J. OPERATOR THEORY
78:1(2017), 71–88
doi: 10.7900/jot.2016may30.2120

COMPLETIONS OF QUANTUM GROUP ALGEBRAS IN CERTAIN NORMS AND OPERATORS WHICH COMMUTE WITH MODULE ACTIONS

MEHDI NEMATI

Communicated by Stefaan Vaes

ABSTRACT. Let $L^1_{\mathrm{cb}}(\mathbb{G})$ (respectively $L^1_{\mathrm{M}}(\mathbb{G})$) denote the closure of the quantum group algebra $L^1(\mathbb{G})$ of a locally compact quantum group $\mathbb{G}$, in the space of completely bounded (respectively bounded) double centralizers of $L^1(\mathbb{G})$. In this paper, we study quantum group generalizations of various results from Fourier algebras of locally compact groups. In particular, left invariant means on $L^1_{\mathrm{cb}}(\mathbb{G})^*$ and on $L^1_{\mathrm{M}}(\mathbb{G})^*$ are defined and studied. We prove that the set of left invariant means on $L^\infty(\mathbb{G})$ and on $L^1_{\mathrm{cb}}(\mathbb{G})^*$ ($L^1_{\mathrm{M}}(\mathbb{G})^*$) have the same cardinality. We also study the left uniformly continuous functionals associated with these algebras. Finally, for a Banach $\mathcal{A}$-bimodule $\mathfrak{X}$ of a Banach algebra $\mathcal{A}$ we establish a contractive and injective representation from the dual of a left introverted subspace of $\mathcal{A}^*$ into $B_{\mathcal{A}}(\mathfrak{X}^*)$. As an application of this result we show that if the induced representation $\Phi : LUC_{\mathrm{cb}}(\mathbb{G})^* \to B_{L^1_{\mathrm{cb}}(\mathbb{G})}(L^\infty(\mathbb{G}))$ is surjective, then $L^1_{\mathrm{cb}}(\mathbb{G})$ has a bounded approximate identity. We also obtain a characterization of co-amenable quantum groups in terms of representations of quantum measure algebras $M(\mathbb{G})$.

KEYWORDS: *Amenability, Arens regularity, co-amenability, double centralizer, locally compact quantum group.*

MSC (2010): Primary 46L89, 43A07; Secondary 46H20, 47L25, 43A20, 46L07.

INTRODUCTION

Let G be a locally compact group and $A(G)$ be the Fourier algebra of G, introduced by Eymard in [8]. We denote the completion of $A(G)$ with respect to the multiplier (cb-multiplier) norm by $A_{\mathrm{M}}(G)$ (respectively $A_{\mathrm{cb}}(G)$), and refer the reader to [6], [9], [12] for more details on these algebras. It is well known that G is amenable if and only if $A(G)$ has a bounded approximate identity. Unlike the Fourier algebra, $A_{\mathrm{cb}}(G)$ can have a bounded approximate identity even if G is non-amenable; see [6], [12]. This characterizes the weakly amenable groups, i.e. groups for which $A(G)$ has an approximate identity which is bounded in cb-multiplier norm. Forrest and Miao in their recent work [11] showed that when G is weakly amenable, the algebras $A_{\mathrm{M}}(G)$ and $A_{\mathrm{cb}}(G)$ have properties that are characteristic of the Fourier algebra of an amenable group. Moreover, they proved that the cardinality of the set of topologically invariant means on $A(G)^* = VN(G)$ is equal to the cardinality of the set of topologically invariant means on $A_{\mathrm{cb}}(G)^*$ and on $A_{\mathrm{M}}(G)^*$.

One of the purposes of this work is to establish that these assertions hold for arbitrary locally compact quantum groups $\mathbb{G}$. In fact, the quantum group algebra $L^1(\mathbb{G})$ embeds canonically into the algebra of completely bounded (bounded) double centralizers on $L^1(\mathbb{G})$. We denote the completion of $L^1(\mathbb{G})$ with respect to the cb-multiplier (multiplier) norm by $L^1_{\mathrm{cb}}(\mathbb{G})$ (respectively $L^1_{\mathrm{M}}(\mathbb{G})$).

The paper is organized as follows: we recall relevant definitions and introduce some notation in the next section. In Section 2, we focus on left invariant means and show that there is a bijection between left invariant means on $L^\infty(\mathbb{G})$ and on $L^1_{\mathrm{cb}}(\mathbb{G})^*$. We then use this to characterize amenability of $\mathbb{G}$ in terms of the existence of a left invariant mean on $L^1_{\mathrm{cb}}(\mathbb{G})^*$. We prove that $L^1_{\mathrm{cb}}(\mathbb{G})$ is a two-sided ideal in its second dual space if and only if $\mathbb{G}$ is compact. We also obtain a number of results about the Arens regularity of $L^1_{\mathrm{cb}}(\mathbb{G})$. The same results can be shown to hold for $L^1_{\mathrm{M}}(\mathbb{G})$.

In Section 3, for a Banach $\mathcal{A}$-bimodule $\mathfrak{X}$ we introduce the left introverted subspace $\mathcal{L}_{\mathfrak{X}}(\mathcal{A})$ of $\mathcal{A}^*$ and establish a contractive injection $\Phi : \mathcal{L}_{\mathfrak{X}}(\mathcal{A})^* \to B_{\mathcal{A}}(\mathfrak{X}^*)$. We show that if $\mathcal{L}_{\mathfrak{X}}(\mathcal{A})^*$ is isomorphic to the Banach algebra $B_{\mathcal{A}}(\mathfrak{X}^*)$, then $\mathfrak{X}$ has a bounded right approximate identity in $\mathcal{A}$. We then use these ideas to give various characterizations for the existence of a bounded approximate identity in $L^1(\mathbb{G})$, $L^1_{\mathrm{cb}}(\mathbb{G})$ or $L^1_{\mathrm{M}}(\mathbb{G})$.

1. PRELIMINARIES

A locally compact quantum group $\mathbb{G}$ is a quadruple $(L^\infty(\mathbb{G}),\Gamma,\varphi,\psi)$, where $L^\infty(\mathbb{G})$ is a von Neumann algebra, $\Gamma : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})\overline{\otimes}L^\infty(\mathbb{G})$ is a co-associative co-multiplication, and φ, ψ are, respectively, left and right invariant normal semifinite faithful Haar weights on $L^\infty(\mathbb{G})$; see for example [17], [18]. The reduced C^*-algebra of $\mathbb{G}$ is denoted by $C_0(\mathbb{G})$, which is a weak* dense C^*-subalgebra of $L^\infty(\mathbb{G})$. Let $L^1(\mathbb{G})$=$L^\infty(\mathbb{G})_*$ be the predual of $L^\infty(\mathbb{G})$. Then the pre-adjoint of Γ induces on $L^1(\mathbb{G})$, a completely contractive Banach algebra product

$$\star = \Gamma_* : h \otimes h' \in L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G}) \mapsto h \star h' = (h \otimes h')\Gamma \in L^1(\mathbb{G}).$$

Since the multiplication $\star$ is a complete quotient map from $L^1(\mathbb{G})\widehat{\otimes}L^1(\mathbb{G})$ onto $L^1(\mathbb{G})$, we get

$$L^1(\mathbb{G}) = \overline{\langle L^1(\mathbb{G}) \star L^1(\mathbb{G})\rangle} = \overline{\mathrm{span}\{f \star g : f, g \in L^1(\mathbb{G})\}}.$$

In the case where $L^\infty(\mathbb{G})$ is $VN(G)$ for a locally compact group G, the algebra $L^1(\mathbb{G})$ is the Fourier algebra $A(G)$. Let $L^2(\mathbb{G},\varphi)$ and $L^2(\mathbb{G},\psi)$ be the Hilbert spaces obtained from the GNS-constructions for φ and ψ, respectively. One can show that $L^2(\mathbb{G},\varphi) \cong L^2(\mathbb{G},\psi)$. We denote this Hilbert space by $L^2(\mathbb{G})$. We also write $M(\mathbb{G}) := C_0(\mathbb{G})^*$, the dual space of $C_0(\mathbb{G})$, and define the completely contractive product $\star$ on $M(\mathbb{G})$ by

$$\langle \omega \star \nu, x\rangle = (\omega \otimes \nu)(\Gamma x) \quad (x \in C_0(\mathbb{G}), \omega, \nu \in M(\mathbb{G}))$$

whence $(M(\mathbb{G}), \star)$ is a completely contractive Banach algebra and contains $L^1(\mathbb{G})$ as a norm closed two-sided ideal; see [17].

