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COMPLEX SYMMETRIC GENERATORS FOR OPERATOR ALGEBRAS

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ABSTRACT. In this paper we explore the complex symmetric generator problem for operator algebras, that is, the problem of determining which operator algebras can be generated by a single complex symmetric operator. For type I von Neumann algebras, properly infinite von Neumann algebras and a large class of finite von Neumann algebras, we give a complete answer. The complex symmetric generator problem for a large class of C^*-algebras, including UHF algebras, AF algebras, irrational rotation algebras and reduced free products, is also studied.

KEYWORDS: *Complex symmetric operator, von Neumann algebra, C^*-algebra, anti-automorphism, generator, single generation.*

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1. INTRODUCTION

Throughout this paper, $\mathcal{H}$ will always denote a complex separable Hilbert space. We let $\mathcal{B}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. An operator $T \in \mathcal{B}(\mathcal{H})$ is called a *complex symmetric operator* (for short CSO) if there exists a conjugation J on $\mathcal{H}$ such that $JTJ = T^*$. Recall that a conjugate-linear map J on $\mathcal{H}$ is called a *conjugation* if J is invertible, $J^{-1} = J$ and $\langle Jx, Jy \rangle = \langle y, x \rangle$ for all $x, y \in \mathcal{H}$. Note that an operator T is complex symmetric if and only if T can be represented as a symmetric matrix with complex coefficients relative to an orthonormal basis for $\mathcal{H}$ (see Lemma 2.16 of [11]). Thus complex symmetric operators can be viewed as a generalization of symmetric matrices in infinite dimensional Hilbert spaces.

Although the study of complex symmetric matrices has classical roots in automorphic functions [20], projective geometry [21], quadratic forms [34], symplectic geometry [37] and function theory [40], the general study of complex symmetric operators was initiated by Garcia, Putinar and Wogen in [12], [13], [15], [16]. Many other people have also made contribution to this subject [10], [17],

[18], [19], [47], [48]. The class of CSOs is surprisingly large and includes normal operators, Hankel operators, binormal operators, some integral operators and many other important operators. In particular, CSOs are closely related to the study of truncated Toeplitz operators, which was initiated in Sarason's seminal paper [33] and has received much attention recently. The reader is referred to [11] for more historical comments about CSOs and their connections to other subjects.

In this paper, we are interested in the question of determining when an operator algebra can be generated by a single complex symmetric operator. By an operator algebra we mean a concrete C^*-algebra or a von Neumann algebra. Given an operator $T \in \mathcal{B}(\mathcal{H})$, we let $W^*(T)$ denote the von Neumann algebra generated by T, that is, the smallest von Neumann algebra containing T. Likewise, we let $C^*(T)$ denote the unital C^*-algebra generated by T, that is, the smallest C^*-algebra containing T and the identity operator on $\mathcal{H}$. It is obvious that the structure of $C^*(T)$ and $W^*(T)$ is completely determined by T and, conversely, $C^*(T)$ and $W^*(T)$ also contain much information about T.

In this paper we will concentrate on the following complex symmetric generator problem.

PROBLEM 1.1. Which operator algebras can be singly generated by complex symmetric operators?

The main motivation of this study lies in the connections between CSOs and their algebras. First of all, the study is closely related to that of real structures in C^*-algebras. Let $\mathcal{A}$ be a C^*-algebra. A *real structure* in $\mathcal{A}$ is a real $*$-subalgebra $\mathcal{R}$ such that $\mathcal{R} \cap \mathrm{i}\mathcal{R} = \{0\}$ and $\mathcal{R} + \mathrm{i}\mathcal{R} = \mathcal{A}$. Each real structure corresponds uniquely to an involutory $*$-anti-automorphism of $\mathcal{A}$. Recall that a *$*$-anti-automorphism* (or just anti-automorphism) of a C^*-algebra $\mathcal{A}$ is a vector space isomorphism $\varphi : \mathcal{A} \to \mathcal{A}$ with $\varphi(a^*) = \varphi(a)^*$ and $\varphi(ab) = \varphi(b)\varphi(a)$ for $a, b \in \mathcal{A}$. An anti-automorphism φ is said to be *involutory* if $\varphi^{-1} = \varphi$. If φ is an involutory anti-automorphism of $\mathcal{A}$, then one can check that $\mathcal{R}_\varphi = \{a \in \mathcal{A} : \varphi(a) = a^*\}$ is a real structure of $\mathcal{A}$. Conversely, each real structure in $\mathcal{A}$ is of this form.

A C^*-algebra may not possess any involutory anti-automorphism on it. Connes [4] constructed type III factors, which do not admit any anti-automorphism on them. Jones [22] constructed a II_1 factor which is anti-isomorphic to itself but has no involutory anti-automorphisms. The reader is referred to [30], [31] for more examples.

For an operator algebra singly generated by a CSO, there exists at least one involutory anti-automorphism on it.

LEMMA 1.2. *Let $T \in \mathcal{B}(\mathcal{H})$ and J be a conjugation on $\mathcal{H}$ satisfying $JTJ = T^*$. For $X \in W^*(T)$, define $\varphi(X) = JX^*J$. Then $\varphi(T) = T$, φ is an involutory anti-automorphism of $W^*(T)$, and $\varphi|_{C^*(T)}$ is an involutory anti-automorphism of $C^*(T)$.*

The proof is a straightforward verification.

This was first observed in [18] and then a C^*-algebra approach has been developed to answer a number of open questions concerning CSOs, including the norm closure problem for CSOs [47]. A better understanding of CSOs was obtained in papers [18], [19], [47] by considering their C^*-algebras. This motivates us to study the operator algebras which are singly generated by CSOs.

Størmer [38] proved that involutory anti-automorphisms of von Neumann algebras are almost, and in many cases exactly, of the form $X \mapsto JX^*J$, where J is a conjugation. In this paper we shall often deal with such anti-automorphisms. The first result of this paper concerns the hereditary property of real structures in a von Neumann algebra. More precisely, we shall study when a real structure in a von Neumann algebra $\mathcal{M}$ can induce real structures in the compression algebras of $\mathcal{M}$. When $\mathcal{M}$ is a properly infinite von Neumann algebra or lies in a large class of II_1 factors, we shall prove that $\mathcal{M}$ admits an involutory anti-automorphism which is induced by a conjugation if and only if $\mathcal{M}$ has a complex symmetric generator (see Theorem 3.14 and Theorem 3.24).

A fundamental question about CSOs is how to develop a model theory [9], [14]. A natural thought is to decompose CSOs into "simple blocks" and then represent them in concrete terms. So what are the "simple blocks"? Recently Guo and the second author [19] gave a decomposition theorem to describe the block structure of general CSOs. Several possible candidates for simple blocks of CSOs were posed.

THEOREM 1.3 ([19]). *An operator $T \in \mathcal{B}(\mathcal{H})$ is complex symmetric if and only if T is unitarily equivalent to a direct sum of (some of the summands may be absent)*

(i) *completely reducible* CSO*s,*

(ii) *irreducible* CSO*s, and*

(iii) *operators of the form $A \oplus JA^*J$, where J is a conjugation, A is irreducible and not complex symmetric.*

Recall that an operator is said to be *completely reducible* if it does not admit any minimal reducing subspace ([8]). If $\mathcal{K}$ is a nonzero reducing subspace of $T \in \mathcal{B}(\mathcal{H})$ and $T|_{\mathcal{K}}$ is irreducible, then $\mathcal{K}$ is called a *minimal reducing subspace* of T. The preceding result provides candidates for simple blocks of CSOs. If T is an irreducible CSO, or has the form as stated in (iii) of Theorem 1.3, then T admits no nontrivial reducing subspace $\mathcal{K}$ so that $T|_{\mathcal{K}}$ is complex symmetric. This motivates a new definition. We say that an operator R is *minimal complex symmetric*, if R is complex symmetric and there exists no nontrivial reducing subspace $\mathcal{K}$ so that $R|_{\mathcal{K}}$ is complex symmetric. Minimal complex symmetric operators are suitable candidates for simple blocks of CSOs, and the statements (ii) and (iii) in Theorem 1.3 provide two kinds of such operator. As for completely reducible CSOs, things become very complicated.

We say that a completely reducible operator T is *completely complex symmetric* if the restrictions of T to its reducing subspaces are always complex symmetric. Note that each normal operator without eigenvalues is completely complex symmetric. There exist non-normal examples of completely complex symmetric operators ([19], Example 4.1). Also there exist completely reducible CSOs which are not completely complex symmetric ([19], Example 4.2).

Note that each operator in a continuous von Neumann algebra is completely reducible. This motivates us to pay attention to CSOs in von Neumann algebras, especially in those continuous ones. By considering complex symmetric generators for von Neumann algebras, we wish to find more concrete examples of completely reducible CSOs. Also we hope that this study will eventually lead to a structure theory of completely reducible CSOs.

Clearly, the study of Problem 1.1 is closely related to the generator problem for operator algebras. The generator problem for von Neumann algebras, which asks if every von Neumann algebra on a separable Hilbert space can be singly generated, was posed by Kadison [23] in 1967. The problem has been solved except for the case of non-hyperfinite type II_1 von Neumann algebras. Note that the generator problem was reduced to the case of a finitely generated II_1-factor by Sherman ([36], Theorem 3.8). The reader is referred to Chapter 7 of [5] for standard notions in von Neumann algebras.

Obviously, we need only consider Problem 1.1 among those singly generated von Neumann algebras. It is known that all separable type I von Neumann algebras, separable properly infinite von Neumann algebras and many separable II_1 factors are singly generated. For a II_1 factor $\mathcal{M}$, an invariant $\mathcal{G}(\mathcal{M})$ was introduced in [35]. It was shown in [35] that if $\mathcal{M}$ lies in a large class of II_1 factors, including hyperfinite II_1 factors, II_1 factors with Cartan subalgebra, non-prime factors, and II_1 factors with property Γ and some II_1 factors with property T, then $\mathcal{G}(\mathcal{M}) = 0$. It was also shown in [35] that if $\mathcal{G}(\mathcal{M}) = 0$, then $\mathcal{M}$ is singly generated. We shall prove that each separable type I von Neumann algebra can be generated by a single complex symmetric operator in Theorem 3.1. If $\mathcal{M}$ is a properly infinite von Neumann algebra or a II_1 factor with $\mathcal{G}(\mathcal{M}) = 0$, then $\mathcal{M}$ has a complex symmetric generator if and only if $\mathcal{M}$ admits an anti-automorphism induced by conjugations on it (see Theorem 3.14 and Theorem 3.23). Thus Problem 1.1 is solved for many von Neumann algebras.

On the other hand, people are also interested in the generator problem for C^*-algebras. A number of authors have made significant contribution to the study [25], [26], [28], [42], [43]. The reader is referred to [42] for more historical comments.

Clearly, it does not make sense to consider whether or not an abstract C^*-algebra has a complex symmetric generator, since the notion of complex symmetric operator is defined in terms of conjugations on Hilbert spaces. Thus, for general C^*-algebras, we will consider the following modified version of complex symmetric generator problem.

PROBLEM 1.4. Which C^*-algebras can be faithfully represented as C^*-algebras with a complex symmetric generator?

The above problem involves the study of involutory anti-automorphisms of C^*-algebras and their fixed points. We shall prove in Section 4 the following result, which reduces the above problem to the study of special generators of C^*-algebras.

THEOREM 1.5. *A unital C^*-algebra $\mathcal{A}$ is $*$-isomorphic to some concrete C^*-algebra with a complex symmetric generator if and only if there exist an involutory anti-automorphism φ of $\mathcal{A}$ and an element $T \in \mathcal{A}$ such that $\varphi(T) = T$ and $C^*(T) = \mathcal{A}$.*

By Theorem 1.5, we need only consider Problem 1.4 among those singly generated C^*-algebras which admit involutory anti-automorphisms. For AF algebras, irrational rotation algebras, reduced free products and those tensor products of separable unital C^*-algebras with UHF algebras, we obtain positive answers. However, the complex symmetric generator problem for general singly generated C^*-algebras is difficult. Given a singly generated C^*-algebra $\mathcal{A}$ and an involutory anti-automorphism φ on it, it is possible that $\mathcal{A}$ contains no any single generator which is invariant under φ, or even $\mathcal{A}$ can not be generated by a fixed point of any anti-automorphism (see Example 4.1, Example 4.2 and Example 4.3).

The rest of this paper is organized as follows.

In Section 2, we study the hereditary property of real structures in von Neumann algebras. In particular, if $\mathcal{M}$ is a II_1 factor, then the existence of a real structure in $\mathcal{M}$ is equivalent to the existence of a real structure in its compression algebras (see Theorem 2.7).

In Section 3, we consider the complex symmetric generator problem for von Neumann algebras. We solve the problem for separable type I von Neumann algebras and separable properly infinite von Neumann algebras. And then, by a reduction result (Theorem 3.9), the complex symmetric generator problem is reduced to the case of type II_1 von Neumann algebras. For those type II_1 factors $\mathcal{M}$ with $\mathcal{G}(\mathcal{M}) = 0$, we solve the complex symmetric generator problem (Theorem 3.24). As applications, several illustrating examples are provided.

In Section 4, we shall consider the complex symmetric generator problem for several classes of C^*-algebras, including UHF algebras, AF algebras, irrational rotation algebras and reduced free products.

2. INVOLUTORY ANTI-AUTOMORPHISMS OF VON NEUMANN ALGEBRAS

In this section, we shall study when a real structure in a von Neumann algebra $\mathcal{M}$ can induce real structures in the compression algebras of $\mathcal{M}$.

Let $\mathcal{A}$ be an operator algebra acting on $\mathcal{H}$. We let $\mathcal{A}'$ denote the commutant of $\mathcal{A}$ in $\mathcal{B}(\mathcal{H})$. For $1 \leqslant n < \infty$, $M_n(\mathcal{A})$ denotes the algebra of $n \times n$ matrices over $\mathcal{A}$ which act on $\bigoplus\limits_{k=1}^{n} \mathcal{H}$.

We first have the following result.

