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INTERPOLATION OF NONCOMMUTATIVE SYMMETRIC MARTINGALE SPACES

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ABSTRACT. Let E_0, E_1 be symmetric spaces on $[0,\infty)$, $0 < \theta < 1$ and $E = (E_0, E_1)_\theta$. We prove that for the Hardy spaces and conditioned Hardy spaces of noncommutative martingales the analogue of this relationship holds under some conditions.

KEYWORDS: *Noncommutative martingale, interpolation, noncommutative symmetric spaces.*

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INTRODUCTION

Interpolation plays a fundamental role in the classical martingale theory and harmonic analysis (for example, see [6]). For more details on interpolation theory, see [3], [4]. Musat [21] studied the noncommutative BMO and its interpolation properties. She proved noncommutative analogues of the classical interpolation results between BMO and L_p spaces (respectively, Hardy spaces). In [2], the authors considered the interpolation of the conditioned Hardy spaces h_p and presented an extension of Musat's results to the conditioned case.

This paper is concerned to the study of interpolation of Hardy spaces of noncommutative martingales associated to symmetric spaces. We will present some extensions of interpolation results in [2], [21] to the symmetric space case. We prove the following result: let E_1, E_2 be symmetric Banach spaces on $[0,\infty)$ satisfying the Fatou property (respectively E_1, E_2 be a fully symmetric Banach spaces on $[0,\infty)$). Suppose that $1 < p_{E_1} \leqslant q_{E_1} < 2$ and $1 < p_{E_2} \leqslant q_{E_2} < 2$ (respectively $2 < p_{E_1} \leqslant q_{E_1} < \infty$ and $2 < p_{E_2} \leqslant q_{E_2} < \infty$). If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then

$$h_E(\mathcal{M}) = (h_{E_0}(\mathcal{M}), h_{E_1}(\mathcal{M}))_\theta, \quad H_E(\mathcal{M}) = (H_{E_0}(\mathcal{M}), H_{E_1}(\mathcal{M}))_\theta$$

hold with equivalent norms.

We extend a classical result of Herz to the noncommutative symmetric space case, i.e., we define an equivalent norm of the conditioned Hardy space $h_E^c(\mathcal{M})$ when the Boyd indices of E are strictly between 1 and 2 and E has some additional property (for the case $E = Lp$ with $p < 2$ see [2]).

The remainder of this paper is divided into three sections. In Section 1 we present some preliminaries and notations on the noncommutative symmetric spaces and various Hardy spaces of noncommutative martingales. In Section 2 we prove the interpolation results. Finally, we define equivalent norms for $H_E^c(\mathcal{M})$ and $h_E^c(\mathcal{M})$, and discuss the description of the dual spaces of $H_E^c(\mathcal{M})$ and $h_E^c(\mathcal{M})$ in Section 3.

1. PREMIMINARIES

1.1. SYMMETRIC SPACES. Let (Ω, Σ, m) be a σ-measure space and $L(\Omega)$ be the linear space of all measurable, a.e. finite functions on Ω. Define $L_0(\Omega)$ as the subspace of $L(\Omega)$ which consists of all functions x such that $m(\{\omega \in (0,\infty) : |x(\omega)| > s\})$ is finite for some s. Let $x \in L_0(\Omega)$. Recall that the decreasing rearrangement function of x is defined by

$$\mu_t(x) = \inf\{s > 0 : m(\{\omega \in \Omega : |x(\omega)| > s\}) \leqslant t\}, \quad t > 0.$$

For $x, y \in L_0(\Omega)$ we say that x is *majorized* by y and write $x \preccurlyeq y$, if

$$\int_0^t \mu_s(x)\mathrm{d}s \leqslant \int_0^t \mu_s(y)\mathrm{d}s, \quad \text{for all } t > 0.$$

Recall the following terminology. Let I be one of the measure spaces $[0,1]$ or $[0,\infty)$ (with the natural measure). A quasi Banach lattice E of measurable functions on I is called a *symmetric quasi Banach space* on I if E satisfies the following properties: if $f \in E$, $g \in L_0(I)$ and $\mu(g) \leqslant \mu(f)$ implies that $g \in E$ and $\|g\|_E \leqslant \|f\|_E$.

Let E be a symmetric Banach space on I. If for every net $(x_i)_{i\in I}$ in E satisfying $0 \leqslant x_i \uparrow$ and $\sup_{i\in I} \|x_i\|_E < \infty$ the supremum $x = \sup_{i\in I} x_i$ exists in E and $\|x_i\|_E \uparrow \|x\|_E$, we say E has the Fatou property.

A symmetric Banach space E on I is called *fully symmetric* if, in addition, for $x \in L_0(I)$ and $y \in E$ with $x \preceq y$ it follows that $x \in E$ and $\|x\|_E \leqslant \|y\|_E$.

The Köthe dual of a symmetric Banach space E on I is the symmetric Banach space $E^\times$ defined by

$$E^\times = \Big\{x \in L_0(I) : \sup\Big\{ \int_I |x(t)y(t)|\mathrm{d}t : \|x\|_E \leqslant 1\Big\} < \infty\Big\};$$

$$\|y\|_{E^\times} = \sup\Big\{ \int_I |x(t)y(t)|\mathrm{d}t : \|x\|_E \leqslant 1 \Big\}, \quad y \in E^\times.$$

A symmetric Banach space E on I is separable if and only if $E = E^\times$ isometrically. Moreover, a symmetric Banach space which is separable or has the Fatou property is automatically fully symmetric.

For any $0 < a < \infty$, let the dilation operator D_a on $L_0(I)$ defined by

$$(D_a f)(s) = f(as)\chi_I(as) \quad (s \in I).$$

If E is a symmetric Banach space on I, then D_a is a bounded linear operator. Define the lower Boyd index p_E of E by

$$p_E = \sup\{p > 0 : \exists\, c > 0\, \forall\, 0 < a \leqslant 1 \|D_a f\|_E \leqslant c a^{-1/p} \|f\|_E\}$$

and the upper Boyd index q_E of E by

$$q_E = \inf\{q > 0 : \exists\, c > 0\, \forall\, a \geqslant 1 \|D_a f\|_E \leqslant c a^{-1/q} \|f\|_E\}.$$

It is clear from the definitions that

$$1 \leqslant p_E \leqslant q_E \leqslant \infty.$$

If E is a symmetric Banach space on I, then

$$\frac{1}{p_E} + \frac{1}{q_{E^\times}} = 1, \quad \frac{1}{p_{E^\times}} + \frac{1}{q_E} = 1. \tag{1.1}$$

For more details on symmetric Banach space we refer to [3], [5], [17], [19].

