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COWEN–DOUGLAS TUPLES AND FIBER DIMENSIONS

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ABSTRACT. Let $T \in L(X)^n$ be a Cowen–Douglas tuple on a Banach space X. We use functional representations of T to associate with each T-invariant subspace $Y \subset X$ an integer called the fiber dimension $\mathrm{fd}(Y)$ of Y. Among other results we prove a limit formula for the fiber dimension, show that it is invariant under suitable changes of Y and deduce a dimension formula for pairs of homogeneous invariant subspaces of graded Cowen–Douglas tuples on Hilbert spaces.

KEYWORDS: *Cowen–Douglas tuples, fiber dimension, Samuel multiplicity, holomorphic model spaces.*

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INTRODUCTION

Let $\mathcal{H} \subset \mathcal{O}(\Omega, \mathbb{C}^N)$ be a functional Hilbert space of $\mathbb{C}^N$-valued analytic functions on a domain $\Omega \subset \mathbb{C}^n$. The number

$$\mathrm{fd}(\mathcal{H}) = \max_{\lambda \in \Omega} \dim \mathcal{H}_\lambda,$$

where $H_\lambda = \{f(\lambda) : f \in \mathcal{H}\}$, is usually referred to as the *fiber dimension* of $\mathcal{H}$. Results going back to Cowen and Douglas [8], Curto and Salinas [9] show that each Cowen–Douglas tuple $T \in L(H)^n$ on a Hilbert space H is locally unitarily equivalent to the tuple $M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(\mathcal{H})^n$ of multiplication operators with the coordinate functions on a suitable analytic functional Hilbert space $\mathcal{H}$. In the present note we use corresponding model theorems for Cowen–Douglas tuples $T \in L(X)^n$ on Banach spaces to associate with each T-invariant subspace $Y \subset X$ an integer $\mathrm{fd}(Y)$ called the fiber dimension of Y. We thus extend results proved by L. Chen, G. Cheng and X. Fang in [5] for single Cowen–Douglas operators on Hilbert spaces to the case of commuting operator systems on Banach spaces.

By definition a commuting tuple $T = (T_1, \ldots, T_n) \in L(X)^n$ of bounded operators on a Banach space X is a *weak Cowen–Douglas tuple* of *rank* $N \in \mathbb{N}$ on

Ω if

$$\dim X/\sum_{i=1}^{n}(\lambda_i - T_i)X = N$$

for each point $\lambda \in \Omega$. We call T a *Cowen–Douglas tuple* if in addition

$$\bigcap_{\lambda\in\Omega}\sum_{i=1}^{n}(\lambda_i - T_i)X = \{0\}.$$

We show that weak Cowen–Douglas tuples $T \in L(X)^n$ admit local representations as multiplication tuples $M_z \in L(\widehat{X})^n$ on suitable functional Banach spaces $\widehat{X}$ and prove that Cowen–Douglas tuples can be characterized as those commuting tuples $T \in L(X)^n$ that are locally jointly similar to a multiplication tuple $M_z \in L(\widehat{X})^n$ on a divisible holomorphic model space $\widehat{X}$. We use the functional representations of weak Cowen–Douglas tuples $T \in L(X)^n$ to associate with each linear subspace $Y \subset X$ invariant for T an integer $\mathrm{fd}(Y)$ called the *fiber dimension* of Y.

Based on the observation that the fiber dimension $\mathrm{fd}(Y)$ of a closed T-invariant subspace $Y \in \mathrm{Lat}(T)$ is closely related to the *Samuel multiplicity* of the quotient tuple $S = T/Y \in L(X/Y)^n$ on Ω we show that the fiber dimension of $Y \in \mathrm{Lat}(T)$ can be calculated by a limit formula

$$\mathrm{fd}(Y) = n!\lim_{k\to\infty}\frac{\dim(Y + M_k(T-\lambda)/M_k(T-\lambda))}{k^n} \quad (\lambda \in \Omega),$$

where $M_k(T-\lambda) = \sum_{|\alpha|=k}(T-\lambda)^{\alpha}X$. Furthermore, we show how to calculate the fiber dimension using the *sheaf model* of T on Ω. We deduce that the fiber dimension is invariant against suitable changes of Y and show that the fiber dimension for graded Cowen–Douglas tuples $T \in L(H)^n$ on Hilbert spaces satisfies the dimension formula

$$\mathrm{fd}(Y_1 \vee Y_2) + \mathrm{fd}(Y_1 \cap Y_2) = \mathrm{fd}(Y_1) + \mathrm{fd}(Y_2)$$

for any pair of homogeneous invariant subspaces $Y_1, Y_2 \in \mathrm{Lat}(T)$. The proof is based on an idea from [6] (see also [5]) where a corresponding result is proved for analytic functional Hilbert spaces given by a complete Nevanlinna–Pick kernel.

1. FIBER DIMENSION FOR INVARIANT SUBSPACES

Let $\Omega \subset \mathbb{C}^n$ be a domain, that is, a connected open set in $\mathbb{C}^n$. Let D be a finite-dimensional vector space and let $M \subset \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$-submodule. We denote the point evaluations on M by

$$\epsilon_\lambda : M \to D, \quad f \mapsto f(\lambda) \quad (\lambda \in \Omega).$$

For $\lambda \in \Omega$, the range of ϵ_λ is a linear subspace

$$M_\lambda = \{f(\lambda) : f \in M\} \subset D.$$

DEFINITION 1.1. The number

$$\mathrm{fd}(M) = \max_{z \in \Omega} \dim M_z$$

is called the *fiber dimension* of M. A point $z_0 \in \Omega$ with $\dim M_{z_0} = \mathrm{fd}(M)$ is called a *maximal point* for M.

For any $\mathbb{C}[z]$-submodule $M \subset \mathcal{O}(\Omega, D)$ and any point $\lambda \in \Omega$, we have

$$\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \subset \ker \epsilon_\lambda.$$

Under the condition that the codimension of $\sum_{i=1}^{n} (\lambda_i - M_{z_i})M$ is constant on Ω, the question whether equality holds here is closely related to corresponding properties of the fiber dimension of M.

LEMMA 1.2. *Consider a $\mathbb{C}[z]$-submodule $M \subset \mathcal{O}(\Omega, D)$ such that there is an integer N with*

$$\dim M / \sum_{i=1}^{n} (\lambda_i - M_{z_i})M \equiv N$$

for all $\lambda \in \Omega$. Then $\mathrm{fd}(M) \leqslant N$. If $\mathrm{fd}(M) < N$, then

$$\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \subsetneq \ker \epsilon_\lambda$$

for all $\lambda \in \Omega$. If $\mathrm{fd}(M) = N$, then there is a proper analytic set $A \subset \Omega$ with

$$\Omega \backslash A \subset \{\lambda \in \Omega : \dim M_\lambda = N\} = \Big\{\lambda \in \Omega : \sum_{i=1}^{n} (\lambda_i - M_{z_i})M = \ker \epsilon_\lambda \Big\}.$$

Proof. Since the maps

$$M / \sum_{i=1}^{n} (\lambda_i - M_{z_i})M \to M / \ker \epsilon_\lambda \cong \mathrm{Im}\, \epsilon_\lambda, \quad [m] \mapsto [m]$$

are surjective for $\lambda \in \Omega$, it follows that $\mathrm{fd}(M) \leqslant N$ and that

$$\{\lambda \in \Omega : \dim M_\lambda = N\} = \Big\{\lambda \in \Omega : \sum_{i=1}^{n} (\lambda_i - M_{z_i})M = \ker \epsilon_\lambda \Big\}.$$

Hence, if $\mathrm{fd}(M) < N$, then $\sum_{i=1}^{n} (\lambda_i - M_{z_i})M \subsetneq \ker \epsilon_\lambda$ for all $\lambda \in \Omega$. A standard argument (cf. Lemma 1.4 in [11] and its proof) shows that there is a proper analytic set $A \subset \Omega$ such that

$$\Omega \backslash A \subset \{\lambda \in \Omega : \dim M_\lambda = \mathrm{fd}(M)\}.$$

This observation completes the proof. ∎

In the following we show that the concept of fiber dimension defined in [5] for invariant subspaces of Cowen–Douglas operators on Hilbert spaces admits a natural extension to the multivariable Banach space setting.

