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COMPOSITION OPERATORS BETWEEN SEGAL–BARGMANN SPACES

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ABSTRACT. For an arbitrary Hilbert space $\mathcal{E}$, the Segal–Bargmann space $\mathcal{H}(\mathcal{E})$ is the reproducing kernel Hilbert space associated with the kernel $K(x,y) = \exp(\langle x,y\rangle)$ for x,y in $\mathcal{E}$. If $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is a mapping between two Hilbert spaces, then the composition operator C_φ is defined by $C_\varphi h = h\circ\varphi$ for all $h \in \mathcal{H}(\mathcal{E}_2)$ for which $h\circ\varphi$ belongs to $\mathcal{H}(\mathcal{E}_1)$. We determine necessary and sufficient conditions for the boundedness and compactness of C_φ. In the special case where $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}^n$, we recover results obtained by Carswell, MacCluer and Schuster. We also compute the spectral radii and the essential norms of a class of operators C_φ.

KEYWORDS: *Segal–Bargmann spaces, composition operators, positive semidefinite kernels, reproducing kernel Hilbert spaces.*

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1. INTRODUCTION

Let $\mathcal{B}$ be a Banach space of functions on a set $\mathcal{X}$ and $\varphi : \mathcal{X} \to \mathcal{X}$ be a mapping. We define the composition operator C_φ by $C_\varphi h = h\circ\varphi$ for any function $h \in \mathcal{B}$ for which the function $h\circ\varphi$ also belongs to $\mathcal{B}$. We are often interested in investigating how the function theoretic properties of φ affect the operator C_φ and vice versa. One of the fundamental problems is to classify the mappings φ which induce bounded or compact operators C_φ. After such classification is obtained, we then try to compute the norms and study the spectral properties of these operators. There is a vast literature on those problems when $\mathcal{B}$ is the Hardy, Bergman or Bloch space over the unit disc on the plane, the unit ball, or the unit polydisc in $\mathbb{C}^n$ (see, just to list a few, [2], [5], [6], [9], [12], [16], [19] and the references therein). In [7], Carswell, MacCluer and Schuster studied composition operators on the Segal–Bargmann space (also known as the Fock space) over $\mathbb{C}^n$. They obtained necessary and sufficient conditions on the mappings φ that give rise to bounded or compact C_φ. They showed that any such mapping must be affine with an additional restriction. They also found a formula for the norm of these operators. The purpose of the current paper is to investigate similar problems for composition operators that act between (possibly different) Segal–Bargmann spaces over arbitrary Hilbert spaces.

Let $n \geqslant 1$ be an integer. We denote by $\mathrm{d}\mu(z) = \pi^{-n}\exp(-|z|^2)\mathrm{d}V(z)$ the Gaussian measure on $\mathbb{C}^n$, where $\mathrm{d}V$ is the usual Lebesgue volume measure on $\mathbb{C}^n \equiv \mathbb{R}^{2n}$. The Segal–Bargmann (Fock) space $\mathcal{H}(\mathbb{C}^n)$ is the space of all entire functions on $\mathbb{C}^n$ that are square integrable with respect to $\mathrm{d}\mu$. For $f, g \in \mathcal{H}(\mathbb{C}^n)$, the inner product of f and g is given by

$$\langle f, g\rangle = \int_{\mathbb{C}^n} f(z)\overline{g(z)}\,\mathrm{d}\mu(z) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} f(z)\overline{g(z)}\exp(-|z|^2)\,\mathrm{d}V(z).$$

It is well known that $\mathcal{H}(\mathbb{C}^n)$ has an orthonormal basis consisting of monomials. In fact, for any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers, if we put $f_\alpha(z) = (\alpha!)^{-1/2}z^\alpha$, where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $z^\alpha = z_1^{\alpha_1}\cdots z_n^{\alpha_n}$, then $\{f_\alpha : \alpha \in \mathbb{Z}^n_{\geqslant 0}\}$ is an orthonormal basis for $\mathcal{H}(\mathbb{C}^n)$. It is also well known that $\mathcal{H}(\mathbb{C}^n)$ is a reproducing kernel Hilbert space of functions on $\mathbb{C}^n$ with kernel $K(z,w) = \exp(\langle z, w\rangle)$. For more details on $\mathcal{H}(\mathbb{C}^n)$, see, for example, Section 1.6 in [11] or Chapter 2 in [22]. We would like to alert the reader that other authors use slightly different versions of the Gaussian measure (for example, $\mathrm{d}\mu(z) = (2\pi)^{-n}$ $\exp(\frac{-|z|^2}{2})\mathrm{d}V(z)$) and hence the resulting reproducing kernels have different formulas (for example, $K(z,w) = \exp(\frac{\langle z,w\rangle}{2})$).

The following theorem ([7], Theorems 1 and 2) characterizes bounded and compact composition operators on $\mathcal{H}(\mathbb{C}^n)$.

THEOREM 1.1 (Carswell, MacCluer and Schuster). *Suppose $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ is a holomorphic mapping. Then*

(i) *C_φ is bounded on $\mathcal{H}(\mathbb{C}^n)$ if and only if $\varphi(z) = Az + b$, where A is an $n \times n$ matrix with $\|A\| \leqslant 1$ and b is an $n \times 1$ vector such that $\langle A\zeta, b\rangle = 0$ whenever $|A\zeta| = |\zeta|$.*

(ii) *C_φ is compact on $\mathcal{H}(\mathbb{C}^n)$ if and only if $\varphi(z) = Az + b$, where $\|A\| < 1$ and b is any $n \times 1$ vector.*

The formula for the norm of C_φ is given in the next theorem, which is Theorem 4 of [7]. Note that the formula presented here is slightly different from the original formula given in [7] because our reproducing kernel is $K(z,w) = \exp(\langle z, w\rangle)$ whereas theirs was $\exp(\frac{1}{2}\langle z, w\rangle)$.

THEOREM 1.2 (Carswell, MacCluer and Schuster). *Suppose $\varphi(z) = Az + b$, where $\|A\| \leqslant 1$ and $\langle A\zeta, b\rangle = 0$ whenever $|A\zeta| = |\zeta|$. Then the norm of C_φ on $\mathcal{H}(\mathbb{C}^n)$ is given by*

$$\|C_\varphi\| = \exp\big(\tfrac{1}{2}(|w_0|^2 - |Aw_0|^2 + |b|^2)\big), \tag{1.1}$$

where w_0 is any solution to the equation $(I - A^*A)w_0 = A^*b$.

Now suppose $\mathcal{E}$ is an arbitrary Hilbert space. There are various approaches that can be used to construct the Segal–Bargmann space $\mathcal{H}(\mathcal{E})$. In Section 2, we shall discuss in more detail such constructions and some elementary properties of $\mathcal{H}(\mathcal{E})$. Motivated by Theorems 1.1 and 1.2, we would like to study composition operators between Segal–Bargmann spaces. More specifically, let $\mathcal{E}_1$ and $\mathcal{E}_2$ be two Hilbert spaces and let $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. We shall characterize bounded and compact operators C_φ from $\mathcal{H}(\mathcal{E}_2)$ to $\mathcal{H}(\mathcal{E}_1)$. We shall also compute the essential norms and the spectral radii of a class of C_φ.

The proof of Theorem 1.1 in [7] makes use of the singular value decomposition of $n \times n$ matrices and the change of variables. Since this approach relies on the assumption that $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}^n$, it does not seem to work when $\mathcal{E}_1 \neq \mathcal{E}_2$ or when theses spaces are infinite dimensional. It turns out that there is an alternative approach, based on the theory of reproducing kernels. The idea of using reproducing kernels to study the boundedness of composition operators appeared in Nordgren's work [18] and it played a main role in [13], where Jury proved the boundedness of composition operators on the Hardy and Bergman spaces of the unit disk without using Littlewood subordination principle.

We shall see that $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded if and only if $\varphi(z) = Az + b$ as in Theorem 1.1 but we need a stronger condition on the vector b when $\mathcal{E}_1$ is an infinite dimensional Hilbert space (in the case $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}^n$, our condition on b is equivalent to the condition in Theorem 1.1). In the course of proving boundedness, we also obtain a formula for $\|C_\varphi\|$. Our formula is stated in a different way but it agrees with the formula in Theorem 1.2 when $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}^n$. For the compactness of C_φ, besides the condition that $\varphi(z) = Az + b$ for some linear operator $A : \mathcal{E}_1 \to \mathcal{E}_2$ with $\|A\| < 1$, it is also necessary that A be a compact operator.

We now state some of our main results. These results, to the best of our knowledge, are new even in the case where the spaces $\mathcal{E}_1$ and $\mathcal{E}_2$ are finite dimensional but have different dimensions.

THEOREM 1.3. *Let $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. Then the composition operator $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded if and only if $\varphi(z) = Az + b$ for all $z \in \mathcal{E}_1$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is linear with $\|A\| \leqslant 1$ and A^*b belongs to the range of $(I - A^*A)^{1/2}$. Furthermore, the norm of $\|C_\varphi\|$ is given by*

$$\|C_\varphi\| = \exp(\tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}\|b\|^2), \tag{1.2}$$

where v is the unique vector in $\mathcal{E}_1$ of minimum norm satisfying $A^*b = (I - A^*A)^{1/2}v$.

If ψ is an arbitrary holomorphic self-map of the open unit disc $\mathbb{D}$, then it is well known that C_ψ is a bounded operator on the Hardy space $H^2(\mathbb{D})$. In [8], Cowen obtained a formula for the spectral radius $r(C_\psi)$. He showed that if $\zeta \in \overline{\mathbb{D}}$ is the Denjoy–Wolff fixed point of ψ, then the spectral radius of C_ψ is 1 if $|\zeta| < 1$ and is $(\psi'(\zeta))^{-1/2}$ if $|\zeta| = 1$. Jury [14] extended this spectral radius formula to composition operators with linear fractional symbols acting on a wide class of Hilbert spaces over the unit ball in higher dimensions. On the other hand, the situation for composition operators on Segal–Bargmann spaces over $\mathbb{C}^n$ is different. This is due to the results in Theorem 1.3, which shows that mappings that give rise to bounded composition operators are quite restrictive.

THEOREM 1.4. *Let $\varphi : \mathbb{C}^n \to \mathbb{C}^n$ be a mapping such that C_φ is a bounded operator on $\mathcal{H}(\mathbb{C}^n)$. Then $r(C_\varphi) = 1$.*

Given the above result, it is natural to ask whether there exist bounded composition operators that have spectral radii strictly bigger than one. The answer is yes and of course we need to consider operators acting on $\mathcal{H}(\mathcal{E})$, where $\mathcal{E}$ has infinite dimension. Details will be presented in Section 3.

The last two theorems that we would like to mention in this section concern the compactness and essential norms. In Section 4, we discuss the proofs and related results.