Let E_i be a quasi symmetric Banach space on $I, i = 1, 2$. We define the pointwise product space $E_1 \odot E_2$ as

$$E_1 \odot E_2 = \{x : x = x_1 x_2, x_i \in E_i, i = 1, 2\} \tag{1.2}$$

with a functional $\|x\|_{E_1 \odot E_2}$ defined by

$$\|x\|_{E_1 \odot E_2} = \inf\{\|x_1\|_{E_1} \|x_2\|_{E_2} : x = x_1 x_2, , x_i \in E_i, i = 1, 2\}.$$

By Theorem 2 in [18], we know that if E_i is a quasi symmetric Banach space on $I, i = 1, 2$ then $E_1 \odot E_2$ is a quasi symmetric Banach space on I.

Let E and F be two Banach lattices on I and let $0 < \theta < 1$. Following Calderòn, we define the lattice $E^\theta F^{1-\theta}$ as the space of those z in $L_0(I)$ such that for some $x \in E, y \in F$ with $\|x\|_E \leqslant 1, \|y\|_F \leqslant 1$ and for some $\lambda > 0$, we have

$$|z| \leqslant \lambda |x|^\theta |y|^{1-\theta} \quad \text{a.e. on } I.$$

We equip this space with the norm $\|z\|_{E^\theta F^{1-\theta}} = \inf\{\lambda\}$ where the infimum is over all such representations. The space $E^\theta F^{1-\theta}$ is again a Banach lattice on I (see Section 13.5 of [7]).

1.2. NONCOMMUTATIVE SYMMETRIC SPACES. We use standard notation and notions from the theory of noncommutative L^p-spaces. Our main references are [23] and [13] (see also [23] for more historical references). Throughout this paper, we denote by $\mathcal{M}$ a semi-finite (respectively finite) von Neumann algebra on the Hilbert space $\mathcal{H}$ with a faithful normal semi-finite (respectively normalized finite) trace τ. We denote by $L_0(\mathcal{M})$ the linear space of all τ-measurable operators. One can show that $L_0(\mathcal{M})$ is a $*$-algebra with sum and product being the respective closure of the algebraic sum and product.

Let $x \in L_0(\mathcal{M})$, and let $e_s^{\perp}(|x|) = 1_{(s,\infty)}(x)$ be the spectral projection of $|x|$ corresponding to the interval (s,∞). Define

$$\lambda_s(x) = \tau(e_s^{\perp}(|x|)) \quad s > 0, \quad \text{and} \quad \mu_t(x) = \inf\{s > 0 : \lambda_s(x) \leqslant t\} \quad t > 0.$$

The function $s \mapsto \lambda_s(x)$ is called the *distribution function* of x and the $\mu_t(x)$ the *generalized singular numbers (decreasing rearrangement)* of x. For more details on generalized singular value function of measurable operators we refer to [13].

Let E be a symmetric Banach space on I. We define

$$L_E(\mathcal{M}) = \{x \in L_0(\mathcal{M}) : \mu.(x) \in E\};$$
$$\|x\|_{L_E(\mathcal{M})} = \|\mu.(x)\|_E, \quad x \in L_E(\mathcal{M}).$$

Then $(L_E(\mathcal{M}), \|\cdot\|_{L_E(\mathcal{M})})$ is a Banach space (cf. [10], [25], [26]).

In what follows, unless otherwise specified, we always denote by E a symmetric Banach space on I.

Let $a = (a_n)_{n\geqslant 0}$ be a finite sequence in $L_E(\mathcal{M})$; define

$$\|a\|_{L_E(\mathcal{M},\ell_c^2)} = \Big\|\Big(\sum_{n\geqslant 0}|a_n|^2\Big)^{1/2}\Big\|_{L_E(\mathcal{M})}, \quad \|a\|_{L_E(\mathcal{M},\ell_r^2)} = \Big\|\Big(\sum_{n\geqslant 0}|a_n^*|^2\Big)^{1/2}\Big\|_{L_E(\mathcal{M})}.$$

This gives two norms on the family of all finite sequences in $L_E(\mathcal{M})$. To see this, denoting by $\mathcal{B}(\ell_2)$ the algebra of all bounded operators on ℓ_2 with its usual trace tr, let us consider the von Neumann algebra tensor product $\mathcal{M} \otimes \mathcal{B}(\ell_2)$ with the product trace $\tau \otimes \mathrm{tr}$ where $\tau \otimes \mathrm{tr}$ is a semi-finite normal faithful trace, and the associated noncommutative L_E space is denoted by $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$. Now, any finite sequence $a = (a_n)_{n\geqslant 1}$ in $L_E(\mathcal{M})$ can be regarded as an element in $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$ via the following map

$$a \longmapsto T(a) = \begin{pmatrix} a_1 & 0 & \dots \\ a_2 & 0 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix},$$

that is, the matrix of $T(a)$ has all vanishing entries except those in the first column which are the a_n's. Such a matrix is called a *column matrix*, and the closure in $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$ of all column matrices is called the *column subspace* of $L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2))$. Then

$$\|a\|_{L_E(\mathcal{M},\ell_c^2)} = \||T(a)|\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))} = \|T(a)\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))}.$$

Therefore, $\|\cdot\|_{L_E(\mathcal{M},\ell^2_{\mathrm{c}})}$ defines a norm on the family of all finite sequences of $L_E(\mathcal{M})$. The corresponding completion is a Banach space, denoted by $L_E(\mathcal{M},\ell^2_{\mathrm{c}})$. Similarly, we may show that $\|\cdot\|_{L_E(\mathcal{M},\ell^2_{\mathrm{r}})}$ is a norm on the family of all finite sequence in $L_E(\mathcal{M})$. As above, it defines a Banach space $L_E(\mathcal{M},\ell^2_{\mathrm{r}})$, which now is isometric to the row subspace of $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))$ consisting of matrices whose nonzero entries lie only in the first row.

We also need $L^{\mathrm{d}}_E(\mathcal{M})$, the space of all sequences $a=(a_n)_{n\geqslant 1}$ in $L_E(\mathcal{M})$ such that

$$\|a\|_{L^{\mathrm{d}}_E(\mathcal{M})}=\|\mathrm{diag}(a_n)\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))}<\infty,$$

where $\mathrm{diag}(a_n)$ is the matrix with the a_n on the diagonal and zeroes elsewhere.