THEOREM 2.1. *Let $\mathcal{M}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$ and P a nonzero projection in $\mathcal{M}$. If there exists an involutory anti-automorphism ψ of $\mathcal{M}$ leaving the center elements of $\mathcal{M}$ fixed, then there exists an involutory anti-automorphism of $P\mathcal{M}P$ leaving the center elements of $P\mathcal{M}P$ fixed.*

Proof. Since there exists an involutory anti-automorphism of $\mathcal{M}$ leaving the center elementwise fixed, by Theorem 3.12 in [38], there exists a conjugation J on $\mathcal{H}$ such that $J\mathcal{M}J = \mathcal{M}$ and $JAJ = A^*$ for A in the center of $\mathcal{M}$.

From Lemma 3.3 in [38], we know that

$$P \sim JPJ \quad \text{and} \quad (I-P) \sim (I-JPJ),$$

where $\sim$ denotes equivalence of projections. It follows that there exists a unitary element U in $\mathcal{M}$ such that $UJPJU^* = P$. Or, $UJP = PUJ$ and $PJU^* = JU^*P$.

CLAIM 2.2. UJP is a conjugate-linear isometry from $P\mathcal{H}$ onto $P\mathcal{H}$ satisfying

$$(UJP)(PJU^*) = P \quad \text{and} \quad (PJU^*)(UJP) = P.$$

The result follows from the fact that UJ is a conjugate-linear isometry from $\mathcal{H}$ onto $\mathcal{H}$ and UJ commutes with P.

Moreover, the following claim is also true.

CLAIM 2.3. (i) $(UJP)(P\mathcal{M}P)(PJU^*) = P\mathcal{M}P$.
(ii) $(UJP)^2$ is in $P\mathcal{M}P$.

Note that $UJP = PUJ$ and $JU^*P = PJU^*$. We have

$$UJP\mathcal{M}PJU^* = PUJ\mathcal{M}JU^*P \subseteq P\mathcal{M}P \quad \text{and} \quad JU^*P\mathcal{M}PUJ = PJU^*\mathcal{M}UJP \subseteq P\mathcal{M}P.$$

Hence $UJP\mathcal{M}PJU^* = P\mathcal{M}P$. This finishes the proof of (i).

Note that $J\mathcal{M}J = \mathcal{M}$ and $U \in \mathcal{M}$. Then $(UJ)^2 = U(JUJ) \in \mathcal{M}$. Therefore, we have the following that ends the proof of (ii) and the proof of whole claim:

$$(UJP)^2 = (UJP)(UJP) = PU(JUJ)P \in P\mathcal{M}P.$$

CLAIM 2.4. Let $\mathcal{Z}(P\mathcal{M}P)$ be the center of $P\mathcal{M}P$. Then

$$(UJP)Z(PJU^*) = Z^*, \quad \forall\, Z \in \mathcal{Z}(P\mathcal{M}P).$$

Denote by $\mathcal{Z}(\mathcal{M})$ the center of $\mathcal{M}$. Then for any $Z \in \mathcal{Z}(P\mathcal{M}P)$, there exists a Z_1 in $\mathcal{Z}(\mathcal{M})$ such that $Z = PZ_1 = Z_1P$ (see Theorem 5.5.6 in [24]). Thus we have the following that ends the proof of the claim:

$$\begin{aligned}(UJP)Z(PJU^*) &= (UJP)PZ_1(PJU^*) = (UJP)Z_1(PJU^*) = (PUJ)Z_1(JU^*P)\\ &= PUZ_1^*U^*P \quad (\text{because } JZ_1J = Z_1^*)\\ &= PZ_1^*P = Z^* \quad (\text{because } UZ_1^* = Z_1^*U).\end{aligned}$$

We continue the proof of the theorem. For $X \in P\mathcal{M}P$, define $\varphi(X) = UJPX^*PJU^*$. Then, from the preceding three claims, we know that φ is an *inner* anti-automorphism of $P\mathcal{M}P$ leaving the center elements of $P\mathcal{M}P$ fixed. It follows from Theorem 4.5 in [38] that there exists a conjugation J_1 on $P\mathcal{H}$ such that $J_1(P\mathcal{M}P)J_1 = (P\mathcal{M}P)$ and $J_1 Z J_1 = Z^*$, where Z is in the center of $P\mathcal{M}P$, i.e. there exists an involutory anti-automorphism of $P\mathcal{M}P$ leaving the center elements of $P\mathcal{M}P$ fixed. ∎

Now we give an example to show that the condition "leaving the center elements of $\mathcal{M}$ fixed" in Theorem 2.1 can not be canceled.

EXAMPLE 2.5. Assume that $\mathcal{M}$ is a von Neumann algebra on $\mathcal{H}$ which admits no involutory anti-automorphism on it. In view of [4] and [22], such a von Neumann algebra must exist. Choose a conjugation J on $\mathcal{H}$. Then $J\mathcal{M}J$ is a von Neumann algebra. Set $\mathcal{N} = \mathcal{M} \oplus J\mathcal{M}J$. Then $\mathcal{N}$ is a von Neumann algebra on $\mathcal{H} \oplus \mathcal{H}$ and one can check that the map φ defined by

$$(X, Y) \longmapsto (JY^*J, JX^*J)$$

is an involutory anti-automorphism of $\mathcal{N}$. However, $\mathcal{M}$ is a compression algebra of $\mathcal{N}$ and admits no involutory anti-automorphism on it. Obviously, φ does not leave the center elements of $\mathcal{N}$ fixed, since $(I, 0)$ lies in the center of $\mathcal{N}$, where I is the identity on $\mathcal{H}$, and $\varphi((I, 0)) = (0, I) \neq (I, 0)$.

THEOREM 2.6. *Let $\mathcal{N}$ be a von Neumann algebra and n a positive integer. Let $\mathcal{M}$ be a tensor product $\mathcal{N} \otimes M_n(\mathbb{C})$ of $\mathcal{N}$ and $M_n(\mathbb{C})$. Then the following two statements are equivalent:*

(i) *There exists an involutory anti-automorphism of $\mathcal{M}$ leaving the center elements of $\mathcal{M}$ fixed.*

(ii) *There exists an involutory anti-automorphism of $\mathcal{N}$ leaving the center elements of $\mathcal{N}$ fixed.*

Proof. In view of Theorem 2.1, the implication (i) ⇒ (ii) is clear.

(ii) ⇒ (i) Assume that ψ is an involutory anti-automorphism of $\mathcal{N}$. We define a mapping $\Psi : \mathcal{M} \to \mathcal{M}$ by

$$\forall\, X = (x_{ij}), \quad \Psi(X) = (y_{ij}) \quad \text{where } y_{ij} = \psi(x_{ji}) \text{ for all } 1 \leqslant i, j \leqslant n.$$

It can be verified that Ψ is an involutory anti-automorphism on $\mathcal{M}$. ∎

Note that if $\mathcal{M}$ is a factor, then each anti-automorphism of $\mathcal{M}$ leaves the center elements in $\mathcal{M}$ fixed. Given two C^*-algebras $\mathcal{A}_1$ and $\mathcal{A}_2$, we write $\mathcal{A}_1 \simeq \mathcal{A}_2$ to denote that $\mathcal{A}_1, \mathcal{A}_2$ are $*$-isomorphic.

THEOREM 2.7. *Let $\mathcal{M}$ be a II_1 factor and P a nonzero projection in $\mathcal{M}$. Then the following two statements are equivalent:*

(i) *There exists an involutory anti-automorphism of $\mathcal{M}$.*

(ii) *There exists an involutory anti-automorphism of $P\mathcal{M}P$.*

Proof. (i) ⇒ (ii) It follows directly from Theorem 2.1.

(ii) ⇒ (i) Let τ be a tracial state of $\mathcal{M}$. Assume that there exists an involutory anti-automorphism of $P\mathcal{M}P$. Note that P is a nonzero projection of $\mathcal{M}$. There exists a positive integer n and a sub-projection Q of P such that $\tau(Q) = 1/n$.

From Theorem 2.1, we know that there exists an involutory anti-automorphism of $Q\mathcal{M}Q$ $(= QP\mathcal{M}PQ)$. Combining with the fact that $\mathcal{M} \simeq Q\mathcal{M}Q \otimes M_n(\mathbb{C})$, we conclude that there exists an involutory anti-automorphism of $\mathcal{M}$. This finishes the proof of the whole theorem. ∎

3. COMPLEX SYMMETRIC GENERATOR PROBLEM FOR VON NEUMANN ALGEBRAS

In this section we consider the complex symmetric generator problem for von Neumann algebras. We solve the problem for type I von Neumann algebras, properly infinite von Neumann algebras and a large class of II_1 factors.

3.1. TYPE I VON NEUMANN ALGEBRAS.

The main result of this subsection is the following theorem.

THEOREM 3.1. *Each von Neumann algebra of type* I *acting on a separable Hilbert space has a complex symmetric generator.*

To give the proof of Theorem 3.1, we need to give several auxiliary results.

First we provide some irreducible CSOs on a separable Hilbert space $\mathcal{H}$. For $e, f \in \mathcal{H}$, we define $f \otimes e \in \mathcal{B}(\mathcal{H})$ by $(f \otimes e)(x) = \langle x, e\rangle f$, $\forall x \in \mathcal{H}$.

EXAMPLE 3.2. When $\dim \mathcal{H} = \infty$, we can choose an operator T on $\mathcal{H}$ which is unitarily equivalent to the classical Volterra integration operator on $L^2([0,1])$ defined by

$$(Vf)(t) = \int_0^t f(s)\mathrm{d}s, \quad \forall\, t \in [0,1].$$

Define a conjugation J_1 on $L^2([0,1])$ as $(J_1 f)(t) = \overline{f(1-t)}$, $\forall t \in [0,1]$. Then one can check that $J_1 V J_1 = V^*$. Thus T, unitarily equivalent to V, is an irreducible CSO. The reader is referred to Theorem 3.14 of [18] for more examples of irreducible CSOs on infinite dimensional Hilbert spaces.

When $\dim \mathcal{H} = 1$, each operator on $\mathcal{H}$ is irreducible and complex symmetric. In the remaining we assume that $1 < \dim \mathcal{H} = n < \infty$. Choose an orthonormal basis $\{e_i\}_{i=1}^n$ of $\mathcal{H}$. Define

$$T = \sum_{i=2}^n e_{i-1} \otimes e_i.$$

Then T is irreducible. For $x \in \mathcal{H}$ with $x = \sum\limits_{i=1}^n \alpha_i e_i$, define $J_2 x = \sum\limits_{i=1}^n \overline{\alpha}_i e_{n-i+1}$. Then one can check that J_2 is a conjugation on $\mathcal{H}$ and $J_2 T J_2 = T^*$. Then T is an irreducible CSO.

Given a conjugation J on $\mathcal{H}$, if $T \in \mathcal{B}(\mathcal{H})$ satisfies $JTJ = T^*$, then we say that T is J-symmetric. Given $A_1, A_2, \ldots, A_n \in \mathcal{B}(\mathcal{H})$, we write $W^*(A_1, A_2, \ldots, A_n)$ for the smallest von Neumann algebra containing $\{A_i : 1 \leqslant i \leqslant n\}$.

LEMMA 3.3. *Let J be a conjugation on $\mathcal{H}$ and $T \in \mathcal{B}(\mathcal{H})$ be J-symmetric. If T is normal, then each operator in $W^*(T)$ is J-symmetric.*

Proof. For $m, n \geqslant 0$, since T is normal and $JTJ = T^*$, we have

$$J(T^{*m}T^n)J = (JT^*J)^m(JTJ)^n = T^mT^{*n} = T^{*n}T^m = (T^{*m}T^n)^*.$$

Then it follows immediately that $Jp(T^*, T)J = p(T^*, T)^*$ for any polynomial $p(\cdot, \cdot)$ in two free variables. Thus $JXJ = X^*$ for any $X \in W^*(T)$. ∎

REMARK 3.4. For general complex symmetric operators, Lemma 3.3 fails to hold. For example, when T is an irreducible CSO on $\mathcal{H}$ with $\dim \mathcal{H} \geqslant 3$, then $W^*(T) = \mathcal{B}(\mathcal{H})$ contains operators that are not complex symmetric.

LEMMA 3.5. *Let J be a conjugation on $\mathcal{H}$ and $A, B \in \mathcal{B}(\mathcal{H})$ be J-symmetric. If $AB = BA$ and $A = A^*$, then there exists J-symmetric R such that $W^*(R) = W^*(A, B)$.*

Proof. Assume that $B = B_1 + \mathrm{i}B_2$, where $B_1, B_2 \in \mathcal{B}(\mathcal{H})$ are both self-adjoint. Since $AB = BA$, it follows that $AB_i = B_iA$, $i = 1, 2$. Set $N_i = A + \mathrm{i}B_i$, $i = 1, 2$. Then each N_i is normal and J-symmetric. For $1 \leqslant i \leqslant 2$, by a classical result of von Neumann [27], there exists self-adjoint $R_i \in \mathcal{B}(\mathcal{H})$ such that $W^*(R_i) = W^*(N_i)$. In view of Lemma 3.3, we deduce that R_1, R_2 are both J-symmetric. So the operator $R = R_1 + \mathrm{i}R_2$ is J-symmetric and one can check the following that ends the proof:

$$W^*(R) = W^*(R_1, R_2) = W^*(N_1, N_2) = W^*(A, B_1, B_2) = W^*(A, B). \quad ∎$$

The following result, whose proof is omitted, first appeared in [12].

