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COMPOSITION OF TOPOLOGICAL CORRESPONDENCES

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ABSTRACT. In our previous work, *J. Operator Theory*, **77**(2017), 217–241, we define a topological correspondence from a locally compact groupoid equipped with a Haar system to another one and we show that a topological correspondence, (X, λ), from a locally compact groupoid with a Haar system (G, α) to another one, (H, β), produces a C^*-correspondence $\mathcal{H}(X)$ from $C^*(G, \alpha)$ to $C^*(H, \beta)$. In the present article, we describe how to form a composite of two topological correspondences when the bispaces are Hausdorff and second countable in addition to being locally compact.

KEYWORDS: *Topological correspondences, composition of topological correspondences, functoriality of topological correspondences.*

MSC (2010): 22D25, 22A22, 46H35, 47L30, 46L08.

1. INTRODUCTION

Let (G, α) and (H, β) be locally compact groupoids with Haar systems. A topological correspondence from (G, α) to (H, β) is a G-H-bispace X which is equipped with a continuous family of measures λ along the momentum map $s_X : X \to H^{(0)}$, and the action of H and the family of measures satisfy certain conditions (see Definition 2.1 of [5]). We need that the action of H is proper, and the condition on λ is that it is H-invariant and each measure in λ is (G, α)-quasi-invariant. The groupoids G and H, and the space X are locally compact but not necessarily Hausdorff. However, in the present work we assume that all the locally compact spaces are Hausdorff.

The main result in [5] says that a topological correspondence (X, λ) from (G, α) to (H, β) produces a C^*-correspondence $\mathcal{H}(X)$ from $C^*(G, \alpha)$ to $C^*(H, \beta)$. Section 3 of [5] discusses many examples of topological correspondences.

Two C^*-correspondences, $\mathcal{K} : A \to B$ and $\mathcal{F} : B \to C$, may be composed to get a correspondence $\mathcal{K}\widehat{\otimes}_B\mathcal{F} : A \to C$. On similar lines, consider two topological correspondences (X, α) and (Y, β) from (G_1, χ_1) to (G_2, χ_2) and (G_2, χ_2)

to (G_3, χ_3), respectively, where (G_i, χ_i) for $i = 1,2,3$ is a locally compact groupoid with a Haar system. We describe the composite $(Y, \beta) \circ (X, \alpha) : (G_1, \chi_1) \to (G_3, \chi_3)$ when X and Y are Hausdorff and second countable in addition to being locally compact. In fact, our construction works when X and Y are Hausdorff and the space $(X \times_{s_X, {G_2}^{(0)}, r_Y} Y)/G_2$ is paracompact; here s_X and r_X denote the momentum maps for the actions of G_2 on X and Y, respectively, and the quotient is taken for the diagonal action of G_2 on $X \times_{s_X, {G_2}^{(0)}, r_Y} Y$.

The composite $(Y, \beta) \circ (X, \alpha)$ should be a pair (Ω, μ) where Ω is a G_1-G_3-bispace, μ is a continuous family of measures along the momentum map

$$s_\Omega : \Omega \to {G_3}^{(0)},$$

and the conditions in Definition 2.1 of [5] are satisfied. Furthermore, we must have an isomorphism $\mathcal{H}(\Omega) \simeq \mathcal{H}(X) \widehat{\otimes}_{C^*(G_2,\chi_2)} \mathcal{H}(Y)$ of C^*-correspondences.

The construction of Ω is well-known — it is the quotient space $(X \times_{s_X, {G_2}^{(0)}, r_Y} Y)/G_2$ for the diagonal action of G_2 on $X \times_{s_X, {G_2}^{(0)}, r_Y} Y$. The diagonal action is proper, since the action of G_2 on X is proper. Thus the quotient space inherits all the *nice* properties of the fibred product such as Hausdorffness. The harder task is to get the continuous family of measures μ satisfying the required conditions.

We need that $\mu := \{\mu_u\}_{u \in {G_3}^{(0)}}$ is G_3-invariant and each μ_u is (G_1, χ_1)-quasi-invariant. We explain how to get *one* such family of measures. The reason to write "one such family of measures" is that the family is not unique; it depends on the choice of a certain continuous function on $X \times_{s_X, {G_2}^{(0)}, r_Y} Y$. However, for any two such families of measures the corresponding C^*-correspondences are naturally isomorphic to $\mathcal{H}(X) \widehat{\otimes}_{C^*(G_2,\chi_2)} \mathcal{H}(Y)$.

The construction of μ is one of the most technical parts of this article. To explain the problem, motivation and idea of constructing the composite of families of measures, we have to do a computation and discuss some technical ideas. Denote the space $X \times_{s_X, {G_2}^{(0)}, r_Y} Y$ by Z. Then Z carries a G_3-invariant continuous family of measures $m := \{m_u\}_{u \in G^{(0)}{}_3}$ which is given by

$$\int_Z f \, \mathrm{d}m_u = \int_Y \int_X f(x,y) \, \mathrm{d}\alpha_{r_Y(y)}(x) \, \mathrm{d}\beta_u(y) \tag{1.1}$$

for $f \in C_c(Z)$. Let $\pi : Z \to \Omega$ be the quotient map and λ the continuous family of measures along it defined as

$$\int_{\pi^{-1}([x,y])} f \, \mathrm{d}\lambda^{[x,y]} := \int_{G_2^{r_Y(y)}} f(x\gamma, \gamma^{-1}y) \, \mathrm{d}\chi_2^{r_Y(y)}(\gamma) \tag{1.2}$$

for $f \in C_c(Z)$, and $[x,y] \in \Omega$ which is the equivalence class of $(x,y) \in Z$. A very natural choice for μ is that it is the family of measures on Ω which gives the disintegration $m = \mu \circ \lambda$. Furthermore, one may expect that the isomorphism $\mathcal{H}(X) \widehat{\otimes}_{C^*(G_2,\chi_2)} \mathcal{H}(Y) \simeq \mathcal{H}(\Omega)$ is induced by the map $\Psi : C_c(Z) \to C_c(\Omega)$ where

$\Psi(F)([x,y]) = \int\limits_{G_2} F(x\gamma, \gamma^{-1}y)\, \mathrm{d}\chi_2^{s_X(x)}(\gamma)$. To be more explicit, we view $C_c(Z)$ and $C_c(\Omega)$ as pre-Hilbert $C^*(G_3, \chi_3)$-modules which complete to the Hilbert $C^*(G_3, \chi_3)$-modules $\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)$ and $\mathcal{H}(\Omega)$. And the map $\Psi : C_c(Z) \to C(\Omega)$ is expected to induce the required isomorphism of Hilbert $C^*(G_3, \chi_3)$-modules which also gives the desired isomorphism of C^*-correspondence.

However, that is not *exactly* the case. Consider the following example: let G be a group and H a closed proper subgroup of G. Let α and κ be the Haar measures on G and H, respectively. Then (G, α^{-1}) is a topological correspondence from (G, α) to (H, κ) which is called the *induction correspondence* in Example 3.13 of [5]. The constant function 1 is the adjoining function for this correspondence.

Let X be a left H-space carrying an (H, κ)-quasi-invariant measure β. Let Δ_X denote the 1-cocycle on the transformation groupoid $H \ltimes X$ that gives the quasi-invariance. Assume that Δ_X is continuous. Then (X, β) is a topological correspondence from (H, κ) to the trivial group Pt, see Example 3.6 of [5]. The adjoining function of this correspondence is Δ_X. Furthermore, $\mathcal{H}(X) = \mathcal{L}^2(X, \beta)$ and the action of H induces the representation of $C^*(H)$ on $\mathcal{L}^2(X, \beta)$. Thus we have

$$(G, \alpha) \xrightarrow{(G,\alpha^{-1})} (K, \kappa) \xrightarrow{(X,\beta)} \mathrm{Pt}.$$

Let $Z, \Omega, \pi, m, \lambda$ and μ have the similar meaning as in the above discussion. Then in this situation, $Z = G \times X$, $\Omega = (G \times X)/K$, $m = \alpha^{-1} \times \beta$ and $\pi : G \times X \to (G \times X)/K$ the quotient map. For $f \in C_c(Z)$ equation (1.2) now reads

$$\int\limits_{\pi^{-1}([\gamma,x])} f \,\mathrm{d}\lambda^{[\gamma,x]} = \int\limits_H f(\gamma\eta, \eta^{-1}x)\, \mathrm{d}\kappa(\eta).$$

What are the necessary and sufficient conditions to get a measure μ on $(G \times X)/K$ satisfying $\alpha^{-1} \times \beta = m = \mu \circ \lambda$?

We may draw a square as in Figure 3 comprising the spaces, maps and measures discussed above. And then (i) of Proposition 3.1 implies that if there is such a measure μ, then the equality $m \circ \kappa = m \circ \kappa^{-1}$ must hold — this is the necessary condition. Recall from the discussion above that $m = \alpha^{-1} \times \beta$. Thus we must have $(\alpha^{-1} \times \beta) \circ \kappa = (\alpha^{-1} \times \beta) \circ \kappa^{-1}$ on $Z \times K = G \times X \times K$. Let $f \in C_c(G \times X \times K)$, then a direct computation gives that

$$(\alpha^{-1} \times \beta) \circ \kappa(f) = \int\limits_G \int\limits_X \int\limits_K f(\gamma, x, \eta)\, \mathrm{d}\kappa(\eta)\, \mathrm{d}\beta(x)\, \mathrm{d}\alpha^{-1}(\gamma).$$

On the other hand,

$$(\alpha^{-1} \times \beta) \circ \kappa^{-1}(f) = \int\limits_G \int\limits_X \int\limits_K f(\gamma, x, \eta^{-1})\, \mathrm{d}\kappa(\eta)\, \mathrm{d}\beta(x)\, \mathrm{d}\alpha^{-1}(\gamma).$$

Now (i) first apply Fubini's theorem to $\mathrm{d}\kappa\,\mathrm{d}\beta$, (ii) then change variable (γ, x, η^{-1}) to $(\gamma\eta, \eta^{-1}x, \eta)$, (iii) then use the (H, κ)-quasi-invariance of β and the right invariance of α^{-1} and (iv) finally apply Fubini's theorem to $\mathrm{d}\beta\,\mathrm{d}\kappa$ to see that the last term equals

$$\int_G \int_X \int_K f(\gamma, \eta^{-1}x, \eta)\, \Delta_X(\eta, \eta^{-1}x)\, \mathrm{d}\kappa(\eta)\, \mathrm{d}\beta(x)\, \mathrm{d}\alpha^{-1}(\gamma).$$

Thus the 1-cocycle Δ_X on the transformation groupoid $H \ltimes X$ is the obstruction for the measures $(\alpha^{-1} \times \beta) \circ \kappa$ and $(\alpha^{-1} \times \beta) \circ \kappa^{-1}$ to be equal. (ii) of Proposition 3.1 says that equality of these measures, $m \circ \kappa = m \circ \kappa^{-1}$, is also a sufficient condition in our situation for the measure μ to exist.

A similar problem appears in the general setting; there the cocycle Δ_X is replaced by the adjoining function of the second correspondence involved in the composition. How to overcome this obstruction?

Let $(X, \alpha) : (G_1, \chi_1) \to (G_2, \chi_2)$ and $(Y, \beta) : (G_2, \chi_2) \to (G_3, \chi_3)$ be topological correspondences and let Δ_2 be the adjoining function of (Y, β). Then we realise Δ_2 as a 1-cocycle on the proper groupoid $Z \rtimes G_2$ and *decompose* it into a quotient $\Delta_2 = b \circ s_{Z \rtimes G_2} / b \circ r_{Z \rtimes G_2}$ for a continuous real-valued function b on the unit space of the groupoid $Z \rtimes G_2$; thus b is a 0-cochain. Here $s_{Z \rtimes G_2}$ and $r_{Z \rtimes G_2}$ denote the source and the range maps of the transformation groupoid $Z \rtimes G_2$, respectively. Using Proposition 3.1 we show that there is a unique measure μ which gives the disintegration $bm = \mu \circ \lambda$. We modify the map $\Psi : C_c(Z) \to C_c(\Omega)$ discussed above (the discussion following equation (1.2) on page 90) to consider the 0-cochain b. Then this μ and modified Ψ produce the desired isomorphism of C^*-correspondences.

In this construction, the function b defined on Z, the space of units of the transformation groupoid $Z \rtimes G_2$, is not unique. However, as mentioned earlier, for any two such functions the C^*-correspondences associated with the composites are isomorphic to $\mathcal{H}(X) \widehat{\otimes}_{C^*(G_2, \chi_2)} \mathcal{H}(Y)$. Given two such functions b and b' on Z which decompose Δ_2 as above, with some slight work, one may show that there is a positive continuous function c on Z with $b = cb'$. This explains the isomorphism of C^*-correspondences associated with the two composites.