1.3. NONCOMMUTATIVE MARTINGALES. Let us now recall the general setup for noncommutative martingales. In the sequel, we always denote by $(\mathcal{M}_n)_{n\geqslant 1}$ an increasing sequence of von Neumann subalgebras of $\mathcal{M}$ whose union $\bigcup\limits_{n\geqslant 1}\mathcal{M}_n$ generates $\mathcal{M}$ (in the w*-topology). $(\mathcal{M}_n)_{n\geqslant 1}$ is called a *filtration* of $\mathcal{M}$. For $n\geqslant 1$, we assume that there exists a trace preserving conditional expectation $\mathcal{E}_n$ from $\mathcal{M}$ onto $\mathcal{M}_n$. The restriction of τ to $\mathcal{M}_n$ is still denoted by τ. It is well-known that $\mathcal{E}_n$ extends to a contractive projection from $L_p(\mathcal{M},\tau)$ onto $L_p(\mathcal{M}_n,\tau_n)$ for all $1\leqslant p\leqslant\infty$. More generally, if E is a symmetric Banach function space on I then $\mathcal{E}_n$ is a contraction from $L_E(\mathcal{M},\tau)$ onto $L_E(\mathcal{M}_n,\tau)$.

A noncommutative L_E-martingale with respect to $(\mathcal{M}_n)_{n\geqslant 1}$ is a sequence $x=(x_n)_{n\geqslant 1}$ such that $x_n\in L_E(\mathcal{M}_n)$ and

$$\mathcal{E}_n(x_{n+1})=x_n$$

for any $n\geqslant 1$. Set $\|x\|_{L_E(\mathcal{M})}=\sup\limits_{n\geqslant 1}\|x_n\|_{L_E(\mathcal{M})}$. If $\|x\|_{L_E(\mathcal{M})}<\infty$, then x is said to be a bounded L_E-martingale.

Let x be a noncommutative martingale. The martingale difference sequence of x, denoted by $dx=(dx_n)_{n\geqslant 1}$, is defined as

$$dx_1=x_1,\quad dx_n=x_n-x_{n-1},\quad n\geqslant 2.$$

For any finite martingale $x=(x_n)_{n\geqslant 1}$ in $L_E(M)$, we set

$$S^{\mathrm{c}}(x)=\Big(\sum_{k\geqslant 1}|dx_k|^2\Big)^{1/2}\quad\text{and}\quad S^{\mathrm{r}}(x)=\Big(\sum_{k\geqslant 1}|dx_k^*|^2\Big)^{1/2},$$

and define

$$\|x\|_{H^{\mathrm{c}}_E(\mathcal{M})}=\|S^{\mathrm{c}}(x)\|_{L_E(\mathcal{M})}=\|dx\|_{L_E(\mathcal{M},\ell^2_{\mathrm{c}})}$$
$$(\text{respectively } \|x\|_{H^{\mathrm{r}}_E(\mathcal{M})}=\|S^{\mathrm{r}}(x)\|_{L_E(\mathcal{M})}=\|dx\|_{L_E(\mathcal{M},\ell^2_{\mathrm{r}})}).$$

Let $H^{\mathrm{c}}_E(\mathcal{M})$ (respectively $H^{\mathrm{r}}_E(\mathcal{M})$) be the corresponding completions. Then $H^{\mathrm{c}}_E(\mathcal{M})$ (respectively $H^{\mathrm{r}}_E(\mathcal{M})$) is a Banach space.

We now consider the conditioned versions of square functions and Hardy spaces developed in [15]. Let $a = (a_n)_{n\geqslant 1}$ be a finite sequence in $\mathcal{M}$. We define (recalling $\mathcal{E}_0 = \mathcal{E}_1$)

$$\|a\|_{L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{c}}^2)} = \Big\|\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(|a_n|^2)\Big)^{1/2}\Big\|_{L_E(\mathcal{M})}.$$

It is shown in [24] that $\|\cdot\|_{L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{c}}^2)}$ is a norm on the vector space of all finite sequences in $\mathcal{M}$. We define $L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{c}}^2)$ to be the corresponding completion. Similarly, we define the conditioned row space $L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{r}}^2)$. Note that $L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{c}}^2)$ (respectively $L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{r}}^2)$) is the conditioned version of $L_E(\mathcal{M},\ell_{\mathrm{c}}^2)$ (respectively $L_E(\mathcal{M},\ell_{\mathrm{r}}^2)$). The space $L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{c}}^2)$ (respectively $L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{r}}^2)$) can be viewed as a closed subspace of the column (respectively row) subspace of $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2(\mathbb{N}^2)))$. We refer to [14], [15], [24] for more details on this.

For a finite noncommutative L_E-martingale $x=(x_n)_{n\geqslant 1}$ define (with $\mathcal{E}_0=\mathcal{E}_1$)

$$\|x\|_{h_E^{\mathrm{c}}(\mathcal{M})} = \|dx\|_{L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{c}}^2)} \quad\text{and}\quad \|x\|_{h_E^{\mathrm{r}}(\mathcal{M})} = \|dx\|_{L_E^{\mathrm{cond}}(\mathcal{M},\ell_{\mathrm{r}}^2)}.$$

Let $h_E^{\mathrm{c}}(\mathcal{M})$ and $h_E^{\mathrm{r}}(\mathcal{M})$ be the corresponding completions. Then $h_E^{\mathrm{c}}(\mathcal{M})$ and $h_E^{\mathrm{r}}(\mathcal{M})$ are Banach spaces. We define the column and row conditioned square functions as follows. For any finite martingale $x = (x_n)_{n\geqslant 1}$ in $L_E(M)$, we set

$$s^{\mathrm{c}}(x) = \Big(\sum_{k\geqslant 1}\mathcal{E}_{k-1}(|dx_k|^2)\Big)^{1/2} \quad\text{and}\quad s^{\mathrm{r}}(x) = \Big(\sum_{k\geqslant 1}\mathcal{E}_{k-1}(|dx_k^*|^2)\Big)^{1/2}.$$

Then

$$\|x\|_{h_E^{\mathrm{c}}(\mathcal{M})} = \|s^{\mathrm{c}}(x)\|_{L_E(\mathcal{M})} \quad\text{and}\quad \|x\|_{h_E^{\mathrm{r}}(\mathcal{M})} = \|s^{\mathrm{r}}(x)\|_{L_E(\mathcal{M})}.$$

Let $x = (x_n)_{n\geqslant 0}$ be a finite L_E-martingale, we set

$$s^{\mathrm{d}}(x) = \mathrm{diag}(|dx_n|)$$

We note that

$$\|s^{\mathrm{d}}(x)\|_{L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))} = \|dx_n\|_{L_E^{\mathrm{d}}(\mathcal{M})}$$

Let $h_E^{\mathrm{d}}(\mathcal{M})$ be the subspace of $L_E^{\mathrm{d}}(\mathcal{M})$ consisting of all martingale difference sequences.