Let $T = (T_1, \dots, T_n) \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X. For $z \in \mathbb{C}^n$, we use the notation $z - T$ both for the commuting tuple $z - T = (z_1 - T_1, \dots, z_n - T_n)$ and for the row operator

$$z - T : X^n \to X, \quad (x_i)_{i=1}^n \mapsto \sum_{i=1}^n (z_i - T_i)x_i.$$

With this notation, we have $\sum\limits_{i=1}^n (z_i - T_i)X = \operatorname{Im}(z - T)$. We denote by $\operatorname{Lat}(T)$ the set of all closed subspaces $Y \subseteq X$ which are invariant under each component T_i of T. For $Y \in \operatorname{Lat}(T)$, we write $T|_Y = (T_1|_Y, \dots, T_n|_Y) \in L(Y)^n$ for the restriction of T to Y and $T/Y = (T_1/Y, \dots, T_n/Y) \in L(X/Y)^n$, where

$$T_i/Y : X/Y \to X/Y, \quad [x] \mapsto [T_i x],$$

for the induced quotient tuple on the quotient space X/Y. Note that, when X is a Hilbert space, the tuple T/Y is unitarily equivalent to the tuple of compressions $P_{Y^\perp} T_i|_{Y^\perp} \in L(Y^\perp)$ on the orthogonal complement of Y.

DEFINITION 1.3. Let $T \in L(X)^n$ be a commuting tuple of bounded operators on X and let $\Omega \subset \mathbb{C}^n$ be a fixed domain. We call T a *weak Cowen–Douglas tuple of rank* $N \in \mathbb{N}$ on Ω if

$$\dim \Big(X / \sum_{i=1}^n (z_i - T_i)X \Big) = N$$

for all $z \in \Omega$. If in addition the condition

$$\bigcap_{z \in \Omega} \operatorname{Im}(z - T) = \{0\}$$

holds, then T is called a *Cowen–Douglas tuple* of rank N on Ω.

If $X = H$ is a Hilbert space, then a tuple $T \in L(H)^n$ is a Cowen–Douglas tuple on Ω if and only if the adjoint $T^* = (T_1^*, \dots, T_n^*)$ is a tuple of class $B_n(\Omega^*)$ on the complex conjugate domain $\Omega^* = \{\bar{z} : z \in \Omega\}$ in the sense of Curto and Salinas [9]. One can show ([24], Theorem 4.12) that, for a weak Cowen–Douglas tuple $T \in L(X)^n$ on a domain $\Omega \subset \mathbb{C}^n$, the identity

$$\bigcap_{z \in \Omega} \operatorname{Im}(z - T) = \bigcap_{k=0}^{\infty} \sum_{|\alpha| = k} (\lambda - T)^\alpha X$$

holds for every point $\lambda \in \Omega$. In particular, if $T \in L(X)^n$ is a Cowen–Douglas tuple on Ω, then it is a Cowen–Douglas tuple on each smaller domain $\emptyset \neq \Omega_0 \subset \Omega$.

DEFINITION 1.4. Let $\Omega \subset \mathbb{C}^n$ be open. A *holomorphic model space* of rank N over Ω is a Banach space $\widehat{X} \subset \mathcal{O}(\Omega, D)$ such that D is an N-dimensional complex vector space and

(i) $M_z \in L(\widehat{X})^n$,

(ii) for each $\lambda \in \Omega$, the point evaluation $\epsilon_\lambda : \widehat{X} \to D, \widehat{x} \mapsto \widehat{x}(\lambda)$, is continuous and surjective.

A holomorphic model space $\widehat{X}$ on Ω is called *divisible* if in addition, for $\widehat{x} \in \widehat{X}$ and $\lambda \in \Omega$ with $\widehat{x}(\lambda) = 0$, there are functions $\widehat{y}_1,\ldots,\widehat{y}_n \in \widehat{X}$ with

$$\widehat{x} = \sum_{i=1}^{n} (\lambda_i - M_{z_i})\widehat{y}_i.$$

The multiplication tuple $M_z = (M_{z_1}, \ldots, M_{z_n})$ on a divisible holomorphic model space $\widehat{X} \subset \mathcal{O}(\Omega, D)$ is easily seen to be a Cowen–Douglas tuple of rank $N = \dim D$ on Ω.

In the following let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a fixed domain $\Omega \subset \mathbb{C}^n$. We equip X with the $\mathbb{C}[z]$-module structure defined by $\mathbb{C}[z] \times X \to X, (p, x) \mapsto p(T)x$. For single Cowen–Douglas operators on Hilbert spaces, the following notion was defined in [5].

DEFINITION 1.5. Let $\emptyset \neq \Omega_0 \subset \Omega$ be a connected open set. A CF-*representation* of T on Ω_0 is a $\mathbb{C}[z]$-module homomorphism

$$\rho : X \to \mathcal{O}(\Omega_0, D)$$

with a finite-dimensional complex vector space D such that:

(i) $\ker \rho = \bigcap_{z \in \Omega} (z - T)X^n$,

(ii) the submodule $\widehat{X} = \rho X \subset \mathcal{O}(\Omega_0, D)$ satisfies

$$\operatorname{fd}(\widehat{X}) = \dim \widehat{X} / \sum_{i=1}^{n} (\lambda_i - M_{z_i})\widehat{X}$$

for all $\lambda \in \Omega_0$.

Let $\mathcal{O}(\Omega_0, D)$ be equipped with its canonical Fréchet space topology. Our first aim is to show that weak Cowen–Douglas tuples possess sufficiently many CF-representations that are continuous and satisfy certain additional properties.

THEOREM 1.6. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on Ω. For each point $\lambda_0 \in \Omega$, there is a* CF*-representation $\rho : X \to \mathcal{O}(\Omega_0, D)$ of T on a connected open neighbourhood $\Omega_0 \subset \Omega$ of λ_0 such that:*

(i) *$\rho : X \to \mathcal{O}(\Omega_0, D)$ is continuous;*

(ii) *$\widehat{X} = \rho(X)$ equipped with the norm $\|\rho(X)\| = \|x + \ker \rho\|$ is a divisible holomorphic model space of rank N on Ω_0.*

Proof. Let $\lambda_0 \in \Omega$ be arbitrary. Choose a linear subspace $D \subset X$ such that

$$X = (\lambda_0 - T)X^n \oplus D.$$

Then $\dim D = N$. The analytic operator-valued function

$$T(z) : X^n \oplus D \to X, \quad ((x_i)_{i=1}^n, y) \mapsto \sum_{i=1}^n (z_i - T_i)x_i + y$$

of bounded operators between Banach spaces is onto at $z = \lambda_0$. By Lemma 2.1.5 in [15] there is an open polydisc $\Omega_0 \subset \Omega$ such that the induced map

$$\mathcal{O}(\Omega_0, X^n \oplus D) \to \mathcal{O}(\Omega_0, X), \quad ((g_i)_{i=1}^n, h) \mapsto \sum_{i=1}^n (z_i - T_i)g_i + h$$

is onto. In particular, for each $z \in \Omega_0$, the linear map

$$D \to X / \sum_{i=1}^n (z_i - T_i)X, \quad x \mapsto [x]$$

is surjective between N-dimensional complex vector spaces. Hence these maps are isomorphisms and, for each $x \in X$ and $z \in \Omega_0$, there is a unique vector $x(z) \in D$ with $x - x(z) \in \sum_{i=1}^n (z_i - T_i)X$. By construction, for each $x \in X$, the mapping $\Omega_0 \to D, z \mapsto x(z)$, is analytic. The induced mapping

$$\rho : X \to \mathcal{O}(\Omega_0, D), \quad x \mapsto x(\cdot)$$

is linear with

$$\ker \rho = \bigcap_{z \in \Omega_0} \sum_{i=1}^n (z_i - T_i)X = \bigcap_{z \in \Omega} \sum_{i=1}^n (z_i - T_i)X.$$

For $x \in X, z \in \Omega_0$ and $j = 1, \ldots, n$,

$$T_j x - z_j x(z) = T_j(x - x(z)) - (z_j - T_j)x(z) \in \sum_{i=1}^n (z_i - T_i)X.$$

Hence ρ is a $\mathbb{C}[z]$-module homomorphism. Equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$, the space $\widehat{X} = \rho(X)$ is a Banach space and $M_z \in L(\widehat{X})^n$ is a commuting tuple of bounded operators on $\widehat{X}$. By definition

$$\rho(x) \equiv x \quad \text{for } x \in D.$$

Hence the point evaluations $\epsilon_z : \widehat{X} \to D$ ($z \in \Omega_0$) are surjective. Since the mappings

$$q_z : D \to X / \sum_{i=1}^n (z_i - T_i)X, \quad x \mapsto [x] \quad (z \in \Omega_0)$$

are topological isomorphisms and since the compositions

$$X \to X / \sum_{i=1}^n (z_i - T_i)X, \quad x \mapsto q_z(\epsilon_z(\rho(x))) = [x]$$

are continuous, it follows that the point evaluations $\epsilon_z : \widehat{X} \to D$ ($z \in \Omega_0$) are continuous. Thus we have shown that $\widehat{X} \subset \mathcal{O}(\Omega_0, D)$ with the norm induced by ρ is a holomorphic model space.

