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ERGODIC INVARIANT STATES AND IRREDUCIBLE REPRESENTATIONS OF CROSSED PRODUCT C^*-ALGEBRAS

HUICHI HUANG and JIANCHAO WU

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ABSTRACT. Motivated by reformulating Furstenberg's $\times p, \times q$ conjecture via representations of a crossed product C^*-algebra, we show that in a discrete C^*-dynamical system (A, Γ), the space of (ergodic) Γ-invariant states on A is homeomorphic to a subspace of (pure) state space of $A \rtimes \Gamma$. Various applications of this in topological dynamical systems and representation theory are obtained.

KEYWORDS: *Invariant state, crossed product C^*-algebra, irreducible representation.*

MSC (2010): 46L30, 46L55, 37B99.

1. INTRODUCTION

Let $S, T : X \to X$ be two commuting continuous maps on a compact Hausdorff space X. A Borel probability measure μ on X is called *S, T-invariant* if $\mu(S^{-1}A) = \mu(T^{-1}A) = \mu(A)$ for every Borel subset A of X. An S, T-invariant measure μ is called *ergodic* if every Borel set E with $S^{-1}E = E = T^{-1}E$ satisfies that $\mu(E) = 0$ or 1.

Assume that p, q are two positive integers greater than 1 with $\frac{\log p}{\log q}$ irrational. Denote the unit circle by $\mathbb{T}$. Define maps $T_p, T_q : \mathbb{T} \to \mathbb{T}$ as $T_p(z) = z^p$ and $T_q(z) = z^q$ for all $z \in \mathbb{T}$.

A Borel probability measure μ on $\mathbb{T}$ is called *$\times p, \times q$-invariant* if it is T_p, T_q-invariant. A Borel set $E \subset \mathbb{T}$ is called $\times p, \times q$-invariant if $E = T_pE = T_qE$.

H. Furstenberg gives the classification of closed $\times p, \times q$-invariant subsets of $\mathbb{T}$, which says that such a set is either finite or $\mathbb{T}$ ([7], Theorem IV.1). He also gives the following conjecture concerning the classification of ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$.

FURSTENBERG'S $\times p, \times q$ CONJECTURE. *An ergodic $\times p, \times q$-invariant Borel probability measure on $\mathbb{T}$ is either finitely supported or the Lebesgue measure.*

Furstenberg's conjecture is the simplest case of conjectures concerning classifications of invariant measures, and there are vast literatures about its general versions and their applications in number theory. See [6] for a survey.

For Furstenberg's conjecture, the best known result is the following theorem, which is proven by D. J. Rudolph under the assumption that p, q is coprime ([16], Theorem 4.9), later improved by A.S.A. Johnson ([9], Theorem A).

RUDOLPH–JOHNSON'S THEOREM. *If μ is an ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$, then either $h_\mu(T_p) = h_\mu(T_q) = 0$ or μ is the Lebesgue measure.*

Here $h_\mu(T_p)$ and $h_\mu(T_q)$ stand for the measure-theoretic entropy of $\times p$ and $\times q$ with respect to μ, respectively. See Chapter 4 of [18] for the definition of entropy for measure preserving maps.

For a $\times p, \times q$-invariant measure on $\mathbb{T}$, denote the two isometries on $L^2(\mathbb{T}, \mu)$ induced by continuous maps $\times p, \times q : \mathbb{T} \to \mathbb{T}$ by V_p, V_q.

By Rudolph–Johnson's theorem, to classify ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$, it suffices to classify such ergodic measures with zero entropy for T_p or T_q.

J. Cuntz notices that when $h_\mu(T_p) = h_\mu(T_q) = 0$, the operators V_p and V_q are two commuting unitary operators on $L^2(\mathbb{T}, \mu)$ ([18], Corollary 4.14.3).

For the unitary operator $M_z : L^2(\mathbb{T}, \mu) \to L^2(\mathbb{T}, \mu)$ given by $M_z f(z) = zf(z)$ for all $f \in L^2(\mathbb{T}, \mu)$ and $z \in \mathbb{T}$, one has $V_p M_z = M_z^p V_p$ and $V_q M_z = M_z^q V_q$. So a $\times p, \times q$ invariant measure μ with zero entropy gives rise to a representation π_μ of the universal unital C^*-algebra $C^*(s, t, z)$ generated by three unitaries s, t and z with the relations

$$st = ts, \quad sz = z^p s, \quad tz = z^q t$$

in the following way:

$$\pi_\mu(s) = V_p, \quad \pi_\mu(t) = V_q, \quad \pi_\mu(z) = M_z.$$

With the above observation, Cuntz suggests that one can consider ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$ via representations of $C^*(s, t, z) \cong C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$, where the two generators of $\mathbb{Z}^2$ act on $C^*(\mathbb{Z}[\frac{1}{pq}])$ by automorphisms induced by $\times p, \times q$ maps on $\mathbb{Z}[\frac{1}{pq}]$, and the isomorphism $\Phi : C^*(s, t, z) \to C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ is given by $\Phi(s) = a, \Phi(t) = b$ and $\Phi(z) = 1$. Here $a = (1, 0)$ and $b = (0, 1)$ are in $\mathbb{Z}^2$ and 1 is in $\mathbb{Z}[\frac{1}{pq}]$ ([3]).

Motivated by Cuntz's observation, firstly one has to answer the following question:

What kind of representation of $C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ is induced by a $\times p, \times q$-invariant measure on $\mathbb{T}$?

Denote the dual of $\mathbb{Z}[\frac{1}{pq}]$ by S_{pq}, the pq-solenoid ([13], A.1). The $\times p, \times q$ isomorphisms on $\mathbb{Z}[\frac{1}{pq}]$ give rise to $\times p, \times q$ isomorphisms on S_{pq}.

We answer the above question in the following way.

