarrow G \times G$ sending $(\gamma, \eta) \mapsto (\gamma, \gamma\phi(\eta))$ is proper. The maps

$\text{Id}_G \times \phi : G \times H \rightarrow G \times G$, and $m : (\eta, \eta') \mapsto (\eta, \eta\eta')$ from $G \times G \rightarrow G \times G$ are proper, and $\Psi = m \circ (\text{Id}_G \times \phi)$. Hence Ψ is proper.

EXAMPLE 3.6. Let G be a locally compact group and α the Haar measure on G . Let X be a locally compact proper left G -space. Let λ be a *strongly* G -quasi-invariant measure on X , that is, there is a continuous function $\Delta : G \times X \rightarrow \mathbb{R}^+$ such that $d(g\lambda)(x) = \Delta(g, x)d\lambda(x)$ for every g in G . In this setting, (X, λ) is a correspondence from (G, α) to $(\text{Pt}, \delta_{\text{Pt}})$, with Δ as the adjoining function. The C^* -algebra for Pt is \mathbb{C} , the Hilbert module $\mathcal{H}(X)$ is the Hilbert space $L^2(X, \lambda)$ and the action of $C^*(G)$ on this Hilbert module is the representation of $C^*(G)$ obtained from the representation of G on $C_c(X)$.

An example of this situation is: when X is a homogeneous space for G , X carries a G -strongly quasi-invariant measure. For details, see Section 2.6 of [4].

EXAMPLE 3.7 (Macho Stadler and O’uchi’s correspondences). In [8], Macho Stadler and O’uchi present a notion of groupoid correspondences. We change the direction of correspondence in their definition to fit our construction and reproduce the definition here.

DEFINITION 3.8. A *Macho Stadler–O’uchi correspondence* from a locally compact, Hausdorff groupoid with Haar system (G, α) to a groupoid with Haar system (H, β) is a G - H -bispaces X such that:

- (i) the action of H is proper and the momentum map for the right action s_X is open;
- (ii) the action of G is proper;
- (iii) the actions of G and H commute;
- (iv) the right momentum map induces a bijection from $G \backslash X$ to $H^{(0)}$.

Macho Stadler and O’uchi do not assume that the left momentum map is open. We do the same. Condition (iv) above is equivalent to saying that $G \backslash X$ and $H^{(0)}$ are homeomorphic.

Macho Stadler and O’uchi do not require a family of measures on the G - H -bispaces X . We show that a correspondence of Macho Stadler and O’uchi carries a canonical H -invariant continuous family of measures λ which is given by

$$\int_{X_u} f \, d\lambda^u := \int_G f(\gamma^{-1}x) \, d\alpha^{r_X(x)}(\gamma) \quad \text{for } f \in C_c(X),$$

where $u = s_X(x)$. Note that this family of measures is the family of measures α_X^{-1} along the quotient map $X \rightarrow G \backslash X$ as in Example 1.7. Condition (iv) in the above definition identifies $G \backslash X \approx H^{(0)}$ to give the desired result.

Since λ is invariant for the G -actions, we get $\Delta = 1$. Thus (X, λ) is a *topological correspondence* from (G, α) to (H, β) in our sense.

Macho Stadler and O’uchi prove that such a correspondence from (G, α) to (H, β) induces a C^* -correspondence from the reduced C^* -algebra $C_r^*(G, \alpha)$ to $C_r^*(H, \beta)$.

EXAMPLE 3.9 (Equivalence of groupoids).

DEFINITION 3.10 (Equivalence of groupoids, a slight modification of Definition 2.1 [9]). Let G and H be locally compact groupoids. A locally compact space X is a G - H -equivalence if

- (i) X is a left free and proper G -space;
- (ii) X is a right free and proper H -space;
- (iii) the momentum maps r_X and s_X are open;
- (iv) the actions of G and H commute, that is, X is a G - H -bispaces;
- (v) the left momentum map $r_X : X \rightarrow G^{(0)}$ induces a bijection of X/H onto $G^{(0)}$;

(vi) the right momentum map $s_X: X \rightarrow H^{(0)}$ induces a bijection of $G \backslash X$ onto $H^{(0)}$.

Equivalences of Hausdorff groupoids ([9]) are a special case of the correspondences of Macho Stadler–O’uchi. Hence equivalences of groupoids are topological correspondences as well. Similarly, one can check that an equivalence of locally compact groupoids defined in [13] is also a topological correspondence. Furthermore, an equivalence of groupoids is an invertible correspondence.

EXAMPLE 3.11 (Generalised morphisms of Buneci and Stachura). Buneci and Stachura define *generalised morphisms* in [3]. We modify this definition to fit our conventions and repeat it here.