To see that $\widehat{X}$ is divisible, fix a vector $x \in X$ and a point $\lambda \in \Omega_0$ such that $x(\lambda) = 0$. Then there are vectors $x_1, \dots, x_n \in X$ with $x = \sum_{i=1}^{n} (\lambda_i - T_i)x_i$. Hence

$$\rho(x) = \sum_{i=1}^{n} (\lambda_i - z_i)\rho(x_i) \in \sum_{i=1}^{n} (\lambda_i - M_{z_i})\widehat{X}.$$

To conclude the proof, it suffices to observe that

$$\dim\left(\widehat{X}/\sum_{i=1}^{n}(\lambda_i - M_{z_i})\widehat{X}\right) = \dim(\widehat{X}/\ker \epsilon_\lambda) = \dim(\operatorname{Im} \epsilon_\lambda) = \dim D = N$$

for all $z \in \Omega_0$. ∎

Note that, for a Cowen–Douglas tuple $T \in L(X)^n$ on a Banach space X, the mappings $\rho : X \to \widehat{X} \subset \mathcal{O}(\Omega_0, D)$ constructed in the previous proof are isometric joint similarities between $T \in L(X)^n$ and the tuples $M_z \in L(\widehat{X})^n$ on the divisible holomorphic model space $\widehat{X} \subset \mathcal{O}(\Omega_0, D)$.

COROLLARY 1.7. *A commuting tuple $T \in L(X)^n$ is a Cowen–Douglas tuple of rank N on a given domain $\Omega \subset \mathbb{C}^n$ if and only if, for each $\lambda \in \Omega$, there exist a connected open neighbourhood $\Omega_0 \subset \Omega$ of λ and a joint similarity between T and the multiplication tuple $M_z \in L(\widehat{X})^n$ on a divisible holomorphic model space $\widehat{X}$ of rank N on Ω_0.*

Proof. The necessity of the stated condition follows from Theorem 1.6 and the subsequent remarks. Since the tuple $M_z \in L(\widehat{X})^n$ on a divisible holomorphic model space of rank N is a Cowen–Douglas tuple of rank N and since similarity preserves this property, also the sufficiency is clear. ∎

The preceding result should be compared with Corollary 4.39 in [24], where a characterization of Cowen–Douglas tuples on suitable admissible domains in $\mathbb{C}^n$ is obtained.

There is a canonical way to associate with each weak Cowen–Douglas tuple of rank N on $\Omega \subset \mathbb{C}^n$ a Cowen–Douglas tuple of rank N.

COROLLARY 1.8. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$. Then the quotient tuple*

$$T^{\mathrm{CD}} = T/\bigcap_{z\in\Omega}\sum_{i=1}^{n}(z_i - T_i)X$$

defines a Cowen–Douglas tuple of rank N on Ω.

Proof. Fix $z_0 \in \Omega$. Choose a CF-representation $\rho : X \to \mathcal{O}(\Omega_0, D)$ as in Theorem 1.6. Then $\widehat{X} = \rho(X) \subset \mathcal{O}(\Omega_0, D)$ is a divisible holomorphic model space of rank N on Ω_0. Since

$$\ker \rho = \bigcap_{z\in\Omega}\sum_{i=1}^{n}(z_i - T_i)X,$$

the map ρ induces a similarity between T^{CD} and $M_z \in L(\widehat{X})^n$. By Corollary 1.7 the tuple T^{CD} is a Cowen–Douglas tuple of rank N on Ω. ∎

As before, let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$. Our next aim is to show that, for each closed T-invariant subspace $Y \in \mathrm{Lat}(T)$, the fiber dimension of Y can be defined as

$$\mathrm{fd}(Y) = \mathrm{fd}(\rho(Y)),$$

where ρ is an arbitrary CF-representation of T. To show that the number $\mathrm{fd}(\rho(Y))$ is independent of the chosen CF-representation ρ, we first observe that the equation $\mathrm{fd}(\rho_1(Y)) = \mathrm{fd}(\rho_2(Y))$ holds for each pair of CF-representations ρ_1, ρ_2 over domains $\Omega_1, \Omega_2 \subset \Omega$ with non-trivial intersection.

LEMMA 1.9. *Let $\Omega_1, \Omega_2 \subset \mathbb{C}^n$ be domains with $\Omega_1 \cap \Omega_2 \neq \emptyset$. Let $M_i \subset \mathcal{O}(\Omega_i, D_i)$ be $\mathbb{C}[z]$-submodules with finite-dimensional vector spaces D_i such that*

$$\mathrm{fd}(M_i) = \dim M_i/(\lambda - M_z)M_i^n \quad (i = 1, 2, \lambda \in \Omega_i).$$

Suppose that there is a $\mathbb{C}[z]$-module isomorphism $U : M_1 \to M_2$. Then, for any submodule $M \subset M_1$, we have

$$\mathrm{fd}(M) = \mathrm{fd}(UM).$$

Proof. Using Lemma 1.4 in [11] as well as elementary properties of analytic sets, we can choose a proper analytic subset $A \subset \Omega_1 \cap \Omega_2$ such that each point $\lambda \in (\Omega_1 \cap \Omega_2) \setminus A$ is maximal for M, M_1 and UM. Fix such a point λ. For $f, g \in M$ with $f(\lambda) = g(\lambda)$, by Lemma 1.2 there are functions $h_1, \ldots, h_n \in M_1$ such that $f - g = \sum_{i=1}^{n} (\lambda_i - M_{z_i})h_i$. But then also

$$U(f - g) = \sum_{i=1}^{n} (\lambda_i - M_{z_i})Uh_i.$$

Hence we obtain a well-defined surjective linear map $U_\lambda : M_\lambda \to (UM)_\lambda$ by setting

$$U_\lambda x = (Uf)(\lambda) \quad \text{if } f \in M \text{ with } f(\lambda) = x.$$

It follows that $\mathrm{fd}(M) = \dim M_\lambda \geqslant \dim(UM)_\lambda = \mathrm{fd}(UM)$. By applying the same argument to U^{-1} and UM instead of U and M we find that also $\mathrm{fd}(UM) \geqslant \mathrm{fd}(M)$. ∎

If $\rho_i : X \to \mathcal{O}(\Omega_i, D_i)$ $(i = 1, 2)$ are CF-representations on domains $\Omega_i \subset \Omega$ with non-trival intersection $\Omega_1 \cap \Omega_2 \neq \emptyset$, then the submodules $M_i = \rho_i X \subset \mathcal{O}(\Omega_i, D_i)$ are canonically isomorphic

$$M_1 \cong X/\ker \rho_1 = X/\ker \rho_2 \cong M_2$$

as $\mathbb{C}[z]$-modules. As an application of the previous result one obtains that

$$\mathrm{fd}(\rho_1 Y) = \mathrm{fd}(\rho_2 Y)$$

for each linear subspace $Y \subset X$ which is invariant for T.

THEOREM 1.10. *Let $\rho_i : X \to \mathcal{O}(\Omega_i, D_i)$ $(i = 1, 2)$ be* CF-*representations of T on domains $\Omega_i \subset \Omega$. Then*

$$\mathrm{fd}(\rho_1 Y) = \mathrm{fd}(\rho_2 Y)$$

for each linear subspace $Y \subset X$ which is invariant for T.