THEOREM 1.5. *Let $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. Then $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is compact if and only if there is a compact linear operator $A : \mathcal{E}_1 \to \mathcal{E}_2$ with $\|A\| < 1$ and a vector $b \in \mathcal{E}_2$ such that $\varphi(z) = Az + b$ for all $z \in \mathcal{E}_1$.*

THEOREM 1.6. *Suppose $\varphi(z) = Az + b$ is a mapping from $\mathbb{C}^n$ into $\mathbb{C}^m$ such that $C_\varphi : \mathcal{H}(\mathbb{C}^m) \to \mathcal{H}(\mathbb{C}^n)$ is a bounded operator. If $\|A\| < 1$, then $\|C_\varphi\|_{\mathrm{e}} = 0$. If $\|A\| = 1$, then $\|C_\varphi\|_{\mathrm{e}} = \|C_\varphi\| \geqslant 1$.*

To conclude the section, we note that several mathematicians have also studied the boundedness and compactness of composition operators between Hardy and weighted Bergman spaces over the unit balls and the unit polydiscs in different dimensions. The situations there are quite different and many problems remain unsolved. See, for example, [15], [20] and the references therein for more details.

2. THE SPACES $\mathcal{H}(\mathcal{E})$ AND THEIR COMPOSITION OPERATORS

In the first half of this section we study the space $\mathcal{H}(\mathcal{E})$, where $\mathcal{E}$ is an arbitrary Hilbert space. Our construction of $\mathcal{H}(\mathcal{E})$ is similar to that of the Drury–Arveson space given in [4]. In the second half of the section, we consider composition operators acting between these spaces. Using the reproducing kernels, we provide a criterion for the boundedness of such operators.

2.1. THE CONSTRUCTION OF $\mathcal{H}(\mathcal{E})$.

For each integer $m \geqslant 1$, we write $\mathcal{E}^m$ for the symmetric tensor product of m copies of $\mathcal{E}$. We also define $\mathcal{E}^0$ to be $\mathbb{C}$ with its usual inner product. We have $\mathcal{E}^1 = \mathcal{E}$ and for $m \geqslant 2$, $\mathcal{E}^m$ is the closed subspace of the full tensor product $\mathcal{E}^{\otimes m}$ which consists of all elements that are invariant under the natural action of the symmetric group S_m. The action of S_m on $\mathcal{E}^{\otimes m}$ is defined on elementary tensors by

$$\pi \cdot (x_1 \otimes \cdots \otimes x_m) = x_{\pi(1)} \otimes \cdots \otimes x_{\pi(m)} \quad \text{for } \pi \in S_m \text{ and } x_1, \ldots, x_m \in \mathcal{E}.$$

By definition, $\mathcal{E}^m = \{x \in \mathcal{E}^{\otimes m} : \pi \cdot x = x \text{ for all } \pi \in S_m\}$. Each $\mathcal{E}^m$ is a Hilbert space with an inner product inherited from the inner product on $\mathcal{E}$. To the end of the paper, we shall generally write $\langle \cdot, \cdot \rangle$ for any inner product without referring to the space on which it is defined. The defining space will be clear from the context.

For any $z \in \mathcal{E}$, we use $z^m = z \otimes \cdots \otimes z \in \mathcal{E}^m$ to denote the tensor product of m copies of z (here z^0 denotes the number 1 in $\mathcal{E}^0 = \mathbb{C}$). A function $p : \mathcal{E} \to \mathbb{C}$ is called a *continuous m-homogeneous polynomial* on $\mathcal{E}$ if there exists an element ζ in $\mathcal{E}^m$ such that $p(z) = \langle z^m, \zeta \rangle$ for $z \in \mathcal{E}$. A function $f : \mathcal{E} \to \mathbb{C}$ is called a *continuous polynomial* if f can be written as a finite sum of continuous homogeneous polynomials. In other words, there is an integer $m \geqslant 0$ and there are elements $a_0 \in \mathbb{C}, a_1 \in \mathcal{E}^1, \ldots, a_m \in \mathcal{E}^m$ such that

$$f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle = a_0 + \langle z, a_1 \rangle + \cdots + \langle z^m, a_m \rangle. \tag{2.1}$$

When $\mathcal{E} = \mathbb{C}^n$ for some positive integer n, the notion of polynomials that we have just given coincides with the usual definition of polynomials in n complex variables. In fact, each polynomial in $z = (z_1, \ldots, z_n)$ is a linear combination of monomials of the form $z_1^{j_1} \cdots z_n^{j_n}$ for non-negative integers $j_1, \ldots, j_n$. Let $\{e_1, \ldots, e_n\}$ denote the standard basis for $\mathbb{C}^n$, where $e_k = (0, \ldots, 0, 1, 0, \ldots)$ with the number 1 in the k^{th} component. Then

$$z_1^{j_1} \cdots z_n^{j_n} = \langle z, e_1 \rangle^{j_1} \cdots \langle z, e_n \rangle^{j_n} = \langle z^l, e_1^{\otimes j_1} \otimes \cdots \otimes e_n^{\otimes j_n} \rangle_{E^{\otimes l}} = \langle z^l, e_1^{j_1} \cdots e_n^{j_n} \rangle_{E^l},$$

where $l = j_1 + \cdots + j_n$ and $e_1^{j_1} \cdots e_n^{j_n}$ denotes the orthogonal projection of $e_1^{\otimes j_1} \otimes \cdots \otimes e_n^{\otimes j_n}$ on $\mathcal{E}^l$. This implies that any polynomial in the variables $z_1, \ldots, z_n$ can be written in the form (2.1).

We denote by $\mathcal{P}_m(\mathcal{E})$ the space of all continuous m-homogeneous polynomials and $\mathcal{P}(\mathcal{E})$ the space of all continuous polynomials on $\mathcal{E}$. For more detailed discussions of polynomials between Banach and locally convex spaces, see [10], [17].

For two continuous polynomials f, g in $\mathcal{P}(\mathcal{E})$, we can find an integer $m \geqslant 0$ and elements $a_j, b_j \in \mathcal{E}^j$ for $0 \leqslant j \leqslant m$ such that $f(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle$ and $g(z) = \sum_{j=0}^{m} \langle z^j, b_j \rangle$. We then define

$$\langle f, g \rangle = \sum_{j=0}^{m} j!\, \langle b_j, a_j \rangle. \tag{2.2}$$

It is not difficult to check that (2.2) defines an inner product on $\mathcal{P}(\mathcal{E})$. We denote by $\mathcal{H}(\mathcal{E})$ the completion of $\mathcal{P}(\mathcal{E})$ in the norm induced by this inner product.

There is a natural anti-unitary operator from $\mathcal{H}(\mathcal{E})$ onto the symmetric (boson) Fock space $\mathcal{F}(\mathcal{E}) = \mathcal{E}^0 \oplus \mathcal{E}^1 \oplus \mathcal{E}^2 \oplus \cdots$, where the sum denotes the infinite direct sum of Hilbert spaces. We skip the proof which is straightforward from the definition of $\mathcal{H}(\mathcal{E})$ and $\mathcal{F}(\mathcal{E})$.

PROPOSITION 2.1. *For each element $f \in \mathcal{P}(\mathcal{E})$ given by formula* (2.1)*, we define an element in $\mathcal{F}(\mathcal{E})$ by*

$$Jf = (a_0, \sqrt{1!}\,a_1, \sqrt{2!}\,a_2, \sqrt{3!}\,a_3, \ldots),$$

where $a_j = 0$ for $j > m$. Then J is an anti-unitary operator from $\mathcal{P}_m(\mathcal{E})$ onto $\mathcal{E}^m$ for each $m \geqslant 0$ and it extends uniquely to an anti-unitary operator from $\mathcal{H}(\mathcal{E})$ onto $\mathcal{F}(\mathcal{E})$.

As in the case of the Drury–Arveson space [4], we can realize the elements of $\mathcal{H}(\mathcal{E})$ in more concrete terms, as entire functions on $\mathcal{E}$.

PROPOSITION 2.2. *Each element f in $\mathcal{H}(\mathcal{E})$ can be identified as an entire function on $\mathcal{E}$ having a power expansion of the form*

$$f(z) = \sum_{j=0}^{\infty} \langle z^j, a_j \rangle \quad \text{for all } z \in \mathcal{E},$$

where $a_0 \in \mathbb{C}$, $a_1 \in \mathcal{E}$, $a_2 \in \mathcal{E}^2, \ldots$. Furthermore, $\|f\|^2 = \sum_{j=0}^{\infty} j!\,\|a_j\|^2$.

Conversely, if $\sum_{j=0}^{\infty} j!\,\|a_j\|^2 < \infty$, then the power series $\sum_{j=0}^{\infty} \langle z^j, a_j \rangle$ defines an element in $\mathcal{H}(\mathcal{E})$.

Proof. By Proposition 2.1, each element f has a formal power series of the form

$$f(z) = \sum_{j=0}^{\infty} \langle z^m, a_m \rangle, \tag{2.3}$$

where a_j belongs to $\mathcal{E}^j$ for $j \geqslant 0$ and $\sum_{j=0}^{\infty} j!\,\|a_j\|^2 = \|f\|^2 < \infty$. For any $z \in \mathcal{E}$, since $\|z^m\| = \|z\|^m$, we have

$$\begin{aligned}\sum_{j=0}^{\infty} |\langle z^j, a_j \rangle| &\leqslant \sum_{j=0}^{\infty} \|z^j\| \|a_j\| = \sum_{j=0}^{\infty} \|z\|^j \|a_j\| = \sum_{j=0}^{\infty} \frac{\|z\|^j}{\sqrt{j!}} \sqrt{j!}\,\|a_j\| \\ &\leqslant \Big(\sum_{j=0}^{\infty} \frac{\|z\|^{2j}}{j!} \Big)^{1/2} \Big(\sum_{j=0}^{\infty} j!\,\|a_j\|^2 \Big)^{1/2} = \exp\big(\tfrac{\|z\|^2}{2}\big) \|f\|.\end{aligned}$$

This shows that the power series (2.3) converges uniformly on any bounded ball in $\mathcal{E}$. It follows that f can be regarded as an entire function on $\mathcal{E}$.

The converse follows from the fact that the sequence $\{p_m\}_{m=1}^{\infty}$ of polynomials defined by $p_m(z) = \sum_{j=0}^{m} \langle z^j, a_j \rangle$ is a Cauchy sequence in $\mathcal{H}(\mathcal{E})$. ∎

For every w in $\mathcal{E}$, we put

$$K_w(z) = \exp(\langle z, w\rangle) = \sum_{j=0}^{\infty} \frac{1}{j!}\langle z, w\rangle^j = \sum_{j=0}^{\infty} \Big\langle z^j, \frac{w^j}{j!}\Big\rangle \text{ for } z \in \mathcal{E}.$$

By Proposition 2.2, K_w belongs to $\mathcal{H}(\mathcal{E})$. For any f given by (2.3), we have

$$\langle f, K_w\rangle = \sum_{j=0}^{\infty} j!\Big\langle \frac{w^j}{j!}, a_j\Big\rangle = f(w).$$

Therefore, the function $K(z, w) = K_w(z)$ for $z, w \in \mathcal{E}$ is the reproducing kernel function for $\mathcal{H}(\mathcal{E})$ and the linear span of the set $\{K_w : w \in \mathcal{E}\}$ is dense in $\mathcal{H}(\mathcal{E})$. As a result, $\mathcal{H}(\mathcal{E})$ is a reproducing kernel Hilbert space. For a general theory of these spaces, see, for example, [3] or Chapter 2 of [1].