LEMMA 3.6 ([12], Lemma 6). *Let $A \in \mathcal{B}(\mathcal{H}_1)$ and $B \in \mathcal{B}(\mathcal{H}_2)$. If A is J_1-symmetric and B is J_2-symmetric, where J_i is a conjugation on $\mathcal{H}_i$, $i = 1, 2$, then $J_1 \otimes J_2$ is a conjugation and $A \otimes B$ is $J_1 \otimes J_2$-symmetric.*

Recall that a von Neumann algebra $\mathcal{M}$ is said to be *homogeneous* if there exists in $\mathcal{M}$ a family of pairwise orthogonal abelian projections that are mutually equivalent and whose sum is the identity. Note that each type I von Neumann algebra is the direct sum of a family of homogeneous von Neumann algebras. Given a family $\{\mathcal{A}_i\}_{i\in\Lambda}$ of von Neumann algebras, we denote by $\bigoplus_{i\in\Lambda} \mathcal{A}_i$ their direct sum, that is,

$$\bigoplus_{i\in\Lambda} \mathcal{A}_i = \Big\{ \bigoplus_{i\in\Lambda} A_i : A_i \in \mathcal{A}_i \text{ and } \sup_i \|A_i\| < \infty \Big\}.$$

LEMMA 3.7. *Each homogeneous von Neumann algebra on a separable Hilbert space has a complex symmetric generator.*

Proof. Let $\mathcal{A}$ be a homogeneous von Neumann algebra on a separable Hilbert space. Then, by Proposition 50.15 of [5], there exist separable Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ and an abelian von Neumann algebra $\mathcal{M}$ on $\mathcal{H}_1$ such that $\mathcal{A}$ is spatially isomorphic to $\mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2)$, where $1 \leqslant \dim \mathcal{H}_i \leqslant \aleph_0$, $i = 1, 2$. Since $\mathcal{M}$ is abelian, we can choose a self-adjoint generator A of $\mathcal{M}$. Assume that J_1 is a conjugation on $\mathcal{H}_1$ such that $J_1 A = A J_1$. By Example 3.2, we can choose a conjugation J_2 on $\mathcal{H}_2$ and an irreducible $B \in \mathcal{B}(\mathcal{H}_2)$ which is J_2-symmetric. It is obvious that $W^*(B) = \mathcal{B}(\mathcal{H}_2)$. Set $R = A \otimes I_{\mathcal{H}_2}$ and $S = I_{\mathcal{H}_1} \otimes B$. Then $R, S \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, R is self-adjoint and $RS = SR$; in addition, it follows from Lemma 3.6 that R, S are $J_1 \otimes J_2$-symmetric. In view of Lemma 3.5, there is a $J_1 \otimes J_2$-symmetric operator $T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ such that $W^*(T) = W^*(R, S)$. Noting that $W^*(R, S) = \mathcal{M} \otimes \mathcal{B}(\mathcal{H}_2)$, we complete the proof. ∎

LEMMA 3.8. *Let $\{\mathcal{A}_n\}_{n=1}^{\infty}$ be von Neumann algebras on separable Hilbert spaces and set $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$. If each $\mathcal{A}_n$ has a complex symmetric generator for $n \geqslant 1$, then so does $\mathcal{A}$.*

Proof. Without loss of generality, we may assume that $\mathcal{A}_n$ acts on $\mathcal{H}_n$ and $A_n \in \mathcal{B}(\mathcal{H}_n)$ is a complex symmetric generator of $\mathcal{A}_n$ for $n \geqslant 1$. Moreover, it can be required that $\sup_n \|A_n\| < \infty$. Assume that J_n is a conjugation on $\mathcal{H}_n$ such that A_n is J_n-symmetric for $n \geqslant 1$. Set

$$A = \bigoplus_{n=1}^{\infty} A_n, \quad B = \bigoplus_{n=1}^{\infty} \frac{I_n}{n}, \quad J = \bigoplus_{n=1}^{\infty} J_n,$$

where I_n is the identity operator on $\mathcal{H}_n$, $n \geqslant 1$. It is obvious that $A, B \in \mathcal{A}$, $B = B^*$, $AB = BA$ and J is a conjugation on $\bigoplus_{n \geqslant 1} \mathcal{H}_n$. Moreover, one can check that A, B are both J-symmetric. By Lemma 3.5, there exists J-symmetric R such that $W^*(R) = W^*(A, B)$.

One can easily check that $W^*(B) = \bigoplus_{n=1}^{\infty} \mathbb{C} I_n$ and $W^*(A_n) = \mathcal{A}_n$ for $n \geqslant 1$. Thus $W^*(A, B) = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$. Hence $W^*(R) = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$. This ends the proof. ∎

Now the proof of Theorem 3.1 is a direct consequence of previous lemmas.

Proof of Theorem 3.1. Let $\mathcal{A}$ be a von Neumann algebra of type I on $\mathcal{H}$. By Theorem 50.19 of [5], each von Neumann algebra of type I is the direct sum of certain homogenous von Neumann algebra. Without loss of generality, we may assume that $\mathcal{A} = \bigoplus_{n=1}^{\infty} \mathcal{A}_n$, where $\mathcal{A}_n$ is a homogenous von Neumann algebra for $n \geqslant 1$. By Lemma 3.7, each $\mathcal{A}_n$ has a complex symmetric generator. By Lemma 3.8, $\mathcal{A}$ has a complex symmetric generator. ∎

3.2. A REDUCTION RESULT. By the type decomposition theorem of von Neumann algebras, each von Neumann algebra is the direct sum of several special von Neumann algebras with various types. This simplifies the study of von Neumann algebras. Inspired by this result, we wish to reduce the complex symmetric generator problem for von Neumann algebras to several special cases.

The main result of this subsection is the following theorem.

THEOREM 3.9. *Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{M} = \bigoplus_{i=1}^{4} \mathcal{M}_i$, where $\mathcal{M}_1$ is of type* I, *$\mathcal{M}_2$ is of type* II_1, *$\mathcal{M}_3$ is of type* II_∞ *and $\mathcal{M}_4$ is of type* III. *Then $\mathcal{M}$ has a complex symmetric generator if and only if each $\mathcal{M}_i$ has a complex symmetric generator for $1 \leqslant i \leqslant 4$.*

Proof. In view of Lemma 3.8, the sufficiency is obvious.

$\Rightarrow$ Assume that $\mathcal{M}$ acts on $\mathcal{H}$ and $\mathcal{M}_i$ acts on $\mathcal{H}_i$ for $1 \leqslant i \leqslant 4$. Then $\mathcal{H} = \bigoplus_{i=1}^{4} \mathcal{H}_i$. Given $A \in \mathcal{M}$, we write C_A for the central cover of A. Denote

$$
\begin{aligned}
&P_1 = \sup\{C_E : E \text{ is an abelian projection}\},\\
&P_2 = \sup\{C : C \text{ is a finite central projection and } CP_1 = 0\},\\
&P_3 = \sup\{C_E : E \text{ is a finite projection and } E(P_1 + P_2) = 0\} \quad \text{and}\\
&P_4 = I - \sum_{i=1}^{3} P_i.
\end{aligned}
$$

Then one can see that $\mathcal{H}_i = \operatorname{ran} P_i$ for $1 \leqslant i \leqslant 4$.

Assume that $T \in \mathcal{B}(\mathcal{H})$ is a complex symmetric generator of $\mathcal{M}$ and J is a conjugation on $\mathcal{H}$ satisfying $JTJ = T^*$. Note that each $\mathcal{H}_i$ reduces T. Denote $T_i = T|_{\mathcal{H}_i}$ for $1 \leqslant i \leqslant 4$. Thus $T_i \in \mathcal{M}_i$ and $W^*(T_i) = \mathcal{M}_i$ for $1 \leqslant i \leqslant 4$. We shall show that each T_i is complex symmetric. Obviously, it suffices to prove that each $\mathcal{H}_i$ reduces J for $1 \leqslant i \leqslant 4$.

For $X \in \mathcal{M}$, define $\varphi(X) = JX^*J$. Note that the map φ is an involutory anti-automorphism of $\mathcal{M}$.

CLAIM 3.10. $JC_EJ = C_{JEJ}$ for each projection $E \in \mathcal{M}$.

By definitions, we have

$$
\begin{aligned}
\operatorname{ran} C_{JEJ} &= \bigvee\{Ah : h \in \operatorname{ran} JEJ, A \in \mathcal{M}\} = \bigvee\{JAJh : h \in \operatorname{ran} JEJ, A \in \mathcal{M}\}\\
&= \bigvee\{JAh : h \in \operatorname{ran} E, A \in \mathcal{M}\} = J\Big(\bigvee\{Ah : h \in \operatorname{ran} E, A \in \mathcal{M}\}\Big)\\
&= J(\operatorname{ran} C_E) = \operatorname{ran} JC_E = \operatorname{ran} JC_EJ,
\end{aligned}
$$

where $\bigvee$ denotes the closed linear span. Thus we obtain $C_{JEJ} = JC_EJ$. This proves Claim 3.10.

The preceding claim shows that φ maps central projections to central projections. Also, one can check that φ maps abelian projections to abelian projections, and finite projections to finite projections.

If E is an abelian projection, then JEJ is abelian and, by Claim 3.10, we have $JC_EJ = C_{JEJ}$. It follows that $J(\text{ran } C_E) \subset \text{ran } P_1$. Moreover we have $J(\mathcal{H}_1) \subset \mathcal{H}_1$. Since J is a conjugation, we obtain $J(\mathcal{H}_1) = \mathcal{H}_1$ and $JP_1 = P_1J$.

If C is a finite central projection and $CP_1 = 0$, then JCJ is a finite central projection and $JCJP_1 = JCP_1J = 0$. Then

$$J(\text{ran } C) = \text{ran } JC = \text{ran } JCJ \subset \text{ran } P_2.$$

This means that $J(\mathcal{H}_2) \subset \mathcal{H}_2$. Since J is a conjugation, we obtain $J(\mathcal{H}_2) = \mathcal{H}_2$ and $JP_2 = P_2J$.

If E is a finite projection and $E(P_1 + P_2) = 0$, then JEJ is finite and $JEJ(P_1 + P_2) = JE(P_1 + P_2)J = 0$. Thus

$$J(\text{ran } C_E) = \text{ran } JC_E = \text{ran } JC_EJ = \text{ran } C_{JEJ} \subset \text{ran } P_3.$$

It follows that $J(\text{ran } P_3) \subset \text{ran } P_3$, that is, $J(\mathcal{H}_3) \subset \mathcal{H}_3$. Since J is a conjugation, we obtain $J(\mathcal{H}_3) = \mathcal{H}_3$ and $JP_3 = P_3J$.

According to the preceding discussion, we deduce that $JP_4 = P_4J$. Then $\mathcal{H}_i = \text{ran } P_i$ reduces J and $J_i := J|_{\mathcal{H}_i}$ is a conjugation on $\mathcal{H}_i$ for $1 \leqslant i \leqslant 4$. Noting that $J_iT_iJ_i = T_i^*$, we complete the proof. ∎

The following result shows that the complex symmetric generator problem for von Neumann algebras is independent of the choice of Hilbert spaces on which von Neumann algebras act.

LEMMA 3.11. *Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$ satisfying:*

(i) *there exists an involutory anti-automorphism ψ of $\mathcal{M}$ fixing the center elements of $\mathcal{M}$, and*

(ii) *there also exists an element T in $\mathcal{M}$ that generates $\mathcal{M}$ and satisfies $\psi(T) = T$.*

If $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{K})$ is a faithful normal $*$-representation of $\mathcal{M}$ on a separable Hilbert space $\mathcal{K}$, then $\pi(\mathcal{M})$ admits a complex symmetric generator.

Proof. Since $\pi : \mathcal{M} \to \mathcal{B}(\mathcal{K})$ is a faithful normal $*$-representation of $\mathcal{M}$ on a separable Hilbert space $\mathcal{K}$, $\pi(\mathcal{M})$ is also a von Neumann algebra. Thus there exist central projections P_1, P_2 and P_3 in $\pi(\mathcal{M})$ such that

$$\pi(\mathcal{M}) = P_1\pi(\mathcal{M}) \oplus P_2\pi(\mathcal{M}) \oplus P_3\pi(\mathcal{M}),$$

where $P_1\pi(\mathcal{M}), P_2\pi(\mathcal{M})$ and $P_3\pi(\mathcal{M})$ are von Neumann algebras of type I or zero, type II or zero, and type III or zero, respectively. We let $Q = P_2 \oplus P_3$.

From Theorem 3.1, there exist a conjugation J_1 on $P_1\mathcal{K}$ and an element S_1 in $P_1\pi(\mathcal{M})$ such that $P_1\pi(\mathcal{M}) = W^*(S_1)$ and $J_1S_1^*J_1 = S_1$.

Since ψ fixes the center elements of $\mathcal{M}$ and π is a faithful normal representation, we conclude that

$$\pi \circ \psi \circ \pi^{-1} : Q\pi(\mathcal{M}) \to Q\pi(\mathcal{M})$$

is an involutory anti-automorphism ψ fixing the center elements of $Q\pi(\mathcal{M})$. From Theorem 4 in [39], there exists a conjugation J_2 on $Q\mathcal{K}$ such that

$$\pi \circ \psi \circ \pi^{-1}(A) = J_2 A^* J_2, \quad \forall\, A \in Q\pi(\mathcal{M}).$$

In particular, $\pi \circ \psi \circ \pi^{-1}(Q\pi(T)) = J_2(Q\pi(T))^* J_2$. Or

$$Q\pi(T) = Q\pi(\psi(T)) = J_2(Q\pi(T))^* J_2.$$

And we can quickly verify that $W^*(Q\pi(T)) = Q\pi(\mathcal{M})$.

Without loss of generality, we can further assume that S_1 and $Q\pi(T)$ are invertible operators in $\mathcal{B}(P_1\mathcal{K})$, and in $\mathcal{B}(Q\mathcal{K})$ respectively, satisfying

$$\sigma(S_1^* S_1) \cap \sigma((Q\pi(T))^*(Q\pi(T)) = \emptyset,$$

where $\sigma(\cdot)$ denotes the spectrum of an operator. Let

$$S = S_1 \oplus (Q\pi(T)) \quad \text{and} \quad J = J_1 \oplus J_2.$$

Then $\pi(\mathcal{M}) = W^*(S)$, J is a conjugation on $\mathcal{K}$ and $S = JS^*J$. ∎

COROLLARY 3.12. *Let $\mathcal{M}$ be a factor acting on a separable Hilbert space $\mathcal{H}$. Then $\mathcal{M}$ admits a complex symmetric generator if and only if*

(i) *there exists an involutory anti-automorphism ψ of $\mathcal{M}$, and*

(ii) *there also exists an element T in $\mathcal{M}$ that generates $\mathcal{M}$ and satisfies $\psi(T) = T$.*

Proof. The necessity follows from Lemma 1.2. Since $\mathcal{M}$ is a factor, each anti-automorphism of $\mathcal{M}$ leaves the center elements in $\mathcal{M}$ fixed. Thus the sufficiency is a direct consequence of Lemma 3.11. ∎

COROLLARY 3.13. *Let $\mathcal{M}$ be a factor acting on a separable Hilbert space $\mathcal{H}$, and $\mathcal{N}$ be a von Neumann algebra $*$-isomorphic to $\mathcal{M}$. If $\mathcal{M}$ has a single complex symmetric generator, then $\mathcal{N}$ also has a single complex symmetric generator.*

Proof. Note that every $*$-isomorphism between von Neumann algebras is normal. Then the desired result is clear from Lemma 3.11. ∎

3.3. PROPERLY INFINITE VON NEUMANN ALGEBRAS. Recall that a von Neumann algebra is said to be *properly infinite* if it contains no nonzero central projections that are finite. The main result of this subsection is the following theorem, which solves the complex symmetric generator problem for properly infinite von Neumann algebras.