Proof. Since Ω is connected, there is a continuous path $\gamma : [0,1] \to \Omega$ with $\gamma(0) \in \Omega_1$ and $\gamma(1) \in \Omega_2$. By Theorem 1.6 there is a family $(\rho_z)_{z \in \mathrm{Im}\gamma}$ of CF-representations $\rho_z : X \to \mathcal{O}(\Omega_z, D_z)$ of T on connected open neighbourhoods $\Omega_z \subset \Omega$ of the points z in $\mathrm{Im}\gamma$ such that $\rho_{\gamma(0)} = \rho_1$ and $\rho_{\gamma(1)} = \rho_2$. Let $\delta > 0$ be a positive number such that each set $A \subset [0,1]$ of diameter less than δ is contained in one of the sets $\gamma^{-1}(\Omega_z)$ (see e.g. Lemma 3.7.2 in [22]). Then we can choose points $z_1 = \gamma(0), z_2, \ldots, z_r = \gamma(1)$ in $\mathrm{Im}\gamma$ such that $\Omega_{z_i} \cap \Omega_{z_{i+1}} \neq \emptyset$ for $i = 1, \ldots, r-1$. Let $Y \subset X$ be a linear T-invariant subspace. By the remarks following Lemma 1.9 we obtain that

$$\mathrm{fd}(\rho_1 Y) = \mathrm{fd}(\rho_{z_2} Y) = \cdots = \mathrm{fd}(\rho_2 Y)$$

as was to be shown. ∎

Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$. Let $Y \subset X$ be a linear subspace that is invariant for T. In view of Theorem 1.10 we can define the *fiber dimension* of Y by

$$\mathrm{fd}(Y) = \mathrm{fd}(\rho Y),$$

where $\rho : X \to \mathcal{O}(\Omega_0, D)$ is an arbitrary CF-representation of T. We are mainly interested in the fiber dimension of closed T-invariant subspaces Y, but the reader should observe that the definition makes perfect sense for linear T-invariant subspaces $Y \subset X$. Since by Theorem 1.6 there are always continuous CF-representations $\rho : X \to \mathcal{O}(\Omega_0, D)$ and since in this case the inclusions

$$\epsilon_\lambda(\rho(\overline{Y})) \subset \overline{\epsilon_\lambda(\rho(Y))} = \epsilon_\lambda(\rho(Y))$$

hold for all $\lambda \in \Omega_0$, it follows that $\mathrm{fd}(Y) = \mathrm{fd}(\overline{Y})$ for each linear T-invariant subspace $Y \subset X$.

It follows from Theorem 1.6 that $\mathrm{fd}(X) = N$. In general, the fiber dimension $\mathrm{fd}(Y)$ of a linear T-invariant subspace $Y \subset X$ is an integer in $\{0, \ldots, N\}$ which depends on Y in a monotone way. Obviously, $\mathrm{fd}(Y) = 0$ if and only if

$$Y \subset \ker \rho = \bigcap_{z \in \Omega} (z - T) X^n.$$

We conclude this section with an alternative characterization of CF-representations.

COROLLARY 1.11. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$ and let $\rho : X \to \mathcal{O}(\Omega_0, D)$ be a $\mathbb{C}[z]$-module homomorphism on a* *domain* $\emptyset \neq \Omega_0 \subset \Omega$ *with a finite-dimensional vector space* D *such that*

$$\ker \rho = \bigcap_{z \in \Omega} (z - T)X^n.$$

Then ρ *is a* CF-*representation of* T *if and only if* $\mathrm{fd}(\rho X) = N$.

Proof. Suppose that $\mathrm{fd}(\rho X) = N$. Define $\widehat{X} = \rho(X)$. Since the maps

$$X/(\lambda - T)X^n \to \widehat{X}/(\lambda - M_z)\widehat{X}^n, \quad [x] \mapsto [\rho x] \quad \text{and}$$
$$\widehat{X}/(\lambda - M_z)\widehat{X}^n \to \widehat{X}_\lambda, \quad [f] \mapsto f(\lambda)$$

are surjective for each $\lambda \in \Omega_0$, it follows that

$$\dim \widehat{X}/(\lambda - M_z)\widehat{X}^n \leqslant N$$

for all $\lambda \in \Omega_0$ and that equality holds on $\Omega_0 \backslash A$ with a suitable proper analytic subset $A \subset \Omega_0$. Equipped with the norm $\|\rho(x)\| = \|x + \ker \rho\|$, the space $\widehat{X}$ is a Banach space and $M_z \in L(\widehat{X})^n$ is a commuting tuple of bounded operators on $\widehat{X}$. A result of Kaballo ([20], Satz 1.5) shows that

$$\left\{ \lambda \in \Omega_0 : \dim \widehat{X}/(\lambda - M_z)\widehat{X}^n > \min_{\mu \in \Omega_0} \dim \widehat{X}/(\mu - M_z)\widehat{X}^n \right\}$$

is a proper analytic subset of Ω_0. Combining these results we find that

$$\dim \widehat{X}/(\lambda - M_z)\widehat{X}^n = N$$

for all $\lambda \in \Omega_0$. Hence ρ is a CF-representation of T.

Conversely, if ρ is a CF-representation of T, then $\mathrm{fd}(\rho X) = N$ by the remarks preceding the corollary. ∎

2. A LIMIT FORMULA FOR THE FIBER DIMENSION

In [17] (Lemma 4) Xiang Fang proved a limit formula for the fiber dimension of submodules of suitable analytic Hilbert modules on domains in $\mathbb{C}^n$. The proof given in [17] is easily seen to extend to the following more general setting (see Lemma 1.4 in [11] for details). Let $\Omega \subset \mathbb{C}^n$ be a domain with $0 \in \Omega$ and let D be a finite-dimensional complex vector space. For $k \in \mathbb{N}$, consider the map $T_k : \mathcal{O}(\Omega, D) \to \mathcal{O}(\Omega, D)$ which associates with each function $f \in \mathcal{O}(\Omega, D)$ its k-th Taylor polynomial, that is,

$$T_k(f)(z) = \sum_{|\alpha| \leqslant k} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha.$$

For a given $\mathbb{C}[z]$-submodule $M \subseteq \mathcal{O}(\Omega, D)$, there is a proper analytic subset A in Ω such that

$$\dim M_z = \max_{w \in \Omega} \dim M_w = n! \lim_{k \to \infty} \frac{\dim T_k(M)}{k^n}$$

holds for all $z \in \Omega \backslash A$.

Based on this observation, we will deduce a similar limit formula for the fiber dimension of invariant subspaces of weak Cowen–Douglas tuples on Ω.

For a commuting tuple $T \in L(X)^n$ of bounded operators on a Banach space X, we write

$$K^\bullet(T,X) : 0 \to \Lambda^0(X) \xrightarrow{\delta_T^0} \Lambda^1(X) \xrightarrow{\delta_T^1} \cdots \xrightarrow{\delta_T^{n-1}} \Lambda^n(X) \to 0$$

for the *Koszul complex* of T (cf. Section 2.2 in [15]). For $i = 0, \ldots, n$, let

$$H^i(T,X) = \ker(\delta_T^i)/\mathrm{Im}(\delta_T^{i-1})$$

be the i-th cohomology group of $K^\bullet(T,X)$. There is a canonical isomorphism $H^n(T,X) \cong X/\sum_{i=1}^n T_i X$ of complex vector spaces.

In the following, given a commuting operator tuple $T \in L(X)^n$ and an invariant subspace $Y \in \mathrm{Lat}(T)$, we denote by

$$R = T|_Y \in L(Y)^n, \quad S = T/Y \in L(Z)^n$$

the restriction of T to Y and the quotient of T modulo Y on $Z = X/Y$. The inclusion $i : X \to Y$ and the quotient map $q : X \to Z$ induce a short exact sequence of complexes

$$0 \to K^\bullet(z-R,Y) \xrightarrow{i} K^\bullet(z-T,X) \xrightarrow{q} K^\bullet(z-S,Z) \to 0.$$

It is a standard fact from homological algebra that there are connecting homomorphisms $d_z^i : H^i(z-S,Z) \to H^{i+1}(z-R,Y)$ $(i = 0, \ldots, n-1)$ such that the induced sequence of cohomology spaces

$$\begin{aligned} 0 &\to H^0(z-R,Y) \xrightarrow{i} H^0(z-T,X) \xrightarrow{q} H^0(z-S,Z) \\ &\xrightarrow{d_z^0} H^1(z-R,Y) \xrightarrow{i} H^1(z-T,X) \xrightarrow{q} H^1(z-S,Z) \\ &\xrightarrow{d_z^1} H^2(z-R,Y) \to \quad \cdots \\ &\xrightarrow{d_z^{n-1}} H^n(z-R,Y) \xrightarrow{i} H^n(z-T,X) \xrightarrow{q} H^n(z-S,Z) \to 0 \end{aligned}$$

is exact again. In particular, we obtain

$$\mathrm{Im}(d_z^{n-1}) = \ker(H^n(z-R,Y) \xrightarrow{i} H^n(z-T,X)) = (Y \cap (z-T)X^n)/(z-R)Y^n.$$

LEMMA 2.1. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$ and let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace of T. Then there is a proper analytic subset $A \subset \Omega$ such that, for all $\lambda \in \Omega \setminus A$,*

$$\dim H^n(\lambda - S, Z) = N - \mathrm{fd}(Y).$$

Proof. Choose a CF-representation $\rho : X \to \mathcal{O}(\Omega_0, D)$ of T on some domain $\Omega_0 \subset \Omega$ as in Theorem 1.6. Let $Y \in \mathrm{Lat}(T)$ be arbitrary. Define $\widehat{X} = \rho(X)$ and