The elements of groupoid cohomology which we need are developed in Section 1 of [5] and we prove the other necessary results in this article. The main result we need is Proposition 2.7 which asserts that the first $\mathbb{R}$-valued cohomology of a proper groupoid is trivial. Lemma 2.5 is the main tool to prove this proposition. This lemma says that a locally compact proper groupoid G equipped with a Haar system and $G \backslash G^{(0)}$ paracompact carries an invariant family of probability measures.

As shown in Examples 3.10 and 3.7 of [5], the generalized morphism defined by Buneci and Stachura in [3], and the topological correspondences for the groupoids with Hausdorff space of units introduced by Tu in [9], respectively, are topological correspondences. Our construction of composite matches the ones described by Tu [9], and Buneci and Stachura [3], see Examples 4.5 and 4.6, respectively.

We discuss some examples at the end of the article. A continuous map $f : X \to Y$ of spaces gives a topological correspondence (X, δ_X) from Y to X ([5], Example 3.1). Here $\delta_X = \{\delta_x\}_{x \in X}$ is the family of measures where each δ_x is the point mass at $x \in X$. A continuous group homomorphism $\phi : G \to H$ gives a topological correspondences (H, β^{-1}) from G to H ([5], Example 3.4) where β is the Haar measure on H. If $g : Y \to Z$ is a function, then Example 4.1 shows that the composite of the topological correspondences (X, δ_X) and (Y, δ_Z) is the correspondence obtained from the function $g \circ f : X \to Z$. Example 4.3 shows that a similar result holds for group homomorphisms. Both these examples agree with the well-known behaviour of the C^*-functor for spaces and groups.

Let G be a locally compact group, and let H and K be closed subgroups of G. Example 4.7 shows how to use topological correspondences to induce a *topological* representation of K to H.

Most of our terminology, definitions, hypotheses and notation are defined in [5]. Now we describe the structure of the article briefly. In the first section, we revise few definitions, notation and results in [5]. We prove that every locally compact proper groupoid equipped with a Haar system carries an invariant continuous family of probability measures. Then using this result we prove that every real valued 1-cocycle on a proper groupoid is a coboundary.

In the second section, we describe the composition of topological correspondences and prove the main result, Theorem 3.14, which says that the C^*-correspondence associated with a composite is isomorphic to the composite of the C^*-correspondences.

The last section contains examples. Most of the examples are related to the ones in [5].

2. PRELIMINARIES

2.1. REVISION. The symbols $\simeq$, $\approx$, $\mathbb{R}^+$ and $\mathbb{R}^+_*$ stand for isomorphic, homeomorphic, the set of positive real numbers and the multiplicative group of positive real numbers, respectively. The symbols $\otimes_{\mathbb{C}}$ and $\widehat{\otimes}$ indicate the algebraic tensor product of modules and the interior tensor product of Hilbert modules, respectively. To denote elements in the algebraic tensor product we use $\otimes$ and for the ones in the interior tensor product of Hilbert modules $\widehat{\otimes}$ is used.

We work with continuous families of measures and all the measures are assumed to be positive, Radon and σ-finite. The families of measures are denoted by small Greek letters and the corresponding integration function that appears in the continuity condition is denoted by the Greek upper case letter used to denote the family of measures. For example, if λ is a family of measures along a map

$f : X \to Y$, then $\Lambda : C_c(X) \to C_c(Y)$ is the function $\Lambda(f)(y) = \int\limits_X f \, d^y$. The capitalisations of α, β, χ and μ are A, B, χ and M, respectively.

However, for a single measure on a space, which is a family of measures along the constant map onto a point, we follow the traditional convention, that is, the same letter is used to denote the measure and the corresponding integration functional. For example, if α is a measure on X, then $\alpha(f) = \int\limits_X f \, d\alpha$ for $f \in C_c(X)$.

Let G be a groupoid, then r_G, s_G and inv_G denote the source, range and the inversion maps for G. Given a left G-space X, we tacitly assume that the momentum map for the action is r_X. If X is a right G-space, then s_X is the momentum map for the action.

We denote $G \times_{s_G, G^{(0)}, r_X} X$, the fibred product for G and X over $G^{(0)}$ along s_G and r_X, by $G \times_{G^{(0)}} X$. If X is a right G-space, then $X \times_{G^{(0)}} G$ has a similar meaning. For a left G-space X, $G \ltimes X$ is the transformation groupoid and its set of arrows is the fibred product $G \times_{G^{(0)}} X$. Similar is the meaning of $X \rtimes G$ for a right G-space X.

For $A, B \subseteq G^{(0)}$ we define $G^A = r_G^{-1}(A), G_B = s_G^{-1}(B)$ and $G_B^A = G^A \cap G_B$. When $A = \{u\}$ and $B = \{v\}$ are singletons, we simply write G^u, G_v and G_v^u instead of $G^{\{u\}}$, $G_{\{v\}}$ and $G_{\{v\}}^{\{u\}}$, respectively. Let X and Y be left and right G-spaces, respectively, and let $A \subseteq G^{(0)}$ and $u \in G^{(0)}$. Then X^A, X^u, Y_B and Y_u have the similar obvious meanings.

Lance's book [6] is our reference for Hilbert modules hence most of our terminology comes from this book. We denote a C^*-correspondence only by the Hilbert module involved in it; we do not write the representation of the left C^*-algebra. Thus we say "$\mathcal{H}$ is a C^*-correspondence from a C^*-algebra A to B", and not "$(\mathcal{H}, \phi)$ is a C^*-correspondence from a C^*-algebra A to B" where $\phi : A \to \mathbb{B}_B(\mathcal{H})$ is the nondegenerate $*$-representation involved in the definition of the correspondence. We also write "$\mathcal{H} : A \to B$ is a C^*-correspondence".

Now we sketch the process of composing the C^*-correspondences briefly and explain a few notation along the way. Let A, B and C be C^*-algebras, and let $\mathcal{H} : A \to B$ and $\mathcal{F} : B \to C$ be C^*-correspondences. Endow $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{F}$ with the sesquilinear map $\langle \zeta \otimes \xi , \zeta' \otimes \xi' \rangle = \langle \xi , \langle \zeta , \zeta' \rangle \xi' \rangle$. Then $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{F}$ becomes a semi-inner product C-module, see page 3 of [6] for the definition of a semi-inner product C^*-module.

Let $N \subseteq \mathcal{H} \otimes_{\mathbb{C}} \mathcal{F}$ be the closed vector subspace of the vectors of zero norm, that is, $N = \{z \in \mathcal{H} \otimes_{\mathbb{C}} \mathcal{F} : \langle z , z \rangle = 0\}$. The proof of Proposition 4.5 in [6] shows that the subspace N is the same as the subspace spanned by the elements of the form $\zeta b \otimes \xi - \zeta \otimes b\xi$ where $\zeta \in \mathcal{H}, \xi \in \mathcal{F}$ and $b \in B$. The C-module $(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{F})/N$ carries a well-defined inner product induced by $\langle \cdot , \cdot \rangle$ which we denote by $[\langle \cdot , \cdot \rangle]$. The C-module $(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{F})/N$ equipped with the inner product $[\langle \cdot , \cdot \rangle]$ is a pre-Hilbert C-module. The Hilbert C-module $\mathcal{H} \widehat{\otimes}_B \mathcal{F}$ is the completion of $(\mathcal{H} \otimes_{\mathbb{C}} \mathcal{F})/N$ in the norm induced by $[\langle \cdot , \cdot \rangle]$.

The notation $[\langle\cdot\,,\cdot\rangle]$ is used in the above discussion for the sake of clarity; in the rest of the article we will not use this notation. We prefer writing $\langle\cdot\,,\cdot\rangle$. We denote the equivalence class of an elementary tensor $\zeta\otimes\xi\in\mathcal{H}\otimes_{\mathbb{C}}\mathcal{F}$ in $\mathcal{H}\widehat{\otimes}_B\mathcal{F}$ by $\zeta\widehat{\otimes}\xi$. The action of A on $\mathcal{H}\widehat{\otimes}_B\mathcal{F}$ is $a(\zeta\widehat{\otimes}\xi)=a\zeta\widehat{\otimes}\xi$ where $a\in A$ and $\zeta\widehat{\otimes}\xi\in\mathcal{H}\widehat{\otimes}\mathcal{F}$. We call the map $\xi\otimes_{\mathbb{C}}\zeta\mapsto\xi\widehat{\otimes}\zeta$, $\mathcal{H}\otimes_{\mathbb{C}}\mathcal{F}\to\mathcal{H}\widehat{\otimes}\mathcal{F}$, the obvious map of Hilbert C^*-modules which, clearly, has a dense image. It can be easily checked that the obvious map is a map of A-modules, that is, $a(\xi\otimes_{\mathbb{C}}\zeta)=(a\xi)\otimes_{\mathbb{C}}\zeta\mapsto(a\xi)\widehat{\otimes}\zeta=a(\xi\widehat{\otimes}\zeta)$ for all $a\in A,\xi\in\mathcal{H}$ and $\zeta\in\mathcal{F}$.

Finally, it is worth mentioning that the polarisation identity holds for a Hilbert module. That is, if $\mathcal{H}$ is a Hilbert B-module over a C^*-algebra B, then for every $\xi,\zeta\in\mathcal{H}$ we have

$$\langle\xi\,,\zeta\rangle=\frac{1}{4}(\langle\xi+\zeta\,,\xi+\zeta\rangle-\langle\xi-\zeta\,,\xi-\zeta\rangle-\mathrm{i}\langle\xi+\mathrm{i}\zeta\,,\xi+\mathrm{i}\zeta\rangle+\mathrm{i}\langle\xi-\mathrm{i}\zeta\,,\xi-\mathrm{i}\zeta\rangle). \tag{2.1}$$

Equation (2.1) follows from a direct computation.

DEFINITION 2.1 (Topological correspondence). A *topological correspondence* from a locally compact groupoid G with a Haar system α to a locally compact groupoid H equipped with a Haar system β is a pair (X,λ), where:

(i) X is a locally compact G-H-bispace,

(ii) the action of H is proper,

(iii) $\lambda=\{\lambda_u\}_{u\in H^{(0)}}$ is an H-invariant proper continuous family of measures along the momentum map $s_X: X\to H^{(0)}$,

(iv) there exists a continuous function $\Delta: G\ltimes X\to\mathbb{R}^+$ such that for each $u\in H^{(0)}$ and $F\in C_c(G\times_{G^{(0)}}X)$,

$$\int_{X_u}\int_{G^{r_X(x)}}F(\gamma^{-1},x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x)=\int_{X_u}\int_{G^{r_X(x)}}F(\gamma,\gamma^{-1}x)\Delta(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma)\,\mathrm{d}\lambda_u(x).$$

The function Δ is unique and is called *the adjoining function* of the correspondence.

For $\phi\in C_c(G)$, $f\in C_c(X)$ and $\psi\in C_c(H)$ define the functions $\phi\cdot f$ and $f\cdot\psi$ on X as follows:

$$\begin{cases}(\phi\cdot f)(x):=\displaystyle\int_{G^{r_X(x)}}\phi(\gamma)f(\gamma^{-1}x)\,\Delta^{1/2}(\gamma,\gamma^{-1}x)\,\mathrm{d}\alpha^{r_X(x)}(\gamma),\\[2ex](f\cdot\psi)(x):=\displaystyle\int_{H^{s_X(x)}}f(x\eta)\psi(\eta^{-1})\,\mathrm{d}\beta^{s_X(x)}(\eta).\end{cases} \tag{2.2}$$

For $f,g\in C_c(X)$ define the function $\langle f,g\rangle$ on H by

$$\langle f,g\rangle(\eta):=\int_{X_{r_H(\eta)}}\overline{f(x)}g(x\eta)\,\mathrm{d}\lambda_{r_H(\eta)}(x). \tag{2.3}$$

Very often we write ϕf and $f\psi$ instead of $\phi \cdot f$ and $f \cdot \psi$, respectively. Lemma 2.10 of [5] proves that $\phi f, f\psi \in C_c(X)$ and $\langle f, g\rangle \in C_c(H)$.

THEOREM 2.2 ([5], Theorem 2.39). *Let (G,α) and (H,β) be locally compact groupoids with Haar systems. Then a topological correspondence (X,λ) from (G,α) to (H,β) produces a C^*-correspondence $\mathcal{H}(X)$ from $C^*(G,\alpha)$ to $C^*(H,\beta)$.*

2.2. COHOMOLOGY OF PROPER GROUPOIDS. In this subsection, we show that the first continuous cohomology group with real coefficients is trivial. It can be readily checked that the result is valid for the groupoid equivariant continuous cohomology introduced in Section 1 of [5], and also for the (equivariant) Borel cohomology of a proper (topological) groupoid.