We recall from [15] that a linear map μ on $L^1(\mathbb{G})$ is called a *left (respectively right) centralizer* of $L^1(\mathbb{G})$ if it satisfies

$$\mu(f \star g) = \mu(f) \star g \quad (\mu(f \star g) = f \star \mu(g)) \quad (f, g \in L^1(\mathbb{G})).$$

A pair of maps (μ_l, μ_r) on $L^1(\mathbb{G})$ is a *double centralizer* if

$$f \star \mu_l(g) = \mu_r(f) \star g \quad (f, g \in L^1(\mathbb{G})).$$

If (μ_l, μ_r) is a double centralizer of $L^1(\mathbb{G})$, then μ_l is a left centralizer and μ_r is a right centralizer of $L^1(\mathbb{G})$. Let $C_l(L^1(\mathbb{G}))$, $C_r(L^1(\mathbb{G}))$ and $C(L^1(\mathbb{G}))$ denote the spaces of left, right and double centralizers of $L^1(\mathbb{G})$, respectively. As norm closed subalgebras of $B(L^1(\mathbb{G}))$ and $B(L^1(\mathbb{G}))^{\mathrm{op}}$, respectively, $C_l(L^1(\mathbb{G}))$ and $C_r(L^1(\mathbb{G}))$ are Banach algebras. The norm on $C(L^1(\mathbb{G}))$ is given by

$$\|\mu\|_{\mathrm{M}} = \|(\mu_l, \mu_r)\|_{\mathrm{M}} = \max\{\|\mu_l\|_{\mathrm{M}}, \|\mu_r\|_{\mathrm{M}}\}$$

and the multiplication is given by

$$(\mu_l, \mu_r) \star (\nu_l, \nu_r) = (\mu_l \circ \nu_l, \nu_r \circ \mu_r).$$

Similarly, using the natural operator space structure on $L^1(\mathbb{G})^* = L^\infty(\mathbb{G})$, let $C_{\mathrm{cb}}(L^1(\mathbb{G}))$ be the set of completely bounded double centralizers of $L^1(\mathbb{G})$. Then $C_{\mathrm{cb}}(L^1(\mathbb{G}))$ is a completely contractive Banach algebra with multiplication defined as above and the operator space matrix norm $\|\cdot\|_{\mathrm{cb}}$ defined by

$$\|\mu\|_{\mathrm{cb}} = \|(\mu_l, \mu_r)\|_{\mathrm{cb}} = \max\{\|\mu_l\|_{\mathrm{cb}}, \|\mu_r\|_{\mathrm{cb}}\} \quad ((\mu_l, \mu_r) \in M_{\mathrm{cb}}(L^1(\mathbb{G}))).$$

For each $\omega \in M(G)$, we obtain a pair of completely bounded maps

$$\omega_l(f) = \omega \star f \quad \omega_r(f) = f \star \omega \quad (f \in L^1(\mathbb{G}))$$

on $L^1(\mathbb{G})$ with $\|(\omega_l, \omega_r)\|_{\mathrm{cb}} \leqslant \|\omega\|$. It turns out that the pair (ω_l, ω_r) is a completely bounded double centralizer of $L^1(\mathbb{G})$. Since the multiplication on $L^1(\mathbb{G})$ (respectively $M(\mathbb{G})$) is faithful (see Proposition 1 of [14] and Proposition 2.2 of [13]) we obtain the natural inclusions

$$L^1(\mathbb{G}) \hookrightarrow M(\mathbb{G}) \hookrightarrow C_{\mathrm{cb}}(L^1(\mathbb{G})) \hookrightarrow C(L^1(\mathbb{G})).$$

Therefore, $L^1(\mathbb{G})$ (respectively $M(\mathbb{G})$) can naturally be identified with a two-sided ideal in both $C_{\mathrm{cb}}(L^1(\mathbb{G}))$ and $C(L^1(\mathbb{G}))$. Moreover, for each $f \in L^1(\mathbb{G})$ we have

$$\|f\|_1 = \|f\|_{M(\mathbb{G})} \geqslant \|f\|_{\mathrm{cb}} \geqslant \|f\|_{\mathrm{M}}.$$

By Proposition 3.1 of [13] if $\mathbb{G}$ is *co-amenable*, i.e. $L^1(\mathbb{G})$ has a bounded right (equivalently, left or two-sided) approximate identity, then we have

$$M(\mathbb{G}) = C_{\mathrm{cb}}(L^1(\mathbb{G})) = C(L^1(\mathbb{G})).$$

However, in the non-co-amenable case, these algebras are typically not equal. In fact, in the general case $\|\cdot\|_{\mathrm{M}}$ and $\|\cdot\|_{\mathrm{cb}}$ need not to be equivalent to $\|\cdot\|_1$. For a locally compact quantum group $\mathbb{G}$ we let

$$L^1_{\mathrm{cb}}(\mathbb{G}) \overset{\mathrm{def}}{=} \overline{L^1(\mathbb{G})}^{\|\cdot\|_{\mathrm{cb}}} \subseteq C_{\mathrm{cb}}(L^1(\mathbb{G})) \quad \text{and} \quad L^1_{\mathrm{M}}(\mathbb{G}) \overset{\mathrm{def}}{=} \overline{L^1(\mathbb{G})}^{\|\cdot\|_{\mathrm{M}}} \subseteq C(L^1(\mathbb{G})).$$

Clearly, $L^1_{\mathrm{cb}}(\mathbb{G})$ and $L^1_{\mathrm{M}}(\mathbb{G})$ are Banach algebras under the multiplication $\star$ defined as above and both contain $L^1(\mathbb{G})$ as a norm dense two-sided ideal.

Let $\mathcal{A}$ be a Banach algebra. Then $X = \mathcal{A}^*$ is a Banach right $\mathcal{A}$-module by the following module action

$$\langle x \cdot \mu, \nu \rangle = \langle x, \mu \star \nu \rangle \quad (x \in \mathcal{A}^*, \mu, \nu \in \mathcal{A}),$$

where $\star$ stands for the multiplication in $\mathcal{A}$. We define the space of *left uniformly continuous functionals* on $\mathcal{A}$ as follows

$$LUC(\mathcal{A}^*) := \overline{\langle \mathcal{A}^* \cdot \mathcal{A} \rangle}^{\|\cdot\|_{\mathcal{A}^*}},$$

where $\langle \mathcal{A}^* \cdot \mathcal{A} \rangle$ denotes the linear span of $\mathcal{A}^* \cdot \mathcal{A}$. It is known that there are two multiplications $\Box$ and $\Diamond$ on the second dual $\mathcal{A}^{**}$ of $\mathcal{A}$, called, respectively, the left and the right Arens products, each extending the multiplication on $\mathcal{A}$. The *left Arens product* $\Box$ is induced by the left $\mathcal{A}$-module structure on $\mathcal{A}$. That is, for $m, n \in \mathcal{A}^{**}$, $x \in \mathcal{A}^*$ and $\mu \in \mathcal{A}$ we have

$$\langle m \Box n, x \rangle = \langle m, n \cdot x \rangle, \quad \langle n \cdot x, \mu \rangle = \langle n, x \cdot \mu \rangle.$$

Similarly, one can define the *right Arens product*.

A Banach right $\mathcal{A}$-submodule X of $\mathcal{A}^*$ is called *left introverted* if $m \cdot x \in X$ for all $m \in X^*$ and $x \in X$. In this case, X^* is a Banach algebra with the multiplication induced by the left Arens product $\Box$ inherited from $\mathcal{A}^{**}$. Examples of left introverted subspace of $\mathcal{A}^*$ are $WAP(\mathcal{A}^*)$, the space of *weakly almost periodic functionals* on $\mathcal{A}$; that is, those functionals $x \in \mathcal{A}^*$ such that the set $\{x \cdot \mu : \mu \in \mathcal{A}, \|\mu\| \leqslant 1\}$ is relatively weakly compact in $\mathcal{A}^*$, and the space $LUC(\mathcal{A}^*)$. For brevity, we use the following notations:

$$\begin{aligned} LUC(\mathbb{G}) &:= LUC(L^\infty(\mathbb{G})), & WAP(\mathbb{G}) &:= WAP(L^\infty(\mathbb{G})), \\ LUC_{\mathrm{cb}}(\mathbb{G}) &:= LUC(L^1_{\mathrm{cb}}(\mathbb{G})^*), & WAP_{\mathrm{cb}}(\mathbb{G}) &:= WAP(L^1_{\mathrm{cb}}(\mathbb{G})^*), \\ LUC_M(\mathbb{G}) &:= LUC(L^1_{\mathrm{M}}(\mathbb{G})^*), & WAP_M(\mathbb{G}) &:= WAP(L^1_{\mathrm{M}}(\mathbb{G})^*). \end{aligned}$$

2. COMPLETION OF QUANTUM GROUP ALGEBRA AND AMENABILITY

We start this section with the next definition which is the main object of the section.

DEFINITION 2.1. Let $\mathcal{A}$ be $L^1(\mathbb{G})$, $L^1_{\mathrm{cb}}(\mathbb{G})$ or $L^1_{\mathrm{M}}(\mathbb{G})$ and let X be a Banach right $\mathcal{A}$-submodule of $\mathcal{A}^*$ containing the identity operator 1 in $\mathcal{A}^*$. Then $m \in X^*$ is called a *left invariant mean (LIM)* on X if $\|m\| = m(1) = 1$ and $\langle m, x \cdot \mu \rangle = \mu(1)\langle m, x \rangle$ for all $x \in X$ and $\mu \in \mathcal{A}$. We denote the set of all left invariant means on X by $LIM(X^*)$.

Right invariant and (two-sided) invariant means are defined similarly. Recall that a locally compact quantum group $\mathbb{G}$ is called *amenable* if there exists a left invariant mean on $L^\infty(\mathbb{G})$. Note that the existence of a right invariant and of an invariant mean on $L^\infty(\mathbb{G})$ is equivalent to the amenability of $\mathbb{G}$ [5]. In the sequel, we will restrict our attention to $L^1_{\mathrm{cb}}(\mathbb{G})$. We note that all results below remain valid if we replace $L^1_{\mathrm{cb}}(\mathbb{G})$ by $L^1_{\mathrm{M}}(\mathbb{G})$, except perhaps Theorem 2.17.