$\widehat{Y} = \rho(Y)$. Since the compositions

$$Y^n \xrightarrow{\lambda - R} Y \xrightarrow{\rho} \mathcal{O}(\Omega_0, D) \xrightarrow{\epsilon_\lambda} D \quad (\lambda \in \Omega_0)$$

are zero, we obtain well-defined surjective linear maps

$$\delta_\lambda : H^n(\lambda - R, Y) \to \widehat{Y}_\lambda, \quad [y] \mapsto \rho(y)(\lambda).$$

Obviously, for each $\lambda \in \Omega_0$, the inclusion

$$\mathrm{Im} d_\lambda^{n-1} = (Y \cap (\lambda - T)X^n)/(\lambda - R)Y^n \subset \ker \delta_\lambda$$

holds. To prove the reverse inclusion, fix an element $y \in Y$ with $\rho(y)(\lambda) = 0$. Since $\widehat{X}$ is divisible, there are vectors $x_1, \ldots, x_n \in X$ with

$$\rho(y) = \sum_{i=1}^{n} (\lambda_i - M_{z_i})\rho(x_i) = \rho\Big(\sum_{i=1}^{n} (\lambda_i - T_i)x_i\Big).$$

But then

$$y - \sum_{i=1}^{n} (\lambda_i - T_i)x_i \in \bigcap_{z \in \Omega} (z - T)X^n$$

and hence $y \in Y \cap (\lambda - T)X^n$. Thus, for each $\lambda \in \Omega_0$, we obtain an exact sequence

$$H^{n-1}(\lambda - S, Z) \xrightarrow{d_\lambda^{n-1}} H^n(\lambda - R, Y) \xrightarrow{\delta_\lambda} \widehat{Y}_\lambda \to 0.$$

Using the exactness of these sequences and of the long exact cohomology sequences explained in the section leading to Lemma 2.1, we find that

$$\begin{aligned} \dim H^n(\lambda - S, Z) &= \dim H^n(\lambda - T, X) - \dim H^n(\lambda - R, Y)/d_\lambda^{n-1} H^{n-1}(\lambda - S, Z) \\ &= N - \dim \widehat{Y}_\lambda \end{aligned}$$

for all $\lambda \in \Omega_0$. By the cited result of Kaballo ([20], Satz 1.5) the set

$$A = \Big\{\lambda \in \Omega : \dim H^n(\lambda - S, Z) > \min_{\mu \in \Omega} \dim H^n(\mu - S, Z)\Big\}$$

is a proper analytic subset of Ω. Since the identity $\dim \widehat{Y}_\lambda = \mathrm{fd}(Y)$ holds for each point in a non-empty open subset of Ω_0, the assertion follows with A as defined above. ∎

It is well known that, in the setting of Lemma 2.1, the minimum

$$\min_{\mu \in \Omega} \{\dim H^n(\mu - S, Z)\}$$

can be interpreted as a suitable Samuel multiplicity of the tuples $S - \mu$ for $\mu \in \Omega$. Let us recall the necessary details.

For simplicity, we only consider the case where Ω is a domain in $\mathbb{C}^n$ with $0 \in \mathbb{C}^n$. For an arbitrary tuple $T \in L(X)^n$ of bounded operators on a Banach space X with

$$\dim H^n(T, X) < \infty,$$

all the spaces $M_k(T) = \sum_{|\alpha|=k} T^\alpha X$ ($k \in \mathbb{N}$) are finite codimensional in X and the limit

$$c(T) = n! \lim_{k\to\infty} \frac{\dim X/M_k(T)}{k^n}$$

exists. This number is referred to as the *Samuel multiplicity* of T. The idea to use this algebraic concept in the Fredholm theory of several commuting operators goes back to a paper [10] of Douglas and Yan. The algebraic Samuel multiplicity of semi-Fredholm operator tuples defined above and its analytic counterpart, which will be considered in Section 4, have been intensely studied in papers of Xiang Fang (see e.g. [16], [17], [18]) and later by the first-named author of the present paper ([11], [12], [13]). One can show that, for each domain $\Omega \subset \mathbb{C}^n$ with $0 \in \Omega$ and $\dim H^n(\lambda - T, X) < \infty$ for all $\lambda \in \Omega$, there is a proper analytic subset $A \subset \Omega$ such that

$$c(T) = \dim H^n(\lambda - T, X) < \dim H^n(\mu - T, X)$$

for all $\lambda \in \Omega \backslash A$ and $\mu \in A$ (see Corollary 3.6 in [13]). In particular, if $S \in L(Z)^n$ is as in Lemma 2.1 and $0 \in \Omega$, then the formula

$$c(S) = N - \mathrm{fd}(Y)$$

holds (see also Theorem 2 in [17]). Hence the following result from [11] allows us to deduce the announced limit formula for the fiber dimension.

LEMMA 2.2 ([11], Lemma 1.6). *Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X, let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace and let $S = T/Y \in L(Z)^n$ be the induced quotient tuple on $Z = X/Y$. Suppose that*

$$\dim H^n(T, X) < \infty.$$

Then the Samuel multiplicities of T and S satisfy the relation

$$c(S) = c(T) - n! \lim_{k\to\infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n}.$$

As a direct application we obtain a corresponding formula for the fiber dimension.

COROLLARY 2.3. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$ with $0 \in \Omega$, and let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace for T. Then the formula*

$$\mathrm{fd}(Y) = n! \lim_{k\to\infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n}$$

holds.

Proof. It suffices to observe that in the setting of Corollary 2.3 the identity $c(T) = N$ holds and then to compare the formula from Lemma 2.2 with the formula

$$c(S) = N - \mathrm{fd}(Y)$$

deduced in the section leading to Lemma 2.2. ∎

For weak Cowen–Douglas tuples $T \in L(X)^n$ on general domains $\Omega \subset \mathbb{C}^n$ (not necessarily containing 0), the above formula for $\mathrm{fd}(Y)$ remains true if on the right-hand side the spaces $M_k(T)$ are replaced by the spaces $M_k(T-\lambda_0)$ with $\lambda_0 \in \Omega$ arbitrary. This follows by an elementary translation argument.

If in Corollary 2.3 the space X is a Hilbert space and if we write P_k for the orthogonal projections onto the subspaces $M_k(T)^\perp$, then there are canonical vector space isomorphisms

$$(Y + M_k(T))/M_k(T) \to P_kY, \quad [y] \mapsto P_kY.$$

Thus the resulting formula

$$\mathrm{fd}(Y) = n! \lim_{k\to\infty} \frac{\dim(P_kY)}{k^n}$$

extends Theorem 19 in [5].

In the final result of this section we show that the fiber dimension $\mathrm{fd}(Y)$ is invariant under sufficiently small changes of the space Y. For given invariant subspaces $Y_1, Y_2 \in \mathrm{Lat}(T)$ with $Y_1 \subset Y_2$, we write $\sigma(T, Y_2/Y_1)$ for the Taylor spectrum of the quotient tuple induced by T on Y_2/Y_1.

COROLLARY 2.4. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain $\Omega \subset \mathbb{C}^n$. If $Y_1, Y_2 \in \mathrm{Lat}(T)$ are closed T-invariant subspaces with $Y_1 \subset Y_2$ and $\Omega \cap (\mathbb{C}^n \backslash \sigma(T, Y_2/Y_1)) \neq \emptyset$, then* $\mathrm{fd}(Y_1) = \mathrm{fd}(Y_2)$.

Proof. By Lemma 2.1 there is a point $\lambda \in \Omega \cap (\mathbb{C}^n \backslash \sigma(T, Y_1/Y_2))$ with

$$\dim H^n(\lambda - T/Y_i, X/Y_i) = N - \mathrm{fd}(Y_i)$$

for $i = 1, 2$. Using the long exact cohomology sequences induced by the canonical exact sequence

$$0 \to Y_2/Y_1 \to Y/Y_1 \to Y/Y_2 \to 0$$

one finds that the n-th cohomology spaces of $\lambda - T/Y_1$ and $\lambda - T/Y_2$ are isomorphic. Hence we obtain that $\mathrm{fd}(Y_1) = \mathrm{fd}(Y_2)$. ∎

To make the above proof work, it suffices that there is a point in Ω which is not contained in the right spectrum of the quotient tuple induced by T on Y_2/Y_1 (cf. Section 2.6 in [15]). The hypotheses of Corollory 2.4 are satisfied for instance if $\dim(Y_2/Y_1) < \infty$. Thus Corollary 2.4 can be seen as an extension of Proposition 2.5 in [7].