LEMMA 2.3 ([2], Lemma 1, Appendix I). *Let X be a locally compact Hausdorff space, R an open equivalence relation in X, such that the quotient space X/R is paracompact; let π be the canonical mapping of X onto X/R. There is a continuous real-valued function $F \geqslant 0$ on X such that:*

(i) *F is not identically zero on any equivalence class with respect to R;*

(ii) *for every compact subset K of X/R, the intersection of $\pi^{-1}(K)$ with* $\operatorname{supp}(F)$ *is compact.*

A continuous map $f : X \to Y$ is proper, if for each $y \in Y$, $f^{-1}(y) \subseteq X$ is quasi-compact. We call a locally compact groupoid proper if the map $(r_G, s_G) : G \to G^{(0)} \times G^{(0)}$, $(r_G, s_G)(\gamma) = (r_G(\gamma), s_G(\gamma))$, is proper.

PROPOSITION 2.4 ([9], Proposition 2.10). *Let G be a locally compact groupoid. Then the following assertions are equivalent:*

(i) *G is proper;*

(ii) *the map $(r_G, s_G) : G \to G^{(0)} \times G^{(0)}$ is closed and for each $u \in G^{(0)}$, $G_u^u \subseteq G$ is quasi-compact;*

(iii) *for all quasi-compact subsets $K, L \subseteq G^{(0)}$, G_K^L is quasi-compact;*

(iv) *for all compact subsets $K, L \subseteq G^{(0)}$, G_K^L is compact;*

(v) *for every quasi-compact subsets $K \subseteq G^{(0)}$, G_K^K is quasi-compact;*

(vi) *for all $x, y \in G^{(0)}$, there are compact neighbourhoods K_x and L_y of x and y, respectively, such that $G_{L_y}^{K_x}$ is quasi-compact.*

Proposition 2.10 of [9] is stated for groupoids which do not, necessarily, have Hausdorff space of units in which case (i)–(v) are equivalent and (v) $\Rightarrow$ (vi).

LEMMA 2.5. *Let (G,α) be a locally compact proper groupoid with a Haar system. If $G\backslash G^{(0)}$ is paracompact then there is a continuous invariant family of probability measures on G. Furthermore, each measure in this family has a compact support.*

Proof. Since G has a Haar system, the range map of G is open, hence the quotient map $\pi : G^{(0)} \to G\backslash G^{(0)}$ is open. Since G is proper, $G\backslash G^{(0)}$ is locally compact and Hausdorff. By hypothesis $G\backslash G^{(0)}$ is paracompact. Now we apply

Lemma 2.3 to get a function F on $G^{(0)}$ such that F is not identically zero on any G-orbit in $G^{(0)}$ and for every compact $K \subseteq G\backslash G^{(0)}$ the intersection $\operatorname{supp}(F) \cap \pi^{-1}(K)$ is compact. Define $h : G^{(0)} \to \mathbb{R}^+$ by

$$h(u) = \int_{G^u} F \circ s_G(\gamma)\, \mathrm{d}\alpha^u(\gamma).$$

Property (ii) of F from Lemma 2.3 and the full support condition of α^u imply that $h(u) > 0$. To see that $h(u) < \infty$, notice that $\operatorname{supp}(F \circ s_G) \cap G^u \subseteq G$ is compact:

$$\gamma \in \operatorname{supp}(F \circ s_G) \cap G^u \Rightarrow s_G(\gamma) \in \operatorname{supp}(F) \quad \text{and} \quad r_G(\gamma) = u.$$

Thus if $\widetilde{u} \subseteq G^{(0)}$ denotes the orbit of u, then $\operatorname{supp}(F \circ s_G) \cap G^u \subseteq (r_G, s_G)^{-1}(\{u\} \times \operatorname{supp}(F|_{\widetilde{u}}))$. Property (ii) of F from Lemma 2.3 says that $\operatorname{supp}(F|_{\widetilde{u}})$ is compact. As G is a proper groupoid, the set $(r_G \times s_G)^{-1}(\{u\} \times \operatorname{supp}(F|_{\widetilde{u}}))$ is compact which implies that $\operatorname{supp}(F \circ s_G) \cap G^u$ is compact.

Using the invariance of α, it is not hard to see that the function h is constant on the orbits of $G^{(0)}$. Put $F' = (F/h) \circ s_G$, then

$$\int_{G^u} F'(\gamma)\, \mathrm{d}\alpha^u(\gamma) = 1. \tag{2.4}$$

Denote $F'\alpha^u$ by p^u, then $p := \{p^u\}_{u \in G^{(0)}}$ is a family of probability measures on G. Explicitly, p is given by

$$\int_{G^u} f\, \mathrm{d}p^u = \int_{G^u} f(\gamma)\, F'(\gamma)\, \mathrm{d}\alpha^u(\gamma)$$

for $f \in C_c(G)$. It follows from the definition of measure p^u, where $u \in G^{(0)}$, that the compact set $\operatorname{supp}(F \circ s_G) \cap G^u$ is the support of p_u.

To check that p is invariant, let $f \in C_c(G)$ and $\eta \in G$; by the definition of p we have

$$\int_{G^{s_G(\eta)}} f(\eta\gamma)\, \mathrm{d}p^{s_G(\eta)}(\gamma) = \int_{G^{s_G(\eta)}} f(\eta\gamma)\, F'(\gamma)\, \mathrm{d}\alpha^{s_G(\eta)}(\gamma).$$

Now change the variable $\eta\gamma \mapsto \gamma$ so that the previous term equals

$$\int_{G^{r_G(\eta)}} f(\gamma)\, F'(\eta^{-1}\gamma)\, \mathrm{d}\alpha^{r_G(\eta)}(\gamma).$$

Use the invariance of α and the fact that

$$F'(\eta^{-1}\gamma) := \frac{F \circ s_G \eta^{-1}\gamma}{h \circ s_G(\eta^{-1}\gamma)} = \frac{F \circ s_G \gamma}{h \circ s_G(\gamma)} = F'(\gamma)$$

and compute further:

$$\int_{G^{r_G(\eta)}} f(\gamma) F'(\eta^{-1}\gamma) \mathrm{d}\alpha^{r_G(\eta)}(\gamma) = \int_{G^{s_G(\eta)}} f(\gamma) F'(\gamma) \mathrm{d}\alpha^{r_G(\eta)}(\gamma) = \int_{G^{s_G(\eta)}} f(\gamma) \mathrm{d}p^{r_G(\eta)}(\gamma). \quad \blacksquare$$

REMARK 2.6. Anantharaman-Delaroche and Renault define *topological* amenability for a locally compact topological groupoid ([1], Definition 2.2.7). Lemma 2.5 says that every proper groupoid G with a Haar system and $G\backslash G^{(0)}$ paracompact is topologically amenable.

The following proposition, namely Proposition 2.7, appears as Lemma 2.5 in [7] with the same proof.

PROPOSITION 2.7. *Let G be a locally compact proper groupoid and α be a Haar system on G. Then every $\mathbb{R}$-valued 1-cocycle is a coboundary, that is,* $\mathrm{H}^1(G;\mathbb{R}) = 0$.

Proof. Let $p = \{p^u\}_{u \in G^{(0)}}$ be an invariant family of probability measures on G which is obtained using Lemma 2.5. We claim that for a 1-cocycle $c : G \to \mathbb{R}$, the function

$$\underline{b}(u) = \int_G c(\gamma)\,\mathrm{d}p^u(\gamma) \quad \text{for } u \in G^{(0)}$$

satisfies $c = \underline{b} \circ s - \underline{b} \circ r$. Lemma 2.5 says that the support of each measure in p is compact, hence the above integral is well-defined. To see that $\underline{b}$ is the desired cochain, let $\eta \in G$ and compute:

$$\begin{aligned}(\underline{b} \circ s - \underline{b} \circ r)(\eta) &= \int_{G^{s_G(\eta)}} c(\gamma)\,\mathrm{d}p^{s(\eta)}(\gamma) - \int_{G^{r_G(\eta)}} c(\gamma)\,\mathrm{d}p^{r(\eta)}(\gamma)\\ &= \int_{G^{s_G(\eta)}} c(\gamma)\,\mathrm{d}p^{s(\eta)}(\gamma) - \int_{G^{s_G(\eta)}} c(\eta\gamma)\,\mathrm{d}p^{s(\eta)}(\gamma)\\ &= \int_{G^{s_G(\eta)}} (c(\gamma) - c(\eta) + c(\gamma))\mathrm{d}p^{s(\eta)}(\gamma) = c(\eta)\int \mathrm{d}p^{s(\eta)}(\gamma) = c(\eta).\end{aligned}$$

We used the invariance of p to get the second equality above. ∎

3. COMPOSITION OF CORRESPONDENCES

3.1. PREPARATION FOR COMPOSITION. Let G be a locally compact proper groupoid with $G/G^{(0)}$ paracompact. Since G is proper the action of G on $G^{(0)}$ given by $\gamma \cdot s_G(\gamma) = r_G(\gamma)$ for $\gamma \in G$ is proper. Let λ be a Haar system on G. Then λ induces a family of measures $[\lambda]$ along $\pi : G^{(0)} \to G/G^{(0)}$. And $[\lambda]$ is defined as

$$\int_{G^{(0)}} f\,\mathrm{d}[\lambda]^{[u]} = \int_G f(\gamma^{-1} \cdot u)\,\mathrm{d}\lambda^u(\lambda) = \int_G f \circ s_G(\gamma)\,\mathrm{d}\lambda^u(\gamma) \tag{3.1}$$

for $f \in C_c(G^{(0)})$. Note that in equation (3.1), $\gamma^{-1} \cdot u$ does not stand for the composite of γ^{-1} and u but for the action of G on $G^{(0)}$, that is, $\gamma^{-1} \cdot u = r_G(\gamma^{-1}) = s_G(\gamma)$. We draw Figure 1 which contains all this data.

$$\begin{array}{ccc} G & \xrightarrow[s_G]{\lambda^{-1}} & G^{(0)} \\ {\scriptstyle \lambda}\big\downarrow{\scriptstyle r_G} & & {\scriptstyle \pi}\big\downarrow{\scriptstyle [\lambda]} \\ G^{(0)} & \xrightarrow[\pi]{[\lambda]} & G/G^{(0)} \end{array}$$

FIGURE 1.

A measure m on $G^{(0)}$ induces measures $m \circ \lambda$ and $m \circ \lambda^{-1}$ on G. For $f \in C_c(G)$

$$\int_G f \,\mathrm{d}(m \circ \lambda^{-1}) := \int_{G^{(0)}} \int_G f(\gamma^{-1}) \,\mathrm{d}\lambda^u(\gamma) \,\mathrm{d}m(u).$$

The measure $m \circ \lambda$ is defined similarly. We call the measure m on $G^{(0)}$ *invariant* with respect to (G, λ) if $m \circ \lambda = m \circ \lambda^{-1}$ and the measure $m \circ \lambda$ on G is called *symmetric*.