Firstly, the space of ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$ is homeomorphic to the space of ergodic $\times p, \times q$-invariant measures on S_{pq}, hence the classification of ergodic $\times p, \times q$-invariant measures on $\mathbb{T}$ amounts to classification of ergodic $\times p, \times q$-invariant measures on S_{pq}. Secondly, ergodic $\times p, \times q$-invariant measures on S_{pq} 1-1 corresponds to irreducible representations of $C^*(\mathbb{Z}[\frac{1}{pq}]) \rtimes \mathbb{Z}^2$ whose restriction to $\mathbb{Z}^2$ contains the trivial representation.

Moreover, in a more general context, we prove the following which briefly shows how the problem of invariant states relates to crossed product C^*-algebras.

Assume that a discrete group Γ acts on a unital C^*-algebra A as automorphisms. Denote this action by α, which is a group homomorphism from Γ to the automorphism group $\mathrm{Aut}(A)$ of A.

A state φ on A is *Γ-invariant* if $\varphi(\alpha_s(a)) = \varphi(a)$ for all s in Γ and a in A. An extreme point of the set of Γ-invariant states on A (this is a closed convex set when equipped with weak* topology, hence when nonempty, the set of extreme points is also nonempty) is called *ergodic*.

Denote by $A \rtimes \Gamma$ the full crossed product of the C^*-dynamical system (A, Γ, α).

THEOREM 1.1. *The space of (ergodic) Γ-invariant states on A is homeomorphic to the space of (pure) states on $A \rtimes \Gamma$ whose restriction to Γ is the trivial character.*

We give some applications of Theorem 1.1 to topological dynamical systems and representation theory.

Suppose a discrete group Γ acts on a compact Hausdorff space X as homeomorphisms (this is the same as Γ acting on the unital C^*-algebra $C(X)$, the space of continuous functions on X, by automorphisms). For a representation $\pi : C(X) \rtimes \Gamma \to B(H)$, denote the space of Γ-invariant vectors in H by H_Γ.

THEOREM 1.2. *Every irreducible representation π of $C(X) \rtimes \Gamma$ on a Hilbert space H satisfies that* $\dim H_\Gamma \leqslant 1$. *When* $\dim H_\Gamma = 1$, *the representation π is uniquely induced by an ergodic Γ-invariant regular Borel probability measure μ on X.*

A special case of Theorem 1.2 is the following corollary.

COROLLARY 1.3. *Suppose that a discrete group Γ acts on a discrete abelian group G by group automorphisms.*

Every irreducible unitary representation π of $G \rtimes \Gamma$ on a Hilbert space H satisfies that $\dim H_\Gamma \leqslant 1$.

When $\dim H_\Gamma = 1$, *the representation π is uniquely induced by an ergodic Γ-invariant regular Borel probability measure μ on the Pontryagin dual $\widehat{G}$ of G.*

The paper is organized as follows.

In the preliminary section, we recall some background of crossed product C^*-algebras. The proof of Theorem 1.1 is given in Section 3. At the end of that section, we include two immediate applications of Theorem 1.1 to C^*-dynamical systems, namely, Proposition 3.6 and Proposition 3.4. In Section 3.2, we prove

Proposition 3.8 and Theorem 1.2. In the last section we prove Theorem 4.2 which enables us to reformulate Furstenberg's $\times p, \times q$ problem in terms of representation theory of the semidirect product group $\mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2$.

2. PRELIMINARIES

In this section, we list some background for C^*-dynamical systems.

Within this article Γ stands for a discrete group and A stands for a unital C^*-algebra whose state space and pure state space are denoted by $S(A)$ and $P(A)$, respectively.

Denote the GNS representation of A with respect to a $\varphi \in S(A)$ by $\pi_\varphi : A \to B(L^2(A,\varphi))$ where $L^2(A,\varphi)$ stands for the Hilbert space corresponding to π_φ. Let $I_\varphi = \{a \in A : \varphi(a^*a) = 0\}$. Denote $a + I_\varphi$ by $\widehat{a}$ for all $a \in A$.

DEFINITION 2.1. An action of Γ on A as automorphisms is a group homomorphism $\alpha : \Gamma \to \mathrm{Aut}(A)$, where $\mathrm{Aut}(A)$ stands for the set of $*$-isomorphisms from A to A (this is a group under composition). We call (A, Γ, α) a dynamical system.

A *Γ-invariant state* is a state φ on A such that $\varphi(\alpha_s(a)) = \varphi(a)$ for all $s \in \Gamma$ and $a \in A$ [17]. Denote the set of Γ-invariant states on A by $S_\Gamma(A)$. It is clear that $S_\Gamma(A)$ is a convex closed set under weak* topology. If $S_\Gamma(A)$ is nonempty, then it contains at least one extreme point. We call an extreme point of $S_\Gamma(A)$ an *ergodic Γ-invariant state* on A. The set of ergodic Γ-invariant states on A is denoted by $E_\Gamma(A)$.

A representation of a C^*-algebra B on a Hilbert space H is a $*$-homomorphism $\pi : B \to B(H)$ and it is called irreducible if the commutant $C(\pi(B))$ consisting of elements in $B(H)$ commuting with every element in $\pi(B)$ contains only scalar multiples of identity operator.

A *covariant* representation (π, U, H) of a dynamical system (A, Γ, α) consists of a representation π of A and a unitary representation U of Γ on a Hilbert space H such that, for all $a \in A$ and $s \in \Gamma$,

$$\pi(\alpha_s(a)) = \mathrm{U}_s \pi(a) \mathrm{U}_s^*.$$

Let $C_c(\Gamma, A)$ be the space of finitely supported A-valued functions on Γ. For $f, g \in C_c(\Gamma, A)$, the product $f * g$ is given by

$$f * g(t) = \sum_{s_1 s_2 = t} f(s_1)\alpha_{s_1}(g(s_2))$$

and f^* is given by

$$f^*(t) = \alpha_t(f(t^{-1})^*)$$

for every $t \in \Gamma$. Then $C_c(\Gamma, A)$ is a $*$-algebra . Given a covariant representation (π, U, H) of a dynamical system (A, Γ, α), one can construct a representation $\widetilde{\pi}$ of $C_c(\Gamma, A)$ on H.