3. ANALYTIC SAMUEL MULTIPLICITY

We briefly indicate an alternative way to calculate fiber dimensions which extends a corresponding idea from [5]. Let $T \in L(X)^n$ be a commuting tuple of bounded operators on a Banach space X. Let $\Omega \subset \mathbb{C}^n$ be a domain with

$$\dim H^n(\lambda - T, X) < \infty$$

for all $\lambda \in \Omega$. For simplicity, we again assume that $0 \in \Omega$. By Corollary 2.2 in [13] the quotient sheaf

$$\mathcal{H}_T = \mathcal{O}_\Omega^X / (z - T)\mathcal{O}_\Omega^{X^n}$$

of the sheaf of all analytic X-valued functions on Ω is a coherent analytic sheaf on Ω. Let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace for T. As before denote by $R = T|_Y \in L(Y)^n$ the restriction of T and by $S = T/Y \in L(Z)^n$ the quotient tuple induced by T on $Z = X/Y$. Let $i : Y \to X$ and $q : X \to Z$ be the inclusion and quotient map, respectively. Then

$$0 \to K^\bullet(z - R, \mathcal{O}_\Omega^Y) \xrightarrow{i} K^\bullet(z - T, \mathcal{O}_\Omega^X) \xrightarrow{q} K^\bullet(z - S, \mathcal{O}_\Omega^Z) \to 0$$

is a short exact sequence of complexes of analytic sheaves on Ω. Passing to stalks and using the induced long exact cohomology sequences, one finds that the upper horizontal in the commutative diagram

$$\begin{array}{ccccc} \mathcal{H}_R & \xrightarrow{\;i\;} & \mathcal{H}_T & \xrightarrow{\;q\;} & \mathcal{H}_S \to 0 \\ \pi_Y \big\uparrow & & \big\uparrow \pi_X & & \\ \mathcal{O}_\Omega^Y & \xrightarrow{\;i\;} & \mathcal{O}_\Omega^X & & \end{array}$$

is an exact sequence of analytic sheaves. Here π_Y and π_X denote the canonical quotient maps. The sheaf $\mathcal{M} = \pi_X(i\mathcal{O}_\Omega^Y)$ is the kernel of the surjective sheaf homomorphism

$$\mathcal{H}_T \xrightarrow{q} \mathcal{H}_S.$$

Since $\mathcal{H}_T$ and $\mathcal{H}_S$ are coherent, also the sheaf $\mathcal{M}$ is a coherent analytic sheaf on Ω ([21], Satz 26.13). Hence

$$0 \to \mathcal{M}_0 \xrightarrow{i} \mathcal{H}_{T,0} \xrightarrow{q} \mathcal{H}_{S,0} \to 0$$

is an exact sequence of Noetherian $\mathcal{O}_0$-modules. For a Noetherian $\mathcal{O}_0$-module E, let us denote by $e_{\mathcal{O}_0}(E)$ its *analytic Samuel multiplicity*, that is, the multiplicity of E with respect to the multiplicity system $(z_1, \ldots, z_n)$ on E (see Section 7.4 in [23]). Since the analytic Samuel multiplicity is additive with respect to short exact sequences of Noetherian $\mathcal{O}_0$-modules ([23], Theorem 7.5), it follows that

$$e_{\mathcal{O}_0}(\mathcal{H}_{T,0}) = e_{\mathcal{O}_0}(\mathcal{M}_0) + e_{\mathcal{O}_0}(\mathcal{H}_{S,0}).$$

By Corollary 4.1 in [13] the analytic Samuel multiplicities $e_{\mathcal{O}_0}(\mathcal{H}_{T,0})$ and $e_{\mathcal{O}_0}(\mathcal{H}_{S,0})$ coincide with the Samuel multiplicities $c(T)$ and $c(S)$ as defined in Section 2. Thus we obtain the identity

$$c(T) = e_{\mathcal{O}_0}(\mathcal{M}_0) + c(S).$$

By Theorem 8.5 in [23] the analytic Samuel multiplicity $e_{\mathcal{O}_0}(\mathcal{M}_0)$ can also be calculated as the Euler characteristic $\chi(K^\bullet(z, \mathcal{M}_0))$ of the Koszul complex of the multiplication operators with $z_1, \ldots, z_n$ on $\mathcal{M}_0$. Summarizing we obtain the following result.

THEOREM 3.1. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple on a domain $\Omega \subset \mathbb{C}^n$ with $0 \in \Omega$. Let $Y \in \mathrm{Lat}(T)$ be a closed invariant subspace for T. The fiber dimension of Y can be calculated as*

$$\mathrm{fd}(Y) = n! \lim_{k \to \infty} \frac{\dim(Y + M_k(T))/M_k(T)}{k^n} = e_{\mathcal{O}_0}(\mathcal{M}_0),$$

where $\mathcal{M}_0$ is the stalk at $z = 0$ of the subsheaf $\pi_X(i\mathcal{O}_Y) \subseteq \mathcal{O}_X/(z - T)\mathcal{O}_X^n$.

4. A LATTICE FORMULA FOR THE FIBER DIMENSION

Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple of rank N on a domain Ω in $\mathbb{C}^n$ and let $Y_1, Y_2 \in \mathrm{Lat}(T)$ be closed invariant subspaces. A natural problem studied in [5] is to find conditions under which the dimension formula

$$\mathrm{fd}(Y_1) + \mathrm{fd}(Y_2) = \mathrm{fd}(Y_1 \vee Y_2) + \mathrm{fd}(Y_1 \cap Y_2)$$

holds. Note that by the remarks following Theorem 1.10 the fiber dimensions of the algebraic sum $Y_1 + Y_2$ and of its closure $Y_1 \vee Y_2 = \overline{\mathrm{span}}(Y_1 \cup Y_2)$ coincide. For a Cowen–Douglas tuple of rank 1, the validity of the above formula for all closed invariant subspaces Y_1, Y_2 is equivalent to the condition that any two non-zero closed invariant subspaces Y_1, Y_2 have a non-trivial intersection. As in the one-variable case basic linear algebra can be used to obtain at least an inequality.

LEMMA 4.1. *Let $T \in L(X)^n$ be a weak Cowen–Douglas tuple on a domain $\Omega \subset \mathbb{C}^n$ and let $Y_1, Y_2 \subset X$ be linear T-invariant subspaces. Then the inequality*

$$\mathrm{fd}(Y_1) + \mathrm{fd}(Y_2) \geqslant \mathrm{fd}(Y_1 + Y_2) + \mathrm{fd}(Y_1 \cap Y_2)$$

holds.

Proof. Let $\rho : X \to \mathcal{O}(\Omega_0, D)$ be a CF-representation of T on a domain $\Omega_0 \subset \Omega$. It suffices to observe that, for each point $\lambda \in \Omega_0$, the estimate

$$\begin{aligned} \dim \epsilon_\lambda \rho(Y_1 + Y_2) &= \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim(\epsilon_\lambda \rho(Y_1) \cap \epsilon_\lambda \rho(Y_2)) \\ &\leqslant \dim \epsilon_\lambda \rho(Y_1) + \dim \epsilon_\lambda \rho(Y_2) - \dim \epsilon_\lambda \rho(Y_1 \cap Y_2) \end{aligned}$$

holds and then to choose λ as a common maximal point for the submodules $\rho(Y_1 + Y_2)$, $\rho(Y_1)$, $\rho(Y_2)$ and $\rho(Y_1 \cap Y_2)$. ∎

In the following we prove that in Lemma 4.1 also the reverse inequality holds in some particular cases. For this purpose, we closely follow ideas from [6] where a corresponding result is proved for analytic functional Hilbert spaces given by a complete Nevanlinna–Pick kernel. We give a shortened proof under weakened hypotheses and obtain further applications. An alternative proof for the Nevanlinna–Pick case can also be found in the recent paper [4].

Let $\Omega \subset \mathbb{C}^n$ be a domain and let D be an N-dimensional complex vector space. We shall say that a function $f \in \mathcal{O}(\Omega, D)$ has coefficients in a given subalgebra $A \subset \mathcal{O}(\Omega)$ if the coordinate functions of f with respect to some, or equivalently, every basis of D belong to A. Let $M \subset \mathcal{O}(\Omega, D)$ be a $\mathbb{C}[z]$-submodule. We say that A is dense in M if every function $f \in M$ is the pointwise limit of a sequence $(f_k)_{k\in\mathbb{N}}$ of functions in M such that each f_k has coordinate functions in A.