PROPOSITION 3.1. *Let G be a proper groupoid with $G/G^{(0)}$ paracompact, let λ be a Haar system for G and $\pi : G^{(0)} \to G/G^{(0)}$ be the quotient map.*

(i) *Let μ be a measure on $G/G^{(0)}$ and let m denote the measure $\mu \circ [\lambda]$ on $G^{(0)}$. Then m is an invariant measure.*

(ii) *Let m be a measure on $G^{(0)}$. If m is invariant, then there is a measure μ on $G/G^{(0)}$ with $\mu \circ [\lambda] = m$.*

(iii) *The measure μ in* (ii), *with $\mu \circ [\lambda] = m$, is unique.*

Proof. (i) Let $f \in C_c(G)$, then

$$\begin{aligned}\int_G f \,\mathrm{d}(m \circ \lambda) = \int_G f \,\mathrm{d}(\mu \circ [\lambda] \circ \lambda) &= \int_{G/G^{(0)}} \int_{G^{(0)}} \Lambda(f)(u) \,\mathrm{d}[\lambda]^{[u]}(u) \,\mathrm{d}\mu([u]) \\ &= \int_{G/G^{(0)}} \int_{G^u} \Lambda(f) \circ s_G(\gamma) \,\mathrm{d}\lambda^u(\gamma) \,\mathrm{d}\mu([u]) \\ &= \int_{G/G^{(0)}} \int_{G^u} \Big(\int_{G^{s_G(\gamma)}} f(\eta) \,\mathrm{d}\lambda^{s_G(\gamma)}(\eta) \Big) \mathrm{d}\lambda^u(\gamma) \,\mathrm{d}\mu[u].\end{aligned}$$

We know that $s_G(\gamma) = r_G(\gamma^{-1})$. Now change the variable $\eta \mapsto \gamma^{-1}\eta$, and use the left invariance of λ to see that the previous term equals

$$\int_{G/G^{(0)}} \int_G \int_G f(\gamma^{-1}\eta) \,\mathrm{d}\lambda^{r_G(\gamma)}(\eta) \mathrm{d}\lambda^u(\gamma) \,\mathrm{d}\mu([u]).$$

We have removed the superscripts of G in the above equation for simplicity. Now apply Fubini's theorem to $\mathrm{d}\lambda^{r_G(\gamma)}(\eta) \,\mathrm{d}\lambda^u(\gamma)$ which is allowed since $r_G(\gamma) = u$, and f is a compactly supported continuous function. Moreover, note that $u =$

$r_G(\gamma) = s_G(\gamma^{-1})$, use the right invariance of λ^{-1} and compute further:

$$\int_{G/G^{(0)}} \int_G \int_G f(\gamma^{-1}\eta)\, \mathrm{d}\lambda^{s_G(\gamma^{-1})}(\gamma)\, \mathrm{d}\lambda^u(\eta)\, \mathrm{d}\mu([u])$$
$$= \int_{G/G^{(0)}} \int_G \int_G f(\gamma^{-1})\, \mathrm{d}\lambda^{s_G(\eta)}(\gamma)\, \mathrm{d}\lambda^u(\eta)\, \mathrm{d}\mu([u])$$
$$= \int_{G/G^{(0)}} \int_G \Lambda^{-1}(f) \circ s_G(\eta)\, \mathrm{d}\lambda^u(\eta)\, \mathrm{d}\mu([u])$$
$$= \int_{G/G^{(0)}} f\, \mathrm{d}(\mu \circ [\lambda] \circ \lambda^{-1}) = \int_G f\, \mathrm{d}(m \circ \lambda^{-1}).$$

To be precise, in the last line, the first equality is obtained by using the right invariance of λ^{-1}.

(ii) Let e be a function on $G^{(0)}$ which is similar to function F/h in Lemma 2.5, and thus $\Lambda(e \circ s_G) = 1$. Function $e \circ s_G$ is like function F' in equation (2.4). Let μ be the measure on $G/G^{(0)}$ which is defined as $\mu(g) = m((g \circ \pi) \cdot e)$ for $g \in C_c(G/G^{(0)})$. Let $[\Lambda]$ denote the integration function corresponding to $[\lambda]$. For $f \in C_c(G^{(0)})$

$$\int_{G^{(0)}} f\, \mathrm{d}(\mu \circ [\lambda]) = \int_{G/G^{(0)}} [\Lambda](f)([u])\, \mathrm{d}\mu([u]) = \int_{G^{(0)}} [\Lambda](f) \circ \pi(u)\, e(u)\, \mathrm{d}m(u)$$
$$= \int_{G^{(0)}} \int_G f \circ s_G(\gamma)\, e(r_G(\gamma))\, \mathrm{d}\lambda^u(\gamma)\, \mathrm{d}m(u).$$

We change $\gamma \mapsto \gamma^{-1}$, then use the symmetry of the measure $m \circ \lambda$ and continue the computation:

$$\int_{G^{(0)}} \int_G f \circ r_G(\gamma)\, e(s_G(\gamma))\, \mathrm{d}\lambda^u(\gamma)\, \mathrm{d}m(u) = \int_{G^{(0)}} f(u)\, \Lambda(e \circ s_G)(\gamma)\, \mathrm{d}m(u) = \int f\, \mathrm{d}m.$$

The last equality is due to the property of e that $\Lambda(e \circ s_G) = 1$.

(iii) Let μ' be another measure on $G/G^{(0)}$ which satisfies the condition $\mu' \circ [\lambda] = m$. Since the integration map $[\Lambda] : C_c(G^{(0)}) \to C_c(G/G^{(0)})$ is surjective, $\mu \circ [\lambda] = \mu' \circ [\lambda]$ implies $\mu = \mu'$. ∎

Now we study the case when the measure m is not invariant, but *strongly quasi-invariant*; m is called *quasi-invariant* with respect to (G, λ) if $m \circ \lambda \sim m \circ \lambda^{-1}$. Following Folland, see Chapter 2, Section 6, page 58 of [4], we call m *strongly quasi-invariant* with respect to (G, λ) if there is a continuous homomorphism $\Delta : G \to \mathbb{R}^+_*$ with $m \circ \lambda = \Delta \cdot (m \circ \lambda^{-1})$, that is, m is quasi-invariant with respect to (G, λ) and the Radon–Nikodym derivative implementing the equivalence of the measures $m \circ \lambda$ and $m \circ \lambda^{-1}$ is continuous. Very often, when there is no chance of confusion, we drop the phrase "with respect to (G, λ)" while talking about a (strongly) quasi-invariant measure. The cohomology theory of groupoids tells us that a homomorphism from G to an abelian group R is the same as an R-valued 1-cocycle (see Section 1 of [5]).

Let (G, λ) be as in Proposition 3.1. Let m be a strongly quasi-invariant measure on $G^{(0)}$, and let Δ be the $\mathbb{R}^+_*$-valued continuous 1-cocycle which implements the quasi-invariance. Then Δ gives an $\mathbb{R}$-valued 1-cocycle $\log \circ \Delta : G \to \mathbb{R}$. Proposition 2.7 says that $\log \circ \Delta = \underline{b} \circ s_G - \underline{b} \circ r_G$ for some continuous function $\underline{b} : G^{(0)} \to \mathbb{R}$. Thus

$$\Delta = \frac{\mathrm{e}^{\underline{b} \circ s_G}}{\mathrm{e}^{\underline{b} \circ r_G}}.$$

Write $b = \mathrm{e}^{\underline{b}}$, then $b > 0$ and it can be checked that

$$\Delta = \frac{\mathrm{e}^{\underline{b} \circ s_G}}{\mathrm{e}^{\underline{b} \circ r_G}} = \frac{\mathrm{e}^{\underline{b}} \circ s_G}{\mathrm{e}^{\underline{b}} \circ r_G} = \frac{b \circ s_G}{b \circ r_G}.$$

Rewriting the definition of (G, λ)-quasi-invariance of m using the above value of Δ gives that $m \circ \lambda = ((b \circ s_G)/(b \circ r_G)) m \circ \lambda^{-1}$ which is equivalent to $(b \circ r_G)(m \circ \lambda) = (b \circ s_G)(m \circ \lambda^{-1})$. A straightforward calculation shows that $(b \circ r_G)(m \circ \lambda) = (bm) \circ \lambda$ and $(b \circ s_G)(m \circ \lambda^{-1}) = (bm) \circ \lambda^{-1}$. Thus we get the following proposition.

PROPOSITION 3.2. *Let (G, λ) be a locally compact proper groupoid with a Haar system. Assume that $G/G^{(0)}$ is paracompact. Let m be a strongly (G, λ)-quasi-invariant measure on $G^{(0)}$. Let Δ be the $\mathbb{R}^+_*$-valued continuous 1-cocycle which implements the quasi-invariance. Then there is a continuous function $b : G^{(0)} \to \mathbb{R}^+$ with*

(i) $(b \circ s_G(\gamma))/(b \circ r_G(\gamma)) = \Delta(\gamma)$ *for all $\gamma \in G$;*

(ii) *the measure bm on $G^{(0)}$ is (G, λ)-invariant, that is, $bm \circ \lambda = bm \circ \lambda^{-1}$.*

3.2. COMPOSITION OF TOPOLOGICAL CORRESPONDENCES. Let (X, α) and (Y, β) be correspondences from (G_1, χ_1) to (G_2, χ_2) and from (G_2, χ_2) to (G_3, χ_3), respectively. Let Δ_1 and Δ_2 be the adjoining functions of (X, α) and (Y, β), respectively. Additionally, assume that X and Y are Hausdorff, and second countable. We draw Figure 2 that comprises this data.

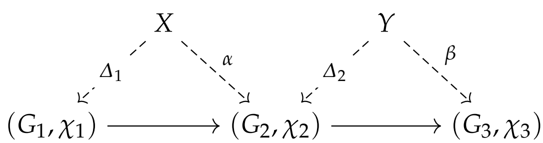

FIGURE 2.

We need to create a G_1-G_3-bispace Ω equipped with a G_3-invariant continuous family of measures $\mu = \{\mu_u\}_{u \in H_3^{(0)}}$ with each μ_u G_1-quasi-invariant. And the

$C^*(G_1,\chi_1)$-$C^*(G_3,\chi_3)$-Hilbert module $\mathcal{H}(\Omega)$ should be isomorphic to the Hilbert module $\mathcal{H}(X)\widehat{\otimes}_{C^*(G_2,\chi_2)}\mathcal{H}(Y)$.

Denote the fibred product $X \times_{G_2{}^{(0)}} Y$ by Z. Then Z carries the diagonal action of G_2. Since the action of G_2 on X is proper, its action on Z is proper. Thus the transformation groupoid $Z \rtimes G_2$ is proper. We define the space $\Omega = Z/G_2 = (Z \rtimes G_2)^{(0)}/(Z \rtimes G_2)$. Since Z is locally compact, Hausdorff and second countable, so is Ω. Being a locally compact, Hausdorff and second countable space, Ω is paracompact.

The following discussion in this section goes through under a milder hypothesis, namely, X and Y are locally compact Hausdorff and Ω is paracompact.

OBSERVATION 3.3. The space Z is a G_1-G_3-bispace. The momentum maps are $r_Z(x,y) = r_X(x)$ and $s_Z(x,y) = s_Y(y)$. For $(\gamma_1,(x,y)) \in G_1 \times_{G_1{}^{(0)}} Z$ and $((x,y),\gamma_3) \in Z \times_{G_3^{(0)}} G_3$, the actions are $\gamma_1 \cdot (x,y) = (\gamma_1 x, y)$ and $(x,y)\cdot\gamma_3 = (x, y\gamma_3)$, respectively. These actions descend to Ω and make it a G_1-G_3-bispace. Thus $r_\Omega([x,y]) = r_X(x)$ and $s_\Omega([x,y]) = s_Y(y)$ and $\gamma_1[x,y]\gamma_3 = [\gamma_1 x, y\gamma_3]$ for appropriate $\gamma_1 \in G_1$, $[x,y] \in \Omega$ and $\gamma_3 \in G_3$.

LEMMA 3.4. *The right action of G_3 on Ω defined in Observation 3.3 is proper.*

The proof follows from Lemma 2.33 of [9].

For each $u \in G^{(0)}{}_3$ define a measure m_u on the space Z as follows: for $f \in C_c(Z)$

$$\int\limits_Z f \,\mathrm{d}m_u = \int\limits_Y \int\limits_X f(x,y)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y).$$

LEMMA 3.5. *The family of measures $\{m_u\}_{u\in G^{(0)}{}_3}$ is a G_3-invariant continuous family of measures on Z.*

Proof. It is a routine computation to check that the G_3-invariance of the family of measures β makes $\{m_u\}_{u\in G^{(0)}{}_3}$ G_3-invariant. The computation is similar to that in Proposition 3.10. To check the continuity, let $f \in C_c(X)$ and $g \in C_c(Y)$, then

$$\int\limits_Z f\otimes g \,\mathrm{d}m_u = B((A(f)\circ r_Y)\, g)(u)$$

which is in $C_c(G_3{}^{(0)})$. Now use the theorem of Stone–Weierstraß to see that the set $\{f\otimes g : f \in C_c(X), g \in C_c(Y)\} \subseteq C_c(Z)$ is dense which concludes the lemma. ∎

The Haar system χ_2 of G_2 induces a Haar system χ on $Z \rtimes G_2$; for $f \in C_c(Z \rtimes G_2)$ and $(x,y) \in Z$

$$\int\limits_{(Z\rtimes G_2)^{(x,y)}} f\,\mathrm{d}\chi^{(x,y)} := \int\limits_{G_2^{s_X(x)}} f((x,y),\gamma)\,\mathrm{d}\chi_2^{s_X(x)}(\gamma).$$

The quotient map $\pi : Z \to \Omega$ carries the family of measures $[\chi]$; the definition of which is similar to the one in equation (3.1). We write λ instead of $[\chi]$, and λ^ω instead of $[\chi]^\omega$ for all $\omega \in \Omega$. Recall from Subsection 3.1 that for $f \in C_c(Z)$ and $\omega = [x, y] \in \Omega$

$$\int_{\pi^{-1}(\omega)} f \,\mathrm{d}\lambda^\omega := \int_{G_2^{r_Y(y)}} f(x\gamma, \gamma^{-1}y)\,\mathrm{d}\chi_2^{r_Y(y)}(\gamma).$$

We wish to prove that, up to equivalence, $\{m_u\}_{u \in G_3{}^{(0)}}$ can be pushed down from Z to Ω to a G_3-invariant family of measures $\{\mu^u\}_{u \in G_3{}^{(0)}}$. We use λ to achieve this. To be precise, we find a continuous function $b : Z \to \mathbb{R}^+$ and a family of measures μ on Ω which gives a disintegration $bm = \mu \circ \lambda$. Before we proceed we prove a small lemma.