Using Lemma 6.4 in [15], Theorem 2.3 in [22] and Theorem 3.4 in [11] we obtain the following result.

PROPOSITION 1.1. *Let E be a separable symmetric Banach space on I with $1 < p_E \leqslant q_E < \infty$. Then we have:*
(i) $h_E^{\mathrm{c}}(\mathcal{M}), h_E^{\mathrm{r}}(\mathcal{M})$ *are complemented in* $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2(\mathbb{N}^2)))$;
(ii) $H_E^{\mathrm{c}}(\mathcal{M}), H_E^{\mathrm{r}}(\mathcal{M})$ *are complemented in* $L_E(\mathcal{M}\otimes\mathcal{B}(\ell_2))$.

By Theorem 5.6 in [12] and Proposition 1.1, it follows the following proposition.

THEOREM 1.2. *Let E be a separable symmetric Banach space on $[0,1]$ with $1 < p_E \leqslant q_E < \infty$. Then we have:*

(i) $(h_E^{\mathrm{c}}(\mathcal{M}))^* = h_{E^\times}^{\mathrm{c}}(\mathcal{M})$ *with equivalent norms;*

(ii) $(H_E^{\mathrm{c}}(\mathcal{M}))^* = H_{E^\times}^{\mathrm{c}}(\mathcal{M})$ *with equivalent norms.*

Similarly, $(H_E^{\mathrm{r}}(\mathcal{M}))^* = h_{E^\times}^{\mathrm{r}}(\mathcal{M})$ *and* $(H_E^{\mathrm{r}}(\mathcal{M}))^* = H_{E^\times}^{\mathrm{r}}(\mathcal{M})$ *with equivalent norms.*

2. INTERPOLATION

In this section $(\mathcal{M};\tau)$ always denotes a semi-finite von Neumann algebra equipped with a faithful semi-finite trace, and $(\mathcal{M}_n)_{n\geqslant 1}$ an increasing filtration of subalgebras of M which generate $\mathcal{M}$. We keep all notations introduced in the first section.

We recall interpolation of noncommutative symmetric spaces. Let E_1, E_2 be fully symmetric spaces on $[0,\infty)$ and $0 < \theta < 1$. If E is complex interpolation of E_1 and E_2, i.e. $E = (E_1, E_2)_\theta$, then

$$L_E(\mathcal{M}) = (L_{E_1}(\mathcal{M}), L_{E_2}(\mathcal{M}))_\theta. \tag{2.1}$$

Since $\{\mathrm{diag}(a_n) : (a_n) \subset \mathcal{M}\}$ is a von Neumann subalgebra of $\mathcal{M} \otimes \mathcal{B}(\ell_2)$, we have that

$$L_E^{\mathrm{d}}(\mathcal{M}) = (L_{E_1}^{\mathrm{d}}(\mathcal{M}), L_{E_2}^{\mathrm{d}}(\mathcal{M}))_\theta. \tag{2.2}$$

For more details on interpolation of noncommutative symmetric spaces we refer to [11].

This section is devoted to showing the analogue of (2.1) for the Hardy spaces and conditioned Hardy spaces of noncommutative martingales.

PROPOSITION 2.1. *Let E_j be a fully symmetric Banach spaces on $[0,\infty)$ with $1 < p_{E_j} \leqslant q_{E_j} < 2$ $(j = 1,2)$. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then:*

(i) $h_E^{\mathrm{c}}(\mathcal{M}) = (h_{E_1}^{\mathrm{c}}(\mathcal{M}), h_{E_2}^{\mathrm{c}}(\mathcal{M}))_\theta, h_E^{\mathrm{r}}(\mathcal{M}) = (h_{E_1}^{\mathrm{r}}(\mathcal{M}), h_{E_2}^{\mathrm{r}}(\mathcal{M}))_\theta;$

(ii) $H_E^{\mathrm{c}}(\mathcal{M}) = (H_{E_1}^{\mathrm{c}}(\mathcal{M}), H_{E_2}^{\mathrm{c}}(\mathcal{M}))_\theta, H_E^{\mathrm{r}}(\mathcal{M}) = (H_{E_1}^{\mathrm{r}}(\mathcal{M}), H_{E_2}^{\mathrm{r}}(\mathcal{M}))_\theta.$

Proof. By Theorem 3.4 in [11], we have that

$$L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2(\mathbb{N}^2))) = (L_{E_1}(\mathcal{M} \otimes \mathcal{B}(\ell_2(\mathbb{N}^2))), L_{E_2}(\mathcal{M} \otimes \mathcal{B}(\ell_2(\mathbb{N}^2))))_\theta \quad \text{and}$$
$$L_E(\mathcal{M} \otimes \mathcal{B}(\ell_2)) = (L_{E_1}(\mathcal{M} \otimes \mathcal{B}(\ell_2)), L_{E_2}(\mathcal{M} \otimes \mathcal{B}(\ell_2)))_\theta.$$

Using Proposition 1.1 we obtain the desired result. ∎

LEMMA 2.2. *Let E_j be a symmetric Banach space on $[0,\infty)$ with $1 < p_{E_j} \leqslant q_{E_j} < \infty$ $(j = 1,2)$ and $0 < \theta < 1$. Suppose that $E = (E_1, E_2)_\theta$, then the following holds with equivalent norms:*

$$(h_{E_1}^{\mathrm{d}}(\mathcal{M}), h_{E_2}^{\mathrm{d}}(\mathcal{M}))_\theta = h_E^{\mathrm{d}}(\mathcal{M}).$$

Proof. Since $h_E^{\mathrm{d}}(\mathcal{M})$ consists of martingale difference sequences in $L_E^{\mathrm{d}}(\mathcal{M})$, $h_E^{\mathrm{d}}(\mathcal{M})$ is 2-complemented in $L_E^{\mathrm{d}}(\mathcal{M}))$ via the projection

$$P : \begin{cases} L_E^{\mathrm{d}}(\mathcal{M})) & \longrightarrow & h_E^{\mathrm{d}}(\mathcal{M}), \\ (a_n)_{n\geqslant 1} & \longmapsto & (\mathcal{E}_n(a_n) - \mathcal{E}_{n-1}(a_n))_{n\geqslant 1}. \end{cases}$$

By (2.2), we obtain $(h_{E_1}^{\mathrm{d}}(\mathcal{M}), h_{E_2}^{\mathrm{d}}(\mathcal{M}))_\theta = h_E^{\mathrm{d}}(\mathcal{M})$. ∎