DEFINITION 2.2. For a dynamical system (A, Γ, α), the *crossed product C^*-algebra* $A \rtimes \Gamma$ is the completion of $C_c(\Gamma, A)$ under the norm $\|f\| = \sup \|\widetilde{\pi}(f)\|$ for $f \in C_c(\Gamma, A)$ where the supremum is taken over all representations of $C_c(\Gamma, A)$. Denote by u_s the unitary in $A \rtimes \Gamma$ corresponding to an $s \in \Gamma$.

There is a one-to-one correspondence between representations of $A \rtimes \Gamma$ and covariant representations of (A, Γ, α).

We refer readers to Chapter VIII of [4] for more about discrete crossed products.

3. MAIN RESULTS

If $\varphi \in S_\Gamma(A)$, then there is a unitary representation (the Koopman representation) U_φ of Γ on $L^2(A, \varphi)$ given by

$$\mathrm{U}_\varphi(s)(\widehat{a}) = \widehat{\alpha_s(a)}$$

for all $s \in \Gamma$ and $a \in A$ ([12], [14], [17]).

Given $\varphi \in S_\Gamma(A)$, the triple $(\pi_\varphi, \mathrm{U}_\varphi, L^2(A, \varphi))$ gives a covariant representation of (A, Γ, α). So there is a representation of $A \rtimes \Gamma$ on $L^2(A, \varphi)$, which we denote by ρ_φ, given by the following, for any $\sum\limits_{s \in \Gamma} a_s u_s \in C_c(\Gamma, A)$:

$$\rho_\varphi\Big(\sum_{s \in \Gamma} a_s u_s\Big) = \sum_{s \in \Gamma} \pi_\varphi(a_s)\mathrm{U}_\varphi(s).$$

3.1. Γ-INVARIANT STATES ON A AND STATES ON $A \rtimes \Gamma$. Denote $\{\varphi \in S(A \rtimes \Gamma) : \varphi(u_s) = 1 \text{ for all } s \in \Gamma\}$ by $S^1(A \rtimes \Gamma)$ and $\{\psi \in P(A \rtimes \Gamma) : \psi(u_s) = 1 \text{ for all } s \in \Gamma\}$ by $P^1(A \rtimes \Gamma)$.

We have the following.

THEOREM 3.1. *When equipped with weak* topologies, the restriction maps $R : S^1(A \rtimes \Gamma) \to S_\Gamma(A)$ and $R : P^1(A \rtimes \Gamma) \to E_\Gamma(A)$ are homeomorphisms.*

To prove this theorem, we first prove the following lemma which says that for every φ in $S^1(A \rtimes \Gamma)$, the restriction $\varphi|_A$ belongs to $S_\Gamma(A)$.

LEMMA 3.2. *For any state φ on $A \rtimes \Gamma$ such that $\varphi(u_s) = 1$ for every $s \in \Gamma$, we have $\varphi(u_s a u_t) = \varphi(a)$ for all $a \in A$ and $s, t \in \Gamma$. Consequently the restriction $\varphi|_A$ is a Γ-invariant state on A.*

The proof follows from Proposition 1.5.7 of [1] since by assumption every u_s is contained in the multiplicative domain of φ.

For a Γ-invariant state φ, there is a representation ρ_φ of $A \rtimes \Gamma$ on $L^2(A, \varphi)$ given by the following,

$$\rho_\varphi\Big(\sum_{s \in \Gamma} a_s u_s\Big) = \sum_{s \in \Gamma} \pi_\varphi(a_s)\mathrm{U}_\varphi(s),$$

for every $\sum_{s\in\Gamma} a_s u_s \in C_c(\Gamma, A)$.

Proof of Theorem 3.1. The restriction map $R : S^1(A \rtimes \Gamma) \to S_\Gamma(A)$ given by $R(\varphi) = \varphi|_A$ for every $\varphi \in S^1(A \rtimes \Gamma)$, is well-defined by Lemma 3.2. Since A is a C^*-subalgebra of $A \rtimes \Gamma$, the map R is continuous under weak* topology.

If $R(\varphi_1) = R(\varphi_2)$ for $\varphi_1, \varphi_2 \in S^1(A \rtimes \Gamma)$, then $\varphi_1(a) = \varphi_2(a)$ for all $a \in A$. Also

$$\varphi_1(au_s) = \varphi_2(au_s),$$

for all $a \in A$ and $s \in \Gamma$. Since every element in $C_c(\Gamma, A)$ is a linear combination of au_s and $C_c(\Gamma, A)$ is a dense subspace of $A \rtimes \Gamma$, it follows that $\varphi_1 = \varphi_2$ and R is injective.

Moreover, every $\varphi \in S_\Gamma(A)$ gives a representation ρ_φ of $A \rtimes \Gamma$ on $L^2(A, \varphi)$. Let $\widetilde{\varphi}$ be the state of $A \rtimes \Gamma$ given by $\widetilde{\varphi}(b) = \langle \rho_\varphi(b)(\widehat{1}), \widehat{1}\rangle$ for all $b \in A \rtimes \Gamma$. By the definition of ρ_φ, we see that $\widetilde{\varphi} \in S^1(A \rtimes \Gamma)$ and $\widetilde{\varphi}|_A = \varphi$. This shows the surjectivity of R.

Consequently, R is a bijective continuous map between two compact Hausdorff spaces $S^1(A \rtimes \Gamma)$ and $S_\Gamma(A)$. Therefore R is a homeomorphism.

Note that R is an affine map between two convex spaces $S^1(A \rtimes \Gamma)$ and $S_\Gamma(A)$, so the set of extreme points of $S^1(A \rtimes \Gamma)$ is homeomorphic to $E_\Gamma(A)$.