REMARK 2.3. The space $\mathcal{H}(\mathcal{E})$ can be defined in an abstract way by the kernel function $K(z, w)$. However it is not immediately clear from the abstract definition why $\mathcal{H}(\mathcal{E})$ consists of the power series given in Proposition 2.2. Our construction above also exhibits the decomposition

$$\mathcal{H}(\mathcal{E}) = \bigoplus_{m\geqslant 0} \mathcal{P}_m(\mathcal{E}) = \mathbb{C} \oplus \mathcal{P}_1(\mathcal{E}) \oplus \mathcal{P}_2(\mathcal{E}) \oplus \cdots, \tag{2.4}$$

which will be useful for us later. When $\mathcal{E} = \mathbb{C}^n$, we obtain the space $\mathcal{H}(\mathbb{C}^n)$, which is defined using the Gaussian measure on $\mathbb{C}^n$ as discussed in the Introduction.

In a reproducing kernel Hilbert space, a sequence is weakly convergent if and only if it is bounded in norm and it converges pointwise. Using this, we have the following lemma.

LEMMA 2.4. *The following statements hold in $\mathcal{H}(\mathcal{E})$:*

(i) $\lim_{\|z\|\to\infty} \|K_z\|^{-1}K_z = 0$ *weakly in $\mathcal{H}(\mathcal{E})$;*

(ii) *let $\{u_m\}$ be a sequence converging weakly to 0 in $\mathcal{E}$ (in particular, $\{u_m\}$ is bounded). For each m, put $f_m(z) = \langle z, u_m\rangle$ for $z \in \mathcal{E}$.*
Then $\lim_{m\to\infty} f_m = 0$ weakly in $\mathcal{H}(\mathcal{E})$.

It is well known that $\mathcal{H}(\mathbb{C}^n)$ can be naturally identified as the tensor product of n copies of $\mathcal{H}(\mathbb{C})$. In fact, the map $f_1 \otimes \cdots \otimes f_n \mapsto f_1(z_1)\cdots f_n(z_n)$ extends to a unitary operator from $\mathcal{H}(\mathbb{C}) \otimes \cdots \otimes \mathcal{H}(\mathbb{C})$ onto $\mathcal{H}(\mathbb{C}^n)$. This is an immediate consequence of the fact that the Gaussian measure on $\mathbb{C}^n$ is the product of n copies of the Gaussian measure on $\mathbb{C}$. The situation is less obvious in the general case since Gaussian measure may not be available. Nevertheless, similar tensor product decomposition still exists.

PROPOSITION 2.5. *Suppose $\mathcal{E}_0 = \mathcal{E}_1 \oplus \mathcal{E}_2$ is a decomposition of $\mathcal{E}_0$ as an orthogonal sum of two Hilbert spaces. For $f_1 \in \mathcal{H}(\mathcal{E}_1)$ and $f_2 \in \mathcal{H}(\mathcal{E}_2)$, define the function $f_1 * f_2$ by $(f_1 * f_2)(z) = f_1(z_1)f_2(z_2)$ for all $z = z_1 + z_2$, where $z_1 \in \mathcal{E}_1$ and $z_2 \in \mathcal{E}_2$.*

Then $f_1 * f_2$ belongs to $\mathcal{H}(\mathcal{E}_0)$ and the map $f_1 \otimes f_2 \mapsto f_1 * f_2$ extends to a unitary operator from $\mathcal{H}(\mathcal{E}_1) \otimes \mathcal{H}(\mathcal{E}_2)$ onto $\mathcal{H}(\mathcal{E}_0)$.

Proof. For $j = 0, 1, 2$, put $K^{(j)}_{w_j}(z_j) = \exp(\langle z_j, w_j\rangle)$ for z_j, w_j in $\mathcal{E}_j$. It is clear that $K^{(1)}_{w_1} * K^{(2)}_{w_2} = K^{(0)}_{w_1+w_2}$. Define $W(K^{(1)}_{w_1} \otimes K^{(2)}_{w_2}) = K^{(1)}_{w_1} * K^{(2)}_{w_2}$ and extend it by linearity. It follows from a direct computation that W is isometric on the linear span of $\{K^{(1)}_{w_1} \otimes K^{(2)}_{w_2} : w_1 \in \mathcal{E}_1, w_2 \in \mathcal{E}_2\}$. Since the linear span of $\{K^{(j)}_{w_j} : w_j \in \mathcal{E}_j\}$ is dense in $\mathcal{H}(\mathcal{E}_j)$ for $j = 0, 1, 2$, the operator W can be extended to a unitary as required. ∎

2.2. COMPOSITION OPERATORS. Suppose $\mathcal{E}_1$ and $\mathcal{E}_2$ are two Hilbert spaces. In what follows, we shall use K to denote the kernel functions of both $\mathcal{H}(\mathcal{E}_1)$ and $\mathcal{H}(\mathcal{E}_2)$. This should not cause any confusion since the kernel functions on these spaces have the same form.

For any mapping $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$, we recall that the composition operator C_φ is defined by $C_\varphi h = h \circ \varphi$ for all h in $\mathcal{H}(\mathcal{E}_2)$ for which $h \circ \varphi$ also belongs to $\mathcal{H}(\mathcal{E}_1)$. Since C_φ is a closed operator, it follows from the closed graph theorem that C_φ is bounded if and only if $h \circ \varphi$ belongs to $\mathcal{H}(\mathcal{E}_1)$ for all $h \in \mathcal{H}(\mathcal{E}_2)$.

Now we suppose that C_φ is a bounded operator. A priori we do not impose any condition on φ but the boundedness of C_φ implies that $\langle \varphi(\cdot), a\rangle$ is entire for any $a \in \mathcal{E}_2$. This follows from the identity $\langle \varphi(\cdot), a\rangle = C_\varphi(\langle \cdot, a\rangle)$, which shows that $\langle \varphi(\cdot), a\rangle$ belongs to $\mathcal{H}(\mathcal{E}_1)$. For any $z \in \mathcal{E}_1$ and $h \in \mathcal{H}(\mathcal{E}_2)$, since $\langle h, C^*_\varphi K_z\rangle = \langle C_\varphi h, K_z\rangle = h(\varphi(z)) = \langle h, K_{\varphi(z)}\rangle$, we obtain the well known formula

$$C^*_\varphi K_z = K_{\varphi(z)}. \tag{2.5}$$

This formula was used in [7] for the proof of the necessity of Theorem 1.1. It turns out that this formula plays an important role in our proof of both the necessity and sufficiency on the boundedness of C_φ.

For $j = 1, 2$, let $\mathcal{M}_j$ denote the linear span of the kernel functions $\{K_z : z \in \mathcal{E}_j\}$. We know that $\mathcal{M}_j$ is dense in $\mathcal{H}(\mathcal{E}_j)$. Motivated by (2.5), for any mapping $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ (even when C_φ is not a bounded operator), we define a linear operator S_φ from $\mathcal{M}_1$ to $\mathcal{M}_2$ by the formula

$$S_\varphi\Big(\sum_{j=1}^m c_j K_{x_j}\Big) = \sum_{j=1}^m c_j K_{\varphi(x_j)}. \tag{2.6}$$

Here, the elements $x_1, \ldots, x_m \in \mathcal{E}_1$ are distinct and $c_1, \ldots, c_m$ are complex numbers. Since reproducing kernels at distinct points are linearly independent, the operator S_φ is well defined. Furthermore, formula (2.6) remains valid even if the elements $x_1, \ldots, x_m$ are not distinct. It follows from (2.5) that if C_φ is bounded from $\mathcal{H}(\mathcal{E}_2)$ to $\mathcal{H}(\mathcal{E}_1)$, then $S_\varphi = C^*_\varphi$ on $\mathcal{M}_1$ and hence S_φ extends to a bounded operator from $\mathcal{H}(\mathcal{E}_1)$ to $\mathcal{H}(\mathcal{E}_2)$. On the other hand, if S_φ extends to a bounded operator from $\mathcal{H}(\mathcal{E}_1)$ into $\mathcal{H}(\mathcal{E}_2)$, then

$$(C_\varphi h)(z) = h(\varphi(z)) = \langle h, K_{\varphi(z)}\rangle = \langle h, S_\varphi K_z\rangle = (S_\varphi^* h)(z)$$

for all $h \in \mathcal{H}(\mathcal{E}_2)$ and all $z \in \mathcal{E}_1$. As a result, C_φ is also a bounded operator. Note that $\|C_\varphi\| = \|S_\varphi\|$ whenever they are bounded operators.

For elements $x_1, \ldots, x_m$ in $\mathcal{E}_1$ and complex numbers $c_1, \ldots, c_m$, since

$$\Big\| S_\varphi\Big(\sum_{j=1}^m c_j K_{x_j}\Big)\Big\|^2 = \sum_{j,l=1}^m \bar{c}_l c_j \langle K_{\varphi(x_j)}, K_{\varphi(x_l)}\rangle = \sum_{j,l=1}^m \bar{c}_l c_j K(\varphi(x_l), \varphi(x_j)), \quad \text{and}$$
$$\Big\| \sum_{j=1}^m c_j K_{x_j}\Big\|^2 = \sum_{j,l=1}^m \bar{c}_l c_j K(x_l, x_j),$$

we conclude that S_φ is bounded with norm $\|S_\varphi\| \leqslant M$ if and only if

$$\sum_{j,l=1}^m c_j \bar{c}_l (M^2 K(x_l, x_j) - K(\varphi(x_l), \varphi(x_j))) \geqslant 0. \tag{2.7}$$

Put $\Phi_M(z,w) = M^2 K(z,w) - K(\varphi(z), \varphi(w))$ for $z, w \in \mathcal{E}_1$. Since (2.7) holds for arbitrary $x_1, \ldots, x_m$ in $\mathcal{E}_1$ and arbitrary complex numbers $c_1, \ldots, c_m$, the function Φ_M is called a *positive semi-definite kernel* on $\mathcal{E}_1$. Therefore, S_φ (and hence, C_φ) is bounded with norm at most M if and only if Φ_M is a positive semi-definite kernel. This criterion for boundedness of composition operators on general reproducing kernel Hilbert spaces was discussed in Theorem 2 of [18]. Using the formula $K(z,w) = \exp(\langle z, w\rangle)$, we obtain the following.

LEMMA 2.6. *Let $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ be a mapping. The composition operator $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded with norm at most M if and only if the function*

$$\Phi_M(z,w) = M^2 \exp(\langle z, w\rangle) - \exp(\langle \varphi(z), \varphi(w)\rangle)$$

is positive semi-definite.

In particular, if C_φ is bounded, then $\Phi_{\|C_\varphi\|}(z,z) \geqslant 0$, which is equivalent to

$$2 \ln \|C_\varphi\| \geqslant \|\varphi(z)\|^2 - \|z\|^2 \tag{2.8}$$

for all $z \in \mathcal{E}_1$.

As a corollary, we show that any mapping that induces a bounded composition operator must be affine.