Recall that $L^\infty(\mathbb{G})$ is naturally a Banach $L^1_{\mathrm{cb}}(\mathbb{G})$-bimodule with the module actions given by

$$\langle x \cdot \mu, f \rangle = \langle x, \mu \star f \rangle, \quad \langle \mu \cdot x, f \rangle = \langle x, f \star \mu \rangle$$

for all $x \in L^\infty(\mathbb{G}), \mu \in L^1_{\mathrm{cb}}(\mathbb{G})$ and $f \in L^1(\mathbb{G})$, noticing that $L^1(\mathbb{G})$ is a two-sided ideal in $L^1_{\mathrm{cb}}(\mathbb{G})$. Let $\iota : L^1(\mathbb{G}) \to L^1_{\mathrm{cb}}(\mathbb{G})$ be the canonical embedding of $L^1(\mathbb{G})$ into $L^1_{\mathrm{cb}}(\mathbb{G})$. Then ι is an injective Banach $L^1_{\mathrm{cb}}(\mathbb{G})$-bimodule morphism. So are $\iota^{**} : L^\infty(\mathbb{G})^* \to L^1_{\mathrm{cb}}(\mathbb{G})^{**}$ and $\iota^* : L^1_{\mathrm{cb}}(\mathbb{G})^* \to L^\infty(\mathbb{G})$, since ι has a dense range. It is straightforward to see that ι^* is in fact the restriction map. Therefore, we can identify $L^1_{\mathrm{cb}}(\mathbb{G})^*$ with a subset of $L^\infty(\mathbb{G})$ and $L^\infty(\mathbb{G})^*$ with a subset of $L^1_{\mathrm{cb}}(\mathbb{G})^{**}$. We have the following proposition, but omit the details of the proof as the argument can easily be adapted from Proposition 3.3 of [11].

PROPOSITION 2.2. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following hold:*

(i) $L^\infty(\mathbb{G}) \cdot L^1_{\mathrm{cb}}(\mathbb{G}) \subseteq LUC(\mathbb{G})$;

(ii) $\iota^*(LUC_{\mathrm{cb}}(\mathbb{G})) \subseteq LUC(\mathbb{G})$;

(iii) *if* $L^1_{\mathrm{cb}}(\mathbb{G})$ *has a bounded approximate identity, then* $L^\infty(\mathbb{G}) \cdot L^1_{\mathrm{cb}}(\mathbb{G}) = LUC(\mathbb{G})$;

(iv) $x \cdot f, f \cdot x \in \iota^*(L^1_{\mathrm{cb}}(\mathbb{G})^*)$ *for all* $x \in L^\infty(\mathbb{G})$ *and* $f \in L^1(\mathbb{G})$.

Before giving the next result, let us first note that for each $m \in L^1_{\mathrm{cb}}(\mathbb{G})^*$, we can define a bounded linear map m_{L} on $L^\infty(\mathbb{G})$ by

$$m_{\mathrm{L}} : L^\infty(\mathbb{G}) \ni x \mapsto m \cdot x \in L^\infty(\mathbb{G}),$$

where the product $m \cdot x \in L^\infty(\mathbb{G})$ is given by $\langle m \cdot x, f \rangle = \langle m, x \cdot f \rangle$ for all $f \in L^1(\mathbb{G})$, noticing that $L^\infty(\mathbb{G}) \cdot L^1(\mathbb{G}) \subseteq \iota^*(L^1_{\mathrm{cb}}(\mathbb{G})^*)$. This map is (completely) bounded, with $\|m_{\mathrm{L}}\| \leqslant \|m\|$ ($\|m_{\mathrm{L}}\|_{\mathrm{cb}} \leqslant \|m\|$).

THEOREM 2.3. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have that $\iota^{**}(LIM(L^\infty(\mathbb{G})^*)) \subseteq LIM(L^1_{\mathrm{cb}}(\mathbb{G})^{**})$. Furthermore, the map $\iota^{**} : LIM(L^\infty(\mathbb{G})^*) \to LIM(L^1_{\mathrm{cb}}(\mathbb{G})^{**})$ is a bijection.*

Proof. We prove that $\iota^{**}(LIM(L^\infty(\mathbb{G})^*)) \subseteq LIM(L^1_{\mathrm{cb}}(\mathbb{G})^{**})$. In fact, for each $m \in LIM(L^\infty(\mathbb{G})^*)$ and $\mu \in L^1_{\mathrm{cb}}(\mathbb{G})$, there is a sequence (f_j) in $L^1(\mathbb{G})$ for which $\|f_j - \mu\|_{\mathrm{cb}} \to 0$. Thus,

$$f_j(1) \to \mu(1), \quad \|x \cdot f_j - x \cdot \mu\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*} \to 0$$

for all $x \in L^1_{\mathrm{cb}}(\mathbb{G})^*$. Therefore,

$$\begin{aligned}\langle \iota^{**}(m), x \cdot \mu\rangle &= \lim_{j\to\infty}\langle \iota^{**}(m), x \cdot f_j\rangle = \lim_{j\to\infty}\langle m, \iota^*(x) \cdot f_j\rangle \\ &= \lim_{j\to\infty} f_j(1)\langle m, \iota^*(x)\rangle = \mu(1)\langle \iota^{**}(m), x\rangle.\end{aligned}$$

Hence, $\iota^{**}(LIM(L^\infty(\mathbb{G})^*)) \subseteq LIM(L^1_{\mathrm{cb}}(\mathbb{G})^{**})$. Obviously ι^{**} is injective.

We now claim that ι^{**} is surjective. In fact, suppose that $m \in LIM(L^1_{\mathrm{cb}}(\mathbb{G})^{**})$. Then, for each $x \in L^\infty(\mathbb{G})$ and $f, g \in L^1(\mathbb{G})$, since $x \cdot f \in \iota^*(L^1_{\mathrm{cb}}(\mathbb{G})^*)$ by Proposition 2.2(iv), we have

$$\langle m_{\mathrm{L}}(x), f \star g\rangle = \langle m, (x \cdot f) \cdot g\rangle = g(1)\langle m_{\mathrm{L}}(x), f\rangle.$$

Therefore,

$$\langle m_{\mathrm{L}}(x), f \star g\rangle = 0 \quad (x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G}), g \in I_0(\mathbb{G})),$$

where $I_0(\mathbb{G}) = \{g \in L^1(\mathbb{G}) : g(1) = 0\}$. Since the linear span of $L^1(\mathbb{G}) \star I_0(\mathbb{G})$ is dense in $I_0(\mathbb{G})$ ([1], Theorem 4.4), we get that $m_{\mathrm{L}}(x)|_{I_0(\mathbb{G})} = 0$ for all $x \in L^\infty(\mathbb{G})$. By the fact that $f \star g - g \star f \in I_0(\mathbb{G})$, we conclude

$$\langle m, x \cdot (f \star g)\rangle = \langle m, x \cdot (g \star f)\rangle \quad (f, g \in L^1(\mathbb{G})).$$

Given a state $f_0 \in L^1(\mathbb{G})$, define $\widetilde{m} \in L^\infty(\mathbb{G})^*$ by

$$\widetilde{m}(x) := \langle m, x \cdot f_0\rangle \quad (x \in L^\infty(\mathbb{G})).$$

By Proposition 2.2(iv), the functional $\widetilde{m}$ is well defined. It follows from above that

$$\widetilde{m}(x \cdot f) = f(1)\widetilde{m}(x) \quad (x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G})).$$

Moreover, $\|\widetilde{m}\| \leqslant 1$ and $\widetilde{m}(1) = f_0(1)m(1) = 1$; that is, $\widetilde{m} \in LIM(L^\infty(\mathbb{G})^*)$. Finally, for each $y \in L^1_{\mathrm{cb}}(\mathbb{G})^*$ we have

$$\langle \iota^{**}(\widetilde{m}), y\rangle = \langle \widetilde{m}, \iota^*(y)\rangle = \langle m, \iota^*(y) \cdot f_0\rangle = \langle m, y\rangle.$$

This shows that $\iota^{**}(\widetilde{m}) = m$, as claimed. ∎

For each $n \in LUC(\mathbb{G})^*$, define a bounded linear map n_{L} from $L^\infty(\mathbb{G})$ into $L^\infty(\mathbb{G})$ by

$$\langle n_{\mathrm{L}}(x), f\rangle = \langle n, x \cdot f\rangle$$

for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$.

THEOREM 2.4. *Let $\mathbb{G}$ be a locally compact quantum group. Then the restriction map $\mathcal{R} : LIM(L^\infty(\mathbb{G})^*) \to LIM(LUC(\mathbb{G})^*)$ is a bijection.*

Proof. It is easy to see that the restriction map $\mathcal{R} : LIM(L^\infty(\mathbb{G})^*) \to LIM(LUC(\mathbb{G})^*)$ is well-defined and is injective. We need to prove that $\mathcal{R}$ is surjective. Let $n \in LIM(LUC(\mathbb{G})^*)$. Consider the functional $\widetilde{n} \in L^\infty(\mathbb{G})^*$ defined by

$$\widetilde{n}(x) = \langle n_{\mathrm{L}}(x), f_0\rangle \quad (x \in L^\infty(\mathbb{G})),$$

where f_0 is a state in $L^1(\mathbb{G})$. Since $f_0(1) = 1$, we conclude that

$$\widetilde{n}(1) = n(1) = 1.$$

From this and the fact that $\|f_0\|_1 = 1$, we obtain that $\|\widetilde{n}\| = 1$. Now, for each $x \in L^\infty(\mathbb{G})$ and $f, g \in L^1(\mathbb{G})$ we have

$$\langle n_{\mathrm{L}}(x), f \star g\rangle = \langle n, (x \cdot f) \cdot g\rangle = g(1)\langle n_{\mathrm{L}}(x), f\rangle.$$

Therefore,

$$\langle n_{\mathrm{L}}(x), f \star g\rangle = 0 \quad (x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G}), g \in I_0(\mathbb{G})).$$

Since the linear span of $L^1(\mathbb{G}) \star I_0(\mathbb{G})$ is dense in $I_0(\mathbb{G})$ ([1], Theorem 4.4), this implies that $n_{\mathrm{L}}(x)|_{I_0(\mathbb{G})} = 0$ for all $x \in L^\infty(\mathbb{G})$. By the fact that $f \star f_0 - f_0 \star f \in I_0(\mathbb{G})$ for all $f \in L^1(\mathbb{G})$, we get

$$\langle \widetilde{n}, x \cdot f\rangle = \langle n_{\mathrm{L}}(x), f \star f_0\rangle = \langle n_{\mathrm{L}}(x), f_0 \star f\rangle = \langle n, (x \cdot f_0) \cdot f\rangle = f(1)\langle \widetilde{n}, x\rangle.$$