THEOREM 3.14. *Let $\mathcal{A}$ be a properly infinite von Neumann algebra on a separable Hilbert space $\mathcal{H}$. Then $\mathcal{A}$ has a complex symmetric generator if and only if there exists a conjugation J on $\mathcal{H}$ such that $J\mathcal{A}J = \mathcal{A}$.*

To prove Theorem 3.14, we first make some preparation.

LEMMA 3.15 ([45], Theorem 2). *If $\mathcal{A}$ is a properly infinite von Neumann algebra on a separable Hilbert space, then $\mathcal{A}$ has a single generator.*

The following lemma is inspired by Theorem 1 of [46].

LEMMA 3.16. *If $\mathcal{A}$ is a singly generated von Neumann algebra on $\mathcal{H}$, then $M_\infty(\mathcal{A})$ has a single generator T with $\bigvee\limits_{n\geqslant 1} \ker T^n = \mathcal{H}^{(\infty)}$ and $\ker T^* = \{0\}$.*

Proof. Assume that $\mathcal{A}$ acts on $\mathcal{H}$ and A is a single generator of $\mathcal{A}$. Let $T = (B_{i,j})_{i,j=1}^\infty \in M_\infty(\mathcal{A})$ be defined by $B_{i,i+1} = I$ for $i \geqslant 1$, $B_{1,3} = A$ and $B_{i,j} = 0$ otherwise, where I is the identity on $\mathcal{H}$. So T can be written as the following upper triangular operator matrix

$$T = \begin{array}{c} \begin{pmatrix} 0 & I & A & & & \\ & 0 & I & & & \\ & & 0 & I & & \\ & & & 0 & \ddots & \\ & & & & \ddots & \end{pmatrix} \end{array} \begin{array}{l} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \\ \vdots \end{array},$$

where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}_3 = \cdots = \mathcal{H}$ and each omitted entry is 0. It is obvious that $\bigvee\limits_{n\geqslant 1} \ker T^n = \mathcal{H}^{(\infty)}$ and $\ker T^* = \{0\}$. Now it remains to check that T generates $M_\infty(\mathcal{A})$. It suffices to prove that each projection commuting with both T and T^* has the form of $Q^{(\infty)}$, where $Q \in \mathcal{A}'$ is a projection.

Let P be a projection commuting with both T and T^*. We may assume that $P = (P_{i,j})_{i,j=1}^\infty$. Since $\ker T^n = \bigoplus\limits_{i=1}^{n} \mathcal{H}_i$ is hyperinvariant under T (that is, $R(\ker T^n) \subset \ker T^n$ for all R commuting with T) for each $n \geqslant 1$, it follows that $P_{i,j} = 0$ whenever $i > j$. Since P is self-adjoint, we obtain $P_{i,j} = 0$ whenever $i < j$. So $P = \bigoplus\limits_{i=1}^{\infty} P_{i,i}$. A direct calculation shows that $P_{1,1} = P_{i,i}$ for all $i \geqslant 2$ and $P_{1,1}$ commuting with both A and A^*, since $PT = TP$. Noting that $\mathcal{A} = W^*(A)$, we obtain $P_{1,1} \in \mathcal{A}'$. This ends the proof. ∎

LEMMA 3.17. *Let $\mathcal{A}$ be a von Neumann algebra on $\mathcal{H}$ and A be a single generator of $\mathcal{A}$ satisfying $\ker A^* = \{0\}$ and $\bigvee\limits_{n\geqslant 1} \ker A^n = \mathcal{H}$. If J is a conjugation on $\mathcal{H}$ and $J\mathcal{A}J = \mathcal{A}$, then $M_2(\mathcal{A})$ is generated by the following complex symmetric operator,*

$$T = \begin{pmatrix} A & I \\ 0 & JA^*J \end{pmatrix},$$

where I is the identity operator on $\mathcal{H}$.

Proof. For convenience, we write $T = \begin{pmatrix} A & I \\ 0 & JA^*J \end{pmatrix} \begin{matrix} \mathcal{H}_1 \\ \mathcal{H}_2 \end{matrix}$, where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$. Since $J\mathcal{A}J = \mathcal{A}$, it follows that $JA^*J \in \mathcal{A}$ and $T \in M_n(\mathcal{A})$. Set $J_0 = \begin{pmatrix} 0 & J \\ J & 0 \end{pmatrix}$. Then one can verify that J_0 is a conjugation on $\mathcal{H}_1 \oplus \mathcal{H}_2$ and $J_0TJ_0 = T^*$. It remains to check that $W^*(T) = M_2(\mathcal{A})$. It suffices to prove that each projection commuting with both T and T^* has the form of $Q^{(2)}$, where $Q \in \mathcal{A}'$ is a projection.

Let P be a projection commuting with both T and T^*. Since $\bigvee_{n\geqslant 1} \ker A^n = \mathcal{H}_1$ and $\dim \ker A^* = \dim \ker JA^*J = 0$, we deduce that

$$\bigvee_{n\geqslant 1} \ker T^n = \mathcal{H}_1.$$

Then $\mathcal{H}_1$ is hyperinvariant under T, and P can be written as

$$P = \begin{pmatrix} P_1 & P_{1,2} \\ 0 & P_2 \end{pmatrix}.$$

Since $P = P^*$, we have $P_{1,2} = 0$. Noting that P commutes with both T and T^*, a direct calculation shows that $P_1 = P_2$ and P_1 is a projection commuting with both A and A^*. Noting that $W^*(A) = \mathcal{A}$, we obtain $P_1 \in \mathcal{A}'$. This ends the proof. ∎

Now we are going to give the proof of Theorem 3.14.

Proof of Theorem 3.14. By Lemma 1.2, the necessity is obvious. We need only prove the sufficiency.

Obviously, the conjugation $J_0 = J^{(\infty)}$ satisfies $J_0M_\infty(\mathcal{A})J_0 = M_\infty(\mathcal{A})$. Since $\mathcal{A}$ is properly infinite, by Lemma 3.15, $\mathcal{A}$ has a single generator. Thus it follows from Lemmas 3.16 and 3.17 that $M_\infty(\mathcal{A}) \otimes M_2(\mathbb{C})$ has a complex symmetric generator. Since $\mathcal{A}$ is properly infinite, in view of page 458 in [29], $\mathcal{A}$ is spatially isomorphic to $M_\infty(\mathcal{A})$, and hence spatially isomorphic to $M_\infty(\mathcal{A}) \otimes M_2(\mathbb{C})$. This completes the proof. ∎

COROLLARY 3.18. *If $\mathcal{A}$ is a type* III *von Neumann algebra, or a type* II_∞ *von Neumann algebra on a separable Hilbert space $\mathcal{H}$, then $\mathcal{A}$ is singly generated by a* CSO *if and only if there exists a conjugation J on $\mathcal{H}$ such that $J\mathcal{A}J = \mathcal{A}$.*

REMARK 3.19. (i) In view of Theorems 3.9 and 3.14, the complex symmetric generator problem for von Neumann algebras is reduced to the case of type II_1 von Neumann algebras.

(ii) In view of [4], there exist type III factors which do not admit any anti-automorphism on them. Then, by Lemma 1.2, there exist type III factors which can not be singly generated by CSOs.

3.4. FINITE VON NEUMANN ALGEBRAS. As stated in Remark 3.19, the complex symmetric generator problem for von Neumann algebras is reduced to the type II_1

ones. In this subsection we shall solve the problem for a large class of II_1 factors (see Theorem 3.24). Also we shall provide some concrete examples.

LEMMA 3.20. *Suppose that $\mathcal{A}$ is a singly generated von Neumann algebra and ψ is an involutory anti-automorphism of $\mathcal{A}$. Then there exist self-adjoint elements X_1, X_2, X_3 and X_4 in $\mathcal{A}$ such that:*

(i) *X_1, X_2, X_3 and X_4 generate $\mathcal{A}$ as a von Neumann algebra, and*

(ii) *$\psi(X_1) = X_1$, $\psi(X_2) = -X_2$, $\psi(X_3) = X_3$, and $\psi(X_4) = -X_4$.*

Proof. Let $\mathcal{R} = \{A \in \mathcal{A} : \psi(A^*) = A\}$. It follows from Lemma 3.2 in [38] that $\mathcal{R} + \mathrm{i}\mathcal{R} = \mathcal{A}$. Assume that $\mathcal{A}$ is generated by an element B in $\mathcal{A}$ as a von Neumann algebra. Then $B = B_1 + \mathrm{i}B_2$ for some B_1, B_2 in $\mathcal{R}$. Therefore, there exist self-adjoint elements X_1, X_2, X_3 and X_4 in $\mathcal{A}$ such that $B_1 = X_1 + \mathrm{i}X_2$ and $B_2 = X_3 + \mathrm{i}X_4$. It is obvious that X_1, X_2, X_3 and X_4 generate $\mathcal{A}$ as a von Neumann algebra. From the fact that B_1, B_2 are in $\mathcal{R}$, we quickly deduce that $\psi(X_1) = X_1$, $\psi(X_2) = -X_2$, $\psi(X_3) = X_3$, and $\psi(X_4) = -X_4$. This ends the proof of the lemma. ∎

THEOREM 3.21. *Suppose that $\mathcal{A}$ is a II_1 factor with a single generator and $\mathcal{M} = \mathcal{A} \otimes M_4(\mathbb{C})$. Assume that $\mathcal{M}$ acts on a Hilbert space $\mathcal{H}$. If there exists an involutory anti-automorphism of $\mathcal{M}$, then there exist a conjugation J on $\mathcal{H}$ and an element T in $\mathcal{M}$ such that:*

(i) *T generates $\mathcal{M}$ as a von Neumann algebra, and*

(ii) *$JTJ = T^*$.*

Proof. Let τ be a tracial state of $\mathcal{M}$ and P a projection in $\mathcal{M}$ such that $\tau(P) = 1/4$. Then $\mathcal{A} \simeq P\mathcal{M}P$ acts naturally on the Hilbert subspace $\mathcal{H}_1 = P\mathcal{H}$. We will identify $\mathcal{H}$ with $\mathbb{C}^4 \otimes \mathcal{H}_1$.

Since there exists an involutory anti-automorphism of $\mathcal{M}$, from Theorem 2.1, we know that there exists an involutory anti-automorphism of $\mathcal{A}$. From Theorem 4.5 in [38], there exists a conjugation J_1 on $\mathcal{H}_1$ such that $J_1\mathcal{A}J_1 = \mathcal{A}$. Then the mapping $\psi : \mathcal{A} \to \mathcal{A}$, defined by $\psi(A) = J_1A^*J_1$ for all A in $\mathcal{A}$, is in fact an involutory anti-automorphism of $\mathcal{A}$. Note that $\mathcal{A}$ is singly generated. It follows from Lemma 3.20 that there exist self-adjoint elements X_1, X_2, X_3 and X_4 in $\mathcal{A}$ such that:

(a) X_1, X_2, X_3 and X_4 generate $\mathcal{A}$ as a von Neumann algebra, and

(b) $J_1X_1J_1 = X_1$, $J_1X_2J_1 = -X_2$, $J_1X_3J_1 = X_3$, and $J_1X_4J_1 = -X_4$.

Define

$$J = \begin{pmatrix} J_1 & 0 & 0 & 0 \\ 0 & J_1 & 0 & 0 \\ 0 & 0 & -J_1 & 0 \\ 0 & 0 & 0 & J_1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1 & 1 & -\mathrm{i} & 1 \\ 1 & X_1 & X_2 & 0 \\ \mathrm{i} & X_2 & X_3 & X_4 \\ 1 & 0 & X_4 & 0 \end{pmatrix}$$

and $\quad T = Y_1 + \mathrm{i}Y_2.$

Then J is a conjugation on $\mathcal{H}$. Moreover Y_1 and Y_2 are self-adjoint elements in $\mathcal{M}$ such that $JY_1J = Y_1$ and $JY_2J = Y_2$. Hence T is an element in $\mathcal{M}$ such that:

(i) T generates $\mathcal{M}$ as a von Neumann algebra, and
(ii) $JTJ = T^*$.

This ends the proof of the theorem. ∎

COROLLARY 3.22. *Suppose that $\mathcal{A}$ is a* II_1 *factor acting on $\mathcal{H}$ with a single generator and $\mathcal{M} = \mathcal{A} \otimes M_n(\mathbb{C})$, where $n \geqslant 4$ is an integer. Then the following are equivalent:*

(i) *$\mathcal{M}$ has a complex symmetric generator.*
(ii) *There exists an involutory anti-automorphism on $\mathcal{M}$.*
(iii) *There exists an involutory anti-automorphism on $\mathcal{A}$.*

Proof. The implication (i) ⇒ (ii) follows from Lemma 1.2, and the equivalence (ii) ⇔ (iii) follows from Theorem 2.7.

(iii) ⇒ (i) Assume that J_1 is a conjugation on $\mathcal{H}_1$ and there exist self-adjoint elements X_1, X_2, X_3 and X_4 in $\mathcal{A}$ such that:

(a) X_1, X_2, X_3 and X_4 generate $\mathcal{A}$ as a von Neumann algebra, and
(b) $J_1X_1J_1 = X_1$, $J_1X_2J_1 = -X_2$, $J_1X_3J_1 = X_3$, and $J_1X_4J_1 = -X_4$.