COROLLARY 2.7. *If $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded, then there exists a linear operator $A : \mathcal{E}_1 \to \mathcal{E}_2$ with $\|A\| \leqslant 1$ and a vector $b \in \mathcal{E}_2$ such that $\varphi(z) = Az + b$ for all $z \in \mathcal{E}_1$.*

Proof. For any unit vector a in $\mathcal{E}_2$, we put $f_a(w) = \langle w, a\rangle$ for $w \in \mathcal{E}_2$ and $F_a(z) = \langle \varphi(z), a\rangle$ for $z \in \mathcal{E}_1$. Then f_a belongs to $\mathcal{H}(\mathcal{E}_2)$ and F_a, which is equal to

$C_\varphi(f_a)$, belongs to $\mathcal{H}(\mathcal{E}_1)$. Expand F_a as a power series

$$F_a(z) = F_a(0) + \sum_{m=1}^{\infty} \langle z^m, \zeta_m \rangle \quad \text{for all } z \text{ in } \mathcal{E}_1,$$

where $\zeta_1 \in \mathcal{E}_1, \zeta_2 \in \mathcal{E}_1^2, \ldots$. The inequality $|F_a(z)| \leqslant \|\varphi(z)\|$ together with (2.8) now gives $|F_a(z)|^2 \leqslant \|z\|^2 + 2\ln(\|C_\varphi\|)$ for all z in $\mathcal{E}_1$. It follows that $\|\zeta_1\| \leqslant 1$ and $\zeta_m = 0$ for all $m \geqslant 2$. Thus, $F_a(z) - F_a(0)$ is a linear functional in z with norm at most 1. Since $\langle \varphi(z) - \varphi(0), a \rangle = F_a(z) - F_a(0)$ is linear in z with norm at most 1 for any unit vector $a \in \mathcal{E}_2$, we conclude that $\varphi(z) = Az + \varphi(0)$ for some linear operator A from $\mathcal{E}_1$ to $\mathcal{E}_2$ with $\|A\| \leqslant 1$. Taking b to be $\varphi(0)$, we complete the proof of the corollary. ∎

In Section 3, we discuss in more detail positive semi-definite kernels and obtain an additional condition on the vector b and the operator A. We then complete the characterization of bounded composition operators.

To conclude the section, we show that some bounded composition operators C_φ may be decomposed as a tensor product of two composition operators. Such decompositions will be useful when we compute the spectral radii of certain composition operators.

PROPOSITION 2.8. *Let $\varphi : \mathcal{E} \to \mathcal{E}$ be a mapping such that C_φ is bounded on $\mathcal{H}(\mathcal{E})$. Assume that there is an orthogonal decomposition $\mathcal{E} = \mathcal{E}_1 \oplus \mathcal{E}_2$ such that $\varphi = \varphi_1 \oplus \varphi_2$, where $\varphi_j : \mathcal{E}_j \to \mathcal{E}_j$ for $j = 1, 2$. Then C_φ and $C_{\varphi_1} \otimes C_{\varphi_2}$ are unitarily equivalent.*

Proof. Let W be the unitary operator from $\mathcal{H}(\mathcal{E}_1) \otimes \mathcal{H}(\mathcal{E}_2)$ onto $\mathcal{H}(\mathcal{E})$ in Proposition 2.5. For $f_1 \in \mathcal{H}(\mathcal{E}_1)$ and $f_2 \in \mathcal{H}(\mathcal{E}_2)$, using the identity $W(f_1 \otimes f_2) = f_1 * f_2$, we obtain

$$\begin{aligned} C_\varphi W(f_1 \otimes f_2) &= C_\varphi(f_1 * f_2) = (f_1 * f_2) \circ (\varphi_1 \oplus \varphi_2) = (C_{\varphi_1} f_1) * (C_{\varphi_2} f_2) \\ &= W((C_{\varphi_1} f_1) \otimes (C_{\varphi_2} f_2)) = W(C_{\varphi_1} \otimes C_{\varphi_2})(f_1 \otimes f_2). \end{aligned}$$

As a result, $W^* C_\varphi W$ and $C_{\varphi_1} \otimes C_{\varphi_2}$ agree on the algebraic tensor product of $\mathcal{H}(\mathcal{E}_1)$ and $\mathcal{H}(\mathcal{E}_2)$. Since C_φ is bounded, it follows that C_{φ_1} and C_{φ_2} are both bounded and $W^* C_\varphi W = C_{\varphi_1} \otimes C_{\varphi_2}$. ∎

3. BOUNDEDNESS OF COMPOSITION OPERATORS

3.1. POSITIVE SEMI-DEFINITE KERNELS. Let $\mathcal{X}$ be an arbitrary set. Recall that a complex-valued function F on $\mathcal{X} \times \mathcal{X}$ is a positive semi-definite kernel if for any finite set $\{x_1, \ldots, x_m\}$ of points in $\mathcal{X}$, the matrix $(F(x_l, x_j))_{1 \leqslant l, j \leqslant m}$ is positive semi-definite. That is, for any complex numbers $c_1, \ldots, c_m$, we have

$$\sum_{j,l=1}^{m} \bar{c}_l c_j F(x_l, x_j) \geqslant 0.$$

We shall write $F \gg 0$ to indicate that F is a positive semi-definite kernel. We list here a few elementary facts that follows directly from the definition of positive semi-definite kernels.

(A1) If $F_j \gg 0$ for all $j = 1, 2 \ldots$, then $\sum_{j=1}^{\infty} F_j \gg 0$ provided that the series converges pointwise on $\mathcal{X}$.

(A2) If $F, G \gg 0$, then $FG \gg 0$. This follows from the fact that the Hadamard (entry-wise) product of two positive semi-definite square matrices is a positive semi-definite matrix.

(A3) Let $F \gg 0$. Suppose g is a function holomorphic on an open disk centered at 0 that contains the range of F. If all the coefficients of the Maclaurin series of g are non-negative, then it follows from (A1) and (A2) that $g \circ F \gg 0$. In particular, by choosing $g(\zeta) = \exp(\zeta) - 1$, we have $\exp(F) - 1 \gg 0$.

(A4) Suppose $\mathcal{M}$ is a complex vector space with an inner product $\langle \cdot, \cdot \rangle_{\mathcal{M}}$. If there is a function $f : \mathcal{X} \to \mathcal{M}$ such that $F(x, y) = \langle f(x), f(y) \rangle_{\mathcal{M}}$ for x, y in X, then $F \gg 0$. In fact, for any $x_1, \ldots, x_m$ in $\mathcal{X}$ and any complex numbers $c_1, \ldots, c_m$, we have

$$\sum_{j,l=1}^{m} \bar{c}_l c_j F(x_l, x_j) = \sum_{j,l=1}^{m} \langle \bar{c}_l f(x_l), \bar{c}_j f(x_j) \rangle_{\mathcal{M}} = \Big\| \sum_{j=1}^{m} \bar{c}_j f(x_j) \Big\|_{\mathcal{M}}^2 \geqslant 0.$$

It turns out ([1], Theorem 2.53; see also [3]) that any positive semi-definite kernel arises in this way.

Now suppose $\mathcal{E}$ is a Hilbert space and T is a bounded linear operator on $\mathcal{E}$. Define $F(z, w) = \langle Tz, w \rangle$ for $z, w \in \mathcal{E}$. If $F \gg 0$ on $\mathcal{E}$, then $F(z, z) \geqslant 0$ for all $z \in \mathcal{E}$. This implies that T is a positive operator. Conversely, if T is positive, then since $F(z, w) = \langle T^{1/2} z, T^{1/2} w \rangle$ (here $T^{1/2}$ denotes the positive square root of T), it follows from (A4) that $F \gg 0$. The following proposition provides a generalization of this observation.

PROPOSITION 3.1. *Let T be a self-adjoint operator on $\mathcal{E}$. Let u be a vector in $\mathcal{E}$ and M a nonnegative real number. Define the function*

$$F(z, w) = \langle Tz, w \rangle - \langle z, u \rangle - \langle u, w \rangle + M^2 \quad \text{for } z, w \in \mathcal{E}. \tag{3.1}$$

Then the following are equivalent:

(i) *the function F is a positive semi-definite kernel;*

(ii) *$F(z, z) \geqslant 0$ for all $z \in \mathcal{E}$;*

(iii) *the operator T is positive and $u = T^{1/2} v$ for some $v \in \mathcal{E}$ with $\|v\| \leqslant M$.*

Furthermore, if the conditions in (iii) *are satisfied and $v_{\min}$ is a unique vector of smallest norm in the set $\{v \in \mathcal{E} : T^{1/2} v = u\}$, then*

$$\inf\{F(z, z) : z \in \mathcal{E}\} = -\|v_{\min}\|^2 + M^2. \tag{3.2}$$

The vector $v_{\min}$ is characterized by two conditions: (a) *$T^{1/2} v_{\min} = u$ and* (b) *$v_{\min}$ belongs to $\overline{\operatorname{ran}}(T^{1/2})$, the closure of the range of $T^{1/2}$.*

Proof. It is immediate from the definition of positive semi-definite kernels that (a) implies (b). Now suppose (b) holds. Let z be in $\mathcal{E}$. Choose a complex number γ of modulus one such that $\gamma\langle z,u\rangle = |\langle z,u\rangle|$. For any real number r, since $F(r\gamma z, r\gamma z) \geqslant 0$, we obtain

$$r^2\langle Tz,z\rangle - 2r|\langle z,u\rangle| + M^2 \geqslant 0.$$

Because this inequality holds for all real r, we conclude that $\langle Tz,z\rangle \geqslant 0$ and $|\langle z,u\rangle|^2 \leqslant M^2\langle Tz,z\rangle$. As a result, T is a positive operator and we have $|\langle z,u\rangle| \leqslant M\|T^{1/2}z\|$ for all $z \in \mathcal{E}$. From this, one can manage to apply Douglas's lemma to conclude that u belongs to the range of $T^{1/2}$. But for completeness, we include here a direct proof. Define a linear functional on the range of $T^{1/2}$ by $\Lambda(T^{1/2}z) = \langle z,u\rangle$. Then Λ is well defined and bounded on $T^{1/2}(\mathcal{E})$ with $\|\Lambda\| \leqslant M$. Extending Λ to all $\mathcal{E}$ by the Hahn–Banach theorem and using the Riesz's representation theorem, we obtain an element v in $\mathcal{E}$ with $\|v\| = \|\Lambda\| \leqslant M$ such that $\Lambda(w) = \langle w,v\rangle$ for all $w \in \mathcal{E}$. It then follows that for any $z \in \mathcal{E}$,

$$\langle z,u\rangle = \Lambda(T^{1/2}z) = \langle T^{1/2}z, v\rangle = \langle z, T^{1/2}v\rangle.$$

Thus $u = T^{1/2}v$ and hence (iii) follows.