Suppose that φ is an extreme point of $S^1(A \rtimes \Gamma)$ and $\varphi = \lambda\varphi_1 + (1-\lambda)\varphi_2$ for two states φ_1, φ_2 on $A \rtimes \Gamma$ and some $0 < \lambda < 1$. Then $1 = \varphi(u_s) = \lambda\varphi_1(u_s) + (1-\lambda)\varphi_2(u_s)$ for every $s \in \Gamma$. It follows that $\varphi_1(u_s) = \varphi_2(u_s) = 1$, that is, $\varphi_1, \varphi_2 \in S^1(A \rtimes \Gamma)$. Hence $\varphi = \varphi_1 = \varphi_2$ and φ is a pure state, which means $P^1(A \rtimes \Gamma)$ is the set of extreme points of $S^1(A \rtimes \Gamma)$. ∎

For a character ξ on Γ (a group homomorphism from Γ to $\mathbb{T}$), denote $\{\varphi \in S(A \rtimes \Gamma) : \varphi(u_s) = \xi(s) \text{ for all } s \in \Gamma\}$ by $S^\xi(A \rtimes \Gamma)$ and $\{\varphi \in P(A \rtimes \Gamma) : \varphi(u_s) = \xi(s) \text{ for all } s \in \Gamma\}$ by $P^\xi(A \rtimes \Gamma)$.

For a representation of $A \rtimes \Gamma$ on a Hilbert space H, define $H_\xi = \{x \in H : \pi(u_s)(x) = \xi(s)x \text{ for all } s \in \Gamma\}$.

We have the following improvement of Theorem 3.1.

COROLLARY 3.3. *Let ξ be a character on Γ. When equipped with weak* topologies, $S_\Gamma(A) \cong S^\xi(A \rtimes \Gamma)$ and $E_\Gamma(A) \cong P^\xi(A \rtimes \Gamma)$.*

Proof. By Theorem 3.1, $S_\Gamma(A) \cong S^1(A \rtimes \Gamma)$. Note that $S^1(A \rtimes \Gamma) \cong S^\xi(A \rtimes \Gamma)$ ($P^1(A \rtimes \Gamma) \cong P^\xi(A \rtimes \Gamma)$ follows) and the homeomorphism is induced by the isomorphism $\Lambda : A \rtimes \Gamma \to A \rtimes \Gamma$ given by $\Lambda\Big(\sum_{s\in\Gamma} a_s u_s\Big) = \sum_{s\in\Gamma} \xi(s) a_s u_s$ for all $\sum_{s\in\Gamma} a_s u_s \in C_c(\Gamma, A)$. ∎

For a representation $\pi : A \rtimes \Gamma \to B(H)$, denote $\{x \in H : \pi(u_s)(x) = x \text{ for all } s \in \Gamma\}$ by H_Γ.

PROPOSITION 3.4. *For a C^*-dynamical system (A, Γ, α), the following are equivalent:*

(i) *The set $S_\Gamma(A)$ is nonempty.*

(ii) *The canonical homomorphism $C^*(\Gamma) \to A \rtimes \Gamma$ is an embedding.*

(iii) *There exists a representation $\pi : A \rtimes \Gamma \to B(H)$ such that $H_\Gamma \neq 0$, or equivalently, there exists a covariant representation (π, U, H) of (A, Γ, α) such that* U *contains the trivial representation of Γ.*

Proof. (i) $\Rightarrow$ (ii) Take a $\varphi \in S_\Gamma(A)$. Let Γ act on $\mathbb{C}$ trivially. By the invariance, the map $\varphi : A \to \mathbb{C}$ is a Γ-equivariant contractive completely positive map. By Exercise 4.1.4 of [1], there exists a contractive completely positive map $\widetilde{\varphi}$: $A \rtimes \Gamma \to C^*(\Gamma)$ such that $\widetilde{\varphi}\Big(\sum\limits_{s\in\Gamma} a_s u_s\Big) = \sum\limits_{s\in\Gamma} \varphi(a_s) u_s$. Immediately one can check that the composition of maps

$$C^*(\Gamma) \to A \rtimes \Gamma \underset{\widetilde{\varphi}}{\to} C^*(\Gamma)$$

is the identity map. Hence the canonical homomorphism $C^*(\Gamma) \to A \rtimes \Gamma$ is an embedding.

(ii) $\Rightarrow$ (i) By Theorem 3.1, it suffices to show $S^1(A \rtimes \Gamma)$ is nonempty.

Let $\pi_0 : \Gamma \to \mathbb{C}$ be the trivial unitary representation of Γ on $\mathbb{C}$. Then π_0 is a state on $C^*(\Gamma)$ such that $\pi_0(u_s) = 1$ for every $s \in \Gamma$. Note that $C^*(\Gamma)$ is a Banach subspace of $A \rtimes \Gamma$. By the Hahn–Banach theorem, one can extend π_0 to a bounded linear functional φ on $A \rtimes \Gamma$ without changing its norm ([2], Corollary 6.5). Hence $\|\varphi\| = \|\pi_0\| = 1 = \pi_0(1) = \varphi(1)$. So φ is a state on $A \rtimes \Gamma$ ([10], Theorem 4.3.2), and satisfies that $\varphi(u_s) = 1$ for all $s \in \Gamma$.

(i) $\Rightarrow$ (iii) The GNS representation with respect to $\varphi \in S_\Gamma(A)$ gives the required covariant representation of (A, Γ, α).

(iii) $\Rightarrow$ (i) Take a unit vector x in H_Γ and define a state φ on $A \rtimes \Gamma$ by $\varphi(b) = \langle \pi(b)x, x\rangle$ for all $b \in A \rtimes \Gamma$. It follows that $\varphi \in S^1(A \rtimes \Gamma)$. ∎

REMARK 3.5. The equivalence of (i) and (ii) may be well-known. When A is commutative and Γ is locally compact, this is mentioned in Remark 7.5 of [19]. To the best of our knowledge, it does not appear elsewhere in the literature.