Define

$$J = \begin{pmatrix} J_1 & & & & & & \\ & J_1 & & & & & \\ & & -J_1 & & & & \\ & & & J_1 & & & \\ & & & & J_1 & & \\ & & & & & \ddots & \\ & & & & & & J_1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 1 & & & & & & \\ & 2 & & & & & \\ & & 3 & & & & \\ & & & 4 & & & \\ & & & & 5 & & \\ & & & & & \ddots & \\ & & & & & & n \end{pmatrix},$$

$$Y_2 = \begin{pmatrix} 1 & 1 & -\mathrm{i} & 1 & 1 & \cdots & 1 \\ 1 & X_1 & X_2 & 0 & 0 & \cdots & 0 \\ \mathrm{i} & X_2 & X_3 & X_4 & 0 & \cdots & 0 \\ 1 & 0 & X_4 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \quad \text{and} \quad T = Y_1 + \mathrm{i}Y_2,$$

where each omitted entry is 0. Then J is a conjugation on $\bigoplus_{i=1}^{n} \mathcal{H}$, T is J-symmetric and $W^*(T) = M_n(\mathcal{A})$. Using a similar argument as in Theorem 3.21, one can see the desired result. ∎

For a II_1 factor $\mathcal{M}$, an invariant $\mathcal{G}(\mathcal{M})$ was introduced in [35]. It was shown in [35] that if $\mathcal{M}$ is in a large class of II_1 factors, including hyperfinite II_1 factor and many others, then $\mathcal{G}(\mathcal{M}) = 0$. It was also shown in [35] that if $\mathcal{G}(\mathcal{M}) = 0$, then $\mathcal{M}$ is singly generated. If $\mathcal{M}$ is a II_1 factor with a trace τ and P is a projection in $\mathcal{M}$ such that $\tau(P) = 1/n$ for a positive integer, then it was proved in [7] that $\mathcal{G}(P\mathcal{M}P) = n^2\mathcal{G}(\mathcal{M})$. In particular, if $\mathcal{M}$ is a II_1 factor such that $\mathcal{G}(\mathcal{M}) = 0$, then for any projection P in $\mathcal{M}$, $P\mathcal{M}P$ is singly generated. Now we have the following result.

THEOREM 3.23. *Suppose that $\mathcal{M}$ is a II_1 factor acting on a Hilbert space $\mathcal{H}$. If there exists an involutory anti-automorphism of $\mathcal{M}$ and $\mathcal{G}(\mathcal{M}) = 0$, then there exist a conjugation J on $\mathcal{H}$ and an element T in $\mathcal{M}$ such that:*

(i) *T generates $\mathcal{M}$ as a von Neumann algebra, and*

(ii) *$JTJ = T^*$.*

Proof. Assume that $\{E_{i,j}\}_{1\leqslant i,j\leqslant 4}$ is a system of self-adjoint matrix units in $\mathcal{M}$ such that $\sum_{i=1}^{4} E_{i,i} = I$. Then $\mathcal{M} \simeq E_{11}\mathcal{M}E_{11} \otimes M_4(\mathbb{C})$. Since $\mathcal{G}(\mathcal{M}) = 0$, we have that $\mathcal{G}(E_{11}\mathcal{M}E_{11}) = 0$, which implies that $E_{11}\mathcal{M}E_{11}$ is singly generated. From Theorem 2.7, there exists an involutory anti-automorphism of $E_{11}\mathcal{M}E_{11}$. By Theorem 4.5 in [38], there exists a conjugation J_1 on $E_{11}\mathcal{H}$ such that $J_1E_{11}\mathcal{M}E_{11}J_1 = E_{11}\mathcal{M}E_{11}$. Combining with Theorem 3.21, we get that $\mathcal{M}$ admits a complex symmetric generator, i.e. there exist a conjugation J on $\mathcal{H}$ and an element T in $\mathcal{M}$ such that:

(i) T generates $\mathcal{M}$ as a von Neumann algebra, and

(ii) $JTJ = T^*$.

This ends the proof of the result. ∎

According to the preceding results, the following result is clear.

THEOREM 3.24. *Suppose that $\mathcal{M}$ is a II_1 factor acting on a Hilbert space $\mathcal{H}$ and $\mathcal{G}(\mathcal{M}) = 0$. Then the following statements are equivalent:*

(i) *There exists an involutory anti-automorphism of $\mathcal{M}$.*

(ii) *There exists a conjugation J on $\mathcal{H}$ such that $J\mathcal{M}J = \mathcal{M}$.*

(iii) *$\mathcal{M}$ admits a complex symmetric generator.*

As applications of our results, we give some examples of von Neumann algebras which are singly generated by CSOs.

EXAMPLE 3.25. Let $F_2 \times F_2$ be the direct sum of two free groups and $L(F_2 \times F_2)$ be the group von Neumann algebra associated with $F_2 \times F_2$. Then $L(F_2 \times F_2)$ is a II_1 factor and $\mathcal{G}(L(F_2 \times F_2)) = 0$ (see Corollary 5.7 in [35]). It is known that there exists an involutory anti-automorphism of $L(F_2 \times F_2)$ (see p. 350 of [38]). If $L(F_2 \times F_2)$ acts on a Hilbert space $\mathcal{H}$, then it follows from Theorem 3.23 that $L(F_2 \times F_2)$ is generated by a complex symmetric operator.

EXAMPLE 3.26. Let G be an I.C.C. amenable group and $L(G)$ be the group von Neumann algebra associated with G. Then $L(G)$ is the hyperfinite II_1 factor and $\mathcal{G}(L(G)) = 0$ (see Corollary 5.16(i) in [35]). It is known that there exists an involutory anti-automorphism of $L(G)$ (see p. 350 of [38]). If $L(G)$ acts on a Hilbert space $\mathcal{H}$, then it follows from Theorem 3.23 that $L(G)$ is generated by a complex symmetric operator.

EXAMPLE 3.27. Let $SL_3(\mathbb{Z})$ be special linear group of order 3 with integer entries and $L(SL_3(\mathbb{Z}))$ be the group von Neumann algebra associated with $SL_3(\mathbb{Z})$. Then $L(SL_3(\mathbb{Z}))$ is a II_1 factor and $\mathcal{G}(L(SL_3(\mathbb{Z}))) = 0$ (see Corollary 5.6 in [35]). It is known that there exists an involutory anti-automorphism of $L(SL_3(\mathbb{Z}))$. If $L(SL_3(\mathbb{Z}))$ acts on a Hilbert space $\mathcal{H}$, then it follows from Theorem 3.23 that $L(SL_3(\mathbb{Z}))$ is generated by a complex symmetric operator.

EXAMPLE 3.28. Let $n \geqslant 4$ be a positive integer and $L(F_2) \otimes M_n(\mathbb{C}) \simeq L(F_{(n^2+1)/n^2})$ be an interpolated free group factor (see [6]). It is known that there exists an involutory anti-automorphism of $L(F_2)$. Since $L(F_2)$ has a single generator, if $L(F_{(n^2+1)/n^2})$ acts on a Hilbert space $\mathcal{H}$, then it follows from Corollary 3.22 that $L(F_{(n^2+1)/n^2})$ is generated by a complex symmetric operator for $n \geqslant 4$.

EXAMPLE 3.29. In this example, we will show that a free group factor on two generators, acting on a Hilbert space $\mathcal{H}$, is generated by a complex symmetric operator.

Let F_2 be the free group with two generators a and b. We let

$$l^2(F_2) = \Big\{ \sum_{g\in F_2} \alpha_g \widehat{g} : \forall g \in F_2, \alpha_g \in \mathbb{C} \text{ and } \sum_{g\in F_2} |\alpha_g|^2 < \infty \Big\}$$

be the canonical Hilbert space associated with F_2, where $\widehat{g}$ denotes the indicator function of $\{g\}$ on F_2. Then $\{\widehat{g} : g \in F_2\}$ forms an orthonormal basis of $l^2(F_2)$.

We let λ and ρ be the left, and right respectively, regular representations of F_2 on the Hilbert space $l^2(F_2)$. We let $L(F_2)$, $R(F_2)$ be the group von Neumann algebras generated by $\lambda(F_2)$, and $\rho(F_2)$ respectively, in $\mathcal{B}(l^2(F_2))$. So

$$\lambda(h)\widehat{g} = \widehat{hg} \quad \text{and} \quad \rho(h)\widehat{g} = \widehat{gh^{-1}}, \quad \forall\, h, g \in F_2.$$

We introduce a map $\tau : F_2 \to F_2$ by

$$\tau(g_1^{\epsilon_1} \cdots g_n^{\epsilon_n}) = g_1^{-\epsilon_1} \cdots g_n^{-\epsilon_n}, \quad \forall\, g_1, \ldots, g_n \in \{a, b\}, \text{ and } \epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}.$$

Then τ is a bijective map on F_2, $\tau^{-1} = \tau$ and $\tau(g_1 g_2) = \tau(g_1)\tau(g_2)$ for $g_1, g_2 \in F_2$. We define a conjugate-linear map $J : l^2(F_2) \to l^2(F_2)$ by

$$J\Big(\sum_{g\in F_2} \alpha_g \widehat{g} \Big) = \sum_{g\in F_2} \overline{\alpha}_g \widehat{\tau(g)}.$$

Then one can verify that J is a conjugation on $l^2(F_2)$.

CLAIM 3.30. $\lambda(a)$ and $\rho(a)$ are both J-symmetric.

Choose an $x \in F_2$ with $x = g_1^{\epsilon_1} \cdots g_k^{\epsilon_k}$, where $g_1, \dots, g_k \in \{a, b\}$ and $\epsilon_1, \dots, \epsilon_k \in \{-1, 1\}$. Compute to see that

$$J\lambda(a)\widehat{x} = J(\widehat{ax}) = \widehat{a^{-1}\tau(x)} = \lambda(a^{-1})\widehat{\tau(x)} = \lambda(a)^* J\widehat{x} \quad \text{and}$$
$$J\rho(a)\widehat{x} = J(\widehat{xa^{-1}}) = \widehat{\tau(x)a} = \rho(a^{-1})\widehat{\tau(x)} = \rho(a)^* J\widehat{x}.$$

It follows that $J\lambda(a) = \lambda(a)^* J$ and $J\rho(a) = \rho(a)^* J$. So $\lambda(a)$ and $\rho(a)$ are both J-symmetric. Likewise one can check that $\lambda(b)$ and $\rho(b)$ are J-symmetric.

Let $\mathcal{A}$ and $\mathcal{B}$ be the abelian von Neumann subalgebra generated by $\lambda(a)$, and $\lambda(b)$ respectively in $L(F_2)$. Then there exist self-adjoint elements A in $\mathcal{A}$ and B in $\mathcal{B}$ such that $\mathcal{A} = W^*(A)$ and $\mathcal{B} = W^*(B)$. Since $\lambda(a)$ and $\lambda(b)$ are unitary, by Lemma 3.3, A, B are J-symmetric. Set $T = A + \mathrm{i}B$. Then T is J-symmetric and

$$W^*(T) = W^*(A, B) = W^*(\lambda(a), \lambda(b)) = L(F_2).$$

That is, $L(F_2)$ is generated by the complex symmetric operator T. Similarly, one can check that $R(F_2)$ has a complex symmetric generator.

EXAMPLE 3.31. Let $\mathcal{A}=\mathcal{B}\rtimes_\alpha\mathbb{Z}_5$ be the II_1 factor as constructed in [22], where $\mathcal{B}=\mathcal{P}\overline{\otimes}\mathcal{R}$ for a II_1 factor $\mathcal{P}$ and a hyperfinite II_1 factor $\mathcal{R}$. From Corollary 5.16 in [35], it follows that $\mathcal{G}(\mathcal{A})=0$. However, as shown in [22], $\mathcal{A}$ has no involutory anti-automorphism and contains no single complex symmetric generator.

4. COMPLEX SYMMETRIC GENERATOR PROBLEM FOR C^*-ALGEBRAS

In this section, we shall consider the complex symmetric generator problem for some C^*-algebras, including AF algebras, irrational rotation algebras, reduced free products and tensor products of C^*-algebras with UHF algebras.

4.1. PROOF OF THEOREM 1.5.

In this subsection, we first give the proof of Theorem 1.5. Then we provide some illustrating examples.

Let $A_i \in \mathcal{B}(\mathcal{H}_i), i = 1, 2$. Recall that A_1, A_2 are said to be *approximately unitarily equivalent*, denoted by $A_1 \cong_{\mathrm{a}} A_2$, if there exist unitary operators $\{U_n\}_{n=1}^{\infty}$ from $\mathcal{H}_2$ onto $\mathcal{H}_1$ such that $U_n^* A_1 U_n \to A_2$ as $n \to \infty$. Let $\mathcal{A}$ be a C^*-algebra and ρ_1, ρ_2 be two representations of $\mathcal{A}$ on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We also write $\rho_1 \cong_{\mathrm{a}} \rho_2$ to denote that ρ_1, ρ_2 are approximately unitarily equivalent, that is, there exist unitary operators $\{V_n\}_{n=1}^{\infty}$ from $\mathcal{H}_2$ onto $\mathcal{H}_1$ such that

$$\lim_n V_n^* \rho_1(a) V_n = \rho_2(a), \quad \forall a \in \mathcal{A}.$$

Proof of Theorem 1.5. The necessity is clear from Lemma 1.2.

$\Leftarrow$ Without loss of generality, we may directly assume that $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{K})$ containing the identity I. Denote $A = T^{(\infty)}$. Then the map $\psi_1 : \mathcal{A} \to C^*(A)$ defined as $X \mapsto X^{(\infty)}$ is a $*$-isomorphism. For $Y \in C^*(A)$, define $\psi_2(Y) = \psi_1 \circ \varphi \circ \psi_1^{-1}(Y)$. Then one can see that ψ_2 is an anti-automorphism of $C^*(A)$ such that $\psi_2(A) = A$.