We define the Hardy space of noncommutative martingales and its conditioned version as follows. For $1 \leqslant p_E \leqslant q_E < 2$,

$$H_E(\mathcal{M}) = H_E^{\mathrm{c}}(\mathcal{M}) + H_E^{\mathrm{r}}(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{H_E(\mathcal{M})} = \inf\{\|y\|_{H_E^{\mathrm{c}}(\mathcal{M})} + \|z\|_{H_E^{\mathrm{r}}(\mathcal{M})} : x = y+z, y \in H_E^{\mathrm{c}}(\mathcal{M}), z \in H_E^{\mathrm{r}}(\mathcal{M})\} \quad \text{and}$$
$$h_E(\mathcal{M}) = h_E^{\mathrm{c}}(\mathcal{M}) + h_E^{\mathrm{r}}(\mathcal{M}) + h_E^{\mathrm{d}}(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{h_E(\mathcal{M})} = \inf\{\|y\|_{h_E^{\mathrm{c}}(\mathcal{M})} + \|z\|_{h_E^{\mathrm{r}}(\mathcal{M})} + \|w\|_{h_E^{\mathrm{d}}(\mathcal{M})} :$$
$$x = y + z + w, y \in h_E^{\mathrm{c}}(\mathcal{M}), z \in h_E^{\mathrm{r}}(\mathcal{M}), w \in h_E^{\mathrm{d}}(\mathcal{M})\}.$$

For $2 \leqslant p_E \leqslant q_E < \infty$,

$$H_E(\mathcal{M}) = H_E^{\mathrm{c}}(\mathcal{M}) \cap H_E^{\mathrm{r}}(\mathcal{M}),$$

equipped with the norm

$$\|x\| = \max\{\|x\|_{H_E^{\mathrm{c}}(\mathcal{M})}, \|x\|_{H_E^{\mathrm{r}}(\mathcal{M})}\} \quad \text{and} \quad h_E(\mathcal{M}) = h_E^{\mathrm{c}}(\mathcal{M}) \cap h_E^{\mathrm{r}}(\mathcal{M}) \cap h_E^{\mathrm{d}}(\mathcal{M}),$$

equipped with the norm

$$\|x\|_{h_E(\mathcal{M})} = \max\{\|x\|_{h_E^{\mathrm{c}}(\mathcal{M})}, \|x\|_{h_E^{\mathrm{r}}(\mathcal{M})}, \|x\|_{h_E^{\mathrm{d}}(\mathcal{M})}\}.$$

THEOREM 2.3. *Let E_j be a symmetric Banach space on $[0,\infty)$ satisfying the Fatou property, and let $1 < p_{E_j} \leqslant q_{E_j} < 2$ $(j = 1,2)$. Suppose that either E_1 or E_2 has order continuous norm. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then:*
(i) *$h_E(\mathcal{M}) = (h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta$ holds with equivalent norms;*
(ii) *$H_E(\mathcal{M}) = (H_{E_1}(\mathcal{M}), H_{E_2}(\mathcal{M}))_\theta$ holds with equivalent norms.*

Proof. We prove only the first equivalence. The proof of the second one is similar. From the definition of complex interpolation, it follows that

$$(h_{E_1}^{\mathrm{d}}(\mathcal{M}), h_{E_2}^{\mathrm{d}}(\mathcal{M}))_\theta + (h_{E_1}^{\mathrm{c}}(\mathcal{M}), h_{E_2}^{\mathrm{c}}(\mathcal{M}))_\theta + (h_{E_1}^{\mathrm{r}}(\mathcal{M}), h_{E_2}^{\mathrm{r}}(\mathcal{M}))_\theta$$
$$\subset (h_{E_1}^{\mathrm{d}}(\mathcal{M}) + h_{E_1}^{\mathrm{c}}(\mathcal{M}) + h_{E_1}^{\mathrm{r}}(\mathcal{M}), h_{E_2}^{\mathrm{d}}(\mathcal{M}) + h_{E_2}^{\mathrm{c}}(\mathcal{M}) + h_{E_2}^{\mathrm{r}}(\mathcal{M}))_\theta.$$

By Proposition 2.1 and Lemma 2.2, we deduce that

$$h_E(\mathcal{M}) = h_E^{\mathrm{d}}(\mathcal{M}) + h_E^{\mathrm{r}}(\mathcal{M}) + h_E^{\mathrm{c}}(\mathcal{M}) \subset (h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta.$$

On the other hand, it was shown by Calderòn [7] (see also [17], Theorem 4.1.14) that

$$(E_1, E_2)_\theta = E_1^{1-\theta} E_2^\theta$$

holds with equality of norms. Using the result in [20] we obtain that

$$E^{\times\times} = ((E_1^{1-\theta} E_2^\theta)^\times)^\times = ((E_1^\times)^{1-\theta}(E_2^\times)^\theta)^\times = (E_1^{\times\times})^{1-\theta}(E_2^{\times\times})^\theta = E.$$

i.e., E has the Fatou property. By definition of the Boyd indices and interpolation, we deduce that $1 < p_E \leqslant q_E < 2$. Hence, by Theorem 3.1 in [24], it follows that $h_E(\mathcal{M}) = L_E(\mathcal{M})$. Therefore, (2.1) gives the reverse inclusion

$$(h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta \subset h_E(\mathcal{M}). \quad \blacksquare$$

THEOREM 2.4. *Let E_j be a fully symmetric Banach space on $[0, \infty)$ with $2 < p_{E_j} \leqslant q_{E_j} < \infty$ $(j = 1, 2)$. If $0 < \theta < 1$ and $E = (E_1, E_2)_\theta$, then*

(i) *$h_E(\mathcal{M}) = (h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta$ holds with equivalent norms;*

(ii) *$H_E(\mathcal{M}) = (H_{E_1}(\mathcal{M}), H_{E_2}(\mathcal{M}))_\theta$ holds with equivalent norms.*

Proof. (i) By definition, we have that

$$(h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta \\ \subset (h^{\mathrm{d}}_{E_1}(\mathcal{M}), h^{\mathrm{d}}_{E_2}(\mathcal{M}))_\theta \cap (h^{\mathrm{c}}_{E_1}(\mathcal{M}), h^{\mathrm{c}}_{E_2}(\mathcal{M}))_\theta \cap (h^{\mathrm{r}}_{E_1}(\mathcal{M}), h^{\mathrm{r}}_{E_2}(\mathcal{M}))_\theta.$$