DEFINITION 3.12. A generalised morphism from (G, α) to (H, β) is a left action Θ of G on the space H with r_{GH} as the momentum map, the action commutes with the right multiplication action of H on itself and there is a continuous positive function Δ_Θ on $G \times_{s_G, H^{(0)}, r_{GH}} H$ such that

$$\iint f(\gamma, \gamma^{-1}\eta) \Delta_\Theta(\gamma, \gamma^{-1}\eta) d\alpha^{r_{GH}(\eta)}(\gamma) d\beta_u^{-1}(\eta) = \iint f(\gamma^{-1}, \eta) d\alpha^{r_{GH}(\eta)}(\gamma) d\beta_u^{-1}(\eta)$$

for all $f \in C_c(G \times_{s_G, H^{(0)}, r_{GH}} H)$ and $u \in H^{(0)}$.

If Θ is a generalised morphism from (G, α) to (H, β) then (H, β^{-1}) is a topological correspondence from (G, α) to (H, β) , where β^{-1} is the family of measures

$$\int f d(\beta^{-1})_u = \int f(\eta^{-1}) d\beta^u(\eta)$$

for $f \in C_c(H)$ and $u \in H^{(0)}$. It is obvious from the definition itself that Δ_Θ is the adjoining function for this correspondence.

In [3], Buneci and Stachura prove that a generalised morphism induces a $*$ -homomorphism from $C^*(G, \alpha)$ to $\mathcal{M}(C^*(H, \beta))$. This is a C^* -correspondence from $C^*(G, \alpha)$ to $C^*(H, \beta)$ with the underlying Hilbert module $C^*(H, \beta)$.

EXAMPLE 3.13. Let X be a locally compact right G -space for a locally compact group G and let λ be the Haar measure on G . Let H and K be subgroups of G . Assume that K is closed and let α and β be the Haar measures on H and K , respectively. Then $X \rtimes H$ and $X \rtimes K$ are subgroupoids of $X \rtimes G$. Denote these three transformation groupoids by \mathbb{H} , \mathbb{K} and \mathbb{G} , respectively. Then \mathbb{G} is an \mathbb{H} - \mathbb{K} -bispaces for the left and the right multiplication actions, respectively. We bestow \mathbb{H} and \mathbb{K} with the Haar systems $\{\alpha^y\}_{y \in X}$ and $\{\beta^z\}_{z \in X}$, respectively, where $\alpha^y = \alpha$ and $\beta^z = \beta$ for each $y, z \in X$. If $\lambda_x^{-1} = \lambda^{-1}$ for all $x \in X$, then the family of measures $\{\lambda_x^{-1}\}_{x \in X}$ on \mathbb{G} is \mathbb{K} -invariant. We show that this family is \mathbb{H} -quasi-invariant with the adjoining function δ_G / δ_H .

For $((x, \gamma), (y, \kappa)) \in \mathbb{G} \rtimes \mathbb{K}$ we have $y = x\gamma$ and the map $(x, \gamma, y, \kappa) \mapsto (x, \gamma, \kappa)$ gives an isomorphism between the groupoids $\mathbb{G} \rtimes \mathbb{K}$ and $X \rtimes (G \times K)$. Using this identification, it can be checked that the right action of \mathbb{K} on \mathbb{G} is

proper, which is implied by the fact that $K \subset G$ is closed. Another quicker way to see this is to observe that $\mathbb{K} \subseteq \mathbb{G}$ is a closed subgroupoid.

Let $f \in C_c(\mathbb{H} \rtimes \mathbb{G})$, $u \in \mathbb{K}^{(0)} = \mathbb{G}^{(0)} \approx X$. Let $b = (u\gamma^{-1}, \gamma) \in s_{\mathbb{G}}^{-1}(u) \subseteq \mathbb{G}$. If $a = (u\gamma^{-1}, \eta) \in \mathbb{H}$, then $a^{-1} = (u\gamma^{-1}, \eta)^{-1} = (u\gamma^{-1}\eta, \eta^{-1})$ is composable with b and $a^{-1}b = (u\gamma^{-1}, \eta)^{-1}(u\gamma^{-1}, \gamma) = (u\gamma^{-1}\eta, \eta^{-1}\gamma)$. Now a computation similar to that in Example 3.4 shows that