Notice that U_φ gives rise to an action of Γ on $B(L^2(A, \varphi))$, also denoted by α for convenience, defined by the following, for every $T \in B(L^2(A, \varphi))$ and $s \in \Gamma$:

$$\alpha_s(T) = \mathrm{U}_\varphi(s) T \mathrm{U}_\varphi(s^{-1}).$$

Denote $\langle T(\widehat{1}), \widehat{1}\rangle$ by $\varphi(T)$ for all $T \in B(L^2(A, \varphi))$. When φ is a Γ-invariant state on A, it is also a Γ-invariant state on $B(L^2(A, \varphi))$ since we have the following, for all $s \in \Gamma$:

$$\begin{aligned}\varphi(\alpha_s(T)) &= \varphi(\mathrm{U}_\varphi(s) T \mathrm{U}_\varphi(s^{-1})) = \langle \mathrm{U}_\varphi(s) T \mathrm{U}_\varphi(s^{-1})(\widehat{1}), \widehat{1}\rangle \\ &= \langle T \mathrm{U}_\varphi(s^{-1})(\widehat{1}), \mathrm{U}_\varphi(s^{-1})(\widehat{1})\rangle = \langle T(\widehat{1}), \widehat{1}\rangle = \varphi(T).\end{aligned}$$

We can also see that

$$\alpha_s(\pi_\varphi(a)) = \pi_\varphi(\alpha_s(a)) \tag{3.1}$$

for every $a \in A$ and $s \in \Gamma$.

We call T in $B(L^2(A,\varphi))$ *Γ-invariant* if $\alpha_s(T) = T$ for all $s \in \Gamma$. Denote the set of Γ-invariant operators in $B(L^2(A,\varphi))$ by $B(L^2(A,\varphi))_\Gamma$. Let

$$\pi_\varphi(A)' = \{T \in B(L^2(A,\varphi)) : T\pi_\varphi(a) = \pi_\varphi(a)T \text{ for all } a \in A\}.$$

PROPOSITION 3.6. *A Γ-invariant state φ on A is ergodic if and only if $\varphi(T^*T) = |\varphi(T)|^2$ for every $T \in B(L^2(A,\varphi))_\Gamma \cap \pi_\varphi(A)'$.*

Proof. Recall that $R : S^1(A \rtimes \Gamma) \to S_\Gamma(A)$ is the restriction map. For $\varphi \in S_\Gamma(A)$, denote $R^{-1}(\varphi)$ by ψ.

Observe that $B(L^2(A,\varphi))_\Gamma \cap \pi_\varphi(A)' = \pi_\psi(A \rtimes \Gamma)'$.

Again by Theorem 3.1, the state φ on A is an ergodic Γ-invariant state if and only if ψ is a pure state on $A \rtimes \Gamma$ if and only if $\pi_\psi(A \rtimes \Gamma)' = \mathbb{C}$. The "only if" part follows immediately.

Now suppose $\varphi(T^*T) = |\varphi(T)|^2$ for every $T \in B(L^2(A,\varphi))_\Gamma \cap \pi_\varphi(A)' = \pi_\psi(A \rtimes \Gamma)'$. A straightforward calculation shows that $T(\widehat{1}) = \varphi(T)\widehat{1}$. Then for every $a \in A$, we have

$$T(\widehat{a}) = T\pi_\varphi(a)(\widehat{1}) = \pi_\varphi(a)T(\widehat{1}) = \pi_\varphi(a)(\varphi(T)\widehat{1}) = \varphi(T)\widehat{a}.$$

This means $T = \varphi(T)$, a scalar multiple of the identity operator. Hence $\pi_\psi(A \rtimes \Gamma)' = \mathbb{C}$ and ψ is a pure state. ∎

REMARK 3.7. The key observation $B(L^2(A,\varphi))_\Gamma \cap \pi_\varphi(A)' = \pi_\psi(A \rtimes \Gamma)'$ in the proof was pointed out to us by Sven Raum.

3.2. ERGODIC Γ-INVARIANT STATES ON A AND IRREDUCIBLE REPRESENTATIONS OF $A \rtimes \Gamma$. We say a representation $\pi_1 : B \to B(H_1)$ of a C^*-algebra B is *unitarily equivalent* to a representation $\pi_2 : B \to B(H_2)$ if there exists a surjective isometry $U : H_1 \to H_2$ such that $U\pi_1(b)(x) = \pi_2(b)U(x)$ for all $b \in B$ and $x \in H_1$.

PROPOSITION 3.8. *A representation $\pi : A \rtimes \Gamma \to B(H)$ is unitarily equivalent to $\rho_\varphi : A \rtimes \Gamma \to B(L^2(A,\varphi))$ for some $\varphi \in E_\Gamma(A)$ if and only if π is irreducible and $H_\Gamma \neq 0$.*

Proof. For a $\varphi \in E_\Gamma(A)$, by Theorem 3.1, there exists a $\psi \in P^1(A \rtimes \Gamma)$ such that $R(\psi) = \varphi$.

Also ρ_φ is unitarily equivalent to the GNS representation $\pi_\psi : A \rtimes \Gamma \to B(L^2(A \rtimes \Gamma, \psi))$ of $A \rtimes \Gamma$ with respect to ψ. Since ψ is a pure state, this shows ρ_φ is irreducible.

Note that $0 \neq \widehat{1} \in L^2(A,\varphi)$ and $\rho_\varphi(u_s)(\widehat{1}) = \widehat{1}$ for all $s \in \Gamma$. Hence $L^2(A,\varphi)_\Gamma \neq 0$.

Conversely, given an irreducible representation $\pi : A \rtimes \Gamma \to B(H)$ with $H_\Gamma \neq 0$, take a unit vector $x \in H_\Gamma$ and define a state ψ on $A \rtimes \Gamma$ by

$$\psi(b) = \langle \pi(b)x, x \rangle$$

for all $b \in A \rtimes \Gamma$. Since π is irreducible, the state ψ is a pure state and the GNS representation of $A \rtimes \Gamma$ with respect to ψ, π_ψ is unitarily equivalent to π ([4], Theorem I.9.8 and [5], 2.4.6). Also $x \in H_\Gamma$ implies that $\psi(u_s) = 1$ for all $s \in \Gamma$. So $\varphi = \psi|_A \in E_\Gamma(A)$, and π_ψ is unitarily equivalent to ρ_φ. This finishes the proof. ∎

Now we consider the case when A is commutative.