Using Proposition 2.1 and Lemma 2.2 we obtain that

$$(h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta \subset h^{\mathrm{d}}_E(\mathcal{M}) \cap h^{\mathrm{r}}_E(\mathcal{M}) \cap h^{\mathrm{c}}_E(\mathcal{M}) = h_E(\mathcal{M}).$$

On the other hand, by Theorem 6.2 in [9] we have $h_{E_j}(\mathcal{M}) = L_{E_j}(\mathcal{M})$ $(j = 1, 2)$. Hence (2.1) gives the reverse inclusion $(h_{E_1}(\mathcal{M}), h_{E_2}(\mathcal{M}))_\theta \supset h_E(\mathcal{M})$. $\blacksquare$

3. AN EQUIVALENT QUASI-NORM

In this section $\mathcal{M}$ always denotes a finite von Neumann algebra equipped with a normalized faithful trace τ, and $(\mathcal{M}_n)_{n\geqslant 1}$ an increasing filtration of subalgebras of M which generate $\mathcal{M}$. We keep all notations introduced in the first section.

Let E be a symmetric quasi Banach space on $[0, 1]$. For $0 < p < \infty$, we define

$$E^{(p)} = \{x : |x|^p \in E\},$$

equipped with the quasi-norm

$$\|x\|_{E^{(p)}} = \||x|^p\|_E^{1/p},$$

then $E^{(p)}$ is a symmetric quasi Banach space on $[0, 1]$ (see [19]).

We need the following results ([1], Lemma 2.1).

LEMMA 3.1. *Let E, E_1, E_2 be symmetric Banach spaces on $[0,1]$ such that $E = E_1 \odot E_2$. If $x \in L_E(\mathcal{M})^+$, then for $\varepsilon > 0$, there exist $a \in L^+_{E_1}(\mathcal{M})$ and $b \in L^+_{E_2}(\mathcal{M})$ such that $x = ab = ba$, $\|a\|_{L_{E_1}(\mathcal{M})}\|b\|_{L_{E_2}(\mathcal{M})} < \|x\|_{L_E(\mathcal{M})} + \varepsilon$ and a is invertible with bounded inverse.*

Let E be a separable symmetric Banach space on $[0,1]$ with $1 < p_E \leqslant q_E < 2$ and suppose that $F = (E^{\times(1/2)})^\times$ is separable. From the proof of Proposition 1.3 in [1], it follows that $E = F \odot E^\times$.

For an L_2-martingale x we set

$$n^{\mathrm{c}}_E(x) = \inf\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(a)^{-1}|dx_n|^2\Big)\Big]^{1/2} : a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1 \text{ and } a \text{ is invertible with bounded inverse}\Big\} \quad \text{and}$$

$$n^{\mathrm{r}}_E(x) = \inf\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(a)^{-1}|dx^*_n|^2\Big)\Big]^{1/2} : a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1 \text{ and } a \text{ is invertible with bounded inverse}\Big\}.$$

PROPOSITION 3.2. *Let E be a separable symmetric Banach space on $[0,1]$ with $1 < p_E \leqslant q_E < 2$ and suppose that $F = (E^{\times(1/2)})^\times$ is separable. Then for any $x \in L_2(M)$ we have $n^{\mathrm{c}}_E(x) \approx \|x\|_{h^{\mathrm{c}}_E(\mathcal{M})}$. A similar statement holds for $n^{\mathrm{r}}_E(x)$ and $h^{\mathrm{r}}_E(\mathcal{M})$.*

Proof. Applying Corollary 2.3 of [8] we obtain that $\mathcal{E}_{n-1}(a^{-1}) \geqslant \mathcal{E}_{n-1}(a)^{-1}$, for all $n \geqslant 1$. Hence

$$n^{\mathrm{c}}_E(x) = \inf_a \Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(a)^{-1}|dx_n|^2\Big)\Big]^{1/2} \leqslant \inf_a \Big[\tau\Big(\sum_{n\geqslant 1}a^{-1}\mathcal{E}_{n-1}(|dx_n|^2)\Big)\Big]^{1/2}.$$

By Lemma 3.1, for any $\varepsilon > 0$, there exist $a \in L^+_F(\mathcal{M})$ and $b \in L^+_{E^\times}(\mathcal{M})$ such that $s^{\mathrm{c}}(x) = ab$, $\|a\|_{L_F(\mathcal{M})} = 1$, $\|b\|_{L_{E^\times}(\mathcal{M})} < \|s^{\mathrm{c}}(x)\|_{L_E(\mathcal{M})} + \varepsilon$ and a is invertible with bounded inverse. Using Hölder inequality we find that

$$\begin{aligned} n^{\mathrm{c}}_E(x) &\leqslant \Big[\tau\Big(\sum_{n\geqslant 1}a^{-1}\mathcal{E}_{n-1}(|dx_n|^2)\Big)\Big]^{1/2} = \Big[\tau\Big(a^{-1}\sum_{n\geqslant 1}\mathcal{E}_{n-1}(|dx_n|^2)\Big)\Big]^{1/2} \\ &= [\tau(a^{-1}(s^{\mathrm{c}}(x))^2)]^{1/2} = [\tau(ab^2)]^{1/2} \leqslant [\|a\|_{L_F(\mathcal{M})}\|b^2\|_{L_{E^{\times(1/2)}}(\mathcal{M})}]^{1/2} \\ &= \|a\|^{1/2}_{L_F(\mathcal{M})}\|b\|_{L_{E^\times}(\mathcal{M})} \leqslant \|x\|_{h^{\mathrm{c}}_E(\mathcal{M})} + \varepsilon. \end{aligned}$$

Applying Theorem 1.2 we obtain that

$$\|x\|_{h^{\mathrm{c}}_E(\mathcal{M})} \lesssim \sup_{\|y\|_{h^{\mathrm{c}}_{E^\times}(\mathcal{M})}\leqslant 1} |\tau(y^*x)|. \tag{3.1}$$

By the Cauchy–Schwarz inequality and the tracial property of τ, we have

$$\begin{aligned}|\tau(y^*x)| &= \Big|\sum_{n\geqslant 1}\tau(dy_n^* dx_n)\Big| = \Big|\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{1/2}dy_n^* dx_n \mathcal{E}_{n-1}(a)^{-1/2})\Big|\\ &\leqslant \Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{1/2}|dy_n|^2\mathcal{E}_{n-1}(a)^{1/2})\Big]^{1/2}\\ &\quad\cdot\Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1/2}|dx_n|^2\mathcal{E}_{n-1}(a)^{-1/2})\Big]^{1/2}\\ &= \Big[\sum_{n\geqslant 1}\tau(a\mathcal{E}_{n-1}(|dy_n|^2))\Big]^{1/2}\Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1}|dx_n|^2)\Big]^{1/2}.\end{aligned}$$