$$\begin{aligned} & \int_{\mathbb{G} \mathbb{H}} f(a, a^{-1}b) \frac{\delta_H(\eta)}{\delta_G(\eta)} d\alpha^{r_{\mathbb{G}}(b)}(a) d\lambda_u^{-1}(b) \\ &= \int_{\mathbb{G} \mathbb{H}} f((u\gamma^{-1}, \eta), (u\gamma^{-1}\eta, \eta^{-1}\gamma)) \frac{\delta_H(\eta)}{\delta_G(\eta)} d\alpha^{r_{\mathbb{G}}(u\gamma^{-1}, \gamma)}(u\gamma^{-1}, \eta) d\lambda_u^{-1}(u\gamma^{-1}, \gamma) \\ &= \int_{G H} f((u\gamma^{-1}, \eta), (u\gamma^{-1}\eta, \eta^{-1}\gamma)) \frac{\delta_H(\eta)}{\delta_G(\eta)} d\alpha(\eta) d\lambda^{-1}(\gamma) \\ &= \int_{G H} f((u\gamma^{-1}, \eta^{-1}), (u\gamma^{-1}\eta^{-1}, \eta\gamma)) \frac{1}{\delta_G(\eta)} d\alpha(\eta) d\lambda^{-1}(\gamma) \text{ (by changing } \eta \mapsto \eta^{-1}\text{)}. \end{aligned}$$

Now we change $\gamma \mapsto \eta^{-1}\gamma$. Then we use the relation $d\lambda^{-1}(\eta^{-1}\gamma) = d\lambda^{-1}(\gamma) / \delta_G(\eta)$ to see that the previous term equals

$$\begin{aligned} & \int_{G H} f((u\gamma^{-1}\eta, \eta^{-1}), (u\gamma^{-1}, \gamma)) d\alpha(\eta) d\lambda^{-1}(\gamma) \\ &= \int_{\mathbb{G} \mathbb{H}} f((u\gamma^{-1}, \eta)^{-1}, (u\gamma^{-1}, \gamma)) d\alpha(u\gamma^{-1}) d\alpha^{r_{\mathbb{G}}(u\gamma^{-1}, \gamma)}(u\gamma^{-1}, \eta) d\lambda_u^{-1}(u\gamma^{-1}, \gamma) \\ &= \int_{\mathbb{G} \mathbb{H}} f(a^{-1}, b) \frac{\delta_H(\eta)}{\delta_G(\eta)} d\alpha^{r_{\mathbb{G}}(b)}(a) d\lambda_u^{-1}(b). \end{aligned}$$

Thus $\{\lambda_x^{-1}\}_{x \in X}$ is an \mathbb{H} -quasi-invariant family of measures on \mathbb{G} with δ_G / δ_H as the adjoining function.

Let $f' : X \rightarrow Y$ be a proper map between locally compact spaces. Let $B \subseteq Y$, $A \subseteq f'^{-1}(B)$ and $f : A \rightarrow B$ be the map obtained by restricting the domain and the codomain of f' . Assume that $A \subseteq X$ is closed. Then we claim that f is proper. Let $K \subseteq B$ be compact, then K is compact in Y also. Thus it is enough to consider the case when $B = Y$, in which case f is the restriction of f' to the closed subspace A . Since the inclusion map $i_A : A \hookrightarrow X$ is closed, Chapter I, Section 10.1, Proposition 2 of [2] shows that i_A is proper. Hence $f = f' \circ i_A$ is proper.

EXAMPLE 3.14 (The induction correspondence). Let (G, α) be a locally compact groupoid with a Haar system, H a closed subgroupoid. Let β be a Haar system for H . Note that $G_{H(0)}$ is a G - H -bispaces where the left and right actions

are multiplication from the left and right, respectively. Both actions are free. We claim that the actions of G and H are proper.

Let $\iota : G^{(0)} \rightarrow G$ be the inclusion map which is continuous. Then $H^{(0)} = \iota^{-1}(H) \subseteq G^{(0)}$ is closed.

Let $\Psi : G_{H^{(0)}} \times_{H^{(0)}} H \rightarrow G_{H^{(0)}} \times G_{H^{(0)}}$ be the map $\Psi(x, \eta) = (x, x\eta)$. Note that Ψ is obtained from the proper map $(x, \eta) \mapsto (x, x\eta)$, $G \times_{s_G, G^{(0)}, r_G} G \rightarrow G \times G$, by restricting the domain and codomain. If we prove that $G_{H^{(0)}} \times_{H^{(0)}} H \subseteq G \times_{s_G, G^{(0)}, r_G} G$ is closed, then, from the discussion preceding this example, it will follow that Ψ is proper.

Since $H^{(0)} \subseteq G^{(0)}$ is closed,

$$(s_G \times r_G)^{-1}(H^{(0)}) = G \times_{s_G, H^{(0)}, r_G} G \subseteq G \times_{s_G, G^{(0)}, r_G} G$$

is closed where $s_G \times r_G : G \times_{s_G, G^{(0)}, r_G} G \rightarrow G^{(0)}$ is the map $(\gamma, \eta) \mapsto s_G(\gamma) = r_G(\eta)$. Now the projection on the second factor $\pi_2 : G \times_{s_G, H^{(0)}, r_G} G \rightarrow G$ is a continuous, due to which $G_{H^{(0)}} \times_{H^{(0)}} H = \pi_2^{-1}(H) \subseteq G \times_{s_G, H^{(0)}, r_G} G$ is closed. Thus $G_{H^{(0)}} \times_{H^{(0)}} H \subseteq G \times_{s_G, H^{(0)}, r_G} G$ is closed and $G \times_{s_G, H^{(0)}, r_G} G \subseteq G \times_{s_G, G^{(0)}, r_G} G$ is closed which implies that $G_{H^{(0)}} \times_{H^{(0)}} H \subseteq G \times_{s_G, G^{(0)}, r_G} G$ is closed.