Now assume that (iii) holds. For z,w in $\mathcal{E}$, we have

$$\begin{aligned} F(z,w) &= \langle T^{1/2}z, T^{1/2}w\rangle - \langle T^{1/2}z, v\rangle - \langle v, T^{1/2}w\rangle + M^2 \\ &= \langle T^{1/2}z - v, T^{1/2}w - v\rangle - \|v\|^2 + M^2. \end{aligned}$$

Since $-\|v\|^2 + M^2 \geqslant 0$, (A1) and (A4) implies that F is positive semi-definite. Furthermore, by Lemma 3.3 below, there exists a unique vector $v_{\min}$ of smallest norm in the set $(T^{1/2})^{-1}(\{u\})$. This vector $v_{\min}$ must necessarily belong to $\overline{\operatorname{ran}}\,(T^{1/2})$. We then have

$$\begin{aligned} \inf\{F(z,z) : z \in \mathcal{E}\} &= \inf\{\|T^{1/2}z - v_{\min}\|^2 : z \in \mathcal{E}\} - \|v_{\min}\|^2 + M^2 \\ &= -\|v_{\min}\|^2 + M^2. \quad \blacksquare \end{aligned}$$

REMARK 3.2. In the case $\mathcal{E} = \mathbb{C}^n$, since $\overline{\operatorname{ran}}\,(T^{1/2}) = \operatorname{ran}\,(T^{1/2})$, the vector $v_{\min}$ in the proposition is given by $v_{\min} = T^{1/2}\zeta$ for any $\zeta \in \mathcal{E}$ that satisfies the equation $T\zeta = u$.

We close this section with an elementary lemma from the theory of Hilbert spaces that we have used in the above proof.

LEMMA 3.3. *Let S be a bounded operator on a Hilbert space $\mathcal{E}$. Suppose y is an element in the range of S. Then there exists a unique $x_{\min} \in \mathcal{E}$ of smallest norm such that $Sx_{\min} = y$. Furthermore, for any $x \in \mathcal{E}$, we have $x = x_{\min}$ if and only if $Sx = y$ and x belongs to $\overline{\operatorname{ran}}(S^*)$, the closure of the range of S^*.*

3.2. BOUNDED COMPOSITION OPERATORS. We now characterize bounded composition operators C_φ between Segal–Bargmann spaces.

Proof of Theorem 1.3. Suppose $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ is a mapping such that the operator $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ is bounded. Lemma 2.6 gives

$$0 \leqslant \|z\|^2 - \|\varphi(z)\|^2 + 2\ln\|C_\varphi\| \quad \text{for all } z \in \mathcal{E}_1. \tag{3.3}$$

Furthermore, by Corollary 2.7, there is a linear operator $A : \mathcal{E}_1 \to \mathcal{E}_2$ with $\|A\| \leqslant 1$ and a vector $b \in \mathcal{E}_2$ such that $\varphi(z) = Az + b$. Now (3.3) gives $\|z\|^2 - \|Az + b\|^2 + 2\ln(\|C_\varphi\|) \geqslant 0$ for all $z \in \mathcal{E}_1$, which is equivalent to

$$\langle (I - A^*A)z, z\rangle - \langle z, A^*b\rangle - \langle A^*b, z\rangle - \|b\|^2 + 2\ln(\|C_\varphi\|) \geqslant 0. \tag{3.4}$$

Using Proposition 3.1, we conclude that the vector A^*b belongs to the range of the operator $(I - A^*A)^{1/2}$. Now we choose $v \in \mathcal{E}_1$ to be the vector of smallest norm such that $A^*b = (I - A^*A)^{1/2}(v)$. By Proposition 3.1 again, the quantity $2\ln(\|C_\varphi\|) - \|v\|^2 - \|b\|^2$, being the infimum of the left hand side of (3.4), is nonnegative. We then obtain

$$\|C_\varphi\| \geqslant \exp(\tfrac{1}{2}\|v\|^2 + \tfrac{1}{2}\|b\|^2). \tag{3.5}$$

Conversely, suppose $\varphi(z) = Az + b$ such that $\|A\| \leqslant 1$; A^*b belongs to the range of $(I - A^*A)^{1/2}$; and $v \in \mathcal{E}_1$ is of smallest norm satisfying $A^*b = (I - A^*A)^{1/2}(v)$. We shall show that C_φ is bounded with norm at most the quantity on the right hand side of (3.5) (hence the inequality in (3.5) is in fact an equality).

Define, for $z, w \in \mathcal{E}_1$,

$$\begin{aligned} F(z,w) &= \langle z, w\rangle - \langle \varphi(z), \varphi(w)\rangle + \|b\|^2 + \|v\|^2 \\ &= \langle (I - A^*A)z, w\rangle - \langle z, A^*b\rangle - \langle A^*b, w\rangle + \|v\|^2. \end{aligned}$$

By the implication (iii) $\Rightarrow$ (i) in Proposition 3.1, we have $F \gg 0$. It then follows that $\exp(F) - 1 \gg 0$. Put $G(z,w) = \exp(\langle \varphi(z), \varphi(w)\rangle)$ for $z, w \in \mathcal{E}_1$. Then $G \gg 0$ and hence, $G \cdot (\exp(F) - 1) \gg 0$. Since for $z, w \in \mathcal{E}_1$,

$$G(z,w)\big(\exp(F(z,w)) - 1\big) = \exp(\|b\|^2 + \|v\|^2)\exp(\langle z, w\rangle) - \exp(\langle \varphi(z), \varphi(w)\rangle),$$

we conclude, by Lemma 2.6, that C_φ is bounded and

$$\|C_\varphi\| \leqslant \exp(\tfrac{1}{2}\|b\|^2 + \tfrac{1}{2}\|v\|^2). \tag{3.6}$$

This completes the proof of the theorem. ∎

REMARK 3.4. If $\|A\| < 1$, then the operator $I - A^*A$ is invertible, hence $(I - A^*A)^{1/2}$ is also invertible. As a result, A^*b belongs to $(I - A^*A)^{1/2}(\mathcal{E}_1)$ for any b in $\mathcal{E}_2$. Theorem 1.3 shows that C_φ is bounded for every φ of the form $\varphi(z) = Az + b$, where b is an arbitrary vector in $\mathcal{E}_2$.

COROLLARY 3.5. *Suppose $\varphi : \mathcal{E} \to \mathcal{E}$ is a mapping such that $\varphi(0) = 0$ and C_φ is bounded on $\mathcal{H}(\mathcal{E})$. Then $r(C_\varphi) = \|C_\varphi\| = 1$, where $r(C_\varphi)$ is the spectral radius of C_φ.*

Proof. First of all, since 1 is an eigenvalue of C_φ, we always have $r(C_\varphi) \geqslant 1$. Now the assumption that $\varphi(0) = 0$ together with Theorem 1.3 shows that $\varphi(z) = Az$ for all $z \in \mathcal{E}$, where $A : \mathcal{E} \to \mathcal{E}$ is a linear operator with $\|A\| \leqslant 1$. The norm formula in Theorem 1.3 (with $b = v = 0$) gives $\|C_\varphi\| = 1$. The conclusion of the corollary then follows since $r(C_\varphi) \leqslant \|C_\varphi\|$. ∎

3.3. THE FINITE-DIMENSIONAL CASE. We now discuss the case when $\mathcal{E}_1$ is finite dimensional. Suppose $A : \mathcal{E}_1 \to \mathcal{E}_2$ is a linear operator with $\|A\| \leqslant 1$ and b is a vector in $\mathcal{E}_2$. We claim that A^*b belongs to the range of $(I - A^*A)^{1/2}$ if and only if $\langle b, A\zeta\rangle = 0$ whenever $\|A\zeta\| = \|\zeta\|$. In fact, for $\zeta \in \mathcal{E}_1$, we have

$$\|\zeta\|^2 - \|A\zeta\|^2 = \langle \zeta,\zeta\rangle - \langle A^*A\zeta,\zeta\rangle = \langle (I - A^*A)\zeta,\zeta\rangle = \|(I - A^*A)^{1/2}\zeta\|^2.$$

Therefore $\|A\zeta\| = \|\zeta\|$ if and only if ζ belongs to $\ker(I - A^*A)^{1/2}$. This shows that $\langle b, A\zeta\rangle = 0$ for all such ζ if and only if A^*b is in the orthogonal complement of $\ker(I - A^*A)^{1/2}$, which is the same as $\overline{\operatorname{ran}}\,(I - A^*A)^{1/2}$. Since $\mathcal{E}_1$ is finite dimensional, the identity $\overline{\operatorname{ran}}\,(I - A^*A)^{1/2} = \operatorname{ran}\,(I - A^*A)^{1/2}$ holds, so the claim follows.

Let $v \in \mathcal{E}_1$ be the vector of smallest norm such that $A^*b = (I - A^*A)^{1/2}v$. Theorem 1.3 shows that $\|C_\varphi\| = \exp(\frac{\|v\|^2+\|b\|^2}{2})$. On the other hand, by Remark 3.2, we have $v = (I - A^*A)^{1/2}w_0$ for any $w_0 \in \mathcal{E}_1$ satisfying $(I - A^*A)w_0 = A^*b$. It follows that

$$\|v\|^2 + \|b\|^2 = \|(I - A^*A)^{1/2}w_0\|^2 + \|b\|^2 = \|w_0\|^2 - \|Aw_0\|^2 + \|b\|^2.$$

We then obtain

$$\|C_\varphi\| = \exp(\tfrac{1}{2}(\|w_0\|^2 - \|Aw_0\|^2 + \|b\|^2)).$$

In the case $\mathcal{E}_1 = \mathcal{E}_2 = \mathbb{C}^n$, we recover the results in Theorem 1.1 part (a), and Theorem 1.2. In the case $\mathcal{E}_1 \neq \mathcal{E}_2$, our results seem to be new.

3.4. SPECTRAL RADII. Theorem 1.3 involves the requirement that A^*b belongs to the range of $(I - A^*A)^{1/2}$. We first discuss how one may obtain a more direct condition on the vector b. We then compute the spectral radii of a certain class of composition operators.

We shall make use of the identities

$$A(I - A^*A)^{1/2} = (I - AA^*)^{1/2}A, \tag{3.7}$$

$$A^*(I - AA^*)^{1/2} = (I - A^*A)^{1/2}A^*. \tag{3.8}$$

LEMMA 3.6. *Let $A : \mathcal{E}_1 \to \mathcal{E}_2$ be a linear operator with $\|A\| \leqslant 1$ and let b belong to $\mathcal{E}_2$. Then the followings are equivalent:*

(i) *the vector A^*b belongs to the range of $(I - A^*A)^{1/2}$;*

(ii) *the vector b belongs to the range of $(I - AA^*)^{1/2}$.*

Furthermore, if either (i) *or* (ii) *holds (and hence both hold) and if $v_{\min} \in \mathcal{E}_1$ is the vector of smallest norm satisfying $A^*b = (I - A^*A)^{1/2}v_{\min}$ and $u_{\min} \in \mathcal{E}_2$ is the* *vector of smallest norm satisfying* $b = (I - AA^*)^{1/2}u_{\min}$, *then* $v_{\min} = A^*u_{\min}$ *and* $\|u_{\min}\|^2 = \|v_{\min}\|^2 + \|b\|^2$.