This shows that $\widetilde{n} \in LIM(L^\infty(\mathbb{G})^*)$. Moreover, for each $y \in LUC(\mathbb{G})$,

$$\widetilde{n}(y) = \langle n, y \cdot f_0\rangle = \langle n, y\rangle;$$

that is, $\mathcal{R}(\widetilde{n}) = n$. Hence, $\mathcal{R}$ is surjective. ∎

LEMMA 2.5. *Let $\mathbb{G}$ be a locally compact quantum group. Then $x \cdot f \in LUC_{\mathrm{cb}}(\mathbb{G})$, for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. Moreover, $\|x \cdot f\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*} \leqslant \|x\|_{L^\infty(\mathbb{G})}\|f\|_1$ and*

$$LUC_{\mathrm{cb}}(\mathbb{G}) = \overline{\langle L^\infty(\mathbb{G}) \cdot L^1(\mathbb{G})\rangle}^{\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}}.$$

Proof. By Proposition 2.2, $x \cdot f \in \iota^*(L^1_{\mathrm{cb}}(\mathbb{G})^*)$ for all $x \in L^\infty(\mathbb{G})$ and $f \in L^1(\mathbb{G})$. Further, it is easy to check that $\|x \cdot f\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*} \leqslant \|x\|_{L^\infty(\mathbb{G})}\|f\|_1$. We claim that $x \cdot f \in LUC_{\mathrm{cb}}(\mathbb{G})$. To prove this, note that for each $\varepsilon > 0$ there exist $g_i, h_i \in L^1(\mathbb{G})$ $(i = 1, \ldots, n)$ such that

$$\Big\| \sum_{i=1}^{n} g_i \star h_i - f \Big\|_1 < \varepsilon.$$

Moreover, we have $x \cdot g \in LUC_{\mathrm{cb}}(\mathbb{G})$, where $g = \sum_{i=1}^{n} g_i \star h_i$. Therefore,

$$\|x \cdot g - x \cdot f\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*} \leqslant \|x\|_{L^\infty(\mathbb{G})}\|g - f\|_1 < \varepsilon\|x\|_{L^\infty(\mathbb{G})}.$$

This shows that $x \cdot f \in LUC_{\mathrm{cb}}(\mathbb{G})$ and so we have the inclusion

$$\overline{\langle L^\infty(\mathbb{G}) \cdot L^1(\mathbb{G})\rangle}^{\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}} \subseteq LUC_{\mathrm{cb}}(\mathbb{G}).$$

For the reverse inclusion, note that if (f_n) is a sequence in $L^1(\mathbb{G})$ which converges to some μ in $L^1_{\mathrm{cb}}(\mathbb{G})$, then $x \cdot f_n \to x \cdot \mu$ in the $\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}$-norm for all $x \in L^1_{\mathrm{cb}}(\mathbb{G})^*$. This implies that

$$\begin{aligned} LUC_{\mathrm{cb}}(\mathbb{G}) &= \overline{\langle L^1_{\mathrm{cb}}(\mathbb{G})^* \cdot L^1_{\mathrm{cb}}(\mathbb{G})\rangle}^{\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}} = \overline{\langle L^1_{\mathrm{cb}}(\mathbb{G})^* \cdot L^1(\mathbb{G})\rangle}^{\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}} \\ &\subseteq \overline{\langle L^\infty(\mathbb{G}) \cdot L^1(\mathbb{G})\rangle}^{\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}}, \end{aligned}$$

as required. ∎

As an application of Lemma 2.5, we record the LUC version of Theorem 2.3 for later use. The proof is left to the reader.

THEOREM 2.6. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have that $\iota^{**}(LIM(LUC(\mathbb{G})^*)) \subseteq LIM(LUC_{\mathrm{cb}}(\mathbb{G})^*)$. Furthermore, the map $\iota^{**} : LIM(LUC(\mathbb{G})^*) \to LIM(LUC_{\mathrm{cb}}(\mathbb{G})^*)$ is a bijection.*

As an immediate consequence of Theorems 2.3, 2.4 and 2.6 we obtain the following result on the cardinality of the sets of left invariant means. In what follows, $|Y|$ stands for the cardinality of a set Y.

COROLLARY 2.7. *Let $\mathbb{G}$ be a locally compact quantum group. Then we have*

$$|LIM(LUC_{\mathrm{cb}}(\mathbb{G})^*)| = |LIM(LUC(\mathbb{G})^*)| = |LIM(L^\infty(\mathbb{G})^*)| = |LIM(L^1_{\mathrm{cb}}(\mathbb{G})^{**})|.$$

We now present our main result in this section.

COROLLARY 2.8. *Let $\mathbb{G}$ be a locally compact quantum group. Then the following statements are equivalent:*

(i) *$\mathbb{G}$ is amenable;*
(ii) *there is a left invariant mean on $L^1_{\mathrm{cb}}(\mathbb{G})^*$;*
(iii) *there is a left invariant mean on $LUC(\mathbb{G})$;*
(iv) *there is a left invariant mean on $LUC_{\mathrm{cb}}(\mathbb{G})$.*

REMARK 2.9. For co-amenable quantum groups $\mathbb{G}$, Runde showed that $\mathbb{G}$ is amenable if and only if there is a left invariant mean on $LUC(\mathbb{G})$ ([25], Theorem 3.6). This equivalence was later generalized by Crann and Neufang ([3], Theorem 4.2) and independently by the author ([22], Theorem 4.1) to arbitrary locally compact quantum groups.

It is well known that a Banach algebra $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$ if and only if, for every $\mu \in \mathcal{A}$, the operators $\ell_\mu : \nu \mapsto \mu \star \nu$ and $r_\mu : \nu \mapsto \nu \star \mu$ are weakly compact [7].

PROPOSITION 2.10. *Let $\mathbb{G}$ be a locally compact quantum group. Then $L^1_{\mathrm{cb}}(\mathbb{G})$ is an ideal in its second dual if and only if $\mathbb{G}$ is compact.*

Proof. Suppose that $L^1_{\text{cb}}(\mathbb{G})$ is an ideal in its second dual. Therefore, for every $\mu \in L^1_{\text{cb}}(\mathbb{G})$, the operator $\ell_\mu : \nu \mapsto \mu \star \nu$ is weakly compact on $L^1_{\text{cb}}(\mathbb{G})$. Now, let $f \in L^1(\mathbb{G})$ be such that $f = g \star \mu$ for some $g, \mu \in L^1(\mathbb{G})$. Let (f_n) be a bounded sequence in $L^1(\mathbb{G})$. Then (f_n) is also bounded in $L^1_{\text{cb}}(\mathbb{G})$. Therefore, there exists a subsequence (f_{n_j}) of (f_n) such that $(\ell_\mu(f_{n_j}))$ converges weakly in $L^1_{\text{cb}}(\mathbb{G})$. Now, we observe that $x \cdot g \in L^1_{\text{cb}}(\mathbb{G})^*$ for all $x \in L^\infty(\mathbb{G})$. This shows that the sequence $(\ell_{g\star\mu}(f_{n_j}))$ converges weakly in $L^1(\mathbb{G})$; that is, the operator $\ell_f : h \mapsto f \star h$ is weakly compact on $L^1(\mathbb{G})$. Since the linear span of $L^1(\mathbb{G}) \star L^1(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, we conclude that for every $f \in L^1(\mathbb{G})$, the operator $\ell_f : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ is weakly compact. Similarly, we can show that for every $f \in L^1(\mathbb{G})$, the operator $r_f : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ is weakly compact. Equivalently, $L^1(\mathbb{G})$ is an ideal in its second dual. Therefore, $\mathbb{G}$ is compact by Theorem 3.8 of [24].

To prove the converse, suppose that $\mathbb{G}$ is compact. Then for each $g \in L^1(\mathbb{G})$, the operator $\ell_g : L^1(\mathbb{G}) \to L^1(\mathbb{G})$ is weakly compact ([24], Corollary 3.5). Let (μ_n) be a bounded sequence in $L^1_{\text{cb}}(\mathbb{G})$ and let $\mu \in L^1_{\text{cb}}(\mathbb{G})$ be such that $\mu = g \star h$ for some $g, h \in L^1(\mathbb{G})$. Then $(h \star \mu_n)$ is a bounded sequence in $L^1(\mathbb{G})$ and so by weak compactness of ℓ_g there is a subsequence $(h \star \mu_{n_j})$ of $(h \star \mu_n)$ such that $(g \star h \star \mu_{n_j})$ converges weakly in $L^1(\mathbb{G})$. Therefore $(\mu \star \mu_{n_j})$ converges weakly in $L^1_{\text{cb}}(\mathbb{G})$. Since the linear span of $L^1(\mathbb{G}) \star L^1(\mathbb{G})$ is $\|\cdot\|_{\text{cb}}$-dense in $L^1_{\text{cb}}(\mathbb{G})$, it follows that for every $\mu \in L^1_{\text{cb}}(\mathbb{G})$, the operator $\ell_\mu : L^1_{\text{cb}}(\mathbb{G}) \to L^1_{\text{cb}}(\mathbb{G})$ is weakly compact. Similarly, one can check that the operator $r_\mu : L^1_{\text{cb}}(\mathbb{G}) \to L^1_{\text{cb}}(\mathbb{G})$ is weakly compact. Therefore, $L^1(\mathbb{G})$ is an ideal in its second dual. ∎

EXAMPLE 2.11. Let G be a locally compact group and $A_{\text{cb}}(G)$ be the completion of the Fourier algebra $A(G)$ with respect to the cb-multiplier norm. It is known that when $L^\infty(\mathbb{G}) = VN(G)$, the compactness of $\mathbb{G}$ is equivalent to G being discrete. So G is discrete if and only if $A_{\text{cb}}(G)$ is an ideal in its second dual by Proposition 2.10.