LEMMA 3.6. *Let (X, α) and (Y, β) be correspondences from (G_1, χ_1) to (G_2, χ_2) and from (G_2, χ_2) to (G_3, χ_3), respectively, with Δ_1 and Δ_2 as their adjoining functions. Then for each $u \in G^{(0)}{}_3$ there is a function b_u on Z such that the measure $b_u m_u$ on $Z = (Z \rtimes G_2)^{(0)}$ is an invariant measure with respect to $(Z \rtimes G_2, \chi)$. Furthermore, b_u satisfies the relation $b(x\gamma, \gamma^{-1}y)b(x, y)^{-1} = \Delta_2(\gamma^{-1}, y)$.*

We work with a single μ_u at a time, so we prefer to drop the suffix u of b_u and simply write b. Using Lemma 3.6 we define the function $\Delta : Z \times_{G_2{}^{(0)}} G_2 \to \mathbb{R}^+$ as $\Delta((x, y), \gamma) = \Delta_2(\gamma^{-1}, y)$.

Proof. The proof follows the steps below.

Step 1. Firstly, we show that for each $u \in G^{(0)}{}_3$, m_u is strongly quasi-invariant with respect to $(Z \rtimes G_2, \chi)$ and $\Delta((x, y), \gamma) := \Delta_2(\gamma^{-1}, y)$ is the cocycle which implements the quasi-invariance.

Step 2. Since $Z \rtimes G_2$ is proper, we appeal to Proposition 3.2 to get a function $b : Z = (Z \rtimes G_2)^{(0)} \to \mathbb{R}^+$ having the desired properties.

(i) We draw Figure 3 which is similar to Figure 1.

$$\begin{array}{ccc} Z \rtimes G_2 & \xrightarrow[s_{Z \rtimes G_2}]{\chi^{-1}} & Z \\ {\scriptstyle \chi} \big\downarrow {\scriptstyle r_{Z \rtimes G_2}} & & {\scriptstyle \pi} \big\downarrow {\scriptstyle \lambda} \\ Z & \xrightarrow[\pi]{\lambda} & \Omega \end{array}$$

FIGURE 3.

Let $f \in C_c(Z \rtimes G_2)$, then

$$\int_{Z \rtimes G_2} f \,\mathrm{d}(m_u \circ \chi) = \int_Z \int_{G_2} f((x, y), \gamma)\,\mathrm{d}\chi_2^{s_X(x)}(\gamma)\,\mathrm{d}m_u(x, y)$$

$$= \int\limits_{Y}\int\limits_{X}\int\limits_{G_2} f((x,y),\gamma)\,\mathrm{d}\chi_2^{s_X(x)}(\gamma)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y).$$

Change variable $((x,y),\gamma) \mapsto ((x\gamma,\gamma^{-1}y),\gamma^{-1})$. Then use the fact that the family measures α is G_2-invariant and each measures in β is G_2-quasi-invariant to see that the previous term equals

$$\int\limits_{Y}\int\limits_{X}\int\limits_{G_2} f((x\gamma,\gamma^{-1}y),\gamma^{-1})\,\Delta_2(\gamma,\gamma^{-1}y)\;\mathrm{d}\chi_2^{s_X(x)}(\gamma)\,\mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_u(y).$$

The function

$$\Delta : Z \rtimes G_2 \to \mathbb{R}^+, \quad \Delta((x,y),\gamma) = \Delta_2(\gamma^{-1},y),$$

is clearly continuous. Furthermore, Δ is an $\mathbb{R}^+_*$-valued 1-cocycle; for a composable pair $((x,y),\gamma),((x\gamma,\gamma^{-1}y),\eta) \in Z \rtimes G_2$ a small routine computation shows that

$$\Delta((x,y),\gamma)\Delta((x\gamma,\gamma^{-1}y),\eta) = \Delta((x,y),\gamma\eta).$$

Thus Δ implements the $(Z \rtimes G_2,\chi)$-quasi-invariance of the measure m_u.

(ii) Since $Z \rtimes G_2$ is a proper groupoid, we apply Proposition 3.2 which gives a function $b : Z \to \mathbb{R}^+_*$ such that bm_u is an invariant measure on Z (with respect to $(Z \rtimes G_2,\chi)$). The function b also satisfies the relation $b \circ s_{Z\rtimes G_2}/b \circ r_{Z\rtimes G_2} = \Delta$, that is, $b(x\gamma,\gamma^{-1}y)b(x,y)^{-1} = \Delta((x,y),\gamma)$ for all $((x,y),\gamma) \in Z \rtimes G_2$. ∎

REMARK 3.7. For the cocycle $\Delta : Z \rtimes G_2 \to \mathbb{R}^+_*$, $\Delta((x,y),\gamma) = \Delta_2(\gamma^{-1},y)$, we observe:

(i) since Δ does not depend on x, Δ is G_1-invariant;

(ii) Δ_2 is G_3-invariant ([5], Remark 2.5). Hence

$$\Delta(((x,y),\gamma)\gamma_3) = \Delta_2(\gamma^{-1},y\gamma_3) = \Delta_2(\gamma^{-1},y) = \Delta((x,y),\gamma)$$

for all $((x,y),\gamma) \in Z \rtimes G_2$ and appropriate $\gamma_3 \in G_3$.
Thus Δ depends only on γ and $[y] \in Y/G_3$.

The function b appearing in Lemma 3.6 can be computed explicitly. Let $p = \{p^z\}_{z\in Z}$ be a family of probability measures on $Z \rtimes G_2$ as in Lemma 2.5. Then Propositions 2.7 and 3.2 give

$$b(x,y) = (\exp \circ \underline{b})(x,y) = \exp\Big(\int\limits_{Z\rtimes G_2} \log \circ \Delta((x,y),\gamma)\,\mathrm{d}p^{(x,y)}((x,y),\gamma)\Big). \tag{3.2}$$

This implies that b is a continuous positive function on Z.

REMARK 3.8. (i) The G_1-invariance of Δ from Remark 3.7 along with equation (3.2) imply that b is G_1-invariant.

(ii) The G_3-invariance of Δ (Remark 3.7 and equation (3.2)) implies that b is G_3-invariant. Indeed, for composable $((x,y),\gamma_3) \in Z \times G_3$

$$\begin{aligned} b(x,y\gamma_3) = b((x,y)\gamma_3) &= \exp\Big(\int \log\circ\Delta((x,y\gamma_3),\gamma)\, \mathrm{d}p^{(x,y\gamma_3)}((x,y\gamma_3),\gamma)\Big) \\ &= \exp\Big(\int \log\circ\Delta((x,y\gamma_3),\gamma)\, F'((x,y),\gamma)\, \mathrm{d}\chi_2^{r_Y(y\gamma_3)}(\gamma)\Big) \end{aligned}$$

where F' is a function as in equation (2.4) for groupoid $Z \rtimes G_2$ used to get the family of probability measures p. The G_3 invariance of Δ and the fact that $r_Y(y\gamma_3) = r_Y(y)$ give that the previous term equals

$$\exp\Big(\int \log\circ\Delta((x,y),\gamma)\, F'((x,y),\gamma)\, \mathrm{d}\chi_2^{r_Y(y)}(\gamma)\Big) = b(x,y).$$

The last equality is obtained from a computation similar to the one we started with, but in reverse order.

REMARK 3.9. Once we have $bm_u \circ \chi = bm_u \circ \chi^{-1}$, (ii) of Proposition 3.1 gives a measure μ_u on Ω with $bm_u = \mu_u \circ \lambda$. And, as we shall see, $\{\mu_u\}_{u\in G_3}$ is the required family of measures. For $f \in C_c(\Omega)$

$$\int_\Omega f\mathrm{d}\mu_u = \int_Z (f\circ\pi)\cdot e\cdot b\mathrm{d}m_u = \int_Y\int_X f\circ\pi(x,y)e(x,y)b(x,y)\mathrm{d}\alpha_{r_Y(y)}(x)\mathrm{d}\beta_u(y) \tag{3.3}$$

where $\pi : Z \to \Omega$ is the quotient map, and e is the function on Z with $\int e\circ s_{Z\rtimes G_2}\mathrm{d}\chi^z = 1$ for all $z \in Z$. In the discussion that follows, the letter e will always stand for such a function. Due to (iii) of Proposition 3.1 the measure μ_u is independent of the choice of the function e.

Recall that Ω is a G_1-G_3-bispace (Observation 3.3), and the action of G_3 is proper (Lemma 3.4).

PROPOSITION 3.10. *The family of measures $\{\mu_u\}_{u\in G^{(0)}_3}$ is a G_3-invariant continuous family of measures on Ω along the momentum map s_Ω.*

Proof. We check the invariance first and then check the continuity. Let $f \in C_c(\Omega)$ and $\gamma \in G_3$, then

$$\int_\Omega f([x,y\gamma])\,\mathrm{d}\mu_{r_{G_3}(\gamma)}[x,y] = \int_Y\int_X f([x,y\gamma])e(x,y\gamma)b(x,y)\ \mathrm{d}\alpha_{r_Y(y)}(x)\,\mathrm{d}\beta_{r_{G_3}(\gamma')}(y).$$

Change $y\gamma \to y$, then use the G_3-invariance of the family β and that of the function b to see that the last term in the above computation equals

$$\begin{aligned} \int_Y\int_X f([x,y])e(x,y)b(x,y)\mathrm{d}\alpha_{r_Y(y)}(x)\mathrm{d}\beta_{s_{G_3}(\gamma)}(y) &= \int_Z f\cdot e\cdot b\mathrm{d}m_{s_{G_3}(\gamma)} \\ &= \int_\Omega f[x,y]\mathrm{d}\mu_{s_{G_3}(\gamma)}[x,y]. \end{aligned}$$

Thus $\{\mu_u\}_{u\in G_3^{(0)}}$ is G_3-invariant.

Now we check that μ is a continuous family of measures. Let M, μ and Λ denote the integration maps which the families of measures m, μ and λ induce between the corresponding spaces of continuous compactly supported functions, respectively. Remark 3.9 says that $M : C_c(Z) \to C_c(G^{(0)}{}_3)$ is the composite of $C_c(Z) \xrightarrow{\Lambda} C_c(\Omega) \xrightarrow{\mu} C_c(G^{(0)}{}_3)$, that is, Figure 4 commutes.

$$\begin{array}{ccc} & & C_c(Z) \\ & \overset{\Lambda}{\swarrow} & \big\downarrow M \\ C_c(\Omega) & & \\ & \underset{\mu}{\searrow} & \\ & & C_c(G^{(0)}{}_3). \end{array}$$

FIGURE 4.