Proof. Suppose (i) holds. Then there is a vector v in $\mathcal{E}_1$ such that $A^*b = (I - A^*A)^{1/2}v$. Using (3.7), we have

$$AA^*b = A(I - A^*A)^{1/2}v = (I - AA^*)^{1/2}Av.$$

From the identity $b = (I - AA^*)b + AA^*b = (I - AA^*)b + (I - AA^*)^{1/2}Av$, we conclude that b belongs to the range of $(I - AA^*)^{1/2}$.

Conversely, suppose (ii) holds and we have $b = (I - AA^*)^{1/2}u$ for some $u \in \mathcal{E}_2$. Using (3.8), we have

$$A^*b = A^*(I - AA^*)^{1/2}u = (I - A^*A)^{1/2}A^*u,$$

which belongs to the range of $(I - A^*A)^{1/2}$.

Now suppose both (i) and (ii) hold. Let $v_{\min}$ be the unique vector in $\mathcal{E}_1$ of smallest norm that satisfies $A^*b = (I - A^*A)^{1/2}v_{\min}$ and let $u_{\min}$ be the unique vector in $\mathcal{E}_2$ of smallest norm that satisfies $b = (I - AA^*)^{1/2}u_{\min}$. By Lemma 3.3, $u_{\min}$ belongs to $\overline{\mathrm{ran}}((I - AA^*)^{1/2})$. Since

$$A^*(\overline{\mathrm{ran}}((I - AA^*)^{1/2})) \subseteq \overline{\mathrm{ran}}(A^*(I - AA^*)^{1/2}) = \overline{\mathrm{ran}}((I - A^*A)^{1/2}A^*)$$

we conclude that $A^*u_{\min}$ belongs to $\overline{\mathrm{ran}}((I - A^*A)^{1/2})$. Furthermore, we have $(I - A^*A)^{1/2}A^*u_{\min} = A^*b$. Applying Lemma 3.3 again, we see that $v_{\min} = A^*u_{\min}$. As a result,

$$\|u_{\min}\|^2 = \|(I - AA^*)^{1/2}u_{\min}\|^2 + \|A^*u_{\min}\|^2 = \|b\|^2 + \|v_{\min}\|^2. \qquad \blacksquare$$

Combining Lemma 3.6 and Theorem 1.3, we have the following theorem.

THEOREM 3.7. *Let* $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ *be a mapping. Then* $C_\varphi : \mathcal{H}(\mathcal{E}_2) \to \mathcal{H}(\mathcal{E}_1)$ *is bounded if and only if there is a linear operator* $A : \mathcal{E}_1 \to \mathcal{E}_2$ *with* $\|A\| \leqslant 1$ *and a vector* b *in the range of* $(I - AA^*)^{1/2}$ *such that* $\varphi(z) = Az + b$ *for all* $z \in \mathcal{E}_1$. *Furthermore,* $\|C_\varphi\| = \exp(\frac{\|u\|^2}{2})$, *where* u *is the unique vector in* $\mathcal{E}_2$ *of minimum norm that satisfies the equation* $b = (I - AA^*)^{1/2}u$.

We now use the norm formula in Theorem 3.7 to determine the spectral radii of the operators C_φ for a certain class of mappings φ. Recall that for T a bounded operator, $r(T)$ denotes its spectral radius.

PROPOSITION 3.8. *Let* $\varphi(z) = Az + b$ *be a mapping on* $\mathcal{E}$ *such that* C_φ *is bounded on* $\mathcal{H}(\mathcal{E})$. *If* A *is an isometry or a co-isometry, then we have* $r(C_\varphi) = \|C_\varphi\| = \exp(\frac{\|b\|^2}{2})$ *(which equals* 1 *when* A *is a co-isometry).*

Proof. If A is a co-isometry, then $AA^* = I$ and hence by Theorem 3.7, $b = 0$. The conclusion of the proposition follows from Corollary 3.5.

Now consider the case where A is an isometry. Since A^*b belongs to the range of $(I - A^*A)^{1/2} = 0$, we conclude that $A^*b = 0$. This gives $\langle A^k b, A^l b\rangle = 0$

whenever $k \neq l$ and hence,

$$\|A^{s-1}b + \cdots + b\|^2 = \|A^{s-1}b\|^2 + \cdots + \|b\|^2 = s\|b\|^2 \tag{3.9}$$

for any positive integer s.

We shall make use of the formula $r(C_\varphi) = \lim_{m\to\infty} \|C_\varphi^m\|^{1/m}$. For any integer $m \geqslant 1$, $C_\varphi^m = C_{\varphi_m}$, where $\varphi_m = \varphi \circ \cdots \circ \varphi$ is the composition of m copies of φ. We have $\varphi_m(z) = A^m z + A^{m-1}b + \cdots + b$ for $z \in \mathcal{E}$. The norm formula in Theorem 3.7 gives $\|C_{\varphi_m}\| = \exp(\frac{\|u_m\|^2}{2})$, where u_m is the vector of smallest norm satisfying $A^{m-1}b + \cdots + b = (I - A^m(A^m)^*)^{1/2}u_m$. Since A^m is an isometry, $I - A^m(A^m)^*$ is a projection. Minimality then forces $u_m = A^{m-1}b + \cdots + b$. By (3.9), we have $\|u_m\|^2 = m\|b\|^2$. It follows that $\|C_{\varphi_m}\| = \exp(\frac{m\|b\|^2}{2})$ and hence

$$r(C_\varphi) = \lim_{m\to\infty} \|C_{\varphi_m}\|^{1/m} = \exp(\tfrac{\|b\|^2}{2}) = \|C_\varphi\|.$$

This completes the proof of the proposition. ∎

PROPOSITION 3.9. *Let $\varphi(z) = Az + b$ be a mapping on $\mathcal{E}$ such that C_φ is bounded on $\mathcal{H}(\mathcal{E})$. If $r(A) < 1$, then $r(C_\varphi) = 1$.*

Proof. As in the proof of Proposition 3.8, we shall make use of the formula

$$r(C_\varphi) = \lim_{m\to\infty} \|C_{\varphi_m}\|^{1/m} = \lim_{m\to\infty} \exp(\tfrac{\|u_m\|^2}{2m}),$$

where u_m is the vector of smallest norm in $\mathcal{E}$ that satisfies the equation $A^{m-1}b + \cdots + b = (I - A^m(A^m)^*)^{1/2}u_m$.

Assume first $\|A\| < 1$. Since $(I - A^m(A^m)^*)^{1/2}$ is invertible, u_m is uniquely determined by $u_m = (I - A^m(A^m)^*)^{-1/2}(A^{m-1}b + \cdots + b)$. Thus,

$$\begin{aligned}\|u_m\| &\leqslant \|(I - A^m(A^m)^*)^{-1/2}\|(\|A\|^{m-1} + \cdots + 1)\|b\| \\ &\leqslant (1 - \|A\|^{2m})^{-1/2}(1 - \|A\|)^{-1}\|b\| \leqslant (1 - \|A\|)^{-3/2}\|b\|.\end{aligned}$$

It follows that $\lim_{m\to\infty} \frac{\|u_m\|^2}{2m} = 0$, which gives $r(C_\varphi) = 1$.

Now consider the general case, where $r(A) < 1$ but $\|A\|$ may equal 1. Since $\lim_{k\to\infty} \|A^k\|^{1/k} = r(A) < 1$, there is an integer $k \geqslant 1$ such that $\|A^k\| < 1$. From the case considered above, we have $r(C_{\varphi_k}) = 1$. Therefore, $r(C_\varphi) = (r(C_\varphi^k))^{1/k} = (r(C_{\varphi_k}))^{1/k} = 1$. ∎

The composition operators considered in Propositions 3.8 and 3.9 are quite restrictive. However, when the dimension of $\mathcal{E}$ is finite, any bounded composition operator on $\mathcal{H}(\mathcal{E})$ can be decomposed as a tensor product of such operators. Using this, we obtain a proof of Theorem 1.4.

Proof of Theorem 1.4. Since C_φ is bounded on $\mathcal{H}(\mathbb{C}^n)$, Theorem 3.7 implies that $\varphi(z) = Az + b$, where $A : \mathbb{C}^n \to \mathbb{C}^n$ is a linear operator with $\|A\| \leqslant 1$ and b

belongs to the range of $(I - AA^*)^{1/2}$. Since $\|A\| \leqslant 1$, for any vector $z \in \mathbb{C}^n$ and any unimodular complex number λ, we have

$$\|A^*z - \overline{\lambda}z\|^2 - \|Az - \lambda z\|^2 = \|A^*z\|^2 - \|Az\|^2 \leqslant \|z\|^2 - \|Az\|^2.$$

It follows that if $Az = \lambda z$, then $A^*z = \overline{\lambda}z$. We conclude that there is an orthogonal decomposition $\mathbb{C}^n = \mathcal{E}_1 \oplus \mathcal{E}_2$, with respect to which, $A = A_1 \oplus A_2$, where A_1 is unitary and all eigenvalues of A_2 have absolute values strictly less than 1. Note that the case $\mathcal{E}_1 = \{0\}$ or $\mathcal{E}_2 = \{0\}$ is allowed.

Write $b = b_1 \oplus b_2$, where $b_1 \in \mathcal{E}_1$ and $b_2 \in \mathcal{E}_2$. With $j = 1, 2$, we put $\varphi_j(z_j) = A_j z_j + b_j$ for $z_j \in \mathcal{E}_j$. Then $\varphi = \varphi_1 \oplus \varphi_2$. By Proposition 3.8, $r(C_{\varphi_1}) = 1$ (since A_1 is a co-isometry) and by Proposition 3.9, $r(C_{\varphi_2}) = 1$. Since C_φ is unitarily equivalent to $C_{\varphi_1} \otimes C_{\varphi_2}$ by Proposition 2.5, we have $r(C_\varphi) = r(C_{\varphi_1}) r(C_{\varphi_2}) = 1$. ∎

We raise here a question for the infinite dimensional case.

QUESTION 3.10. *Let $\varphi(z) = Az + b$ be a mapping on $\mathcal{E}$ such that C_φ is bounded on $\mathcal{H}(\mathcal{E})$. Suppose that $\dim(\mathcal{E}) = \infty$ and $r(A) = 1$. Find the spectral radius $r(C_\varphi)$. Of course, we only need to consider mappings that cannot be written as a direct sum of mappings in Propositions 3.8 and 3.9.*

We provide here an example which shows that in the case $\dim(\mathcal{E}) = \infty$, the spectral radius $r(C_\varphi)$ could be any number between 1 and $\|C_\varphi\|$. Let $\{\beta_m\}_{m=0}^\infty$ be a non-increasing sequence of positive real numbers with $\beta_0 = 1$. Let $\mathcal{E}$ be a Hilbert space with an orthonormal basis $\{e_m\}_{m=0}^\infty$. Let A be the unilateral weighted shift on $\mathcal{E}$ defined by $Ae_m = \frac{\beta_{m+1}}{\beta_m} e_{m+1}$ for all $m \geqslant 0$. Consider $\varphi(z) = Az + e_0$. Since $\|A\| \leqslant 1$ and $A^*e_0 = 0$, Theorem 1.3 shows that C_φ is a bounded operator on $\mathcal{H}(\mathcal{E})$ with $\|C_\varphi\| = e^{1/2}$.