The proof of the following result is similar to that of the proof of Proposition 4.6 in [25], but we provide the details for the convenience of the reader.

PROPOSITION 2.12. *Let $\mathbb{G}$ be an amenable, locally compact quantum group. Then there is a unique left invariant mean on $WAP_{\text{cb}}(\mathbb{G})$ that is automatically right invariant.*

Proof. As $\mathbb{G}$ is amenable, there is a (two-sided) invariant mean m on $L^\infty(\mathbb{G})$. It follows from Theorem 2.3 that $\iota^{**}(m)$ is a (two-sided) invariant mean on $L^1_{\text{cb}}(\mathbb{G})^*$. It is easy to check that $m_0 := \iota^{**}(m)|_{WAP_{\text{cb}}(\mathbb{G})}$ is an invariant mean on $WAP_{\text{cb}}(\mathbb{G})$. Now, let n be any left invariant mean on $WAP_{\text{cb}}(\mathbb{G})$. Since the two Arens products $\Box$ and $\Diamond$ on $WAP_{\text{cb}}(\mathbb{G})^*$ coincide; see Proposition 3.11 of [4], we conclude the following, as required:

$$m_0 = n(1)m_0 = n\Box m_0 = n\Diamond m_0 = m_0(1)n = n. \quad ∎$$

Recall that a Banach algebra $\mathcal{A}$ is called *Arens regular* if the left and the right Arens products coincide on $\mathcal{A}^{**}$ or, equivalently, by [23], $WAP(\mathcal{A}^*) = \mathcal{A}^*$; see also Theorem 3.14, Proposition 5.7 of [4].

COROLLARY 2.13. *Let $\mathbb{G}$ be an amenable locally compact quantum group such that $L^1(\mathbb{G})$ is separable. If $L^1_{\mathrm{cb}}(\mathbb{G})$ is Arens regular, then $\mathbb{G}$ is compact.*

Proof. Suppose that $L^1_{\mathrm{cb}}(\mathbb{G})$ is Arens regular. Then $WAP_{\mathrm{cb}}(\mathbb{G}) = L^1_{\mathrm{cb}}(\mathbb{G})^*$. Now, if m is a left invariant mean on $L^1_{\mathrm{cb}}(\mathbb{G})^*$, then Proposition 2.12 yields that m is unique. Therefore, there is a unique left invariant mean $\widetilde{m}$ on $L^\infty(\mathbb{G})$ by Corollary 2.7. Since $L^1(\mathbb{G})$ is separable and it is an F-algebra, we conclude from Proposition 4.15(b) of [19] that $\widetilde{m}$ is in $L^1(\mathbb{G})$. This implies that $\mathbb{G}$ is compact by Proposition 3.1 of [2]. ∎

COROLLARY 2.14. *Let $\mathbb{G}$ be an amenable locally compact quantum group such that $L^1(\mathbb{G})$ is separable. If $LUC_{\mathrm{cb}}(\mathbb{G}) \subseteq WAP_{\mathrm{cb}}(\mathbb{G})$, then $\mathbb{G}$ is compact.*

Proof. Suppose that $LUC_{\mathrm{cb}}(\mathbb{G}) \subseteq WAP_{\mathrm{cb}}(\mathbb{G})$. From Corollary 2.7, Proposition 2.12 and the fact that $\mathbb{G}$ is amenable, we conclude that there is a unique left invariant mean on $L^\infty(\mathbb{G})$. As in the proof of Corollary 2.13, we obtain that $\mathbb{G}$ is compact. ∎

LEMMA 2.15. *Let $\mathbb{G}$ be a locally compact quantum group. Then $L^1_{\mathrm{cb}}(\mathbb{G}) \subseteq M(\mathbb{G})$ if and only if $L^1(\mathbb{G}) = L^1_{\mathrm{cb}}(\mathbb{G})$.*

Proof. We need to prove the necessity part of the lemma. Suppose that $L^1_{\mathrm{cb}}(\mathbb{G}) \subseteq M(\mathbb{G})$ and $\mu \in \overline{L^1_{\mathrm{cb}}(\mathbb{G})}^{\|\cdot\|_{M(\mathbb{G})}}$. Then there is a sequence (μ_n) in $L^1_{\mathrm{cb}}(\mathbb{G})$ such that $\|\mu_n - \mu\|_{M(\mathbb{G})} \to 0$. Since $\|\cdot\|_{\mathrm{cb}} \leqslant \|\cdot\|_{M(\mathbb{G})}$, it is easy to check that $\mu \in L^1_{\mathrm{cb}}(\mathbb{G})$. Therefore, $L^1_{\mathrm{cb}}(\mathbb{G})$ is closed in $M(\mathbb{G})$. So the norms $\|\cdot\|_{\mathrm{cb}}$ and $\|\cdot\|_{M(\mathbb{G})}$ are equivalent on $L^1_{\mathrm{cb}}(\mathbb{G})$ and hence on $L^1(\mathbb{G})$, by the open mapping theorem. Thus, $L^1_{\mathrm{cb}}(\mathbb{G}) = L^1(\mathbb{G})$. ∎

REMARK 2.16. For a locally compact group G, it is known from Losert [21] that G is amenable, or equivalently $A(G)$ has a bounded approximate identity whenever $A(G) = A_{\mathrm{cb}}(G)$. However, the question whether this is true for general locally compact quantum groups is still open.

Completely bounded right, left, and double multipliers over locally compact quantum groups have been studied in [16]. Recall that the algebra $M^{\mathrm{r}}_{\mathrm{cb}}(L^1(\mathbb{G}))$ of completely bounded right multipliers of $L^1(\mathbb{G})$, introduced by Junge, Neufang, and Ruan [16], is defined to be the set of all $q \in L^\infty(\widehat{\mathbb{G}})'$ such that $\rho(L^1(\mathbb{G}))q \subseteq \rho(L^1(\mathbb{G}))$ and the induced map

$$m^{\mathrm{r}}_q : f \in L^1(\mathbb{G}) \to \rho^{-1}(\rho(f)q) \in L^1(\mathbb{G})$$

is in $C^{\mathrm{r}}_{\mathrm{cb}}(L^1(\mathbb{G}))$, where $\widehat{\mathbb{G}}$ is the dual quantum group of $\mathbb{G}$, $L^\infty(\widehat{\mathbb{G}})'$ is the commutant of $L^\infty(\widehat{\mathbb{G}})$ in $B(L^2(\mathbb{G}))$, and $\rho : L^1(\mathbb{G}) \to L^\infty(\widehat{\mathbb{G}})'$ is the right regular representation of $\mathbb{G}$.

Similarly, $M^{\mathrm{l}}_{\mathrm{cb}}(L^1(\mathbb{G}))$ denotes *the completely bounded left multiplier algebra of* $L^1(\mathbb{G})$ corresponding to the left regular representation $\lambda : L^1(\mathbb{G}) \to L^\infty(\widehat{\mathbb{G}})$. We can identify $M^{\mathrm{l}}_{\mathrm{cb}}(L^1(\mathbb{G}))$ with $UM^{\mathrm{l}}_{\mathrm{cb}}(L^1(\mathbb{G}))U^*$, the subalgebra of $L^\infty(\widehat{\mathbb{G}})'$ consisting of all $p \in L^\infty(\widehat{\mathbb{G}})'$ such that $p\rho(L^1(\mathbb{G})) \subseteq \rho(L^1(\mathbb{G}))$ and the induced map

$$m^{\mathrm{l}}_p : f \in L^1(\mathbb{G}) \to \rho^{-1}(p\rho(f)) \in L^1(\mathbb{G})$$

is in $C^{\mathrm{l}}_{\mathrm{cb}}(L^1(\mathbb{G}))$. It was shown in Theorem 3.2 of [13] that each $(S,T) \in C_{\mathrm{cb}}(L^1(\mathbb{G}))$ is uniquely associated with an element q of

$$M_{\mathrm{cb}}(L^1(\mathbb{G})) := (UM^{\mathrm{l}}_{\mathrm{cb}}(L^1(\mathbb{G}))U^*) \cap m^{\mathrm{r}}_{\mathrm{cb}}(L^1(\mathbb{G})) \subseteq L^\infty(\widehat{\mathbb{G}})'$$

via $(S,T) = (m^{\mathrm{l}}_q, m^{\mathrm{r}}_q)$. We let $M_{\mathrm{cb}}(L^1(\mathbb{G})$ denote the set of completely bounded multipliers of $L^1(\mathbb{G})$. Therefore, we have $M_{\mathrm{cb}}(L^1(\mathbb{G})) \cong C_{\mathrm{cb}}(L^1(\mathbb{G}))$, completely isometrically and algebraically. Equivalently, $M_{\mathrm{cb}}(L^1(\mathbb{G}))$ can be defined as

$$M^{\mathrm{l}}_{\mathrm{cb}}(L^1(\mathbb{G})) \cap (U^* m^{\mathrm{r}}_{\mathrm{cb}}(L^1(\mathbb{G}))U) \subseteq L^\infty(\widehat{\mathbb{G}}).$$

THEOREM 2.17. *Let $\mathbb{G}$ be a locally compact quantum group. Then $\iota^*(LUC_{\mathrm{cb}}(\mathbb{G})) = LUC(\mathbb{G})$ if and only if $L^1(\mathbb{G}) = L^1_{\mathrm{cb}}(\mathbb{G})$.*