To see that the left action is proper, first note that $G_{H^{(0)}} = s_G^{-1}(H^{(0)}) \subseteq G$ is closed. Thus $\pi_2^{-1}(G_{H^{(0)}}) = G \times_{s_G, G^{(0)}, r_G} G_{H^{(0)}} \subseteq G \times_{s_G, G^{(0)}, r_G} G$ is closed. And then arguing same as the right action shows that the left action is also proper.

Now it is not hard to see that $G \setminus G_{H^{(0)}} \approx H^{(0)}$. By Example 3.7, X produces a topological correspondence from (G, α) to (H, β) .

Though both actions are free and proper, this correspondence need not be a groupoid equivalence as it might fail to satisfy condition (v) of Definition 3.10.

Acknowledgements. I thank Jean Renault and Ralf Meyer for many fruitful discussions. I thank Ralf Meyer specially for proofreading the document carefully. I am thankful to Erasmus Mundus EuroIndia Project and CNPq, Brazil for the support.

REFERENCES

- [1] C. ANANTHARAMAN-DELAROCHE, J. RENAULT, *Amenable Groupoids*, Monogr. Enseignement Math., vol. 36, L'Enseignement Mathématique, Geneva 2000.
- [2] N. BOURBAKI, *Elements of Mathematics, General Topology*. Part 1, Hermann, Paris; Addison-Wesley Publ. Co., Reading, Mass.-London-Don Mills, Ont. 1966.
- [3] M.R. BUNECI, P. STACHURA, Morphisms of locally compact groupoids endowed with Haar systems, arXiv 0511613, 2005.
- [4] G.B. FOLLAND, *A Course in Abstract Harmonic Analysis*, Stud. Adv. Math. Ser., CRC Press, Boca Raton, FL 1995.

- [5] R.D. HOLKAR, Topological construction of C^* -correspondences for groupoid C^* -algebras, Ph.D. Dissertation, Georg-August Universität, Göttingen 2014.
- [6] R.D. HOLKAR, J. RENAULT, Hypergroupoids and C^* -algebras, *C. R. Math. Acad. Sci. Paris* **351**(2013), 911–914.
- [7] E.C. LANCE, *Hilbert C^* -Modules*, London Math. Soc. Lecture Note Ser., vol. 210, Cambridge Univ. Press, Cambridge 1995.
- [8] M. MACHO STADLER, M. O'UCHI, Correspondence of groupoid C^* -algebras, *J. Operator Theory* **42**(1999), 103–119.
- [9] P.S. MUHLY, J.N. RENAULT, D.P. WILLIAMS, Equivalence and isomorphism for groupoid C^* -algebras, *J. Operator Theory* **17**(1987), 3–22.
- [10] P.S. MUHLY, M. TOMFORDE, Topological quivers, *Internat. J. Math.* **16**(2005), 693–755.
- [11] A.L.T. PATERSON, *Groupoids, Inverse Semigroups, and their Operator Algebras*, Progr. Math., vol. 170, Birkhäuser Boston, Inc., Boston, MA 1999.
- [12] J. RENAULT, *A Groupoid Approach to C^* -Algebras*, Lecture Notes in Math., vol. 793, Springer, Berlin 1984.
- [13] J. RENAULT, Représentation des produits croisés d'algèbres de groupoïdes, *J. Operator Theory* **18**(1987), 67–97.
- [14] J. RENAULT, Induced representations and hypergroupoids, *SIGMA Symmetry Integrability Geom. Methods Appl.* **10**(2014), Paper 057, 18.
- [15] M.A. RIEFFEL, Induced representations of C^* -algebras, *Adv. in Math.* **13**(1974), 176–257.
- [16] J.-L. TU, Non-Hausdorff groupoids, proper actions and K -theory, *Doc. Math.* **9**(2004), 565–597 (electronic).
- [17] J.J. WESTMAN, Cohomology for ergodic groupoids, *Trans. Amer. Math. Soc.* **146**(1969), 465–471.

ROHIT DILIP HOLKAR, DEPARTMENT OF MATHEMATICS, FEDERAL UNIVERSITY OF SANTA CATARINA, FLORIANOPÓLIS, 88. 040-900, BRAZIL
E-mail address: rohit.d.holkar@gmail.com

Received March 21, 2016; revised June 9, 2016 and August 14, 2016.