THEOREM 3.9. *For any irreducible representation $\pi : C(X) \rtimes \Gamma \to B(H)$, we have* $\dim H_\Gamma \leqslant 1$. *If $H_\Gamma \neq 0$, then there exists a unique ergodic Γ-invariant state φ on $C(X)$ (or a unique regular Γ-invariant Borel probability measure on X) such that π is unitarily equivalent to ρ_φ.*

Proof. Suppose $H_\Gamma \neq 0$ for an irreducible representation $\pi : C(X) \rtimes \Gamma \to B(H)$.

Take unit vectors $x, y \in H_\Gamma$. Define a state ψ on $A \rtimes \Gamma$ by

$$\psi(b) = \langle \pi(b)x, x \rangle$$

for all $b \in C(X) \rtimes \Gamma$. Then $\varphi = \psi|_{C(X)}$ gives an ergodic Γ-invariant probability measure μ on X with $\varphi(f) = \int\limits_X f \, d\mu$ for all $f \in C(X)$. Also the GNS representation π_ψ of $C(X) \rtimes \Gamma$ with respect to ψ, is unitarily equivalent to $\rho_\varphi : C(X) \rtimes \Gamma \to B(L^2(A, \varphi))$. Note that $L^2(A, \varphi) = L^2(X, \mu)$ and $L^2(A, \varphi)_\Gamma$ consists of Γ-invariant functions in $L^2(X, \mu)$, which are always constant functions ([8], Chapter 3, 3.10). Under surjective isometries $H \cong L^2(A \rtimes \Gamma, \psi) \cong L^2(X, \mu)$, both x and y are mapped to Γ-invariant functions in $L^2(X, \mu)$. Since μ is ergodic, their images in $L^2(X, \mu)$ are both constant functions. Hence there exists a constant λ with absolute value 1 such that $x = \lambda y$. This shows that $\dim H_\Gamma = 1$.

For the second part, the existence of φ follows from Theorem 3.8.

To prove the uniqueness of φ, we show the following claim.

Claim. If $\rho_\varphi \sim \rho_\psi$ for $\varphi, \psi \in S_\Gamma(C(X))$, then $\varphi = \psi$.

Proof of Claim. Let $\Theta : L^2(A, \varphi) \to L^2(A, \psi)$ be an isomorphism such that $\rho_\psi(f) = \Theta^{-1}\rho_\varphi(f)\Theta$ for every f in $C(X)$. It is easy to see that Θ preserves Γ-invariant vectors, i.e., $\Theta : L^2(A, \varphi)_\Gamma \to L^2(A, \psi)_\Gamma$ is also an isomorphism. Hence $\Theta(\widehat{1}) = \lambda\widehat{1}$ for some complex number λ with $|\lambda| = 1$.

By definition of ρ_φ, we have $\varphi(f) = \langle \rho_\varphi(f)\widehat{1}, \widehat{1} \rangle$ for all $f \in C(X)$. It follows, for all $f \in C(X)$, that

$$\varphi(f) = \langle \rho_\varphi(f)\widehat{1}, \widehat{1} \rangle = \langle \Theta^{-1}\rho_\psi(f)\Theta\widehat{1}, \widehat{1} \rangle = \langle \rho_\psi(f)\lambda\widehat{1}, \lambda\widehat{1} \rangle = \psi(f). \quad ∎$$

Hence φ is uniquely determined by the unitary equivalence class of π. ∎

REMARK 3.10. (i) Proposition 3.8 and Theorem 3.9 say that classification of ergodic Γ-invariant regular Borel probability measures on a compact Hausdorff space X amounts to classification of equivalence classes of irreducible representations of $C(X) \rtimes \Gamma$ whose restriction to Γ contains the trivial representation.

(ii) When A is non-commutative, Theorem 3.9 fails. For instance, one can take a noncommutative C^*-algebra A and a discrete group Γ acting on A trivially. An irreducible representation $\pi : A \to B(H)$ with $\dim H > 1$ and the trivial representation $\Gamma \to B(H)$ give rise to an irreducible representation of $\rho : A \rtimes \Gamma \to B(H)$. But $H = H_\Gamma$ is not of dimension 1.

There is an immediate application of Theorem 3.9 to representation theory of semidirect product groups.

COROLLARY 3.11. *Suppose a discrete group Γ acts on a discrete abelian group G by group automorphisms. Every irreducible unitary representation $\pi : G \rtimes \Gamma \to B(H)$ of the semidirect product group $G \rtimes \Gamma$ satisfies $\dim H_\Gamma \leqslant 1$. When $\dim H_\Gamma = 1$, the representation π is induced by an ergodic Γ-invariant regular Borel probability measure μ on the Pontryagin dual $\widehat{G}$ of G.*

Proof. Note that Γ acts on the group C^*-algebra $C^*(G)$ as automorphisms. Also $C^*(G) = C(\widehat{G})$ for the dual group $\widehat{G}$ of G and $C^*(G) \rtimes \Gamma \cong C^*(G \rtimes \Gamma)$. There exists a 1-1 correspondence between irreducible unitary representations of $G \rtimes \Gamma$ and irreducible representations of $C^*(G \rtimes \Gamma)$. Apply Theorem 3.9 to the case $C(X) = C(\widehat{G})$. ∎

4. FURSTENBERG'S $\times p, \times q$ PROBLEM VIA REPRESENTATION THEORY

We can define $\times p, \times q$ maps T_p, T_q on $\mathbb{Z}[\frac{1}{pq}]$ by $T_p(g) = pg, T_q(g) = qg$ for every $g \in \mathbb{Z}[\frac{1}{pq}]$. Note that T_p and T_q are group automorphisms. Hence they induce group automorphisms on the dual group S_{pq} of $\mathbb{Z}[\frac{1}{pq}]$. For convenience we also call them $\times p, \times q$ maps on S_{pq}.