Proof. Let $\iota^*(LUC_{\mathrm{cb}}(\mathbb{G})) = LUC(\mathbb{G})$ and set $\theta := \iota^*|_{LUC_{\mathrm{cb}}(\mathbb{G})}$. Then

$$\theta^* : LUC(\mathbb{G})^* \to LUC_{\mathrm{cb}}(\mathbb{G})^*$$

is an isomorphism. Therefore, if $\mu \in L^1_{\mathrm{cb}}(\mathbb{G}) \subseteq LUC_{\mathrm{cb}}(\mathbb{G})^*$, then we may find $\nu \in LUC(\mathbb{G})^*$ for which $\theta^*(\nu) = \mu$. Furthermore, we can identify $L^1_{\mathrm{cb}}(\mathbb{G})$ with a subalgebra of $L^\infty(\widehat{\mathbb{G}})$ via the completely isometric algebra isomorphism $M_{\mathrm{cb}}(L^1(\mathbb{G})) \cong C_{\mathrm{cb}}(L^1(\mathbb{G}))$ defined as above. Let $\widehat{\lambda} : L^1(\widehat{\mathbb{G}}) \to L^\infty(\mathbb{G})$ be the left regular representation of $\widehat{\mathbb{G}}$. Since $C_0(\mathbb{G}) \subseteq LUC(\mathbb{G})$ and $C_0(\mathbb{G}) = \overline{\widehat{\lambda}(L^1(\widehat{\mathbb{G}}))}^{\|\cdot\|_{L^\infty(\mathbb{G})}}$, under the identification $L^1(\widehat{\mathbb{G}}) \cong \widehat{\lambda}(L^1(\widehat{\mathbb{G}}))$, we can consider μ as a bounded linear functional on $C_0(\mathbb{G})$. This shows that $\mu \in C_0(\mathbb{G})^* = M(\mathbb{G})$. Therefore, $L^1(\mathbb{G}) = L^1_{\mathrm{cb}}(\mathbb{G})$ by Lemma 2.15. The converse is trivial. ∎

3. OPERATORS WHICH COMMUTE WITH MODULE ACTIONS AND CO-AMENABILITY

In this section, for a Banach algebra $\mathcal{A}$ we shall focus on the Banach algebra $B_{\mathcal{A}}(\mathfrak{X})$ of bounded right $\mathcal{A}$-module maps on a Banach $\mathcal{A}$-bimodule $\mathfrak{X}$. When a right $\mathcal{A}$-module $\mathfrak{X}$ is the dual space of a given Banach space, we let $B^{\sigma}_{\mathcal{A}}(\mathfrak{X})$ denote the subalgebra of $B_{\mathcal{A}}(\mathfrak{X})$ consisting of weak*-weak* continuous maps in $B_{\mathcal{A}}(\mathfrak{X})$.

Suppose now that $\mathfrak{X}$ is a Banach right $\mathcal{A}$-module. For each $\phi \in \mathfrak{X}^*$ and $\xi \in \mathfrak{X}$ define the functional $\phi \circ \xi \in \mathcal{A}^*$ by

$$\langle \phi \circ \xi, \mu \rangle = \langle \phi, \xi \cdot \mu \rangle \quad (\mu \in \mathcal{A})$$

and put

$$\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) := \overline{\langle \mathfrak{X}^* \circ \mathfrak{X} \rangle}^{\|\cdot\|_{\mathcal{A}^*}}.$$

Then, it is clear that

$$(\phi \circ \xi) \cdot \mu = \phi \circ (\xi \cdot \mu), \quad \mu \cdot (\phi \circ \xi) = (\mu \cdot \phi) \circ \xi$$

for all $\phi \in \mathfrak{X}^*$, $\xi \in \mathfrak{X}$ and $\mu \in \mathcal{A}$. Thus, $\mathcal{L}_{\mathfrak{X}}(\mathcal{A})$ is a sub-$\mathcal{A}$-bimodule of $\mathcal{A}^*$. Now let X be a left introverted subspace of $\mathcal{A}^*$ such that $\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) \subseteq X$. Then, it is easily verified that $\mathfrak{X}^*$ is a left X^*-module with the following module action

$$m \bullet \phi(\xi) = m(\phi \circ \xi) \quad (m \in X^*, \phi \in \mathfrak{X}^*, \xi \in \mathfrak{X}).$$

Moreover, for each $\phi \in \mathfrak{X}^*$ the map

$$X^* \to \mathfrak{X}^*, \quad m \mapsto m \bullet \phi$$

is weak*-weak* continuous; see for example Proposition 4.2 of [26]. We note that $\mathcal{L}_{\mathfrak{X}}(\mathcal{A})$ is itself a left introverted subspace of $\mathcal{A}^*$. Indeed, for each $m \in \mathcal{A}^{**}$, $\phi \in \mathfrak{X}^*$ and $\xi \in \mathfrak{X}$, we have

$$m \cdot (\phi \circ \xi) = (m \bullet \phi) \circ \xi.$$

By the above notions, it is not hard to see that, if $\mathfrak{X}$ is a Banach $\mathcal{A}$-bimodule, then the map

$$\Phi : \mathcal{L}_{\mathfrak{X}}(\mathcal{A})^* \to B_{\mathcal{A}}(\mathfrak{X}^*), \quad m \mapsto m_{\mathrm{L}}$$

is a weak*-weak* continuous, contractive, injective algebra homomorphism, where m_{L} is given by

$$m_{\mathrm{L}}(\phi) = m \bullet \phi \quad (\phi \in \mathfrak{X}^*).$$

THEOREM 3.1. *Let $\mathcal{A}$ be a Banach algebra and let $\mathfrak{X}$ be a Banach $\mathcal{A}$-bimodule. Consider the following statements:*

(i) $\Phi : \mathcal{L}_{\mathfrak{X}}(\mathcal{A})^* \to B_{\mathcal{A}}(\mathfrak{X}^*)$ *is surjective.*

(ii) $\mathrm{id}_{\mathfrak{X}^*} \in \Phi(\mathcal{L}_{\mathfrak{X}}(\mathcal{A}))$.

(iii) $\mathfrak{X}$ *has a bounded right approximate identity in* $\mathcal{A}$.

Then (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (ii).

Proof. That (i) implies (ii) is trivial. Now suppose that (ii) holds. Then there exists $E \in \mathcal{L}_{\mathfrak{X}}(\mathcal{A})^*$ such that $\Phi(E) = \mathrm{id}_{\mathfrak{X}^*}$. We extend E to a functional $\widetilde{E}$ on $\mathcal{A}^*$ with the same norm. By Goldstine's theorem, there is a net (μ_γ) in $\mathcal{A}$ such that $\mu_\gamma \xrightarrow{\mathrm{w}^*} \widetilde{E}$ and $\|\mu_\gamma\| \leqslant \|\widetilde{E}\|$ for all γ. Therefore, for each $\phi \in \mathfrak{X}^*$ and $\xi \in \mathfrak{X}$ we have

$$\langle \phi, \xi \rangle = \langle \mathrm{id}_{\mathfrak{X}^*}(\phi), \xi \rangle = \langle E \bullet \phi, x \rangle = \langle E, \phi \circ \xi \rangle = \langle \widetilde{E}, \phi \circ \xi \rangle = \lim_{\gamma} \langle \phi, \xi \cdot \mu_\gamma \rangle.$$

This shows that (μ_γ) is a bounded weak right approximate identity for $\mathfrak{X}$ in $\mathcal{A}$. Applying Mazur's theorem, we obtain a bounded right approximate identity for $\mathfrak{X}$ in $\mathcal{A}$.

Suppose that (iii) holds. Let (μ_γ) be a bounded right approximate identity for $\mathfrak{X}$ in $\mathcal{A}$ and let $\widetilde{E}$ be any weak* cluster point of (μ_γ) in $\mathcal{A}^{**}$. If $E := \widetilde{E}|_{\mathcal{L}_{\mathfrak{X}}(\mathcal{A})}$, then it is easily verified that $\Phi(E) = \mathrm{id}_{\mathfrak{X}^*}$. ∎

In the sequel, for a locally compact quantum group $\mathbb{G}$, we consider the algebras $L^1(\mathbb{G})$, $L^1_{\mathrm{cb}}(\mathbb{G})$ and $L^1_{\mathrm{M}}(\mathbb{G})$, and attempt to describe when $\mathfrak{X}$ has a bounded approximate identity in $\mathcal{A}$, where $\mathcal{A}$ and $\mathfrak{X}$ are chosen from above algebras provided that $\mathfrak{X}$ is naturally a Banach $\mathcal{A}$-bimodule.

EXAMPLE 3.2. Let $\mathbb{G}$ be a locally compact quantum group.

(i) Let $\mathcal{A}$ be $L^1(\mathbb{G})$, $L^1_{\mathrm{cb}}(\mathbb{G})$ or $L^1_{\mathrm{M}}(\mathbb{G})$. Considering $\mathfrak{X} = \mathcal{A}$ as a Banach $\mathcal{A}$-bimodule with the module action being given by the product of $\mathcal{A}$, we have $\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) = LUC(\mathcal{A}^*)$. For example if $\mathcal{A} = L^1_{\mathrm{cb}}(\mathbb{G})$, then $\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) = LUC_{\mathrm{cb}}(\mathbb{G})$.