Lemma 3.5 shows that M is continuous, Example 1.8 of [5] shows that Λ is continuous and surjective. Hence μ is continuous. ∎

The family of measures μ on Ω is the required family of measures for the composite correspondence. We still need to show that each μ_u is G_1-quasi-invariant. The following computation shows this quasi-invariance and also yields the adjoining function. Let $f \in C_c(G_1 \times_{G_1{}^{(0)}} \Omega)$ and $u \in G^{(0)}{}_3$, then

$$\int_{\Omega}\int_{G_1} f(\eta^{-1}, [x,y])\, \mathrm{d}\chi_1^{r_\Omega([x,y])}(\eta)\, \mathrm{d}\mu_u[x,y]$$
$$= \int_Y \int_X \int_{G_1} f(\eta^{-1}, [x,y])\, e(x,y)\, b(x,y)\, \mathrm{d}\chi_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y).$$

Now we change variable $(\eta^{-1}, [x,y]) \mapsto (\eta, [\eta^{-1}x, y])$. Then the (G_1, χ_1)-quasi-invariance of α changes

$$\mathrm{d}\chi_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x) \mapsto \Delta_1(\eta, \eta^{-1}x)\, \mathrm{d}\chi_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x).$$

We incorporate this change and continue the computation further:

$$\begin{aligned} \text{R.H.S.} &= \int_Y \int_X \int_{G_1} f(\eta, [\eta^{-1}x, y])\, e(\eta^{-1}x, y)\, b(\eta^{-1}x, y) \\ &\qquad \cdot \Delta_1(\eta, \eta^{-1}x)\, \mathrm{d}\chi_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y) \\ &= \int_Y \int_X \int_{G_1} f(\eta, [\eta^{-1}x, y])\, \frac{b(\eta^{-1}x, y)}{b(x,y)}\, \Delta_1(\eta, \eta^{-1}x) \\ &\qquad \cdot e(\eta^{-1}x, y) b(x,y)\, \mathrm{d}\chi_1^{r_X(x)}(\eta)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_u(y). \end{aligned}$$

We transfer the integration on Ω where the previous term equals

$$\int_{\Omega}\int_{G_1} f(\eta, [\eta^{-1}x, y])\, \frac{b(\eta^{-1}x, y)}{b(x, y)}\, \Delta_1(\eta, \eta^{-1}x)\, \mathrm{d}\chi_1^{r_\Omega([x,y])}(\eta)\, \mathrm{d}\mu_u[x, y].$$

Define $\Delta_{1,2} : G_1 \ltimes \Omega \to \mathbb{R}^+_*$ by

$$\Delta_{1,2}(\eta, [x, y]) = b(\eta x, y)^{-1}\Delta_1(\eta, x)b(x, y), \tag{3.4}$$

then the above computation gives

$$\int_{\Omega}\int_{G_1} f(\eta^{-1}, [x, y])\, \mathrm{d}\chi_1(\eta)\, \mathrm{d}\mu_u[x, y]$$
$$= \int_{\Omega}\int_{G_1} f(\eta, [\eta^{-1}x, y])\, \Delta_{1,2}(\eta, \eta^{-1}[x, y])\, \mathrm{d}\chi_1(\eta)\, \mathrm{d}\mu_u[x, y]$$

for $u \in G^{(0)}{}_3$. To announce that μ_u is G_1-quasi-invariant and $\Delta_{1,2}$ is the adjoining function, we must check that the function $\Delta_{1,2}$ is well-defined which the next lemma does.

LEMMA 3.11. *The function $\Delta_{1,2}$ defined in equation* (3.4) *is a well-defined $\mathbb{R}^+_*$-valued continuous* 1*-cocycle on the groupoid $G_1 \ltimes \Omega$.*

Proof. Let $(x\gamma, \gamma^{-1}y) \in [x, y]$, then

$$\Delta_{1,2}(\eta^{-1}, [x\gamma, \gamma^{-1}y]) = b(\eta^{-1}x\gamma, \gamma^{-1}y)^{-1}\Delta_1(\eta^{-1}, x\gamma)b(x\gamma, \gamma^{-1}y).$$

We multiply and divide the last term by $b(\eta^{-1}x, y)^{-1}b(x, y)$, and use the G_2-invariance of Δ_1, then re-write the term as

$$b(\eta^{-1}x, y)^{-1}\Delta_1(\eta^{-1}, x)b(x, y)\left(\frac{b(\eta^{-1}x, y)}{b(\eta^{-1}x\gamma, \gamma^{-1}y)}\,\frac{b(x\gamma, \gamma^{-1}y)}{b(x, y)}\right).$$

Now use the last claim in Lemma 3.6 which relates b and Δ_2, use the definition of $\Delta_{1,2}$, and compute the above term further:

$$\Delta_{1,2}(\eta^{-1}, [x, y])\,(\Delta_2(\gamma^{-1}, y)\Delta_2(\gamma, \gamma^{-1}y)) = \Delta_{1,2}(\eta^{-1}, [x, y]).$$

To get the equality above, observe that $(\gamma^{-1}, y)^{-1} = (\gamma, \gamma^{-1}y)$ and use the fact that Δ_2 is a homomorphism.

Due to the continuity of b and Δ_1, the cocycle $\Delta_{1,2}$ is continuous. Using a computation as above, it can be checked that $\Delta_{1,2}$ is a groupoid homomorphism. ∎

PROPOSITION 3.12. *The family of measures $\{\mu_u\}_{u\in G^{(0)}{}_3}$, described above, is G_1-quasi-invariant. The adjoining function for the quasi-invariance is given by equation* (3.4).

The proof is clear from the discussion above.

DEFINITION 3.13 (Composition). Let

$$(X,\alpha):(G_1,\chi_1)\to(G_2,\chi_2) \quad\text{and}\quad (Y,\beta):(G_2,\chi_2)\to(G_3,\chi_3)$$

be topological correspondences with Δ_1 and Δ_2 as the adjoining functions, respectively. A composite of these correspondences $(\Omega,\mu):(G_1,\chi_1)\to(G_3,\chi_3)$ is defined by:

(i) the space $\Omega := (X\times_{G_2^{(0)}} Y)/G_2$;

(ii) a family of measures $\mu=\{\mu_u\}_{u\in G^{(0)}_3}$ such that:

(a) let $\Delta\in C^1_{G_3}((X\times_{G_2^{(0)}} Y)\rtimes G_2,\mathbb{R}^+_*)$ be the 1-cocycle $\Delta((x,y),\gamma)=\Delta_2(\gamma^{-1},y)$,

(b) let $b\in C^0_{G_3}((X\times_{G_2^{(0)}} Y)\rtimes G_2,\mathbb{R}^+_*)$ be a cochain with $d^0(b)=\Delta$,

(c) then μ disintegrates the family of measures $\{b(\alpha\times\beta_u)\}_{u\in G_3^{(0)}}$ on $X\times_{G_2^{(0)}} Y$ along the quotient map $\pi: X\times_{G_2^{(0)}} Y\to\Omega$ using λ, that is, $b(\alpha\times\beta_u)=\mu_u\circ\lambda$ for each $u\in G_3^{(0)}$.

In Definition 3.13, $C^n_{G_3}((X\times_{G_2^{(0)}} Y)\rtimes G_2,\mathbb{R}^+_*)$ denotes the n-th cochain group consisting of G_3-invariant $\mathbb{R}^+_*$-valued continuous cochains on groupoid $(X\times_{G_2^{(0)}} Y)\rtimes G_2$. For the composite (Ω,μ) as above, the adjoining function $\Delta_{1,2}$ is given by equation (3.4).

THEOREM 3.14. *Let $(X,\alpha):(G_1,\chi_1)\to(G_2,\chi_2)$ and $(Y,\beta):(G_2,\chi_2)\to(G_3,\chi_3)$ be topological correspondences of locally compact groupoids with Haar systems. In addition, assume that X and Y are Hausdorff, and second countable. Let $(\Omega,\mu):(G_1,\chi_1)\to(G_3,\chi_3)$ be a composite of the correspondences. Then $\mathcal{H}(\Omega)$ and $\mathcal{H}(X)\,\widehat{\otimes}_{C^*(G_2,\chi_2)}\mathcal{H}(Y)$ are isomorphic C^*-correspondences from $C^*(G_1,\chi_1)$ to $C^*(G_3,\chi_3)$.*

Proof. The symbols Z, Ω and the families of measures m,λ and μ continue to have the same meaning as in the earlier discussion. Let b be a fixed zeroth cochain on $Z\rtimes G_2$ with $\Delta=d^0(b)$ as in Definition 3.13. In the calculations below, the subscripts to $\langle\cdot\,,\cdot\rangle$ indicate the Hilbert module on which the inner product is defined. We write $\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)$ instead of $\mathcal{H}(X)\widehat{\otimes}_{C^*(G_2,\chi_2)}\mathcal{H}(Y)$ in this proof to reduce the complexity in writing.

Recall the process of composing two C^*-correspondences in Section 2.1 on page 94. Since $C_c(X)\subseteq\mathcal{H}(X)$ and $C_c(Y)\subseteq\mathcal{H}(Y)$ are, respectively, pre-Hilbert $C^*(G_2,\chi_2)$ and $C^*(G_3,\chi_3)$-modules, the image of $C_c(X)\otimes_{\mathbb{C}} C_c(Y)\to\mathcal{H}(X)\,\widehat{\otimes}\,\mathcal{H}(Y)$ under the obvious map is dense. Here the topologies on $C_c(X)$, $C_c(Y)$ and the image of $C_c(X)\otimes_{\mathbb{C}} C_c(Y)$ in $\mathcal{H}(X)\,\widehat{\otimes}\,\mathcal{H}(Y)$ are obtained by considering them as normed linear subspaces of the complete normed linear vector spaces $\mathcal{H}(X)$, $\mathcal{H}(Y)$ and $\mathcal{H}(X)\,\widehat{\otimes}\,\mathcal{H}(Y)$, respectively (later, for certain arguments, we shall bestow the function spaces with the inductive limit topology to exploit a basic and vital fact that the inductive limit topology on the function spaces is finer than the topologies the functions spaces inherit as subspaces of normed linear spaces; we shall specify the topologies wherever it is necessary).

Recall from the same discussion of composing two C^*-correspondences in Section 2.1 on page 94 that the obvious map sends $f \otimes g \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ to its equivalence class in

$$\frac{C_c(X) \otimes_{\mathbb{C}} C_c(Y)}{C_c(X) \otimes_{\mathbb{C}} C_c(Y) \cap N}$$

where $N \subseteq \mathcal{H}(X) \otimes_{\mathbb{C}} \mathcal{H}(Y)$ is the subspace of vectors of zero norm. The seminorm on $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ that defines the subspace N is induced by the inner product

$$\langle f \otimes g\,, f' \otimes g' \rangle_{\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)} := \langle g\,, \langle f\,, f' \rangle_{\mathcal{H}(X)} g' \rangle_{\mathcal{H}(Y)} \tag{3.5}$$

for $f \otimes g, f' \otimes g' \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$.

The same discussion on page 94 says that when equipped with the sesquilinear form in equation (3.5), $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ becomes a semi-inner product $C^*(G_3, \chi_3)$-module which completes to the Hilbert $C^*(G_3, \chi_3)$-module $\mathcal{H}(X) \widehat{\otimes} \mathcal{H}(Y)$.

On the other hand, $C_c(\Omega) \subseteq \mathcal{H}(\Omega)$ is a pre-Hilbert $C^*(G_3, \chi_3)$-module. We define an inner product preserving $C_c(G_3)$-module map $W' : C_c(X) \otimes_{\mathbb{C}} C_c(Y) \to C_c(\Omega)$ which has a dense image when $C_c(\Omega)$ is equipped with the inductive limit topology. Hence W' induces a unitary isomorphism $W : \mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y) \to \mathcal{H}(\Omega)$ of Hilbert $C^*(G_3, \chi_3)$-modules. This finishes the first part of the proof. Figure 5 shows a commutative triangle which comprises all of the important maps we need.

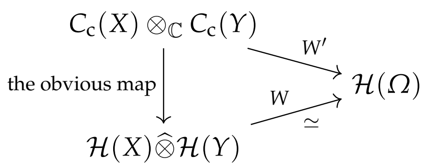

FIGURE 5.

In the latter half of the proof, we show that W' intertwines the representations of $C_c(G)$ on $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ and $C_c(\Omega)$ which implies that W intertwines the representations of the pre-C^*-algebra $C_c(G)$ on $\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)$ and $\mathcal{H}(\Omega)$. Then a standard argument can be used to conclude that W intertwines the representations of $C^*(G, \alpha)$ on $\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)$ and $\mathcal{H}(\Omega)$ which completes the proof of the claim that W' induces an isomorphism $W : \mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y) \to \mathcal{H}(\Omega)$ of C^*-correspondences.

Before starting the main proof, we sketch an argument to shows that if W' intertwines the representations, then so does W. Firstly, note that in Figure 5, $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ is a $C_c(G_1)$-module and the obvious map from it to $\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)$

is a map of $C_c(G_1)$-modules, indeed, the representation $C_c(G_1)$ on $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ is given by $\zeta(f \otimes_{\mathbb{C}} g) = (\zeta \cdot f) \otimes_{\mathbb{C}} g$ where $\zeta \in C_c(G_1)$ and $f \otimes_{\mathbb{C}} g \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$.

Let ρ_1' be the representation of the pre-C^*-algebra $C_c(G)$ on $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ discussed above. Let ρ_1 and ρ_2 denote the representations of $C^*(G_1, \chi_1)$ on $\mathcal{H}(X) \widehat{\otimes} \mathcal{H}(Y)$ and $\mathcal{H}(\Omega)$, respectively, that is, $\rho_1 : C^*(G_1, \chi_1) \to \mathbb{B}(\mathcal{H}(X) \widehat{\otimes} \mathcal{H}(Y))$ and $\rho_2 : C^*(G_1, \chi_1) \to \mathbb{B}(\mathcal{H}(\Omega))$ are the nondegenerate $*$-representations that give the C^*-correspondences from $C^*(G_1, \chi_1)$ to $C^*(G_3, \chi_3)$.