Denote the set of $\times p$, $\times q$-invariant measures on the unit circle by $M_{p,q}(\mathbb{T})$, the set of ergodic $\times p$, $\times q$-invariant measures on the unit circle by $EM_{p,q}(\mathbb{T})$, the set of $\times p$, $\times q$-invariant measures on S_{pq} by $M_{p,q}(S_{pq})$, the set of ergodic $\times p$, $\times q$-invariant measures on S_{pq} by $EM_{p,q}(S_{pq})$.

4.1. $\times p, \times q$-INVARIANT MEASURES ON pq-SOLENOID AND $\times p, \times q$-INVARIANT MEASURES ON THE UNIT CIRCLE.

The following result is well-known for experts. For completeness we give a proof here.

PROPOSITION 4.1. *When equipped with weak* topologies, the restriction map $R : M_{p,q}(S_{pq}) \to M_{p,q}(\mathbb{T})$ defined by $R(\mu)(f) = \mu(f)$ for $\mu \in M_{p,q}(S_{pq})$ and $f \in C(\mathbb{T})$ is a homeomorphism. Also R restricts to a homeomorphism from $EM_{p,q}(S_{pq})$ to $EM_{p,q}(\mathbb{T})$.*

Proof. Take $\mu \in M_{p,q}(S_{pq})$. Since $C(\mathbb{T})$ is a C^*-subalgebra of $C(S_{pq})$ and this inclusion intertwines endomorphisms on $C(\mathbb{T})$ and $C(S_{pq})$ induced by $\times p, \times q$ on $\mathbb{T}$ and S_{pq}, the restriction $R(\mu)$ of μ on $C(\mathbb{T})$ belongs to $M_{p,q}(\mathbb{T})$ and R is also continuous under the weak* topology.

Conversely, assume that $\mu \in M_{p,q}(\mathbb{T})$. Note that the group algebra $\mathbb{C}\mathbb{Z}([\frac{1}{pq}])$ is a dense $*$-subalgebra of $C(S_{pq})$ and define $\nu(z^{kp^m q^n}) = \mu(z^k)$ for $n, m, k \in \mathbb{Z}$. By Bochner's theorem ([15], 1.4.3) ν is a Borel probability measure on S_{pq} if and only if $\{\nu(z^k)\}_{k \in \mathbb{Z}[\frac{1}{pq}]}$ is a positive definite sequence.

For any finite subset F of $\mathbb{Z}[\frac{1}{pq}]$, there exist positive integers k, l such that $F' = p^k q^l F = \{p^k q^l s : s \in F\}$ is a finite subset of $\mathbb{Z}$. Then we have

$$\begin{aligned}\nu\Big(\Big(\sum_{s\in F}\lambda_s z^s\Big)^*\Big(\sum_{t\in F}\lambda_t z^t\Big)\Big) &= \sum_{s,t\in F}\overline{\lambda}_s\lambda_t\nu(z^{t-s}) = \sum_{s,t\in F}\overline{\lambda}_s\lambda_t\mu(z^{p^k q^l(t-s)})\\ &= \mu\Big(\Big(\sum_{s\in F'}\lambda_{sp^{-k}q^{-l}}z^s\Big)^*\Big(\sum_{t\in F'}\lambda_{tp^{-k}q^{-l}}z^t\Big)\Big) \geqslant 0.\end{aligned}$$

Furthermore the $\times p$, $\times q$-invariance of ν follows from the definition. This shows that ν is in $M_{p,q}(S_{pq})$. Moreover, $\mu(z^k) = \nu(z^k)$ for all $k \in \mathbb{Z}$ by the $\times p, \times q$-invariance of μ, hence μ is the restriction of ν on $C(\mathbb{T})$, and this proves the surjectivity of R.

On the other hand, if $R(\mu_1) = R(\mu_2)$ for $\mu_1, \mu_2 \in M_{p,q}(S_{pq})$, then $\mu_1(z^k) = \mu_2(z^k)$ for all $k \in \mathbb{Z}$. Since μ_1 and μ_2 are $\times p, \times q$-invariant, we have $\mu_1(z^{kp^m q^n}) = \mu_2(z^{kp^m q^n})$ for all $n, m, k \in \mathbb{Z}$. This proves the injectivity of R.

So R is a bijective continuous map between two compact Hausdorff spaces $M_{p,q}(S_{pq})$ and $M_{p,q}(\mathbb{T})$, this implies that R is a homeomorphism.

Furthermore R is a homeomorphism from $EM_{p,q}(S_{pq})$ to $EM_{p,q}(\mathbb{T})$ since R is affine. ∎

THEOREM 4.2. *A representation $\pi : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(H)$ is induced by a finitely supported ergodic $\times p, \times q$-invariant measure μ on $\mathbb{T}$ (here π is induced by μ means that π is unitarily equivalent to π_μ) if and only if*

(i) *π is irreducible;*

(ii) *$H_{\mathbb{Z}^2} \neq 0$;*

(iii) *there exists nonzero $N \in \mathbb{Z}$ such that $\pi(z^N)x = x$ for every $x \in H_{\mathbb{Z}^2}$.*

Proof. Let μ be a finitely supported ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$. Since both $\times p$ and $\times q$ maps have zero entropy with respect to μ, there is a representation $\pi_\mu : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(L^2(\mathbb{T}, \mu))$ induced by μ (see Introduction for the definition of π_μ).

Suppose that $\pi : C(S_{pq}) \rtimes \mathbb{Z}^2 \to B(H)$ is unitarily equivalent to π_μ. By Proposition 4.1, $\nu = R^{-1}(\mu)$ is an ergodic $\times p, \times q$-invariant measure on S_{pq}. Hence the representation ρ_ν of $C(S_{pq}) \rtimes \mathbb{Z}^2$ on $L^2(S_{pq}, \nu)$ is irreducible.