(ii) Let $\mathcal{A}$ be $L^1(\mathbb{G})$ and $\mathfrak{X}$ be $L^1_{\mathrm{cb}}(\mathbb{G})$. Then, using the definition of these algebras, $\mathfrak{X}$ is naturally a Banach $\mathcal{A}$-bimodule. Note that in this case

$$\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) = \overline{\langle L^1_{\mathrm{cb}}(\mathbb{G})^* \cdot L^1_{\mathrm{cb}}(\mathbb{G})\rangle}^{\|\cdot\|_{L^\infty(\mathbb{G})}} = \overline{\langle L^1_{\mathrm{cb}}(\mathbb{G})^* \cdot L^1(\mathbb{G})\rangle}^{\|\cdot\|_{L^\infty(\mathbb{G})}} \subseteq LUC(\mathbb{G}).$$

(iii) Let $\mathcal{A}$ be $L^1_{\mathrm{cb}}(\mathbb{G})$ and $\mathfrak{X}$ be $L^1(\mathbb{G})$. If we regard $L^1(\mathbb{G})$ as a Banach $L^1_{\mathrm{cb}}(\mathbb{G})$-bimodule with the module action being given by the product of $L^1_{\mathrm{cb}}(\mathbb{G})$ on $L^1(\mathbb{G})$, then by Lemma 2.5 we conclude that

$$\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) = \overline{\langle L^\infty(\mathbb{G}) \cdot L^1(\mathbb{G})\rangle}^{\|\cdot\|_{L^1_{\mathrm{cb}}(\mathbb{G})^*}} = LUC_{\mathrm{cb}}(\mathbb{G}).$$

The following result generalizes a result of Forrest ([10], Proposition 1) to the quantum group context.

PROPOSITION 3.3. *Let $\mathbb{G}$ be a locally compact quantum group. Then $L^1(\mathbb{G})$ has an approximate identity that is bounded in the $\|\cdot\|_{\mathrm{cb}}$-norm if and only if $L^1_{\mathrm{cb}}(\mathbb{G})$ has a bounded approximate identity.*

Proof. Suppose that $L^1(\mathbb{G})$ has an approximate identity (f_γ) with

$$C = \sup \|f_\gamma\|_{\mathrm{cb}} < \infty.$$

Let $\mu \in L^1_{\mathrm{cb}}(\mathbb{G})$ and $\varepsilon > 0$. Then there exists $f \in L^1(\mathbb{G})$ such that $\|f - \mu\|_{\mathrm{cb}} < \varepsilon$. We can find γ_0 such that if $\gamma_0 \leqslant \gamma$, then

$$\|f - f \star f_\gamma\|_{\mathrm{cb}} \leqslant \|f - f \star f_\gamma\|_1 < \varepsilon.$$

Thus, for each $\gamma_0 \leqslant \gamma$ we have

$$\|\mu - \mu \star f_\gamma\|_{\mathrm{cb}} \leqslant \|f - \mu\|_{\mathrm{cb}} + \|f - f \star f_\gamma\|_{\mathrm{cb}} + \|f \star f_\gamma - \mu \star f_\gamma\|_{\mathrm{cb}} < \varepsilon(2 + C).$$

This shows that (f_γ) is a bounded left approximate identity for $L^1_{\mathrm{cb}}(\mathbb{G})$. Similarly, we can show that (f_γ) is a bounded right approximate identity for $L^1_{\mathrm{cb}}(\mathbb{G})$, as required.

Conversely, suppose that (f_γ) is a bounded approximate identity for $L^1_{\mathrm{cb}}(\mathbb{G})$. As $L^1(\mathbb{G})$ is dense in $L^1_{\mathrm{cb}}(\mathbb{G})$, we can assume that $(f_\gamma) \subseteq L^1(\mathbb{G})$. Let $f \in L^1(\mathbb{G})$ and $\varepsilon > 0$ be given. Since the linear span of $L^1(\mathbb{G}) \star L^1(\mathbb{G})$ is dense in $L^1(\mathbb{G})$, there exist $g_i, h_i \in L^1(\mathbb{G})$ $(i = 1, \dots, n)$ such that $\Big\| \sum_{i=1}^n g_i \star h_i - f \Big\|_1 < \varepsilon$. Thus

$$\begin{aligned}\|f - f \star f_\gamma\|_1 &\leqslant \Big\|f - \sum_{i=1}^n g_i \star h_i\Big\|_1 + \Big\| \sum_{i=1}^n g_i \star h_i - \sum_{i=1}^n g_i \star h_i \star f_\gamma \Big\|_1 \\ &\quad + \Big\| \Big(\sum_{i=1}^n g_i \star h_i \Big) \star f_\gamma - f \star f_\gamma \Big\|_1 \\ &< \varepsilon(1 + C) + \sum_{i=1}^n \|g_i\|_1 \, \|h_i - h_i \star f_\gamma\|_{\mathrm{cb}},\end{aligned}$$

where $C = \sup \|f_\gamma\|_{\mathrm{cb}}$. This shows that $\limsup_\gamma \|f - f \star f_\gamma\|_1 \leqslant \varepsilon(1 + C)$ and consequently (f_γ) is an approximate identity for $L^1(\mathbb{G})$ which is bounded in the $\|\cdot\|_{\mathrm{cb}}$-norm. ∎

Before stating the next result, we recall that every locally compact quantum group $\mathbb{G}$ has a canonical co-involution R, called the unitary antipode of $\mathbb{G}$. That is, $R : L^\infty(\mathbb{G}) \to L^\infty(\mathbb{G})$ is a $*$-anti-homomorphism satisfying $R^2 = \mathrm{id}$ and $\Gamma \circ R = \sigma(R \otimes R) \circ \Gamma$, where σ is the flip map on $L^2(\mathbb{G}) \otimes L^2(\mathbb{G})$; see [18]. Then R induces a completely isometric involution on $L^1(\mathbb{G})$ defined by

$$\langle x, f' \rangle = \overline{\langle f, R(x^*) \rangle} \quad (x \in L^\infty(\mathbb{G}), f \in L^1(\mathbb{G})).$$

Hence, $L^1(\mathbb{G})$ becomes an involutive Banach algebra.

COROLLARY 3.4. *Let $\mathbb{G}$ be a locally compact quantum group and let $\mathcal{A}$ be $L^1(\mathbb{G})$, $L^1_{\mathrm{cb}}(\mathbb{G})$ or $L^1_{\mathrm{M}}(\mathbb{G})$. Then the map*

$$\Phi : LUC(\mathcal{A}^*)^* \to B_{\mathcal{A}}(\mathcal{A}^*)$$

is surjective if and only if $\mathcal{A}$ has a bounded approximate identity.

Proof. We give the proof for the case where $\mathcal{A} = L^1_{\mathrm{cb}}(\mathbb{G})$. The proofs of the other cases are similar. Consider $\mathfrak{X} = \mathcal{A}$ as a natural Banach $\mathcal{A}$-bimodule. By Example 3.2(i) we have that $\mathcal{L}_{\mathfrak{X}}(\mathcal{A}) = LUC_{\mathrm{cb}}(\mathbb{G})$. Moreover, if Φ is surjective, then $L^1_{\mathrm{cb}}(\mathbb{G})$ has a bounded right approximate identity by Theorem 3.1. Now, the proof of Proposition 3.3 shows that $L^1(\mathbb{G})$ has a right approximate identity (f_γ) that is bounded in the cb-multiplier norm. Since the unitary antipode R of $\mathbb{G}$ induces a completely isometric involution on $L^1(\mathbb{G})$, we can construct a left approximate identity (f'_γ) from (f_γ) for $L^1(\mathbb{G})$ that is bounded in the cb-multiplier norm. It is then easily seen that $(f_\gamma + f'_\gamma - f_\gamma \star f'_\gamma)$ is an approximate identity for $L^1(\mathbb{G})$ that is bounded in the $\|\cdot\|_{\mathrm{cb}}$-norm. Thus, $L^1_{\mathrm{cb}}(\mathbb{G})$ has a bounded approximate identity by Proposition 3.3.

Now, suppose that $\mathcal{A} = L^1_{\rm cb}(\mathbb{G})$ has a bounded approximate identity (ν_γ). Let E be any weak* cluster point of (ν_γ) in $\mathcal{A}^{**}$. Then E is a right identity in $(\mathcal{A}^{**}, \Box)$. For $T \in B_{\mathcal{A}}(\mathcal{A}^*)$ if we put $m := T^*(E)|_{LUC_{\rm cb}(\mathbb{G})}$, then we show that $\Phi(m) = T$. In fact, for each $x \in \mathcal{A}^*$ and $\mu \in \mathcal{A}$, we have

$$\begin{aligned}\langle \Phi(m)(x), \mu\rangle &= \langle m_{\rm L}(x), \mu\rangle = \langle m, x\cdot\mu\rangle = \langle T^*(E), x\cdot \mu\rangle \\ &= \langle T(x), \mu\Box E\rangle = \langle T(x), \mu\rangle.\end{aligned}$$

This shows that $\Phi(m) = T$ and consequently Φ is surjective. ∎

COROLLARY 3.5. *Let $\mathbb{G}$ be a locally compact quantum group. Then the map*

$$\Phi : \mathcal{L}_{L^1_{\rm cb}(\mathbb{G})}(L^1(\mathbb{G}))^* \to B_{L^1(\mathbb{G})}(L^1_{\rm cb}(\mathbb{G})^*)$$

is surjective if and only if $\mathbb{G}$ is co-amenable.

Proof. First note that by Example 3.2(ii) we have

$$\mathcal{L}_{L^1_{\rm cb}(\mathbb{G})}(L^1(\mathbb{G})) = \overline{\langle L^1_{\rm cb}(\mathbb{G})^* \cdot L^1_{\rm cb}(\mathbb{G})\rangle}^{\|\cdot\|_{L^\infty(\mathbb{G})}} \subseteq LUC(\mathbb{G}).$$

If Φ is surjective, then it follows from Theorem 3.1 that $L^1_{\rm cb}(\mathbb{G})$ has a right approximate identity in $L^1(\mathbb{G})$ which is bounded in the $\|\cdot\|_1$-norm. Hence, $\mathbb{G}$ is co-amenable by Theorem 3.1 of [2].

For the converse, note that if $\mathbb{G}$ is co-amenable, then $L^1(\mathbb{G}) = L^1_{\rm cb}(\mathbb{G})$ and the result follows from Corollary 3.4. ∎

COROLLARY 3.6. *Let $\mathbb{G}$ be a locally compact quantum group. If the map*

$$\Phi : LUC_{\rm cb}(\mathbb{G})^* \to B_{L^1_{\rm cb}(\mathbb{G})}(L^\infty(\mathbb{G}))$$

is surjective, then $L^1_{\rm cb}(\mathbb{G})$ has a bounded approximate identity.