Assume that we have shown that W' intertwines the representations ρ_1' and ρ_2, this result is proved in the second part of the proof. Let a vector $\sum_{i=1}^{n} f_i \widehat{\otimes} g_i$ in the image of the obvious map (see Figure 5) and $\phi \in C_c(G_1)$ be given. Now choose $\sum_{j=1}^{l} f_j' \otimes g_j' \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ which maps to $\sum_{i=1}^{n} f_i \widehat{\otimes} g_i$ via the obvious map. Then, since the obvious map is a map of $C_c(G_1)$-modules, we see that the image of $\rho_1'(\phi) \sum_{j=1}^{l} f_j' \otimes g_j' = \sum_{j=1}^{l} \rho_1'(\phi)(f_j' \otimes g_j')$ under the obvious map is $\sum_{i=1}^{n} \rho_1(\phi)(f_i \widehat{\otimes} g_i) = \rho_1(\phi) \sum_{i=1}^{n} f_i \widehat{\otimes} g_i$. Now using the commutativity of Figure 5 we see that

$$\begin{aligned} W\Big(\rho_1(\phi) \sum_{i=1}^{n} f_i \widehat{\otimes} g_i\Big) &= W\Big(\sum_{i=1}^{n} \rho_1(\phi)(f_i \widehat{\otimes} g_i)\Big) = W'\Big(\sum_{j=1}^{l} \rho_1'(\phi)(f_j' \otimes g_j')\Big) \\ &= W'\Big(\rho_1'(\phi) \sum_{j=1}^{l} f_j' \otimes g_j'\Big) = \rho_2(\phi)\, W'\Big(\sum_{j=1}^{l} f_j \otimes g_j\Big) \\ &= \rho_2(\phi)\, W\Big(\sum_{i=1}^{n} f_i \widehat{\otimes} g_i\Big). \end{aligned}$$

Since the image of $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ in the Hilbert $C^*(G_3, \chi_3)$-module $\mathcal{H}(X) \widehat{\otimes} \mathcal{H}(Y)$ is a dense normed subspace, we may conclude that W intertwines the representations of the pre-C^*-algebra $C_c(G) \subseteq C^*(G_3, \chi_3)$, hence it can be concluded that W intertwines the desired representations of $C^*(G_3, \chi_3)$.

Thus the proof is divided into two parts, the first one proves the isomorphism of Hilbert modules and the other proves the isomorphism of the representations.

The strategy of the proof is explained and we start the proof by defining W'. Map $f \otimes g \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ to its restriction $(f \otimes g)|_Z \in C_c(Z)$ where $(f \otimes g)|_Z(x, y) = f(x)g(y)$ for $(x, y) \in Z$. Then the Stone–Weierstraß theorem says that the complex vector subspace of $C_c(Z)$ spanned by the set $\{(f \otimes g)|_Z\colon f \otimes g \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)\}$ is dense in the inductive limit topology on $C_c(Z)$. For an elementary tensor $f \otimes g$ in $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$, define

$$W'(f \otimes g)[x, y] = \Lambda((f \otimes g)|_Z\, b^{-1/2})[x, y]$$

$$= \int_{G_2} (f \otimes g)|_Z(x\gamma, \gamma^{-1}y)\, b^{-1/2}(x\gamma, \gamma^{-1}y)\, \mathrm{d}\chi_2^{s_X(x)}(\gamma)$$

and then extend W' linearly to a map $W' : C_c(X) \otimes_{\mathbb{C}} C_c(Y) \to C_c(\Omega)$.

Since b is a positive function, the multiplication by $b^{-1/2}$ is an isomorphism from $C_c(Z)$ to itself. As λ is a continuous family of measure with full support, $\Lambda : C_c(Z) \to C_c(\Omega)$ is a continuous $\mathbb{C}$-linear surjection when $C_c(Z)$ and $C_c(\Omega)$ are equipped with the inductive limit topologies. Thus when $C_c(\Omega)$ is bestowed with the inductive limit topology, the image of W', that is, the image of the composite $C_c(X) \otimes_{\mathbb{C}} C_c(Y) \xrightarrow{f\otimes g \mapsto (f\otimes g)|_Z} C_c(Z) \xrightarrow{\text{multiplication by } b^{-1/2}} C_c(Z) \xrightarrow{\Lambda} C_c(\Omega)$, is dense in $C_c(\Omega)$.

Let $z \in \mathbb{C}, f, f' \in C_c(X)$ and $g, g' \in C_c(Y)$. Then it is a straightforward computation to check that $W'(zf \otimes g + f' \otimes g') = zW'(f \otimes g) + W'(f' \otimes g')$. Furthermore, if $\psi \in C_c(G_3)$, then a computation using Fubini's theorem shows that $W'((f \otimes g)\psi) = W'(f \otimes g)\psi$. These $\mathbb{C}$ and $C_c(G_3)$-linearity results extend to a finite linear combinations of elementary tensors. Thus W' is a homomorphism of $C_c(G_3)$-modules.

3.3. THE ISOMORPHISM OF THE HILBERT MODULES. In this part, we show that W' preserves $C^*(G_3, \chi_3)$-valued inner products. Let $f \otimes g \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ be an elementary tensor and $\underline{\gamma} \in G_3$, then

$$\begin{aligned}
&\langle f \otimes g\,, f \otimes g\rangle_{\mathcal{H}(X)\hat{\otimes}\mathcal{H}(Y)}(\underline{\gamma}) \\
&:= \langle g\,, \langle f\,, f\rangle_{\mathcal{H}(X)} g\rangle_{\mathcal{H}(Y)}(\underline{\gamma}) \\
&= \int_Y \overline{g(y)}\ \big(\langle f\,, f\rangle_{\mathcal{H}(X)}\, g\big)(y\underline{\gamma})\ \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y) \\
&= \int_Y \int_{G_2} \overline{g(y)}\ \langle f, f\rangle_{\mathcal{H}(X)}(\gamma)\ g(\gamma^{-1}y\underline{\gamma})\Delta_2^{1/2}(\gamma, \gamma^{-1}y\underline{\gamma})\ \mathrm{d}\chi_2^{r_Y(y)}(\gamma)\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y) \\
&= \int_Y \int_{G_2} \overline{g(y)}\Big(\int_X \overline{f(x)} f(x\gamma)\mathrm{d}\alpha_{r_{G_2}(\gamma)}(x)\Big) \\
&\qquad \cdot g(\gamma^{-1}y\underline{\gamma})\Delta_2^{1/2}(\gamma, \gamma^{-1}y\underline{\gamma})\mathrm{d}\chi_2^{r_Y(y)}(\gamma)\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

We rearrange the functions, note that $r_{G_2}(\gamma) = r_Y(y)$, and write the last term as

$$\begin{aligned}
\text{(3.6)}\qquad &\int_Y \int_{G_2} \int_X \overline{f(x)}\,\overline{g(y)}\ f(x\gamma)g(\gamma^{-1}y\underline{\gamma}) \\
&\qquad \cdot \Delta_2^{1/2}(\gamma, \gamma^{-1}y\underline{\gamma})\ \mathrm{d}\alpha_{r_Y(y)}(x)\ \mathrm{d}\chi_2^{r_Y(y)}(\gamma)\ \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

Now we calculate the norm of $W'(f\otimes g)\in C_c(\Omega)$:

$$\langle W'(f\otimes g)\,,W'(f\otimes g)\rangle_{\mathcal{H}(\Omega)}(\underline{\gamma}) := \int_{\Omega} \overline{W'(f\otimes g)[x,y]}\; W'(f\otimes g)[x,y\underline{\gamma}]\;\mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x,y].$$

We plug the value of the first $W'(f\otimes g)$ and continue computing further:

$$\begin{aligned}
&\int_{\Omega}\Big(\int_{G_2} \overline{f(x\gamma_*)}\overline{g(\gamma_*^{-1}y)}b^{-1/2}(x\gamma_*,\gamma_*^{-1}y)\;\mathrm{d}\chi_2^{r_Y(y)}(\gamma_*)\Big)\\
&\qquad\cdot W'(f\otimes g)[x,y\underline{\gamma}]\,\mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x,y]\\
&=\int_{\Omega}\int_{G_2} \overline{f(x\gamma_*)}\overline{g(\gamma_*^{-1}y)}b^{-1/2}(x\gamma_*,\gamma_*^{-1}y)\\
&\qquad\cdot W'(f\otimes g)[x,y\underline{\gamma}]\,\mathrm{d}\chi_2^{r_Y(y)}(\gamma_*)\;\mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x,y]\\
&=\int_{Y}\int_{X} \overline{f(x)}\,\overline{g(y)}b^{-1/2}(x,y)W'(f\otimes g)[x,y\underline{\gamma}]b(x,y)\;\mathrm{d}\alpha_{r_Y(y)}(x)\;\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

The last equality above is due to Remark 3.9, which says that

$$\mathrm{d}\chi_2^{r_Y(y)}(\gamma_*)\;\mathrm{d}\mu_{r_{G_3}(\underline{\gamma})}[x,y] = b(x,y)\;\mathrm{d}\alpha_{r_Y(y)}(x)\;\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).$$

After adjusting the powers of the function b, the last term in the previous computation equals

$$\int_{Y}\int_{X} \overline{f(x)}\,\overline{g(y)}W'(f\otimes g)[x,y\underline{\gamma}]\,b^{1/2}(x,y)\,\mathrm{d}\alpha_{r_Y(y)}(x)\;\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).$$

Now, firstly, we plug in the value of $W'(f\otimes g)$ in the above term, and then compute further:

$$\begin{aligned}
&\int_{Y}\int_{X}\overline{f(x)}\,\overline{g(y)}\Big(\int_{G_2} f(x\gamma)g(\gamma^{-1}y\underline{\gamma})b^{-1/2}(x\gamma,\gamma^{-1}y\underline{\gamma})\;\mathrm{d}\chi_2^{r_Y(y)}(\gamma)\Big)\\
&\qquad\cdot b^{1/2}(x,y)\;\mathrm{d}\alpha_{r_Y(y)}(x)\;\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y)\\
&=\int_{Y}\int_{X}\int_{G_2}\overline{f(x)}\,\overline{g(y)}f(x\gamma)g(\gamma^{-1}y\underline{\gamma})\\
&\qquad\cdot\Big(\frac{b(x,y)}{b(x\gamma,\gamma^{-1}y\underline{\gamma})}\Big)^{1/2}\mathrm{d}\chi_2^{r_Y(y)}(\gamma)\;\mathrm{d}\alpha_{r_Y(y)}(x)\;\mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).
\end{aligned}$$

First we use the G_3-invariance of b (Remark 3.8) to write $b(x,y)=b(x,y\underline{\gamma})$. Then we use Lemma 3.2 to relate the factors of b and get a factor of Δ which can be written in terms of Δ_2 using Remark 3.7. At the end of these computations, the last term of the previous equation becomes

$$\int_Y \int_X \int_{G_2} (\overline{f(x)}\,\overline{g(y)} f(x\gamma) g(\gamma^{-1} y \underline{\gamma})) \cdot {\Delta_2}^{1/2}(\gamma, \gamma^{-1} y \underline{\gamma})\, \mathrm{d}\chi_2^{r_Y(y)}(\gamma)\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).$$

Finally, we apply Fubini's theorem to $\chi_2^{r_Y(y)}$ and $\alpha_{r_Y(y)}$ to get

$$\begin{aligned}(3.7)\quad & \langle W'(f \otimes g)\,, W'(f \times g)\rangle_{\mathcal{H}(\Omega)}(\underline{\gamma}) \\ &= \int_Y \int_{G_2} \int_X (\overline{f(x)}\,\overline{g(y)} f(x\gamma) g(\gamma^{-1} y \underline{\gamma})) \\ &\qquad \cdot {\Delta_2}^{1/2}(\gamma, \gamma^{-1} y \underline{\gamma})\, \mathrm{d}\alpha_{r_Y(y)}(x)\, \mathrm{d}\chi_2^{r_Y(y)}(\gamma)\, \mathrm{d}\beta_{r_{G_3}(\underline{\gamma})}(y).\end{aligned}$$

Comparing the values of both inner products, that is, equations (3.6) and (3.7), we conclude that

$$(3.8) \qquad \langle f \otimes g\,, f \otimes g\rangle_{\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)} = \langle W'(f \otimes g)\,, W'(f \otimes g)\rangle_{\mathcal{H}(\Omega)}.$$

Equation (3.8) says that W' is an isometry on the elementary tensors. Now let $\sum_{j=1}^{m} f_j \otimes g_j$ be a general element in $C_c(X) \otimes_{\mathbb{C}} C_c(Y)$ where m is a natural number. Then a standard argument that uses the polarisation identity (that is, equation (2.1)) and the principle of mathematical induction shows that

$$\Big\langle \sum_{j=1}^{m} f_j \otimes g_j\,, \sum_{j=1}^{m} f_j \otimes g_j \Big\rangle_{\mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y)} = \Big\langle W'\Big(\sum_{j=1}^{m} f_j \otimes g_j\Big)\,, W'\Big(\sum_{j=1}^{m} f_j \otimes g_j\Big)\Big\rangle_{\mathcal{H}(\Omega)}.$$

Thus W' is an inner-product preserving $C_c(G_3)$-linear map. Now the discussion on page 110 (the strategy of the proof) allows us to conclude that W' induces a unitary isomorphisms $W : \mathcal{H}(X)\widehat{\otimes}\mathcal{H}(Y) \to \mathcal{H}(\Omega)$ of Hilbert $C^*(G_3, \chi_3)$-modules.