THEOREM 4.2. *Let $A \subset \mathcal{O}(\Omega)$ be a subalgebra and let $M_1, M_2 \subset \mathcal{O}(\Omega, D)$ be $\mathbb{C}[z]$-submodules such that A is dense in M_1 and in M_2 and such that $AM_i \subset M_i$ for $i = 1, 2$. Then we have*

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2).$$

Proof. Exactly as in the proof of Lemma 4.1 it follows that

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) \leqslant \mathrm{fd}(M_1) + \mathrm{fd}(M_2).$$

To prove the reverse inequality, define $M = M_1 + M_2$ and choose a point $\lambda \in \Omega$ which is maximal for M_1, M_2 and M. Define $E = (M_1)_\lambda \cap (M_2)_\lambda$ and choose direct complements E_1 of E in $(M_1)_\lambda$ and E_2 of E in $(M_2)_\lambda$. Fix bases $(e_1, \ldots, e_{d_1})$ of E_1, $(e_{d_1+1}, \ldots, e_{d_1+d_2})$ for E_2 and $(e_{d_1+d_2+1}, \ldots, d_{d_1+d_2+d'})$ for E, where $d_1, d_2, d' \geqslant 0$ are non-negative integers. Set $d = d_1 + d_2 + d'$. An elementary argument shows that $(e_1, \ldots, e_d)$ is a basis of M_λ. Let us complete this basis to a basis $B = (e_1, \ldots, e_d, e_{d+1}, \ldots, e_N)$ of D. Since $\mathrm{fd}(M_1) + \mathrm{fd}(M_2) - \mathrm{fd}(M) = d'$, we have to show that

$$\mathrm{fd}(M_1 \cap M_2) \geqslant d'.$$

We may of course assume that $d' \neq 0$. Since A is dense in M, there are functions $h_1, \ldots, h_d \in M$ with $h_i(\lambda) = e_i$ for $i = 1, \ldots, d$ such that each h_i has coefficients in A. Write

$$h_i = \sum_{j=1}^{N} h_{ij} e_j \quad (i = 1, \ldots, d).$$

Then $\theta = (h_{ij})_{1\leqslant i,j\leqslant d}$ is a $(d \times d)$-matrix with entries in A such that $\theta(\lambda) = E_d$ is the unit matrix. By basic linear algebra there is a $(d \times d)$-matrix (A_{ij}) with entries in A such that $(A_{ij})\theta = \mathrm{diag}(\det\theta)$ is the $(d \times d)$-diagonal matrix with all diagonal terms equal to $\det(\theta)$. Then

$$(A_{ij})_{1\leqslant i,j\leqslant d}(h_{ij})_{\substack{1\leqslant i\leqslant d\\ 1\leqslant j\leqslant N}} = (\mathrm{diag}(\det\theta), (g_{ij})),$$

where (g_{ij}) is a suitable matrix with entries in A. We define functions $H_1, \ldots, H_d \in M$ by setting

$$H_i = \det(\theta)e_i + \sum_{j=1}^{N-d} g_{ij} e_{d+j} = \sum_{j=1}^{N} \Big(\sum_{\nu=1}^{d} A_{i\nu} h_{\nu j} \Big) e_j = \sum_{\nu=1}^{d} A_{i\nu} h_\nu.$$

By construction $H_i(\lambda) = e_i$ and $(H_1(z), \ldots, H_d(z))$ is a basis of M_z for every $z \in \Omega$ with $\det(\theta(z)) \neq 0$. If $f = f_1 e_1 + \cdots + f_N e_N \in M$ is arbitrary, then at each point $z \in \Omega$ not contained in the zero set $Z(\det(\theta))$ of the analytic function $\det(\theta) \in \mathcal{O}(\Omega)$, the function f can be written as a linear combination

$$f(z) = \lambda_1(z, f) H_1(z) + \cdots + \lambda_d(z, f) H_d(z).$$

Using the definition of the functions H_i, we find that

$$f_1 = \lambda_1(\cdot, f) \det(\theta), \ldots, f_d = \lambda_d(\cdot, f) \det(\theta).$$

Hence, for $j = d+1, \ldots, N$ and $z \in \Omega \backslash Z(\det \theta)$, we obtain that

$$\begin{aligned} f_j(z) &= \lambda_1(z, f) g_{1,j-d}(z) + \cdots + \lambda_d(z, f) g_{d,j-d}(z) \\ &= \frac{g_{1,j-d}(z)}{\det \theta(z)} f_1(z) + \cdots + \frac{g_{d,j-d}(z)}{\det \theta(z)} f_d(z). \end{aligned}$$

In particular, each function $f = f_1 e_1 + \cdots + f_N e_N \in M$ is uniquely determined by its first d coordinate functions $(f_1, \ldots, f_d)$.

Since A is dense in M_1 and in M_2, there are functions $F_1, \ldots, F_{d_1+d'} \in M_1$ and $G_1, \ldots, G_{d_2+d'} \in M_2$ with coefficients in A such that

$$\begin{aligned} &(F_i(\lambda))_{i=1,\ldots,d_1+d'} = (e_1, \ldots, e_{d_1}, e_{d_1+d_2+1}, \ldots, e_{d_1+d_2+d'}) \quad \text{and} \\ &(G_i(\lambda))_{i=1,\ldots,d_2+d'} = (e_{d_1+1}, \ldots, e_{d_1+d_2+d'}). \end{aligned}$$

Write the first d coordinate functions of each of the functions

$$F_1, \ldots, F_{d_1}, G_1, \ldots, G_{d_2}, F_{d_1+1}, \ldots, F_{d_1+d'}, G_{d_2+1}, \ldots, G_{d_2+d'}$$

with respect to the basis $(e_1, \ldots, e_N)$ of D as column vectors and arrange these column vectors to a matrix Δ in the indicated order. Then Δ is a $(d \times (d + d'))$-matrix with entries in A. Write $\Delta = (\Delta_0, \Delta_1)$ where Δ_0 is the $(d \times d)$-matrix consisting of the first d columns of Δ and Δ_1 is the $(d \times d')$-matrix consisting of the last d' columns of Δ.

By construction we have $\det(\Delta_0(\lambda)) = 1$. On $\Omega \backslash Z(\det \Delta_0)$, we can write

$$(\det \Delta_0) \Delta_0^{-1} \Delta = (\operatorname{diag}(\det \Delta_0), \Gamma),$$

where $\operatorname{diag}(\det \Delta_0)$ is the $(d \times d)$-diagonal matrix with all diagonal terms equal to $\det \Delta_0$ and $\Gamma = (\gamma_{ij})$ is a $(d \times d')$-matrix with entries in A. The column vectors

$$r_j = (\gamma_{1j}, \ldots, \gamma_{dj}, 0, \ldots, 0, -\det \Delta_0, 0, \ldots, 0)^{\mathrm{t}} \quad (j = 1, \ldots, d'),$$

where $-\det \Delta_0$ is the entry in the $(d+j)$-th position, satisfy the equations

$$(\det \Delta_0) \Delta_0^{-1} \Delta r_j = ((\det \Delta_0) \gamma_{ij} - (\det \Delta_0) \gamma_{ij})_{i=1}^{d} = 0$$

on $\Omega \backslash Z(\det \Delta_0)$. Hence $\Delta r_j = 0$ for $j = 1, \ldots, d'$, or equivalently, for each $j = 1, \ldots, d$, the first d coordinate functions of

$$\gamma_{1j} F_1 + \cdots + \gamma_{d_1 j} F_{d_1} + \gamma_{d_1+d_2+1,j} F_{d_1+1} + \cdots + \gamma_{d_1+d_2+d',j} F_{d_1+d'}$$

with respect to $(e_1, \dots, e_N)$ coincide with those of

$$(\det \Delta_0) G_{d_2+j} - \gamma_{d_1+1,j} G_1 - \cdots - \gamma_{d_1+d_2,j} G_{d_2}.$$

Since, for each j, both functions belong to M, they coincide. But then these functions belong to $M_1 \cap M_2$. Since the vectors

$$G_i(\lambda) = e_{d_1+i} \quad (i = 1, \dots, d_2 + d')$$

are linearly independent and $\det(\Delta_0(\lambda)) = 1$, it follows that $\mathrm{fd}(M_1 \cap M_2) = \dim(M_1 \cap M_2)_\lambda \geqslant d'$. ∎

Recall that a domain $\Omega \subset \mathbb{C}^n$ is called *polynomially-convex* or a *Runge domain* if the polynomial-convex hull of each compact subset $K \subset \Omega$ is contained in Ω. By the Oka–Weil approximation theorem ([1], Corollary 8.3.8) on each Runge domain $\Omega \subseteq \mathbb{C}^n$ the polynomials are dense in $\mathcal{O}(\Omega)$ with respect to the Fréchet space topology of uniform convergence on compact subsets, and hence each $\mathbb{C}[z]$-submodule $M \subset \mathcal{O}(\Omega, D)$ which is closed with respect to the Fréchet space topology of $\mathcal{O}(\Omega, D)$ is automatically an $\mathcal{O}(\Omega)$-submodule. Thus by applying Theorem 4.2 with $A = \mathcal{O}(\Omega)$ we obtain the following general lattice formula for fiber dimensions in the category of Fréchet submodules of $\mathcal{O}(\Omega, D)$. The reader should be aware that this result does not apply to Banach or Hilbert spaces of analytic functions.