Arbitrarily choose a conjugation J on $\mathcal{K}^{(\infty)}$ and define $\rho(Y) = J\psi_2(Y^*)J$ for $Y \in C^*(A)$. Then ρ is a unital faithful representation of $C^*(A)$ on $\mathcal{K}^{(\infty)}$ and $\rho(A) = JA^*J$. Since $C^*(A)$ contains no nonzero compact operators, by a consequence of Voiculescu's theorem (see Theorem II.5.8 of [1]), ρ is approximately unitarily equivalent to the identity representation of $C^*(A)$. So A is approximately unitarily equivalent to JA^*J. Hence

$$A = T^{(\infty)} \cong A \oplus A \cong_{\mathrm{a}} A \oplus JA^*J,$$

where $\cong$ denotes unitary equivalence and $\cong_{\mathrm{a}}$ denotes approximate unitary equivalence. Denote $R = A \oplus JA^*J$. Then R is complex symmetric (see Lemma 3.6 of [18]), and the approximate unitary equivalence of R and A induces a faithful representation ψ_3 of $C^*(A)$ satisfying $\psi_3(A) = R$. Set $\pi = \psi_3 \circ \psi_1$. Then π satisfies all requirements. ∎

EXAMPLE 4.1. Let ψ be an involutory anti-automorphism of $M_2(\mathbb{C})$ defined by

$$\psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}).$$

Suppose that $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an element in $M_2(\mathbb{C})$ such that $\psi(X) = X$. Then we have

$$a = d \quad \text{and} \quad b = c = 0.$$

Thus the C^*-algebra (also a von Neumann algebra in this case) generated by such X is $\mathbb{C}I$, i.e. $M_2(\mathbb{C})$ is not generated by an X in $M_2(\mathbb{C})$ such that $\psi(X) = X$.

Let $\phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be an involutory anti-automorphism of $M_2(\mathbb{C})$ defined by

$$\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{C}).$$

It is easy to verify that $T = \begin{pmatrix} 1 & \mathrm{i} \\ \mathrm{i} & 2 \end{pmatrix}$ is a matrix in $M_2(\mathbb{C})$ such that T generates $M_2(\mathbb{C})$ and $\phi(T) = T$.

EXAMPLE 4.2. Let $\mathcal{A} = C[-1,1]$ and $\psi : \mathcal{A} \to \mathcal{A}$ be an involutory anti-automorphism of $\mathcal{A}$ defined by, $\forall f \in \mathcal{A}$,

$$\psi(f)(t) = f(-t) \quad \text{for all } t \in [-1,1].$$

Suppose that g in $\mathcal{A}$ satisfies that $\psi(g) = g$, i.e.

$$\psi(g)(t) = g(-t) \quad \text{for all } t \in [-1,1].$$

Then

$$C^*(g) \subseteq \{f \in \mathcal{A} : f(t) = f(-t),\ \forall\, t \in [-1,1]\} \neq \mathcal{A}.$$

Therefore, $\mathcal{A}$ is not generated by an element that is invariant under ψ.

Now we give a singly generated C^*-algebra $\mathcal{T}$ which admits an involutory anti-automorphism; however, $\mathcal{T}$ is not $*$-isomorphic to any concrete C^*-algebra singly generated by CSOs.

EXAMPLE 4.3. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis of $\mathcal{H}$ and $S \in \mathcal{B}(\mathcal{H})$ be the unilateral shift defined by

$$Se_n = e_{n+1}, \quad \forall\, n \geqslant 1.$$

Denote $\mathcal{T} = C^*(S)$. Then $\mathcal{T}$ is singly generated and it is easy to see that $\mathcal{T}$ contains all compact operators on $\mathcal{H}$. For $x \in \mathcal{H}$ with $x = \sum\limits_{i=1}^{\infty} \alpha_i e_i$, define $Jx = \sum\limits_{i=1}^{\infty} \overline{\alpha}_i e_i$. Then one can check that J is a conjugation on $\mathcal{H}$ and $JSJ = S$. It is easy to check that the map $X \mapsto JX^*J$ is an involutory anti-automorphism of $\mathcal{T}$. However, we shall show that there exists no anti-automorphism φ of $\mathcal{T}$ such that $\mathcal{T}$ is generated by an element that is invariant under φ.

Assume that φ is an anti-automorphism of $\mathcal{T}$. Since $\mathcal{T}$ contains all compact operators, it follows from Proposition 3.18 of [18] that φ is of the form $X \mapsto DX^*D^{-1}$, where D is an invertible, conjugate-linear isometry on $\mathcal{H}$. If $X \in \mathcal{T}$ and $\varphi(X) = X$, then $DX^*D^{-1} = X$ and it is obvious that $\operatorname{ind}(X-\lambda) = \operatorname{ind}(X-\lambda)^*$ for $\lambda \notin \sigma_{\mathrm{e}}(X)$. Here $\sigma_{\mathrm{e}}(\cdot)$ denotes the essential spectrum. Since S is essentially normal, each operator in $\mathcal{T}$ is essentially normal. Then, by the B-D-F Theorem [3], X has the form "normal plus compact". It follows that each operator in $C^*(X)$ is of the form "normal plus compact". So $C^*(X) \neq \mathcal{T}$. This shows that $\mathcal{T}$ can not be generated by an element that is invariant under φ. Then, by Theorem 1.5, $\mathcal{T}$ is not $*$-isomorphic to any concrete C^*-algebra singly generated by CSOs.

4.2. UHF ALGEBRAS. Olsen and Zame [28] proved that a tensor product of a separable unital C^*-algebra and a UHF algebra is always singly generated. The main result of this subsection is the following theorem, which solves the complex symmetric generator problem for such C^*-algebras.

THEOREM 4.4. *Suppose that $\mathcal{A}$ is a unital C^*-algebra such that $\mathcal{A} \simeq \mathcal{B} \otimes \mathcal{U}$ for a separable C^*-algebra $\mathcal{B}$ and a* UHF *C^*-algebra $\mathcal{U}$. If there exists an involutory anti-automorphism ψ of $\mathcal{A}$, then there exist an element T in $\mathcal{A}$ and an involutory anti-automorphism Ψ of $\mathcal{A}$ such that:*

(i) *T generates $\mathcal{A}$ as a C^*-algebra, and*

(ii) $\Psi(T) = T$.

We first prove several auxiliary results.

LEMMA 4.5. *Suppose that $\mathcal{A}$ is a singly generated C^*-algebra and ψ is an involutory anti-automorphism of $\mathcal{A}$. Then there exist two elements A and B in $\mathcal{A}$ such that:*

(i) *A and B generate $\mathcal{A}$ as a C^*-algebra, and*

(ii) *$\psi(A) = A^*$ and $\psi(B) = B^*$.*

Proof. Assume that $\mathcal{A}$ is generated by an element X in $\mathcal{A}$ as a C^*-algebra. We let

$$A = \frac{1}{2}(X + \psi(X^*)), \quad \text{and} \quad B = \frac{1}{2\mathrm{i}}(X - \psi(X^*)).$$

Then $X = A + \mathrm{i}B$. Thus A and B are two elements in $\mathcal{A}$ that generate $\mathcal{A}$ as a C^*-algebra. Moreover we have $\psi(A) = A^*$ and $\psi(B) = B^*$. This ends the proof of the lemma. ∎

THEOREM 4.6. *Suppose that $\mathcal{A}$ is a singly generated, unital C^*-algebra with an involutory anti-automorphism ψ. Suppose $\mathcal{B} = \mathcal{A} \otimes M_n(\mathbb{C})$ is a tensor product of $\mathcal{A}$ and $M_n(\mathbb{C})$ for some positive integer $n \geqslant 4$. Then there exist an element T in $\mathcal{B}$ and an involutory anti-automorphism Ψ on $\mathcal{B}$ such that:*

(i) *T generates $\mathcal{B}$ as a C^*-algebra, and*

(ii) $\Psi(T) = T$.

Proof. It follows from the preceding lemma that there exist two elements A and B in $\mathcal{A}$ such that (a) A and B generate $\mathcal{A}$ as a C^*-algebra, and (b) $\psi(A) = A^*$ and $\psi(B) = B^*$.

We define a mapping $\Psi : \mathcal{B} \to \mathcal{B}$ by

$$\forall\, X = (x_{ij}), \quad \Psi(X) = (y_{ij}), \quad \text{where } y_{ij} = \psi(x_{ji}) \text{ for all } 1 \leqslant i, j \leqslant n.$$

It can be verified that Ψ is an involutory anti-automorphism on $\mathcal{B}$.

Note that $n \geqslant 4$. We let

$$Y_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & n \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & A & B & \cdots & 0 \\ 1 & A^* & 0 & 0 & \cdots & 0 \\ 1 & B^* & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

From the definition of Ψ, we know that

$$\Psi(Y_1) = Y_1 \quad \text{and} \quad \Psi(Y_2) = Y_2.$$

It is not hard to show that Y_1 and Y_2 are two self-adjoint elements that generate $\mathcal{B}$ as a C^*-algebra. Therefore $T = Y_1 + \mathrm{i}Y_2$ is an element in $\mathcal{B}$ such that T generates $\mathcal{B}$ as a C^*-algebra, and $\Psi(T) = T$. ∎

Proof of Theorem 4.4. From Theorem 9 in [28], we know that $\mathcal{A}$ is singly generated. Observe that the UHF algebra $\mathcal{U} \simeq \mathcal{U} \otimes M_n(\mathbb{C})$ for some integer $n \geqslant 4$. Therefore, $\mathcal{A} \simeq \mathcal{A} \otimes M_n(\mathbb{C})$ for an integer $n \geqslant 4$. Now the result follows from the preceding theorem. ∎

COROLLARY 4.7. *Let G be a discrete countable group and $C^*_{\mathrm{r}}(G)$ (or $C^*(G)$) be the reduced (or full) C^*-algebra of G. Let $\mathcal{U}$ be a* UHF *algebra. Then there exist an* *element T in $C_r^*(G) \otimes \mathcal{U}$ (or $C^*(G) \otimes \mathcal{U}$) and an involutory anti-automorphism Ψ of $C_r^*(G) \otimes \mathcal{U}$ (or of $C^*(G) \otimes \mathcal{U}$) such that:*

(i) *T generates $C_r^*(G) \otimes \mathcal{U}$ (or $C^*(G) \otimes \mathcal{U}$) as a C^*-algebra, and*

(ii) $\Psi(T) = T$.

Proof. Since $C_r^*(G)$ (or $C^*(G)$) always contains involutory anti-automorphisms, the results follow directly from the preceding theorem. ∎

Our next result shows that the complex symmetric generator problem for C^*-algebras depends on the choice of Hilbert spaces on which C^*-algebras are represented.

PROPOSITION 4.8. *Let F_∞ be the free group on countably infinitely many generators and $C^*(F_\infty)$ the full C^*-algebra of F_∞. Let $\mathcal{U}$ be a* UHF *algebra and $C^*(F_\infty) \otimes \mathcal{U}$ be the minimal tensor product of C^*-algebras $C^*(F_\infty)$ and $\mathcal{U}$. Then*

(i) *there exists a faithful $*$-representation π of $C^*(F_\infty) \otimes \mathcal{U}$ on a separable Hilbert space $\mathcal{H}$ such that $\pi(C^*(F_\infty) \otimes \mathcal{U})$ contains a complex symmetric generator;*

(ii) *there exists a faithful $*$-representation ρ of $C^*(F_\infty) \otimes \mathcal{U}$ on a separable Hilbert space $\mathcal{K}$ such that $\rho(C^*(F_\infty) \otimes \mathcal{U})$ does not contain any complex symmetric generator.*

Proof. (i) The result follows readily from Theorem 1.5 and Corollary 4.7.

(ii) Let $\mathcal{A} = \mathcal{B} \rtimes_\alpha \mathbb{Z}_5$ be the II_1 factor as constructed in [22], where $\mathcal{B} = \mathcal{P} \overline{\otimes} \mathcal{R}$ for a II_1 factor $\mathcal{P}$ and a hyperfinite II_1 factor $\mathcal{R}$. It was shown in [22] that $\mathcal{A}$ has no involutory anti-automorphisms. It follows from Proposition 1.11 in [32] that $\mathcal{A}$ is a McDuff factor, i.e. $\mathcal{A} \simeq \mathcal{A} \overline{\otimes} \mathcal{R}$. We might assume that $\mathcal{A} \overline{\otimes} \mathcal{R}$ acts on the standard Hilbert space $\mathcal{K} = L^2(\mathcal{A} \overline{\otimes} \mathcal{R}, \tau)$, where τ is a tracial state of $\mathcal{A} \overline{\otimes} \mathcal{R}$. By Proposition 5.1.1 in [2], there exists a faithful $*$-representation $\rho_1 : C^*(F_\infty) \to \mathcal{A} \otimes I \subseteq \mathcal{B}(\mathcal{K})$ such that $\rho(C^*(F_\infty))$ is dense in $\mathcal{A} \otimes I$ in weak operator topology. It is not hard to see that there exists a faithful $*$-representation $\rho_2 : \mathcal{U} \to I \otimes \mathcal{R} \subseteq \mathcal{B}(\mathcal{K})$ such that $\rho_2(\mathcal{U})$ is dense in $I \otimes \mathcal{R}$ in weak operator topology. Therefore, there exists a faithful $*$-representation $\rho_1 \otimes \rho_2 : C^*(F_\infty) \otimes \mathcal{U} \to \mathcal{A} \overline{\otimes} \mathcal{R} \subseteq \mathcal{B}(\mathcal{K})$ such that $\rho(C^*(F_\infty) \otimes \mathcal{U})$ is dense in $\mathcal{A} \overline{\otimes} \mathcal{R}$ in weak operator topology. Since $\mathcal{A} \simeq \mathcal{A} \overline{\otimes} \mathcal{R}$ has no involutory anti-automorphisms, $\rho(C^*(F_\infty) \otimes \mathcal{U})$ can not be generated by a complex symmetric operator in $\mathcal{B}(\mathcal{K})$. ∎

Now we are going to give two $*$-isomorphic C^*-algebras of operators, one of which can be singly generated by CSOs and the other can not.