Let φ_k denote the iteration of φ with itself k times. Then $C_\varphi^k = C_{\varphi_k}$ and we have for any $z \in \mathcal{E}$,

$$\varphi_k(z) = A^k z + (A^{k-1} + \cdots + I)e_0 = A^k z + \beta_{k-1}e_{k-1} + \cdots + \beta_0 e_0.$$

It follows that $\|\varphi_k(0)\|^2 = \beta_{k-1}^2 + \cdots + \beta_0^2$. Since $A^{*k}\varphi_k(0) = 0$, Theorem 1.3 gives

$$\|C_{\varphi_k}\| = \exp\left(\tfrac{\|\varphi_k(0)\|^2}{2}\right) = \exp\left(\tfrac{\beta_{k-1}^2 + \cdots + \beta_0^2}{2}\right).$$

We now compute $r(C_\varphi)$ as

$$r(C_\varphi) = \lim_{k\to\infty} \|C_\varphi^k\|^{1/k} = \lim_{k\to\infty} \exp\left(\tfrac{\beta_{k-1}^2 + \cdots + \beta_0^2}{2k}\right) = \exp\left(\tfrac{\beta^2}{2}\right).$$

Here $\beta = \lim\limits_{k\to\infty} \beta_k$ and we have used the fact that

$$\beta^2 = \lim_{k\to\infty} \frac{\beta_{k-1}^2 + \cdots + \beta_0^2}{k}.$$

By choosing an appropriate sequence, the limit β may be any number in the interval $[0,1]$. This shows that $r(C_\varphi)$ may be any number in the interval $[1,e^{1/2}] = [1,\|C_\varphi\|]$.

4. COMPACTNESS OF COMPOSITION OPERATORS

In this section we characterize mappings φ that induce compact composition operators C_φ. Before discussing the general case, let us consider first the case $\varphi(z) = Az : \mathcal{E} \to \mathcal{E}$, where A is a linear operator on $\mathcal{E}$ with $\|A\| \leqslant 1$. In what follows, we shall simply write C_A for C_φ.

It turns out that via the anti-unitary J that we have seen in Proposition 2.1, the operator C_A has an easy description. Let f be a continuous m-homogeneous polynomial on $\mathcal{E}$. Then there is an element $a_m \in \mathcal{E}^m$ such that $f(z) = \langle z^m, a_m\rangle$ for $z \in \mathcal{E}$. This gives

$$(C_A f)(z) = \langle (Az)^m, a_m\rangle = \langle A^{\otimes m}(z^m), a_m\rangle = \langle z^m, (A^*)^{\otimes m} a_m\rangle,$$

where $A^{\otimes m}$ denotes the tensor product of m copies of A. We conclude that $C_A f$ is also a continuous m-homogeneous polynomial. Therefore, the space $\mathcal{P}_m(\mathcal{E})$ of continuous m-homogeneous polynomials is invariant under C_A and we have the identity $C_A|_{\mathcal{P}_m(\mathcal{E})} = J^{-1}(A^*)^{\otimes m} J$. This, together with the decomposition in Remark 2.3, gives

$$C_A = J^{-1}(1_{\mathbb{C}} \oplus A^* \oplus (A^*)^{\otimes 2} \oplus (A^*)^{\otimes 3} \oplus \cdots)J, \tag{4.1}$$

where the sum is an infinite direct sum of operators. The identity (4.1) shows that C_A is compact if and only if $(A^*)^{\otimes m}$ is compact for each $m \geqslant 1$ and $\|(A^*)^{\otimes m}\| \to 0$ as $m \to \infty$. Using the fact that $(A^*)^{\otimes m}$ is compact if and only if A^* (and hence A) is compact and the well known identity $\|(A^*)^{\otimes m}\| = \|A^*\|^m = \|A\|^m$, we conclude that C_A is compact if and only if A is compact and $\|A\| < 1$. We have thus proved a special case of Theorem 1.5. A proof of the full version of Theorem 1.5 will be given later.

For T a bounded operator between two Hilbert spaces, we recall that the essential norm of T, denoted $\|T\|_{\mathrm{e}}$, is defined by

$$\|T\|_{\mathrm{e}} = \inf\{\|T+K\| : K \text{ is a compact operator}\}.$$

It is clear that $\|T\|_{\mathrm{e}} \leqslant \|T\|$ and $\|T^*\|_{\mathrm{e}} = \|T\|_{\mathrm{e}}$. It is also standard that if $\{x_m\}_{m=1}^\infty$ is a sequence of unit vectors converging weakly to zero, then we have the inequality $\|T\|_{\mathrm{e}} \geqslant \limsup\limits_{m\to\infty} \|Tx_m\|$.

PROPOSITION 4.1. *Suppose $\varphi(z) = Az + b$ is a mapping from $\mathcal{E}_1$ into $\mathcal{E}_2$ such that C_φ is a bounded operator from $\mathcal{H}(\mathcal{E}_2)$ into $\mathcal{H}(\mathcal{E}_1)$. If $\|A\| = 1$, then $\|C_\varphi\|_{\mathrm{e}} = \|C_\varphi\|$.*

Proof. Since C_φ is bounded, Theorem 1.3 implies that $\|A\| \leqslant 1$ and there is a vector v belonging to $\overline{\mathrm{ran}}(I - A^*A)^{1/2}$ such that $A^*b = (I - A^*A)^{1/2}v$. Furthermore, $\|C_\varphi\|^2 = \exp(\|v\|^2 + \|b\|^2)$.

If $\|A\| = 1$, then there is a sequence $\{w_m\}_{m=1}^\infty$ of vectors in $\mathcal{E}_1$ such that $\|w_m\| = 1$ and $\|Aw_m\| \to 1$ as $m \to \infty$. Passing to a subsequence if necessary, we may assume $\lim\limits_{m\to\infty} m^2(1 - \|Aw_m\|^2) = 0$, which implies

$$\lim_{m\to\infty} m\|(I - A^*A)^{1/2}w_m\| = \lim_{m\to\infty} m(1 - \|Aw_m\|^2)^{1/2} = 0.$$

Let z be a vector in $\mathcal{E}_1$. Put $z_m = m\,w_m + z$. Then we have $\|z_m\| \to \infty$ and hence, by Lemma 2.4, $\|K_{z_m}\|^{-1}K_{z_m} \to 0$ weakly as $m \to \infty$. This gives

$$\begin{aligned} \|C_\varphi\|_{\mathrm{e}}^2 = \|C_\varphi^*\|_{\mathrm{e}}^2 &\geqslant \limsup_{m\to\infty} \|K_{z_m}\|^{-2}\|C_\varphi^* K_{z_m}\|^2 = \limsup_{m\to\infty} \|K_{z_m}\|^{-2}\|K_{\varphi(z_m)}\|^2 \\ &= \limsup_{m\to\infty}\ \exp(\|\varphi(z_m)\|^2 - \|z_m\|^2). \end{aligned} \tag{4.2}$$

Now for each positive integer m, we have

$$\begin{aligned} \|\varphi(z_m)\|^2 - \|z_m\|^2 &= \|Az_m\|^2 + \langle z, A^*b\rangle + \langle A^*b, z_m\rangle + \|b\|^2 - \|z_m\|^2 \\ &= -\|(I - A^*A)^{1/2}z_m - v\|^2 + \|v\|^2 + \|b\|^2 \\ &= -\|m(I-A^*A)^{1/2}w_m + (I-A^*A)^{1/2}z - v\|^2 + \|v\|^2 + \|b\|^2. \end{aligned}$$

Letting $m \to \infty$, we obtain

$$\lim_{m\to\infty}(\|\varphi(z_m)\|^2 - \|z_m\|^2) = -\|(I - A^*A)^{1/2}z - v\|^2 + \|v\|^2 + \|b\|^2.$$

This identity, together with (4.2), gives

$$2\ln\|C_\varphi\|_{\mathrm{e}} \geqslant -\|(I - A^*A)^{1/2}z - v\|^2 + \|v\|^2 + \|b\|^2.$$

Since v belongs $\overline{\mathrm{ran}}(I - A^*A)^{1/2}$, the supremum of the right hand side when z varies in $\mathcal{E}_1$ is $\|v\|^2 + \|b\|^2 = 2\ln\|C_\varphi\|$. As a result, we have $\|C_\varphi\|_{\mathrm{e}} \geqslant \|C_\varphi\|$. Since $\|C_\varphi\|_{\mathrm{e}} \leqslant \|C_\varphi\|$, we conclude that $\|C_\varphi\|_{\mathrm{e}} = \|C_\varphi\|$. ∎

We now have the necessary tools for the proof of Theorem 1.5.

Proof of Theorem 1.5. Assume first that C_φ is a compact operator from $\mathcal{H}(\mathcal{E}_2)$ into $\mathcal{H}(\mathcal{E}_1)$. By Theorem 1.3, $\varphi(z) = Az + b$ for all $z \in \mathcal{E}_1$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is linear with $\|A\| \leqslant 1$ and $b \in \mathcal{E}_2$ with $A^*b \in (I - A^*A)^{1/2}(\mathcal{E}_1)$. By Proposition 4.1, we have $\|A\| < 1$. It now remains to show that A is compact.

Let $\{u_m\}_{m=1}^\infty$ be a sequence in $\mathcal{E}_2$ that converges weakly to zero. For each m, put $f_m(w) = \langle w, u_m\rangle$ for $w \in \mathcal{E}_2$. Then $f_m \to 0$ weakly as $m \to \infty$ by Lemma 2.4. This implies that $\lim\limits_{m\to\infty} \|C_\varphi f_m\| = 0$. But for $z \in \mathcal{E}_1$,

$$(C_\varphi f_m)(z) = f_m(\varphi(z)) = \langle Az + b, u_m\rangle = \langle z, A^*u_m\rangle + \langle b, u_m\rangle,$$

so $\|C_\varphi f_m\|^2 = \|A^*u_m\|^2 + |\langle b, u_m\rangle|^2$. We then obtain $\lim\limits_{m\to\infty} \|A^*u_m\|^2 = 0$. Therefore, A^* is compact and hence, A is also compact.

Now suppose $\varphi(z) = Az + b$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is a compact operator with $\|A\| < 1$ and b is an arbitrary vector in $\mathcal{E}_2$. Let $A = U|A|$ be the polar decomposition of A, where $U : \mathcal{E}_1 \to \mathcal{E}_2$ is a partial isometry and $|A| = (A^*A)^{1/2}$ is a compact operator on $\mathcal{E}_1$. Pick a real number α such that $\|A\| < \alpha < 1$. Put $\varphi_1(z) = \alpha^{-1}|A|z$ and $\varphi_2(z) = \alpha Uz + b$ for $z \in \mathcal{E}_1$. As we have shown, C_{φ_1} is compact. Since $\|\alpha U\| \leqslant \alpha < 1$, Theorem 1.3 implies that C_{φ_2} is bounded. From the identity $\varphi = \varphi_2 \circ \varphi_1$, it follows that $C_\varphi = C_{\varphi_1} C_{\varphi_2}$ and hence, C_φ is a compact operator. ∎

Proposition 4.1 and Theorem 1.5 together give us the essential norms of a class of operators C_φ. In fact, suppose $\varphi(z) = Az + b$, where $A : \mathcal{E}_1 \to \mathcal{E}_2$ is a linear operator with $\|A\| \leqslant 1$ and $b \in \operatorname{ran}(I - AA^*)^{1/2}$. If $\|A\| = 1$, then $\|C_\varphi\|_{\mathrm{e}} = \|C_\varphi\| \geqslant 1$. If $\|A\| < 1$ and A is compact (which is automatic if either $\mathcal{E}_1$ or $\mathcal{E}_2$ has finite dimension), then $\|C_\varphi\|_{\mathrm{e}} = 0$. This gives a proof of Theorem 1.6. The remaining case is when $\|A\| < 1$ and A is not compact. In the following result, we assume $\varphi(0) = 0$.