COROLLARY 4.3. *Let $\Omega \subset \mathbb{C}^n$ be a Runge domain and let D be a finite-dimensional complex vector space. Then the fiber dimension formula*

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds for each pair of closed $\mathbb{C}[z]$-submodules $M_1, M_2 \subset \mathcal{O}(\Omega, D)$.

Suppose that $T \in L(X)^n$ is a Cowen–Douglas tuple of rank N on a domain Ω in $\mathbb{C}^n$. Choose a CF-representation

$$\rho : X \to \mathcal{O}(\Omega_0, D)$$

of T as in the proof of Theorem 1.6. Let $M \in \mathrm{Lat}(T)$ be an invariant subspace of T such that each vector $m \in M$ is the limit of a sequence of vectors in

$$M \cap \mathrm{span}\{T^\alpha x : \alpha \in \mathbb{N}^n \text{ and } x \in D\}.$$

Then $\rho(M) \subset \mathcal{O}(\Omega_0, D)$ is a $\mathbb{C}[z]$-submodule in which the polynomials are dense in the sense explained in the section leading to Theorem 4.2. Hence, for any two invariant subspaces $M_1, M_2 \in \mathrm{Lat}(T)$ of this type, the fiber dimension formula

$$\begin{aligned}\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) &= \mathrm{fd}(\rho(M_1) + \rho(M_2)) + \mathrm{fd}(\rho(M_1) \cap \rho(M_2)) \\ &= \mathrm{fd}(\rho(M_1)) + \mathrm{fd}(\rho(M_2)) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)\end{aligned}$$

holds. The above density condition on M is trivially fulfilled for every closed T-invariant subspace M which is generated by a subset of D. But there are other situations to which this observation applies.

A commuting tuple $T \in L(H)^n$ of bounded operators on a complex Hilbert space H is called *graded* if $H = \bigoplus_{k=0}^{\infty} H_k$ is the orthogonal sum of closed subspaces $H_k \subset H$ such that $\dim H_0 < \infty$ and

(i) $T_j H_k \subset H_{k+1}$ $(k \geqslant 0, j = 1, \ldots, n)$,

(ii) $\sum_{j=1}^{n} T_j H \subset H$ is closed,

(iii) $\bigvee_{\alpha \in \mathbb{N}^n} T^\alpha H_0 = H$.

Under these hypotheses the identities

$$\sum_{|\alpha|=k} T^\alpha H = \bigoplus_{j=k}^{\infty} H_j \quad \text{and} \quad \sum_{|\alpha|=k} T^\alpha H_0 = H_k$$

hold for all integers $k \geqslant 0$ ([14], Lemma 2.4). A closed invariant subspace $M \in \mathrm{Lat}(T)$ of a graded tuple $T \in L(H)^n$ is said to be *homogeneous* if

$$M = \bigoplus_{k=0}^{\infty} M \cap H_k.$$

COROLLARY 4.4. *Let $T \in L(H)^n$ be a graded Cowen–Douglas tuple on a domain Ω in $\mathbb{C}^n$. Then the fiber dimension formula*

$$\mathrm{fd}(M_1 + M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds for any pair of homogeneous invariant subspaces $M_1, M_2 \in \mathrm{Lat}(T)$.

Proof. By the remarks preceding the corollary

$$H = \Big(\sum_{j=1}^{n} T_j H\Big) \oplus H_0.$$

Hence in the proof of Theorem 1.6 we can choose $D = H_0$. Let $\rho : H \to \mathcal{O}(\Omega_0, H_0)$ be a CF-representation of T as constructed in the proof of Theorem 1.6. Let $M \in \mathrm{Lat}(T)$ be a homogeneous invariant subspace for T. Then each element $m \in M$ can be written as a sum $m = \sum_{k=0}^{\infty} m_k$ with

$$m_k \in M \cap \sum_{|\alpha|=k} T^\alpha H_0 \quad (k \in \mathbb{N}).$$

Hence the assertion follows from the remarks preceding Corollary 4.4. ∎

Typical examples of graded Cowen–Douglas tuples are multiplication tuples

$$M_z = (M_{z_1}, \ldots, M_{z_n}) \in L(H)^n$$

with the coordinate functions on functional Hilbert spaces $H = H(K_f, \mathbb{C}^N)$ of analytic functions given by a reproducing kernel

$$K_f : B_r(a) \times B_r(a) \to L(\mathbb{C}^N), \quad K_f(z,w) = f(\langle z, w\rangle) 1_{\mathbb{C}^N},$$

where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a one-variable power series with radius of convergence $R = r^2 > 0$ such that $a_0 = 1$, $a_n > 0$ for all n and

$$0 < \inf_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} \leqslant \sup_{n\in\mathbb{N}} \frac{a_n}{a_{n+1}} < \infty$$

(see [19] or [24]). In this case H is the orthogonal sum

$$H = \bigoplus_{k=0}^{\infty} \mathbb{H}_k \otimes \mathbb{C}^N$$

of the subspaces consisting of all homogeneous $\mathbb{C}^N$-valued polynomials of degree k and every invariant subspace

$$M = \bigvee_{i=1}^{s} \mathbb{C}[z]p_i \in \mathrm{Lat}(M_z)$$

generated by a finite set of homogeneous polynomials $p_i \in \mathbb{H}_{k_i} \otimes \mathbb{C}^N$ is homogeneous. This class of examples contains the Drury–Arveson space, the Hardy space and the weighted Bergman spaces on the unit ball.

Let $H = H(K) \subset \mathcal{O}(\Omega)$ be an analytic functional Hilbert space on a domain $\Omega \subset \mathbb{C}^n$, or equivalently, a functional Hilbert space given by a sesqui-analytic reproducing kernel $K : \Omega \times \Omega \to \mathbb{C}$. Let D be a finite-dimensional complex Hilbert space. Then the D-valued functional Hilbert space $H(K_D) \subset \mathcal{O}(\Omega, D)$ given by the kernel

$$K_D : \Omega \times \Omega \to L(D), \quad K_D(z,w) = K(z,w)1_D$$

can be identified with the Hilbert space tensor product $H(K) \otimes D$. Let us denote by $M(H) = \{\varphi : \Omega \to \mathbb{C} : \varphi H \subset H\}$ the multiplier algebra of H.

COROLLARY 4.5. *Suppose that $H = H(K)$ contains all constant functions and that $z_1, \ldots, z_n \in M(H)$.*

(i) *For any pair of closed subspaces $M_1, M_2 \subset H(K_D)$ with $M(H)M_i \subset M_i$ for $i = 1,2$ and such that $M(H)$ is dense in M_1 and M_2, the fiber dimension formula*

$$\mathrm{fd}(M_1 \vee M_2) + \mathrm{fd}(M_1 \cap M_2) = \mathrm{fd}(M_1) + \mathrm{fd}(M_2)$$

holds.

(ii) *If in addition K is a complete Nevanlinna–Pick kernel, that is, K has no zeros and also the mapping $1 - (1/K)$ is positive definite, then the fiber dimension formula holds for all closed subspaces $M_1, M_2 \subset H(K_D)$ which are invariant for $M(H)$.*

Proof. Part (i) is a direct consequence of Theorem 4.2. If K is a complete Nevanlinna–Pick kernel, then the Beurling–Lax–Halmos theorem proved by McCullough and Trent (see Theorem 8.67 in [2] or Theorem 3.3.8 in [3]) implies that $M(H)$ is dense in every closed subspace $M \subset H(K_D)$ which is invariant for $M(H)$. ∎

Note that the condition that $M(H)$ is dense in a subspace $M \subset H(K_D)$ is satisfied for every closed $M(H)$-invariant subspace $M \subset H(K_D)$ that is generated by an arbitrary family of functions $f_i : \Omega \to D$ $(i \in I)$ with coefficients in $M(H)$. Part (ii) for domains $\Omega \subset \mathbb{C}$ was proved in [5].

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