On the other hand, by Theorem 5.6 in [12], we have

$$\begin{aligned}\sum_{n\geqslant 1}\tau(a\mathcal{E}_{n-1}(|dy_n|^2)) &= \tau\Big(a\sum_{n\geqslant 1}\mathcal{E}_{n-1}(|dy_n|^2)\Big)\\ &\leqslant \|a\|_{L_F(\mathcal{M})}\Big\|\sum_{n\geqslant 1}\mathcal{E}_{n-1}(|dy_n|^2)\Big\|_{L_{E^{\times(1/2)}}(\mathcal{M})}\\ &= \|a\|_{L_F(\mathcal{M})}\Big\|\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(|dy_n|^2)\Big)^{1/2}\Big\|^2_{L_{E^\times}(\mathcal{M})}\\ &= \|a\|_{L_F(\mathcal{M})}\|y\|^2_{h^c_{E^\times}(\mathcal{M})}\leqslant 1.\end{aligned}$$

Hence, $\|x\|_{h^c_E(\mathcal{M})}\lesssim n^c_E(x)$. Passing to adjoints yields $n^r_E(x)\approx\|x\|_{h^r_E(\mathcal{M})}$. ∎

For an L_2-martingale x we define two norms:

$$m^c_E(x) = \sup\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(a)|dx_n|^2\Big)\Big]^{1/2} : a\in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})}\leqslant 1\Big\}\quad\text{and}$$

$$m^r_E(x) = \sup\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_{n-1}(a)|dx_n^*|^2\Big)\Big]^{1/2} : a\in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})}\leqslant 1\Big\}.$$

The space

$$w^c_E(\mathcal{M}) = \{x\in L_2(\mathcal{M}) : m^c_E(x)<\infty\}$$

equipped with the norm m^c_E is a Banach space. Similarly, we set

$$w^r_E(\mathcal{M}) = \{x : x^*\in w^c_E(\mathcal{M})\}$$

equipped with the norm m^r_E.

THEOREM 3.3. *Let E be a separable symmetric Banach space on $[0,1]$ with $1 < p_E\leqslant q_E < 2$ and suppose that $F = (E^{\times(1/2)})^\times$ is separable. Then we have:*
(i) $(h^c_E(\mathcal{M}))^* = w^c_E(\mathcal{M})$ *with equivalent norms;*
(ii) $(h^r_E(\mathcal{M}))^* = w^r_E(\mathcal{M})$ *with equivalent norms.*

Proof. (i) Let $x\in w^c_E(\mathcal{M})$. Then x defines a continuous linear functional on $h^c_E(\mathcal{M})$ by $\phi_x(y) = \tau(yx^*)$ for $y\in L_2(\mathcal{M})$. To see this let $a\in L_F(\mathcal{M})^+$,

$\|a\|_{L_F(\mathcal{M})} \leqslant 1$ and a be invertible with bounded inverse. We fix it and the Cauchy–Schwarz inequality gives

$$\begin{aligned}|\phi_x(y)| &= \Big|\sum_{n\geqslant 1}\tau(dy_n dx_n^*)\Big| = \Big|\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1/2}dy_n^* dx_n)\mathcal{E}_{n-1}(a)^{1/2}\Big| \\ &\leqslant \Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1}|dy_n|^2)\Big]^{1/2}\Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)|dx_n|^2)\Big]^{1/2} \\ &\leqslant \Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1}|dy_n|^2)\Big]^{1/2} m_E^{\mathrm{c}}(x).\end{aligned}$$

Taking the infimum over a we obtain $|\phi_x(y)| \leqslant n_E^{\mathrm{c}}(y)m_E^{\mathrm{c}}(x)$. Conversely, let $\phi \in h_E^{\mathrm{c}}(\mathcal{M})^*$ of norm one. As $L_2(\mathcal{M}) \subset h_E^{\mathrm{c}}(\mathcal{M})$, it follows that ϕ induces a continuous functional $\widetilde{\phi}$ on $L_2(M)$. Consequently, $\widetilde{\phi}$ is given by an element x of $L_2(\mathcal{M})$,

$$\widetilde{\phi}(y) = \tau(yx^*), \quad \forall\, y \in L_2(\mathcal{M}).$$

By the density of $L_2(\mathcal{M})$ in $h_E^{\mathrm{c}}(\mathcal{M})$, we have $\|\phi\|_{h_E^{\mathrm{c}}(\mathcal{M})^*} = \sup\limits_{y\in L_2(\mathcal{M}), \|y\|_{h_E^{\mathrm{c}}(\mathcal{M})}\leqslant 1} |\tau(yx^*)|$ $= 1$. Hence, by Proposition 3.2 we have

$$\|\phi\|_{h_E^{\mathrm{c}}(\mathcal{M})^*} = \sup_{y\in L_2(\mathcal{M}), n_E^{\mathrm{c}}(y)\leqslant 1} |\tau(yx^*)| \lesssim 1. \tag{3.2}$$

We want to show that $m_E^{\mathrm{c}}(x) \lesssim 1$. Let $a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1$ and a be invertible with bounded inverse. We fix a, and let y be the martingale defined by $dy_n = \mathcal{E}_{n-1}(a)^{-1}dx_n$. By (3.2), it follows that

$$\tau(yx^*) = \sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1}|dx_n|^2) \leqslant n_E^{\mathrm{r}}(y) \leqslant \Big[\sum_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1}|dx_n|^2)\Big]^{1/2}.$$

Hence $\Big[\sum\limits_{n\geqslant 1}\tau(\mathcal{E}_{n-1}(a)^{-1}|dx_n|^2)\Big]^{1/2} \leqslant 1$. Taking the supremum over a we obtain $m_E^{\mathrm{c}}(x) \lesssim 1$.

(ii) Passing to adjoint, we obtain the desired result. ∎

Similarly, for the Hardy spaces of noncommutative martingales H_E, we have the following results: for an L_2-martingale x we set

$$N_E^{\mathrm{c}}(x) = \inf\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_n(a)^{-1}|dx_n|^2\Big)\Big]^{1/2} : a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1 \text{ and } a \text{ is invertible with bounded inverse}\Big\} \quad \text{and}$$

$$N_E^{\mathrm{r}}(x) = \inf\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_n(a)^{-1}|dx_n^*|^2\Big)\Big]^{1/2} : a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1 \text{ and } a \text{ is invertible with bounded inverse}\Big\}.$$

PROPOSITION 3.4. *Let E be a separable symmetric Banach space on $[0,1]$ with $1 < p_E \leqslant q_E < 2$ and suppose that $F = (E^{\times(1/2)})^{\times}$ is separable. Then for any $x \in L_2(M)$ we have $N_E^{\mathrm{c}}(x) \approx \|x\|_{h_E^{\mathrm{c}}(\mathcal{M})}$. A similar statement holds for $N_E^{\mathrm{r}}(x)$ and $H_E^{\mathrm{r}}(\mathcal{M})$.*

The proof is similar to the proof of Proposition 3.2.