EXAMPLE 4.9. Let S be the unilateral shift of multiplicity one on a Hilbert space $\mathcal{H}$. Denote $A = S \oplus S^*$ and $B = S \oplus S \oplus S^*$. Then the map $\varphi : p(A^*, A) \mapsto p(B^*, B)$ extends to a $*$-isomorphism of $C^*(A)$ onto $C^*(B)$. Noting that A is complex symmetric (see Theorem 3.14 of [18]), $C^*(A)$ is singly generated by CSOs. However, $C^*(B)$ does not have a complex symmetric generator. In fact, if X is a complex symmetric operator in $C^*(B)$, then one can check that ind $(X - \lambda) = 0$ for $\lambda \notin \sigma_e(X)$. Since B is essentially normal, it follows that X is essentially normal. By the B-D-F Theorem, X is of the form "normal plus compact" and so is each operator in $C^*(X)$. Then $B \notin C^*(X)$, since ind $B = -1$. Hence $C^*(X) \neq C^*(B)$. So $C^*(B)$ does not have a complex symmetric generator.

4.3. AF ALGEBRAS. The main result of this subsection is the following theorem, which shows that each unital AF algebra is $*$-isomorphic to some C^*-algebra with a single complex symmetric generator.

THEOREM 4.10. *Let $\mathcal{A}$ be a unital, separable* AF *algebra. Then there are an involutory anti-automorphism ψ of $\mathcal{A}$ and an element T in $\mathcal{A}$ such that T generates $\mathcal{A}$ and $\psi(T) = T$.*

We first prove some auxiliary results.

DEFINITION 4.11. Assume that $\mathcal{A}$ is a unital C^*-algebra with an involutory anti-automorphism ψ. We define

$$\mathcal{S}(\mathcal{A},\psi) = \{(X,Y) : X, Y \in \mathcal{A} \text{ are self-adjoint such that } \psi(X) = X, \psi(Y) = Y\},$$
$$\mathcal{G}(\mathcal{A},\psi) = \{(X,Y) \in \mathcal{S}(\mathcal{A},\psi) : X, Y \text{ generate } \mathcal{A} \text{ as a } C^*\text{-algebra}\}.$$

THEOREM 4.12. *Assume that $\mathcal{A}$ is a unital separable C^*-algebra with an involutory anti-automorphism ψ. Assume that there exists an increasing sequence of C^*-subalgebras $\{\mathcal{A}_k\}_{k=1}^{\infty}$ satisfying:*

(i) $\mathcal{A} = \overline{\bigcup_k \mathcal{A}_k}$;

(ii) $\psi(\mathcal{A}_k) = \mathcal{A}_k$ *for each* $k \geqslant 1$;

(iii) $\forall\, k \geqslant 1$, $\mathcal{G}(\mathcal{A}_k,\psi)$ *is dense in* $\mathcal{S}(\mathcal{A}_k,\psi)$ *in norm topology.*

Then there exists an element T in $\mathcal{A}$ such that T generates $\mathcal{A}$ as a C^*-algebra and $\psi(T) = T$.

Proof. We adapt the proof of Lemma 2.9 in [41] to obtain our result.

Let $A_1 (= 0), A_2, \ldots$ be a dense sequence of $\mathcal{A}$ such that each element in $\{A_n\}_n$ appears infinitely many times in the sequence.

Next we will construct by induction a sequence of elements $\{(X_n, Y_n)\}_n$ in $\bigcup_k \mathcal{S}(\mathcal{A}_k,\psi)$ and a sequence of positive numbers $\{\delta_n\}_n$ in $\mathbb{R}^+$ satisfying:

(a) $\|X_n - X_{n-1}\| + \|Y_n - Y_{n-1}\| \leqslant (\min\{\delta_1, \delta_2, \ldots, \delta_{n-1}\})/2^{n+1}$ for each $n \geqslant 2$.

(b) For each $n \geqslant 1$, if (X,Y) is an element in $\mathcal{S}(\mathcal{A},\psi)$ such that

$$\|X - X_n\| + \|Y - Y_n\| \leqslant \frac{\delta_n}{2^n},$$

then there is an element Z in the C^*-algebra generated by X and Y such that

$$\|Z - A_n\| < \frac{1}{2^n}.$$

Base Step. Note that $\mathcal{S}(\mathcal{A}_1,\psi)$ is nonempty. We choose an element (X_1, Y_1) from $\mathcal{S}(\mathcal{A}_1,\psi)$ and let $\delta_1 = 1$. Then (b) is satisfied, since $A_1 = 0$.

Inductive Step. Assume that $(X_1, Y_1), \ldots, (X_n, Y_n) \in \bigcup_k \mathcal{S}(\mathcal{A}_k, \psi)$ and δ_1, δ_2, $\ldots$, $\delta_n \in \mathbb{R}^+$ have been chosen. We will now find $(X_{n+1}, Y_{n+1}) \in \bigcup_k \mathcal{S}(\mathcal{A}_k, \psi)$ and $\delta_{n+1} \in \mathbb{R}^+$ with desired properties. Since (X_n, Y_n) is in $\bigcup_k \mathcal{S}(\mathcal{A}_k, \psi)$, there exists an integer k_n such that $(X_n, Y_n) \in \mathcal{S}(\mathcal{A}_{k_n}, \psi)$. For $A_{n+1} \in \mathcal{A}$, by condition (i), there exists an integer $p_n \geqslant k_n$ and an element B_n in $\mathcal{A}_{p_n}$ such that

$$\|A_{n+1} - B_n\| \leqslant \frac{1}{2^{n+3}}. \tag{4.1}$$

Notice that

$$(X_n, Y_n) \in \mathcal{S}(\mathcal{A}_{k_n}, \psi) \subseteq \mathcal{S}(\mathcal{A}_{p_n}, \psi). \tag{4.2}$$

Combing (4.2) with the condition (iii), we know that there exist (X_{n+1}, Y_{n+1}) in $\mathcal{G}(\mathcal{A}_{p_n}, \psi)$ and a noncommutative polynomial f such that

$$\|X_{n+1} - X_n\| + \|Y_{n+1} - Y_n\| \leqslant \frac{\min\{\delta_1, \delta_2, \ldots, \delta_n\}}{2^{n+2}} \quad \text{and} \tag{4.3}$$

$$\|B_n - f(X_{n+1}, Y_{n+1})\| \leqslant \frac{1}{2^{n+3}}. \tag{4.4}$$

From (4.3), we know that (a) is satisfied. From (4.1) and (4.4) we have

$$\|A_{n+1} - f(X_{n+1}, Y_{n+1})\| \leqslant \|A_{n+1} - B_n\| + \|B_n - f(X_{n+1}, Y_{n+1})\| \leqslant \frac{1}{2^{n+2}}. \tag{4.5}$$

From the fact that f is a polynomial, there exists a positive number δ_{n+1} such that if (X, Y) is an element in $\mathcal{S}(\mathcal{A}, \psi)$ satisfying

$$\|X - X_{n+1}\| + \|Y - Y_{n+1}\| \leqslant \frac{\delta_{n+1}}{2^{n+1}}, \tag{4.6}$$

then

$$\|f(X, Y) - f(X_{n+1}, Y_{n+1})\| \leqslant \frac{1}{2^{n+2}}, \tag{4.7}$$

whence, from (4.5) and (4.7) it follows that

$$\begin{aligned} \|A_{n+1} - f(X, Y)\| &\leqslant \|A_{n+1} - f(X_{n+1}, Y_{n+1})\| + \|f(X, Y) - f(X_{n+1}, Y_{n+1})\| \\ &\leqslant \frac{1}{2^{n+1}}. \end{aligned}$$

Thus (b) is satisfied. Therefore $\{(X_n, Y_n)\}_n$ and $\{\delta_n\}_n$ can be constructed such that (a) and (b) hold.

Assume that the sequences $\{(X_n, Y_n)\}_n \subseteq \bigcup_k \mathcal{S}(\mathcal{A}_k, \psi)$ and $\{\delta_n\}_n \subseteq \mathbb{R}^+$ have now been obtained. From (a) we know that $\{X_n\}, \{Y_n\}$ are two Cauchy sequences of self-adjoint elements in $\mathcal{A}$. We let $X = \lim_n X_n$ and $Y = \lim_n Y_n$. Then X, Y are two self-adjoint elements in $\mathcal{A}$ such that

$$\psi(X) = X \quad \text{and} \quad \psi(Y) = Y,$$

since ψ is norm continuous. So $(X,Y) \in \mathcal{S}(\mathcal{A},\psi)$. Again from (a) we have, for each $n \geqslant 1$,

$$\|X-X_n\| = \lim_k \|X_k - X_n\| \leqslant \lim_k \sum_{n<j\leqslant k} \|X_j - X_{j-1}\| \leqslant \lim_k \sum_{n<j\leqslant k} \frac{\delta_n}{2^{j+1}} = \frac{\delta_n}{2^{n+1}} \quad \text{and}$$

$$\|Y-Y_n\| = \lim_k \|Y_k - Y_n\| \leqslant \lim_k \sum_{n<j\leqslant k} \|Y_{j+1} - Y_j\| \leqslant \lim_k \sum_{n<j\leqslant k} \frac{\delta_n}{2^{j+1}} = \frac{\delta_n}{2^{n+1}}.$$

Now it follows from (b) that there exists an element Z_n in the C^*-algebra generated by X and Y such that $\|Z_n - A_n\| < 1/2^n$. Note that $A_1, A_2, \dots$ is a dense sequence of $\mathcal{A}$ such that each element in $\{A_n\}_n$ appears infinitely many times in the sequence. We know that $\mathcal{A} = C^*(X,Y)$. Then $T = X + \mathrm{i}Y$ is a desired element in $\mathcal{A}$ that generates $\mathcal{A}$ as a C^*-algebra and satisfies

$$\psi(T) = \psi(X+\mathrm{i}Y) = \psi(X) + \mathrm{i}\psi(Y) = X + \mathrm{i}Y = T. \quad ∎$$

LEMMA 4.13. *Let $\mathcal{A} \simeq M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$ be a finite dimensional C^*-algebra and ψ be an involutory anti-automorphism of $\mathcal{A}$ induced from the transpose operation on $M_{n_1}(\mathbb{C}) \oplus M_{n_2}(\mathbb{C}) \oplus \cdots \oplus M_{n_k}(\mathbb{C})$. Then*

$$\mathcal{S}(\mathcal{A},\psi) \simeq \{(X,Y) : X, Y \in M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})\}$$

and $\mathcal{G}(\mathcal{A},\psi)$ is dense in $\mathcal{S}(\mathcal{A},\psi)$.

Proof. It is not hard to verify that

$$\mathcal{S}(\mathcal{A},\psi) \simeq \{(X,Y) : X, Y \in M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})\}.$$

Let X, Y be two self-adjoint elements in $M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})$. Then there exists an orthogonal matrix U in $M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})$ such that UXU^{t} is a diagonal matrix, where U^{t}, the transpose of U, is also in $M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})$.

For any $\varepsilon > 0$, we can find two self-adjoint X_1, Y_1 in $M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})$ such that (i) UX_1U^{t} is a diagonal matrix with distinct eigenvalues, (ii) all entries in UY_1U^{t} are not zero, and (iii) $\|X - X_1\| + \|Y - Y_1\| \leqslant \varepsilon$. From (i) and (ii), we know that X_1 and Y_1 generate $M_{n_1}(\mathbb{R}) \oplus M_{n_2}(\mathbb{R}) \oplus \cdots \oplus M_{n_k}(\mathbb{R})$. Since ε is arbitrary, $\mathcal{G}(\mathcal{A},\psi)$ is dense in $\mathcal{S}(\mathcal{A},\psi)$. ∎

Proof of Theorem 4.10. Note that $\mathcal{A}$ is a unital, separable AF algebra. From [1] there exist a family of increasing finite dimensional C^*-algebras $\{\mathcal{A}_k\}_k$ and an involutory anti-automorphism ψ of $\mathcal{A}$ such that:

(i) $\mathcal{A} = \overline{\bigcup_k} \mathcal{A}_k$,

(ii) $\psi(\mathcal{A}_k) = \mathcal{A}_k$ for each $k \geqslant 1$, and

(iii) $\forall\, k \geqslant 1$, $\psi|_{\mathcal{A}_k}$ is induced from the transpose operation on $\mathcal{A}_k$.

Now the desired result follows from the preceding theorem and the lemma. ∎

4.4. IRRATIONAL ROTATION ALGEBRAS. In this subsection we consider the complex symmetric generator problem for irrational rotation algebras. The following theorem is our main result.

THEOREM 4.14. *For any irrational rotation algebra $\mathcal{A}$, there exist an involutory anti-automorphism ψ of $\mathcal{A}$ and $T \in \mathcal{A}$ such that $\psi(T) = T$, and T generates $\mathcal{A}$ as a C^*-algebra.*

Proof. Let $\mathcal{H} = L^2(\mathbb{T})$ and θ be an irrational number, where $\mathbb{T}$ is the unit circle. Denote $\alpha = \mathrm{e}^{2\pi \mathrm{i}\theta}$. For $f \in \mathcal{H}$, define unitary operators U, V on $\mathcal{H}$ as

$$(Uf)(z) = f(\alpha z), \quad \forall\, z \in \mathbb{T}, \quad (Vf)(z) = zf(z), \quad \forall\, z \in \mathbb{T}.$$

Thus one can check that $UV = \alpha VU$. Denote $\mathcal{A}_\theta = C^*(U, V)$ and $T = U(V^* + V + 3I)$, where I is the identity operator on $\mathcal{H}$.

CLAIM 4.15. T is complex symmetric.

We define a conjugate-linear map J on $\mathcal{H}$ as

$$(Jf)(z) = \overline{f(\overline{\alpha z})}, \quad \forall\, z \in \mathbb{T}.$$

Then it is obvious that J is a conjugation. Moreover, for each $f \in \mathcal{H}$, we check that

$$JUf = U^* Jf \quad \text{and} \quad JVf = \alpha VJf,$$

which implies that $JUJ = U^*$ and $JVJ = \alpha V$. On the other hand, one can see from $UV = \alpha VU$ that $U^*V^* = \alpha V^*U^*$ and $U^*V = \overline{\alpha} VU^*$. This implies

$$\begin{aligned} JTJ &= JU(V^* + V + 3)J = U^*J(V^* + V + 3)J \\ &= U^*(\overline{\alpha}V^* + \alpha V + 3) = (V^* + V + 3)U^* = T^*. \end{aligned}$$

That is, T is complex symmetric.