Note that $L^2(\mathbb{T},\mu)$ is a subspace of $L^2(S_{pq},\nu)$ since μ is the restriction of ν from $C(S_{pq})$ onto $C(\mathbb{T})$. Also $L^2(\mathbb{T},\mu)$ is a nonzero invariant subspace of $L^2(S_{pq},\nu)$ under ρ_ν since V_p, V_q and M_z are all unitary operators on $L^2(\mathbb{T},\mu)$. Hence $L^2(\mathbb{T},\mu) = L^2(S_{pq},\nu)$, which implies that π_μ is unitarily equivalent to ρ_ν. So π is irreducible and $H_{\mathbb{Z}^2} \neq 0$.

Moreover, μ is finitely supported in a subset of $\{\frac{i}{N}\}_{i=0}^{N-1} \subset [0,1)$ (here we identify $\mathbb{T}$ with $[0,1)$). Hence $\mu(z^N) = 1$. It follows that $z^N = 1$ μ-a.e, which implies that $\pi(z^N)x = x$ for every $x \in H_{\mathbb{Z}^2}$.

Conversely, assume that π is an irreducible representation satisfying that $H_{\mathbb{Z}^2} \neq 0$ and $\pi(z^N)x = x$ for a nonzero $N \in \mathbb{Z}$ and every $x \in H_{\mathbb{Z}^2}$.

Claim. The Hilbert space H is finite dimensional.

Proof of Claim. Take a unit $y \in H_{\mathbb{Z}^2}$. Then $\overline{\mathrm{span}\pi(\mathbb{Z}[\frac{1}{pq}])y}$ is an invariant subspace of H under π. Since π is irreducible, we have $H = \overline{\mathrm{span}(\pi(\mathbb{Z}[\frac{1}{pq}])y)}$. So it suffices to prove $\mathrm{span}(\pi(\mathbb{Z}[\frac{1}{pq}])y)$ is finite dimensional.

Firstly, we prove that $\pi(z^M)y = y$ for a positive integer M coprime to pq. Without loss of generality we can assume $N > 0$. There exist nonnegative integers i, j, K, M such that $KN = Mp^iq^j$ with M coprime to pq. Then $\pi(z^{Mp^iq^j})y = \pi(z^{KN})y = y$.

Note that $\mathbb{Z}^2$ acts on $\mathbb{Z}[\frac{1}{pq}]$ by $\times p, \times q$, that is, $(m,n)\cdot z^k = z^{kp^mq^n}$ for all $m,n \in \mathbb{Z}$ and every $k \in \mathbb{Z}[\frac{1}{pq}]$. Since y is in $H_{\mathbb{Z}^2}$, we have $\pi((i,j))\pi(z^M)y = \pi(z^{Mp^iq^j})\pi((i,j))y = \pi(z^{Mp^iq^j})y = y$, which implies $\pi(z^M)y = y$.

Secondly, we prove that $\pi(\mathbb{Z}[\frac{1}{pq}])y = \pi(\mathbb{Z})y$. For all nonnegative integers i, j, there exists an integer l such that $lp^iq^j = rM + 1$ since M is coprime to pq. Hence for every positive integer k, we have

$$\pi(z^{k/p^iq^j})y = \pi(z^{k(lp^iq^j - rM)/p^iq^j})y = \pi(z^{kl})\pi(z^{-krM/p^iq^j})y = \pi(z^{kl})y.$$

This shows that $\pi(\mathbb{Z}[\frac{1}{pq}])y \subseteq \pi(\mathbb{Z})y$.

Lastly, we prove that $\mathrm{span}(\pi(\mathbb{Z})y)$ is finite dimensional. Every $k \in \mathbb{Z}$ can be written as $k = lN + r$ for some $l \in \mathbb{Z}$ and $0 \leqslant r < N$. Hence $\pi(z^k)y = \pi(z^{lN+r})y = \pi(z^r)y$. This implies that

$$\pi(\mathbb{Z})y \subset \mathrm{span}\{\pi(z^i)y\}_{i=0}^{N-1}.$$

So $\dim H \leqslant N$. ∎

Define a state ψ on $C(S_{pq}) \rtimes \mathbb{Z}^2$ by $\psi(b) = \langle \pi(b)y, y\rangle$ for every $b \in C(S_{pq}) \rtimes \mathbb{Z}^2$. We have $\psi \in P^1(C(S_{pq}) \rtimes \mathbb{Z}^2)$ since π is irreducible and $y \in H_{\mathbb{Z}^2}$. By Theorem 3.1, $\nu = R(\psi) = \psi|_{C(S_{pq})}$ is an ergodic $\times p, \times q$-invariant measure on S_{pq}.

By Theorem 3.8 and Theorem 3.9, we have $\pi \sim \rho_\nu$. Of course $H \cong L^2(S_{pq},\nu)$. From the claim, $L^2(S_{pq},\nu)$ is finite dimensional.

Hence ν is finitely supported in S_{pq}. Let $\mu = R(\nu) = \nu|_{C(S_{pq})}$. As before, ρ_ν is unitarily equivalent to π_μ. Hence $L^2(\mathbb{T},\mu) \sim L^2(S_{pq},\nu)$ is finite dimensional. Hence μ is a finitely supported ergodic $\times p, \times q$-invariant measure on $\mathbb{T}$ and $\pi \sim \pi_\mu$. ∎

Consequently we have the following corollary.

COROLLARY 4.3. *Furstenberg's conjecture is true if and only if there is a unique irreducible unitary representation* $U : \mathbb{Z}[\frac{1}{pq}] \rtimes \mathbb{Z}^2 \to B(H)$ *such that* $H_{\mathbb{Z}^2} \neq 0$ *and* $\pi(z^k)x \neq x$ *for every nonzero integer* k *and nonzero* $x \in H_{\mathbb{Z}^2}$.

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HUICHI HUANG, COLLEGE OF MATHEMATICS AND STATISTICS, CHONGQING UNIVERSITY, CHONGQING, 401331, CHINA
E-mail address: huanghuichi@cqu.edu.cn

JIANCHAO WU, DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY, UNIVERSITY PARK, PENNSYLVANIA, 16802, U.S.A.
E-mail address: jxw710@psu.edu

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