Proof. This follows by combining the same argument as that given in the proof of Corollary 3.4 and Example 3.2(iii). ∎

Let $\mathbb{G}$ be a locally compact quantum group. Then we have the canonical $M(\mathbb{G})$-bimodule actions on $C_0(\mathbb{G})$ given by

$$\omega\cdot x = ({\rm id}\otimes\omega)\Gamma(x), \quad x\cdot\omega = (\omega\otimes{\rm id})\Gamma(x) \quad (x\in C_0(\mathbb{G}), \omega\in M(\mathbb{G})).$$

It was shown in Proposition 2.2 of [13] that

$$C_0(\mathbb{G}) = \overline{\langle M(\mathbb{G})\cdot C_0(\mathbb{G})\rangle}^{\|\cdot\|_{L^\infty(\mathbb{G})}}.$$

We also note that the inclusion $C_0(\mathbb{G}) \to L^\infty(\mathbb{G})$ is an $M(\mathbb{G})$-bimodule map. Therefore, if we regard $C_0(\mathbb{G})$ as a sub-$L^1(\mathbb{G})$-bimodule of $L^\infty(\mathbb{G})$, then we have

$$\mathcal{L}_{L^1(\mathbb{G})}(C_0(\mathbb{G})) = C_0(\mathbb{G}) \subseteq L^\infty(\mathbb{G}).$$

This implies that the map $\Phi : M(\mathbb{G}) \to B_{L^1(\mathbb{G})}(M(\mathbb{G}))$, $\omega \mapsto \omega_{\rm L}$ induces a contractive, injective algebra homomorphism, where $\omega_{\rm L}$ is given by $\omega_{\rm L}(\nu) = \omega\star\nu$

for all $\nu \in M(\mathbb{G})$. Finally, since $\omega \star \nu = \omega \Box \nu = \omega \Diamond \nu$ for all $\omega, \nu \in M(\mathbb{G})$, we have $\Phi(M(\mathbb{G})) \subseteq B^{\sigma}_{L^1(\mathbb{G})}(M(\mathbb{G}))$.

COROLLARY 3.7. *Let $\mathbb{G}$ be a locally compact quantum group. Then the map*

$$\Phi : M(\mathbb{G}) \to B^{\sigma}_{L^1(\mathbb{G})}(M(\mathbb{G}))$$

is surjective if and only if $\mathbb{G}$ is co-amenable.

Proof. Suppose that Φ is surjective. Replacing $\mathfrak{X}$ and $\mathcal{A}$ in Proposition 3.1 by $C_0(\mathbb{G})$ and $L^1(\mathbb{G})$, respectively, and noticing that $\mathrm{id}_{M(\mathbb{G})} \in B^{\sigma}_{L^1(\mathbb{G})}(M(\mathbb{G}))$, we conclude that $C_0(\mathbb{G})$ has a bounded right approximate identity (f_γ) in $L^1(\mathbb{G})$. Let μ_0 be a weak* cluster point of (f_γ) in $C_0(\mathbb{G})^* = M(\mathbb{G})$. A routine calculation shows that μ_0 is a left identity for $M(\mathbb{G})$. Therefore, $M(\mathbb{G})$ is unital, since $C_0(\mathbb{G}) = \overline{\langle M(\mathbb{G}) \cdot C_0(\mathbb{G})\rangle}^{\|\cdot\|_{L^\infty(\mathbb{G})}}$; see Proposition 2.2 of [13]. Thus, $\mathbb{G}$ is co-amenable by Theorem 3.1 of [2].

For the converse, note that if $\mathbb{G}$ is co-amenable, then $M(\mathbb{G})$ has an identity element, say μ_0, by Theorem 3.1 of [2]. Now let $T \in B^{\sigma}_{L^1(\mathbb{G})}(M(\mathbb{G}))$ and $\nu \in M(\mathbb{G})$. Since $L^1(\mathbb{G})$ is weak*-dense in $M(\mathbb{G})$, there is a net (f_γ) in $L^1(\mathbb{G})$ such that $f_\gamma \xrightarrow{w^*} \nu$ and hence

$$T(\mu_0 \star f_\gamma) = T(\mu_0) \star f_\gamma \xrightarrow{w^*} T(\mu_0) \star \nu.$$

In particular, for each $x \in C_0(\mathbb{G})$, by weak*-weak* continuity of T, we have

$$\langle \Phi(T(\mu_0))(\nu), x\rangle = \langle T(\mu_0) \star \nu, x\rangle = \lim_{\gamma} \langle T(\mu_0 \star f_\gamma), x\rangle = \langle T(\nu), x\rangle.$$

Therefore, $\Phi(T(\mu_0)) = T$ for all $T \in B^{\sigma}_{L^1(\mathbb{G})}(M(\mathbb{G}))$. ∎

Acknowledgements. The author is grateful to the referee for valuable comments on the paper.

REFERENCES

[1] O.Y. ARISTOV, Amenability and compact type for Hopf–von Neumann algebras from the homological point of view, in *Banach Algebras and their Applications*, Contemp. Math., vol. 363, Amer. Math. Soc., Providence, RI 2004, pp. 15–37.

[2] E. BÉDOS, L. TUSET, Amenability and co-amenability for locally compact quantum groups, *Internat. J. Math.* **14**(2003), 865–884.

[3] J. CRANN, M. NEUFANG, Amenability and covariant injectivity of locally compact quantum groups, *Trans. Amer. Math. Soc.* **368**(2016), 495–513.

[4] H.G. DALES, A.T.-M. LAU, The second duals of Beurling algebras, *Mem. Amer. Math. Soc.* **177**(2005), 1–191.

[5] P. DESMEDT, J. QUAEGEBEUR, S. VAES, Amenability and the bicrossed product construction, *Illinois J. Math.* **46**(2002), 1259–1277.

[6] J. DE CANNIÉRE, U. HAAGERUP, Multipliers of the Fourier algebras of some simple Lie groups and their discrete subgroups, *Amer. J. Math.* **107**(1985), 455–500.

[7] J. DUNCAN, S.A.R. HOSSEINIUN, The second dual of a Banach algebra, *Proc. Roy. Soc. Edinburgh Sect. A* **84**(1979), 309–325.

[8] P. EYMARD, L'algèbre de Fourier d'un groupe localement compact, *Bull. Soc. Math. France* **92**(1964), 181–236.

[9] B.E. FORREST, Some Banach algebras without discontinuous derivations, *Proc. Amer. Math. Soc.* **114**(1992), 965–970.

[10] B.E. FORREST, Completely bounded multipliers and ideals in $A(G)$ vanishing on closed subgroups, in *Banach Algebras and their Applications*, Contemp. Math., vol. 363, Amer. Math. Soc., Providence, RI 2004, pp. 89–94.

[11] B.E. FORREST, T. MIAO, Uniformly continuous functionals and M-weakly amenable groups, *Canad. J. Math.* **65**(2013), 1005–1019.

[12] B.E. FORREST, V. RUNDE, N. SPRONK, Operator amenability of the Fourier algebra in the cb-multiplier norm, *Canad. J. Math.* **59**(2007), 966–980.

[13] Z. HU, M. NEUFANG, Z.-J. RUAN, Completely bounded multipliers over locally compact quantum groups, *Proc. London Math. Soc.* **103**(2011), 1–39.

[14] Z. HU, M. NEUFANG, Z.-J. RUAN, Multipliers on a new class of Banach algebras, locally compact quantum groups, and topological centres, *Proc. London Math. Soc.* **100**(2010), 429–458.

[15] B.E. JOHNSON, An introduction to double centralizers, *Proc. London Math. Soc.* **14**(1964), 299–320.

[16] M. JUNGE, M. NEUFANG, Z.-J. RUAN, A representation theorem for locally compact quantum groups, *Internat. J. Math.* **20**(2009), 377–400.

[17] J. KUSTERMANS, S. VAES, locally compact quantum groups, *Ann. Sci. École Norm. Sup. (4)* **33**(2000), 837–934.

[18] J. KUSTERMANS, S. VAES, locally compact quantum groups in the von Neumann algebraic setting, *Math. Scand.* **92**(2003), 68–92.

[19] A.T.-M. LAU, Analysis on a class of Banach algebras with application to harmonic analysis on locally compact groups and semigroups, *Fund. Math.* **118**(1983), 161–175.

[20] V. LOSERT, Properties of the Fourier algebra that are equivalent to amenability, *Proc. Amer. Math. Soc.* **92**(1984), 347–354.

[21] V. LOSERT, Fourier-Algebra und mittelbare Gruppen, Lectures at Heidelberg Univ., Heidelberg 1985.

[22] M. NEMATI, Invariant φ-means on left introverted subspaces with application to locally compact quantum groups, *Arch. Math. (Basel)* **106**(2016), 543–552.

[23] J.S. PYM, The convolution of functionals on spaces of bounded functions, *Proc. London Math. Soc.* **15**(1965), 84–104.

[24] V. RUNDE, Characterizations of compact and discrete quantum groups through second duals, *J. Operator Theory* **60**(2008), 415–428.

[25] V. RUNDE, Uniform continuity over locally compact quantum groups, *J. London Math. Soc.* **80**(2009), 55–71.

[26] R. STOKKE, Amenability and modules for Arens product algebras, *Q. J. Math.* **66**(2015), 295–321.

MEHDI NEMATI, DEPARTMENT OF MATHEMATICAL SCIENCES, ISFAHAN UNIVERSITY OF TECHNOLOGY, ISFAHAN 84156-83111, IRAN
E-mail address: m.nemati@cc.iut.ac.ir

Received May 30, 2016; revised August 21, 2016.