CLAIM 4.16. T generates $\mathcal{A}_\theta$ as a C^*-algebra.

Note that

$$T^*T = (V^* + V + 3I)U^*U(V^* + V + 3I) = (V^* + V + 3I)^2$$

and $V^* + V + 3I$ is an invertible, positive operator. We have $V^* + V + 3I, (V^* + V + 3I)^{-1} \in C^*(T)$. It follows that $C^*(T)$ contains $V + V^*$ and U.

Noting that $UV = \alpha VU$ and $UV^* = \overline{\alpha} V^* U$, we have

$$U(V + V^*) = (\alpha V + \overline{\alpha} V^*)U.$$

Hence $\alpha V + \overline{\alpha} V^* = U(V + V^*)U^* \in C^*(T)$. Since $V + V^* \in C^*(T)$ and $\alpha^2 \neq 1$, one can deduce that $V \in C^*(T)$. Thus we have proved that $U, V \in C^*(T)$, which implies that $C^*(T) = C^*(U, V) = \mathcal{A}_\theta$. This proves Claim 4.16.

Obviously, the map $\varphi(X) = JX^*J$ is an involutory anti-automorphism of $\mathcal{A}_\theta$ and $\varphi(T) = T$. Since each irrational rotation algebra is $*$-isomorphic to such a C^*-algebra $\mathcal{A}_\theta$, we complete the proof. ∎

4.5. REDUCED FREE PRODUCTS OF UNITAL C^*-ALGEBRAS. The concept of reduced free products of unital C^*-algebras was introduced by Voiculescu in the context of his free probability theory (see [44]).

Assume that $\mathcal{A}_i$, $i = 1,2$, is a separable unital C^*-algebra with a faithful tracial state τ_i. For each $i = 1,2$, let $\mathcal{H}_i = L^2(\mathcal{A}_i, \tau_i)$ be the canonical Hilbert space associated with $(\mathcal{A}_i, \tau_i)$. If no confusion arises, we will view an element A in $\mathcal{A}_i$ also as a vector $\widehat{A}$ in $\mathcal{H}_i$. We let $\pi_i^{\mathrm{L}}, \pi_i^{\mathrm{R}}$ be the left, and right respectively, regular representations of $\mathcal{A}_i$ on $\mathcal{H}_i$, i.e.

$$\pi_i^{\mathrm{L}}(A)\widehat{B} = \widehat{AB} \quad \text{and} \quad \pi_i^{\mathrm{R}}(A)\widehat{B} = \widehat{BA^*}, \quad \forall\, A, B \in \mathcal{A}_i. \tag{4.8}$$

Let $\overset{\circ}{\mathcal{H}}_i = \mathcal{H}_i \ominus \mathbb{C}\widehat{I}_{\mathcal{A}_i}$ for $i = 1,2$. The Hilbert space free product of $(\mathcal{H}_1, \widehat{I}_{\mathcal{A}_1})$ and $(\mathcal{H}_2, \widehat{I}_{\mathcal{A}_2})$ is given by

$$\mathcal{H} = (\mathcal{H}_1, \widehat{I}_{\mathcal{A}_1}) * (\mathcal{H}_2, \widehat{I}_{\mathcal{A}_2}) = \mathbb{C}\widehat{I} \oplus \bigoplus_{n\geqslant 1} \Big(\bigoplus_{\substack{j_1\neq j_2\neq\cdots\neq j_n \\ 1\leqslant j_1,\ldots,j_n\leqslant 2}} \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n} \Big),$$

where $\widehat{I}$ is the distinguished unit vector in $\mathcal{H}$. Let, for $i = 1,2$,

$$\mathcal{H}(i) = \mathbb{C}\widehat{I} \oplus \bigoplus_{n\geqslant 1} \Big(\bigoplus_{\substack{i\neq j_1\neq j_2\neq\cdots\neq j_n \\ 1\leqslant j_1,\ldots,j_n\leqslant 2}} \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n} \Big).$$

We can define unitary operators $V_i : \mathcal{H}_i \otimes \mathcal{H}(i) \to \mathcal{H}$ as follows:

$$\begin{gathered}
\widehat{I}_{\mathcal{A}_i} \otimes \widehat{I} \longmapsto \widehat{I}, \\
\overset{\circ}{\mathcal{H}}_i \otimes \widehat{I} \longmapsto \overset{\circ}{\mathcal{H}}_i, \\
\widehat{I}_{\mathcal{A}_i} \otimes (\overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}) \longmapsto \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}, \\
\overset{\circ}{\mathcal{H}}_i \otimes (\overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}) \longmapsto \overset{\circ}{\mathcal{H}}_i \otimes \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}.
\end{gathered}$$

Let λ_i be the representation of $\mathcal{A}_i$ on $\mathcal{H}$ given by

$$\lambda_i(A) = V_i(\pi_i^{\mathrm{L}}(A) \otimes I_{\mathcal{H}(i)})V_i^*, \quad \forall\, A \in \mathcal{A}_i.$$

Then the reduced free product of $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$, or the reduced free product of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to τ_1 and τ_2, is the C^*-algebra generated by $\lambda_1(\mathcal{A}_1)$ and $\lambda_2(\mathcal{A}_2)$ in $\mathcal{B}(\mathcal{H})$, and is denoted by

$$(\mathcal{A}_1, \tau_1) *_{\mathrm{red}} (\mathcal{A}_2, \tau_2).$$

THEOREM 4.17. *Assume that $\mathcal{A}_i$, $i = 1,2$, is a separable abelian unital C^*-algebra with a faithful tracial state τ_i and $(\mathcal{A}_1, \tau_1) *_{\mathrm{red}} (\mathcal{A}_2, \tau_2)$ is the reduced free product of $\mathcal{A}_1$ and $\mathcal{A}_2$ with respect to τ_1 and τ_2. Then the following are true:*

(i) *There exists an involutory anti-automorphism ψ of $(\mathcal{A}_1, \tau_1) *_{\mathrm{red}} (\mathcal{A}_2, \tau_2)$ such that*

$$\psi(T) = T, \quad \forall\, T \in \lambda_i(\mathcal{A}_i),\ i = 1,2.$$

(ii) *If both $\mathcal{A}_1$ and $\mathcal{A}_2$ are generated by a single self-adjoint element, then there exists $T \in (\mathcal{A}_1, \tau_1) *_{\mathrm{red}} (\mathcal{A}_2, \tau_2)$ such that:*

(a) $(\mathcal{A}_1, \tau_1) *_{\text{red}} (\mathcal{A}_2, \tau_2)$ *is generated by* T *as a* C^**-algebra, and*
(b) $\psi(T) = T$.

Proof. (ii) follows directly from (i). So we will need only to prove (i).

We will follow the notations used in the discussion before the theorem. Let

$$\mathcal{A} = (\mathcal{A}_1, \tau_1) *_{\text{red}} (\mathcal{A}_2, \tau_2).$$

If no confusion arises, we will view an element A in $\mathcal{A}_i$ also as a vector $\widehat{A}$ in $\mathcal{H}_i$ for $i = 1, 2$. Let J_i be a conjugation on $\mathcal{H}_i$ such that $J_i(\widehat{A}) = \widehat{A}^*$ for all $A \in \mathcal{A}_i$. Then

$$J_i^2 = I_{\mathcal{A}_i} \quad \text{and} \quad J_i(\pi_i^{\mathrm{L}}(A))J_i = \pi_i^{\mathrm{R}}(A), \quad \forall\, A \in \mathcal{A}_i, \tag{4.9}$$

where π_i^{L} and π_i^{L} are as in equation (4.8). We define a conjugate-linear operator

$$J : \mathcal{H} \to \mathcal{H}$$

as follows: $J(\widehat{I}\,) = \widehat{I}$ and $\forall\, \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} \in \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}$ with $1 \leqslant j_1 \neq j_2 \neq \cdots \neq j_n \leqslant 2$,

$$J(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) = (J_{j_n}\xi_{j_n}) \otimes \cdots \otimes (J_{j_1}\xi_{j_1}).$$

It is not hard to verify that $J^2 = I$.

We also define a linear unitary operator $U : \mathcal{H} \to \mathcal{H}$ as follows: $U(\widehat{I}\,) = \widehat{I}$ and $\forall\, \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} \in \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}$ with $1 \leqslant j_1 \neq j_2 \neq \cdots \neq j_n \leqslant 2$,

$$U(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) = \xi_{j_n} \otimes \cdots \otimes \xi_{j_1}.$$

It is easy to see that

$$U^2 = I \quad \text{and} \quad (UJ)^2 = I. \tag{4.10}$$

CLAIM 4.18. We have

$$UJ\lambda_i(A)JU = \lambda_i(A^*), \quad \forall\, A \in \mathcal{A}_i. \tag{4.11}$$

For $i = 1, 2$, let $\overset{\circ}{\mathcal{A}}_i = \{A \in \mathcal{A}_i : \tau_i(A) = 0\}$. Notice that each $\mathcal{A}_i$ is abelian. We have that

$$\pi_i^{\mathrm{L}}(A) = \pi_i^{\mathrm{R}}(A^*) \in \mathcal{B}(\mathcal{H}_i), \quad \forall\, A \in \mathcal{A}_i. \tag{4.12}$$

Let $A \in \overset{\circ}{\mathcal{A}}_i$. Then

$$\widehat{A} \text{ and } \widehat{A}^* \text{ are in } \overset{\circ}{\mathcal{H}}_i.$$

Therefore

$$UJ\lambda_i(A)JU(\widehat{I}\,) = \widehat{A}^* = \lambda_i(A^*)(\widehat{I}\,).$$

Let $\xi_{j_1} \otimes \cdots \otimes \xi_{j_n} \in \overset{\circ}{\mathcal{H}}_{j_1} \otimes \cdots \otimes \overset{\circ}{\mathcal{H}}_{j_n}$ with $1 \leqslant j_1 \neq j_2 \neq \cdots \neq j_n \leqslant 2$. We will consider the following two cases.

Case 1. If $i \neq j_1 \neq j_2 \neq \cdots \neq j_n$, we have

$$\begin{aligned} UJ\lambda_i(A)JU(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) &= UJ\lambda_i(A)(J_{j_1}\xi_{j_1} \otimes \cdots \otimes J_{j_n}\xi_{j_n}) \\ &= UJ(\widehat{A} \otimes J_{j_1}\xi_{j_1} \otimes \cdots \otimes J_{j_n}\xi_{j_n}) \\ &= U(\xi_{j_n} \otimes \cdots \otimes \xi_{j_1} \otimes J_i\widehat{A}) \\ &= \widehat{A}^* \otimes \xi_{j_1} \otimes \cdots \otimes \xi_{j_n} = \lambda_i(A^*)(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}). \end{aligned}$$

Case 2. If $i = j_1 \neq j_2 \neq \cdots \neq j_n$, we have

$$\begin{aligned} UJ\lambda_i(A)JU(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}) &= UJ\lambda_i(A)(J_{j_1}\xi_{j_1} \otimes \cdots \otimes J_{j_n}\xi_{j_n}) \\ &= UJ((\pi_i^{\mathrm{L}}(A)J_{j_1}\xi_{j_1}) \otimes J_{j_2}\xi_{j_2} \otimes \cdots \otimes J_{j_n}\xi_{j_n}) \\ &= U(\xi_{j_n} \otimes \cdots \otimes \xi_{j_2} \otimes (J_{j_1}\pi_i^{\mathrm{L}}(A)J_{j_1}\xi_{j_1})) \\ &= U(\xi_{j_n} \otimes \cdots \otimes \xi_{j_2} \otimes (\pi_i^{\mathrm{R}}(A)\xi_{j_1})) \quad \text{(because of (4.9))} \\ &= U(\xi_{j_n} \otimes \cdots \otimes \xi_{j_2} \otimes (\pi_i^{\mathrm{L}}(A^*)\xi_{j_1})) \quad \text{(because of (4.12))} \\ &= (\pi_i^{\mathrm{L}}(A^*)\xi_{j_1}) \otimes \xi_{j_2} \otimes \cdots \otimes \xi_{j_n} \\ &= \lambda_i(A^*)(\xi_{j_1} \otimes \cdots \otimes \xi_{j_n}). \end{aligned}$$

As a summary, we have

$$UJ\lambda_i(A)JU(\xi) = \lambda_i(A^*)(\xi), \quad \forall\, A \in \mathring{\mathcal{A}}_i,\ \forall\, \xi \in \mathcal{H},$$

whence

$$UJ\lambda_i(A)JU = \lambda_i(A^*), \quad \forall\, A \in \mathcal{A}_i.$$

This ends the proof of the claim.

We continue the proof of the theorem. We define a mapping $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ as follows:

$$\psi(T) = UJT^*JU, \quad \forall\, T \in \mathcal{A}.$$

Recall that $\mathcal{A}$ is the C^*-algebra generated by $\lambda_1(\mathcal{A}_1)$ and $\lambda_2(\mathcal{A}_2)$ in $\mathcal{B}(\mathcal{H})$. By equation (4.11), we have that

$$(UJ)\mathcal{A}(JU) \subseteq \mathcal{A}. \tag{4.13}$$

Combining equations (4.10), (4.11) and equation (4.13), we know that ψ is an involutory anti-automorphism of $\mathcal{A}$ and $\psi(T) = T$, $\forall\, T \in \lambda_i(\mathcal{A}_i)$ for $i = 1, 2$. This ends the proof of (i). ∎

EXAMPLE 4.19. Let $C[0,1]$ be the C^*-algebra consisting all continuous functions on $[0,1]$ and τ be a faithful state of $C[0,1]$ defined by

$$\tau(f) = \int_0^1 f(t)\mathrm{d}t, \quad \forall\, f \in C[0,1].$$

Let

$$\mathcal{A} = (C[0,1], \tau) *_{\mathrm{red}} (C[0,1], \tau)$$

be a reduced free product C^*-algebra. Then, by Theorem 4.17, there exist an involutory anti-automorphism ψ of $\mathcal{A}$ and an element T in $\mathcal{A}$ such that (i) $\mathcal{A}$ is generated by T as a C^*-algebra, and (ii) $\psi(T) = T$.

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