PROPOSITION 4.2. *Suppose $A : \mathcal{E}_1 \to \mathcal{E}_2$ is a linear operator with $\|A\| < 1$. Then $\|C_A\|_{\mathrm{e}} = \|A\|_{\mathrm{e}}$. (The case $\|A\|_{\mathrm{e}} = 0$ is already covered by Theorem* 1.5 *so here we are only interested in the case $\|A\|_{\mathrm{e}} > 0$.)*

Proof. We consider first $\mathcal{E}_1 = \mathcal{E}_2$. Using the identity (4.1) and the fact that $\|A^*\| = \|A\| < 1$, we obtain

$$\|C_A\|_{\mathrm{e}} = \|1_{\mathbb{C}} \oplus A^* \oplus (A^*)^{\otimes 2} \oplus (A^*)^{\otimes 3} \oplus \cdots\|_{\mathrm{e}} = \|A^*\|_{\mathrm{e}} = \|A\|_{\mathrm{e}}.$$

In the general case, since A^*A is an operator on $\mathcal{E}_1$, the above argument gives $\|C_{A^*A}\|_{\mathrm{e}} = \|A^*A\|_{\mathrm{e}} = \|A\|_{\mathrm{e}}^2$. The conclusion of the proposition follows from the identity $C_A C_A^* = C_{A^*A}$. ∎

Unfortunately, we have not been able to find a formula in the case when $\varphi(0)$ is not zero. We raise here a question.

QUESTION 4.3. *Assume that $A : \mathcal{E} \to \mathcal{E}$ is a linear operator which is not compact, $\|A\| < 1$ and b is an arbitrary non-zero vector. Put $\varphi(z) = Az + b$ for $z \in \mathcal{E}$. Find the essential norm $\|C_\varphi\|_{\mathrm{e}}$.*

The last result we would like to discuss in this section is the spectrum $\sigma(C_\varphi)$ when C_φ is a compact operator. The following theorem is similar to Theorem 7.20 of [9], which describes the spectra of certain compact composition operators acting on weighted Hardy spaces over the unit ball in $\mathbb{C}^n$. In the setting of Segal–Bargmann spaces, our approach is less involved and it works also for the infinite dimensional case.

THEOREM 4.4. *Let $\varphi : \mathcal{E} \to \mathcal{E}$ be a mapping such that C_φ is a compact operator on $\mathcal{H}(\mathcal{E})$. Then we have*

$$\sigma(C_\varphi) = \{0, 1\} \cup \{\lambda_1 \cdots \lambda_s : \lambda_1, \ldots, \lambda_s \in \sigma(A) \textit{ and } s \geqslant 1\}.$$

Here A is a compact operator with $\|A\| < 1$ such that $\varphi(z) = Az + b$ for $z \in \mathcal{E}$.

Proof. Since C_φ is compact, we have $\sigma(C_\varphi) = \{0\} \cup \sigma_p(C_\varphi)$, where $\sigma_p(C_\varphi)$ is the point spectrum of C_φ.

Because $C_\varphi 1 = 1$, we know that $\lambda = 1$ is an eigenvalue. Now suppose that $\lambda \in \mathbb{C}\backslash\{0, 1\}$ is an eigenvalue of C_φ and $f \in \mathcal{H}(\mathcal{E})$ is a corresponding eigenvector. Then $f(Az + b) = \lambda f(z)$ for all $z \in \mathcal{E}$. Let $z_0 = (I - A)^{-1}(b)$ be the unique fixed point of φ. Since $f(z_0) = f(\varphi(z_0)) = \lambda f(z_0)$ and $\lambda \neq 1$, we have $f(z_0) = 0$. Let

$$f(z) = \sum_{j=1}^{\infty} \langle (z - z_0)^j, a_j \rangle$$

be the power expansion of f around z_0. It then follows from the identities $Az_0 + b = z_0$ and $f(A(z + z_0) + b) = \lambda f(z + z_0)$ that $f(Az + z_0) = \lambda f(z + z_0)$ for all $z \in \mathcal{E}$. This gives

$$\sum_{j=1}^{\infty} \langle (Az)^j, a_j \rangle = \lambda \sum_{j=1}^{\infty} \langle z^j, a_j \rangle \iff \sum_{j=1}^{\infty} \langle z^j, (A^*)^{\otimes j} a_j \rangle = \lambda \sum_{j=0}^{\infty} \langle z^j, a_j \rangle.$$

We conclude that $(A^*)^{\otimes j} a_j = \overline{\lambda} a_j$ for all $j \geqslant 1$. Since f is not the zero function, there exists an $l \geqslant 1$ such that $a_l \neq 0$, which shows that $\overline{\lambda}$ an eigenvalue of $(A^*)^{\otimes l}$. (Because $\|A\| < 1$, there are only a finite number of such l. This implies that f is in fact a polynomial.) Since A is compact, we conclude that λ is an eigenvalue of $A^{\otimes l}$. Hence, $\lambda = \lambda_1 \cdots \lambda_l$ for some eigenvalues $\lambda_1, \ldots, \lambda_l$ of A.

Conversely, suppose that $\lambda = \lambda_1 \cdots \lambda_l$ is a product of l (not necessarily distinct) eigenvalues of A. Let v_j be an eigenvector of A^* corresponding to the eigenvalue $\overline{\lambda}_j$ for $j = 1, \ldots, l$. Put $f(z) = \langle z - z_0, v_1 \rangle \cdots \langle z - z_0, v_l \rangle$ for $z \in \mathcal{E}$. Then f is a non-zero polynomial of degree l and we have

$$\begin{aligned} f(Az + b) &= \langle Az + b - z_0, v_1 \rangle \cdots \langle Az + b - z_0, v_l \rangle \\ &= \langle A(z - z_0), v_1 \rangle \cdots \langle A(z - z_0), v_l \rangle \\ &= \langle (z - z_0), A^* v_1 \rangle \cdots \langle (z - z_0), A^* v_l \rangle = \lambda f(z). \end{aligned}$$

Since f clearly belongs to $\mathcal{H}(\mathcal{E})$, we conclude that λ is an eigenvalue of C_φ on $\mathcal{H}(\mathcal{E})$. This completes the proof of the theorem. ∎

5. NORMAL, ISOMETRIC AND CO-ISOMETRIC COMPOSITION OPERATORS

We determine in this section the mappings $\varphi : \mathcal{E} \to \mathcal{E}$ that give rise to normal, isometric or co-isometric operators C_φ on $\mathcal{H}(\mathcal{E})$. (Recall that an operator on the Hilbert space is called co-isometric if its adjoint is an isometric operator.) We shall make use of the identities

$$C_\varphi 1 = 1,\ C_\varphi^* 1 = K_{\varphi(0)}, \quad \text{and} \quad C_\varphi^* C_\varphi 1 = K_{\varphi(0)}, \tag{5.1}$$

where 1 denotes the constant function with value one, which is also the reproducing kernel function K_0.

We first show that if C_φ is either a normal, isometric or co-isometric operator on $\mathcal{H}(\mathcal{E})$, then $\varphi(0) = 0$. The argument is fairly standard. In fact, if C_φ is normal, then we have $\|C_\varphi^* 1\| = \|C_\varphi 1\|$, which, together with (5.1), gives $\|K_{\varphi(0)}\| = \|1\|$. If C_φ is isometric, then $C_\varphi^* C_\varphi 1 = 1$, which gives $K_{\varphi(0)} = 1$ and hence, in particular, $\|K_{\varphi(0)}\| = \|1\|$. If C_φ is co-isometric then we also have $\|1\| = \|C_\varphi^* 1\| = \|K_{\varphi(0)}\|$. Since $\|K_{\varphi(0)}\|^2 = \exp(-\|\varphi(0)\|^2)$ and $\|1\|^2 = 1$, we conclude that in each of the above cases, $\varphi(0) = 0$.

Now since $\varphi(0) = 0$, Theorem 1.3 shows that $\varphi(z) = Az$ for some operator A on $\mathcal{E}$ with $\|A\| \leqslant 1$. Then $C_\varphi = C_A$, $C_\varphi^* = C_{A^*}$, and hence

$$C_\varphi^* C_\varphi = C_{A^*} C_A = C_{AA^*} \quad \text{and} \quad C_\varphi C_\varphi^* = C_A C_{A^*} = C_{A^*A}.$$

As a result, we obtain the following proposition.

PROPOSITION 5.1. *Let $\varphi : \mathcal{E} \to \mathcal{E}$ be a mapping such that C_φ is a bounded operator on $\mathcal{H}(\mathcal{E})$. Then*

(i) *C_φ is normal if and only if there exists a normal operator A on $\mathcal{E}$ with $\|A\| \leqslant 1$ such that $\varphi(z) = Az$ for all $z \in \mathcal{E}$.*

(ii) *C_φ is isometric if and only if there exists a co-isometric operator A on $\mathcal{E}$ such that $\varphi(z) = Az$ for all $z \in \mathcal{E}$.*

(iii) *C_φ is co-isometric if and only if there exists an isometric operator A on $\mathcal{E}$ such that $\varphi(z) = Az$ for all $z \in \mathcal{E}$.*

REMARK 5.2. Statement (i) in Proposition 5.1 holds also for composition operators on the Hardy and Bergman spaces of the unit ball (see Theorem 8.1 of [9]), where a similar result to Theorem 1.3 is not available. (In fact, on the Hardy and Bergman spaces, mappings that are not affine can give rise to bounded composition operators.) The proof of Theorem 8.1 in [9] can be adapted to prove Proposition 5.1(i) without appealing to Theorem 1.3 in the case $\mathcal{E}$ has finite dimension. On the other hand, since that proof relies on the finiteness of the dimension, it does not seem to work when $\mathcal{E}$ is infinite dimensional.

REMARK 5.3. In the case $\mathcal{E} = \mathbb{C}^n$, isometric operators on $\mathcal{E}$ are also co-isometric and vice versa, and all these operators are unitary. Statements (ii) and (iii) in Proposition 5.1 then imply that C_φ is isometric if and only if it is co-isometric if and only if it is unitary.

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ADDED IN PROOFS. It has been brought to our attention that a version of Theorem 1.3 in the case $\mathcal{E}_1 = \mathcal{E}_2$ and Theorem 1.4 were obtained independently via a different approach in [21].