For an L_2-martingale x we define two norms:

$$M_E^{\mathrm{c}}(x) = \sup\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_n(a)|dx_n|^2\Big)\Big]^{1/2} : a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1\Big\} \quad \text{and}$$

$$M_E^{\mathrm{r}}(x) = \sup\Big\{\Big[\tau\Big(\sum_{n\geqslant 1}\mathcal{E}_n(a)|dx_n^*|^2\Big)\Big]^{1/2} : a \in L_F(\mathcal{M})^+, \|a\|_{L_F(\mathcal{M})} \leqslant 1\Big\}.$$

The space

$$W_E^{\mathrm{c}}(\mathcal{M}) = \{x \in L_2(\mathcal{M}) : M_E^{\mathrm{c}}(x) < \infty\}$$

equipped with the norm M_E^{c} is a Banach space. Similarly, we set

$$W_E^{\mathrm{r}}(\mathcal{M}) = \{x : x^* \in W_E^{\mathrm{c}}(\mathcal{M})\}$$

equipped with the norm M_E^{r}.

We use Proposition 3.4 and the same method as in the proof Theorem 3.3 to obtain the following result.

THEOREM 3.5. *Let E be a separable symmetric Banach space on $[0,1]$ with $1 < p_E \leqslant q_E < 2$ and suppose that $F = (E^{\times(1/2)})^{\times}$ is separable. Then we have:*

(i) $(H_E^{\mathrm{c}}(\mathcal{M}))^* = W_E^{\mathrm{c}}(\mathcal{M})$ *with equivalent norms;*

(ii) $(H_E^{\mathrm{r}}(\mathcal{M}))^* = W_E^{\mathrm{r}}(\mathcal{M})$ *with equivalent norms.*

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REFERENCES

[1] T.N. BEKJAN, Duality for symmetric Hardy spaces of noncommutative martingales, arXiv:1510.03967 [math.OA]

[2] T.N. BEKJAN, Z. CHEN, M. PERRIN, Z. YIN, Atomic decomposition and interpolation for Hardy spaces of noncommutative martingales, *J. Funct. Anal.* **258**(2010), 2483–2505.

[3] C. BENNETT, R. SHARPLEY, *Interpolation of Operators*, Academic Press Inc., Boston, MA 1988.

[4] J. BERGH, J. LÖFSTRÖM, *Interpolation Spaces. An Introduction*, Springer, New York 1976.

[5] D. BOYD, Indices of function spaces and their relationship to interpolation, *Canad. J. Math.* **21**(1969), 1245–1254.

[6] D. BURKHOLDER, Martingales and singular integrals in Banach spaces, in *Handbook on the Geometry of Banach Spaces*, Vol. 1, North-Holland, Amsterdam 2001, pp. 233–269.

[7] A.P. CALDERÒN, Intermediate spaces and interpolation, the complex method, *Studia Math.* **24**(1964), 133–190.

[8] M.D. CHOI, A Schwarz inequality for positive linear maps on C^*-algebras, *Illinois J. Math.* **18**(1974), 565–574.

[9] S. DIRKSEN, Noncommutative Boyd interpolation theorems, *Trans. Amer. Math. Soc.* **367**(2015), 4079–4110.

[10] P.G. DODDS, T.K. DODDS, B. DE PAGER, Noncommutative Banach function spaces, *Math. Z.* **201**(1989), 583–587.

[11] P.G. DODDS, T.K. DODDS, B. DE PAGER, Fully symmetric operator spaces, *Integral Equations Operator Theory* **15**(1992), 942–972.

[12] P.G. DODDS, T.K. DODDS, B. DE PAGER, Noncommutative Köthe duality, *Trans. Amer. Math. Soc.* **339**(1993), 717–750.

[13] T. FACK, H. KOSAKI, Generalized s-numbers of τ-measurable operators, *Pacific J. Math* **123**(1986), 269–300.

[14] M. JUNGE, Doob's inequality for noncommutative martingales, *J. Reine Angew. Math.* **549**(2002), 149–190.

[15] M. JUNGE, Q. XU, Noncommutative Burkholder/Rosenthal inequalities, *Ann. Probab.* **31**(2003), 948–995.

[16] M. JUNGE, Q. XU, Noncommutative Burkholder/Rosenthal inequalities II: Applications, *Israel J. Math.* **167**(2008), 227–282.

[17] S.G. KREIN, J.I. PETUNIN, E.M. SEMENOV, Interpolation of linear operators, *Transl. Math. Monogr.* vol. 54, Amer. Math. Soc., Providence, RI 1982.

[18] P. KOLWICZ, K.LEŚNIK, L. MALIGRANDA, Pointwise products of some Banach function spaces and factorization, *J. Funct. Anal.* **266**(2014), 616–659.

[19] J. LINDENTRAUSS, L. TZAFRIRI, *Classical Banach Space*. II, Springer-Verlag, Berlin 1979.

[20] G.Y. LOZANOVSKII, On some Banach lattices, *Siberian Math. J.* **10**(1969), 419–430.

[21] M. MUSAT, Interpolation between noncommutative BMO and noncommutative L^p-spaces, *J. Funct. Anal.* **202**(2003), 195–225.

[22] G. PISIER, Q. XU, Non-commutative martingale inequalities, *Comm. Math. Phys.* **189**(1997), 667–698.

[23] G. PISIER, Q. XU, Noncommutative L^p-spaces, in *Handbook of the Geometry of Banach Spaces*, Vol. 2, North-Holland, Amsterdam 2003, pp. 1459–1517.

[24] N. RANDRIANANTOANINA, L. WU, Martingale inequalities in noncommutative symmetric spaces, *J. Funct. Anal.* **269**(2015), 2222–2253.

[25] F. SUKOCHEV, Completeness of quasi-normed symmetric operator spaces, *Indag. Math. (N.S.)* **25**(2014), 376–388.

[26] Q. XU, Analytic functions with values in lattices and symmetric spaces of measurable operators, *Math. Proc. Cambridge Philos. Soc.* **109**(1991), 541–563.

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