3.4. THE ISOMORPHISM OF REPRESENTATIONS. Let ρ_1', ρ_1 and ρ_2 have the same meaning as in the beginning of the proof, see page 110. Now we show that W' intertwines ρ_1' and ρ_2.

Let $\Delta_{1,2}$ be the adjoining function of (Ω, μ) which is given by equation (3.4). Let $\phi \in C_c(G_1)$ and $f \otimes g \in C_c(X) \otimes_{\mathbb{C}} C_c(Y)$, then

$$\begin{aligned}&(\rho_2(\phi)W')(f \otimes g)[x, y] \\ &\quad = (\phi * W'(f \otimes g))[x, y] \\ &\quad = \int_{G_1} (\phi(\eta) W'(f \otimes g))[\eta^{-1} x, y]\, \Delta_{1,2}^{1/2}(\eta, [\eta^{-1} x, y])\, \mathrm{d}\chi_1^{r_X(x)}(\eta)\end{aligned}$$

$$(3.9)\qquad = \int_{G_1}\int_{G_2} \phi(\eta) f(\eta^{-1}x\gamma) g(\gamma^{-1}y)\, b^{-1/2}(\eta^{-1}x\gamma, \gamma^{-1}y) \cdot \Delta_{1,2}^{1/2}(\eta, [\eta^{-1}x, y])\, \mathrm{d}\chi_2^{s_X(x)}(\gamma)\, \mathrm{d}\chi_1^{r_X(x)}(\eta).$$

Lemma 3.11 and equation (3.4) allows us to write

$$\Delta_{1,2}(\eta, [\eta^{-1}x, y]) = \Delta_{1,2}(\eta, [\eta^{-1}x\gamma, \gamma^{-1}y]) = \Delta_1(\eta, \eta^{-1}x\gamma)\, \frac{b(\eta^{-1}x\gamma, \gamma^{-1}y)}{b(x\gamma, \gamma^{-1}y)}.$$

Substitute this value of $\Delta_{1,2}(\eta, [\eta^{-1}x, y])$ in equation (3.9). Then apply Fubini's theorem and continue computing further:

$$\int_{G_2}\Big(\int_{G_1} \phi(\eta) f(\eta^{-1}x\gamma) \Delta_1^{1/2}(\eta, \eta^{-1}x\gamma) \mathrm{d}\chi_1^{r_X(x)}(\eta)\Big) g(\gamma^{-1}y) b^{-1/2}(x\gamma, \gamma^{-1}y) \mathrm{d}\chi_2^{s_X(x)}(\gamma)$$
$$= \int_{G_2} (\phi * f)(x\gamma) g(\gamma^{-1}y)\, b^{-1/2}(x\gamma, \gamma^{-1}y)\, \mathrm{d}\chi_2^{s_X(x)}(\gamma)$$
$$= W'((\phi * f) \otimes g)[x, y] = W'(\rho_1'(\phi)(f \otimes g))[x, y]. \quad ∎$$

Due to the linearity of W', ρ_1' and ρ_2, this result holds for finite linear combinations of elementary tensors also. The strategy discussion on page 110 explains how does this result extend to the representations ρ_1 and ρ_2 of $C^*(G_1, \chi_1)$.

4. EXAMPLES

EXAMPLE 4.1. Let X, Y and Z be locally compact Hausdorff spaces and let $f : X \to Y$ and $g : Y \to Z$ be a continuous functions. Then Example 3.1 of [5] shows that (X, δ_X) is a topological correspondence from Y to X and (Y, δ_Y) is the one from Y to Z. Here $\delta_X = \{\delta_x\}_{x \in X}$ is the family of measures consisting of point masses along the identity map $X \to X$. Similar is the meaning of δ_Y. The constant function 1 is the adjoining function for both correspondences.

The space involved in the composite of (Y, δ_Y) and (X, δ_X) is $(Y \times_{\mathrm{Id}_Y, Y, f} X) \approx X$, and the homeomorphism $(Y \times_{\mathrm{Id}_Y, Y, f} X) \to X$ is implemented by the function $(f(x), x) \mapsto x$. The inverse of this function is $x \mapsto (f(x), x)$. The left momentum map $Y \times_{\mathrm{Id}_Y, Y, f} X \to Z$ is $(f(x), x) \mapsto g(f(x))$ which we identify with $g \circ f : X \to Z$. Thus the composite of the topological correspondences related to continuous maps is the same as the topological correspondence related to the composite of the maps. The reader may check that the C^*-algebraic counterpart of this example agrees with Theorem 3.14.

EXAMPLE 4.2. Let V, W, X, Y and Z be locally compact Hausdorff spaces and let $f : X \to Z, g : X \to Y, k : V \to Y$ and $l : V \to W$ be continuous maps. Let λ_1 and λ_2 be continuous families of measures along g and l, respectively (see Figure 6 on page 115). Then (X, λ_1) is a topological correspondence from Z to Y

and (V, λ_2) is one from Y to W ([5], Example 3.3). The composite correspondence

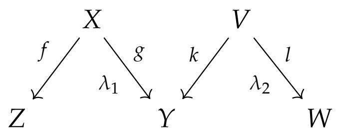

FIGURE 6.

is $(X \times_{g,Y,k} V, \lambda_1 \circ \lambda_2)$ where $(\lambda_1 \circ \lambda_2)_w$ is defined by

$$\int_{X \times_{g,Y,k} V} f \, \mathrm{d}(\lambda_1 \circ \lambda_2)_w = \int_V \int_X f(x, v) \, \mathrm{d}\lambda_{1k(v)}(x) \, \mathrm{d}\lambda_{2w}(v)$$

for $w \in W$ and $f \in C_c(X \times_{g,Y,k} V)$. Note that in this example $\lambda_1 \circ \lambda_2$ is the family of measures m in Lemma 3.5 and, since there are only the trivial actions, it is the same as the family of measures μ in Proposition 3.10.

EXAMPLE 4.3. Let G, H and K be locally compact groups, $\psi : K \to H$ and $\phi : H \to G$ continuous homomorphisms. Let α, β and λ be the Haar measures on G, H and K, respectively. Then (G, α^{-1}) is a correspondence from (H, β) to (G, α), and (H, β^{-1}) is one from (K, λ) to (H, β) ([5], Example 3.4). Let δ_G, δ_H and δ_K denote the modular functions of G, H and K, respectively. Then $(\delta_G \circ \phi)/\delta_H$ and $(\delta_H \circ \phi)/\delta_K$ are the adjoining function for these correspondences, respectively.

The K-G-bispace in the composite of these correspondences is $(H \times G)/H \approx G$. The map $a : \gamma \mapsto [e_H, \gamma]$, $G \to (H \times G)/H$, gives the homeomorphism where e_H is the unit in H. The inverse of this map a^{-1} is $a^{-1} : [\eta, \gamma] \mapsto \phi(\eta)\gamma$, $(H \times G)/H \to G$.

We figure out the action of K on this K-G-bispace: if $\kappa \in K$ then $\kappa\gamma = a^{-1}(\kappa[e_H, \gamma]) := a^{-1}([\psi(\kappa), \gamma]) = \phi(\psi(\kappa))\gamma$. Thus K acts on G via the homomorphism $\phi \circ \psi : K \to G$. Similarly, the right action of G on the composite space $(H \times G)/H \approx G$ is identified with the right multiplication action of G on itself. A computation as in Example 3.4 of [5] gives that $(\delta_G \circ \phi \circ \psi)/\delta_K$ is the adjoining function for the composite correspondence.

This shows that the composite of (H, β^{-1}) and (G, α^{-1}) is the same as the correspondence associated with the homomorphism $\phi \circ \psi : K \to G$.

EXAMPLE 4.4. Let $(G, \alpha), (H, \beta)$ and (K, λ) be locally compact groups with Haar measures, and let $\phi : H \to G$ and $\psi : K \to G$ be continuous homomorphisms. Assume the ψ is a proper map. Then ϕ gives a correspondence (G, α^{-1}) from (H, β) to (G, α) as in Example 3.4 of [5] and ψ gives a correspondence (G, α^{-1}) from (G, α) to (K, λ) as in Example 3.5 of [5]. The adjoining function of the topological correspondence associated with ψ is the constant function 1.

The composite of these correspondences is a correspondence $(H, \beta) \to (K, \lambda)$. On similar lines as of Example 4.3, one may show that the space involved in the composite is homeomorphic to G, the actions of H and K are identified with the left and right multiplication via ϕ and ψ, the K-invariant family of measures on G is α^{-1}. From Example 3.4 of [5] we know that the α^{-1} is (H, β)-quasi-invariant and the function $(\delta_G \circ \phi)/\delta_H$ is the cocycle involved in the quasi-invariance. Hence $(\delta_G \circ \phi)/\delta_H$ is the adjoining function for the composite.

An interesting situation is when $H, K \subseteq G$ are closed subgroups, and ϕ and ψ are the inclusion maps. Then the composite correspondence from (H, β) to (K, λ) is (G, α^{-1}) where G is made into an H-K bispace using the left and right multiplication actions, respectively. The adjoining function in this case is δ_G/δ_H.

EXAMPLE 4.5. Example 3.7 in [5] shows that the correspondences defined by Macho Stadler and O'uchi in [8] are topological correspondences. The same example shows that the topological correspondences defined by Tu in Proposition 7.5 of [9] are also topological correspondences provided that the spaces of the units of the groupoids are Hausdorff. The composition of correspondences of Macho Stadler and O'uchi defined by Tu ([9]) is the same as the composition we define.

Recall from Definition 1.2 of [8] or Definition 7.3 of [9] that a correspondence $(G_1, \chi_1) \to (G_2, \chi_2)$ is a G_1-G_2-bispace, and the actions and the quotient $G_1 \backslash X$ satisfy certain conditions. Since the correspondences of Macho Stadler and O'uchi or Tu do not involve explicit families of measures, the construction of the composite in this is purely topological. If Y is a correspondence in the sense from (G_2, χ_2) to (G_3, χ_3), then Tu shows ([9]) the space Ω in Definition 3.13 is the composite.

EXAMPLE 4.6. Example 3.10 in [5] shows that the generalized morphisms defined by Buneci and Stachura are topological correspondences in our sense. Though it is not as straightforward as in Example 4.5 above, but it may be checked that the composition of the generalized morphisms of Buneci and Stachura defined in Section 2.2 of [5] match our definition of composition.

EXAMPLE 4.7. Let G be a locally compact group, let H and K be closed subgroups of G, and let α, β and λ be the Haar measures on G, H and K, respectively. Let δ_G and δ_H be the modular functions of G and H, respectively. Then (G, α^{-1}) is a correspondence from H to K with δ_G/δ_H as the adjoining function, see Example 4.4.

Let X be a left K-space carrying a strongly (K, λ)-quasi-invariant measure κ, that is, κ is a (K, λ)-quasi-invariant measure on X and the Radon–Nikodym derivative for the quasi-invariance, say $\Delta : K \ltimes X \to \mathbb{R}^+_*$, is a continuous function. Then (X, κ) is a correspondence from K to Pt with Δ as adjoining function. Here Pt stands for the trivial group(oid) which consists of the unit only.

We discuss the composite of these two correspondences. The space in the composite is $(G \times X)/K$, which we denote by Z. In this example, writing the measure ν on Z concretely is not always possible. However, when $(X, \kappa) = (K, \lambda)$, we get $Z \approx G$ and $\nu = \alpha^{-1}$.

The correspondence (X, κ) gives a representation of K on $\mathcal{L}^2(X, \kappa)$ and the composite correspondence is the representation of H *induced* by this representation of K.

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