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THE UNIQUENESS OF THE PREDUALS OF QUANTUM GROUP ALGEBRAS

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ABSTRACT. If G is a compact group, then $\mathcal{C}(G)$ is the unique predual with respect to which the measure algebra $M(G)$ is a dual Banach algebra. In fact, we shall prove a stronger result that even holds for any compact quantum group in the sense of Woronowicz.

KEYWORDS: *Measure algebra, compact quantum group, dual Banach algebra, unique predual.*

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Let $\mathcal{A}$ be a Banach space. A *predual* of $\mathcal{A}$ is a pair (E, κ) consisting of a Banach space E and a linear homeomophism $\kappa : \mathcal{A} \to E^*$. A predual (E, κ) is *isometric* if the mapping κ is isometric. We often omit mentioning of κ when it is clear from the context.

Two preduals (E_1, κ_1) and (E_2, κ_2) are considered to be *the same*, if there exists a linear homeomorphism $T : E_1 \to E_2$ such that $\kappa_1 = T^* \circ \kappa_2$, or equivalently, if the weak*-topologies on $\mathcal{A}$ associated with (E_1, κ_1) and with (E_2, κ_2) are identical. This is also equivalent to the condition that $\kappa_1^*(E_1) = \kappa_2^*(E_2)$ as subsets of $\mathcal{A}^*$, where we identify any Banach space with a canonical subspace of its second dual.

The uniqueness of the predual with a certain property is understood in this context. For example, the combination of classical theorems of Dixmier and of Sakai shows that the canonical predual $\mathfrak{M}_*$ of a von Neumann algebra $\mathfrak{M}$ is the unique *isometric* predual of $\mathfrak{M}$. Moreover, $\mathfrak{M}_*$ is also the unique predual with respect to which the multiplication of $\mathfrak{M}$ is separately weak*-continuous (see Section 5 of [2]).

A *dual Banach algebra* is a Banach algebra $\mathcal{A}$ with a predual (E, κ) with respect to which the multiplication of $\mathcal{A}$ is separately weak*-continuous (see [7]).

Thus dual Banach algebras are generalisations of von Neumann algebras. Other examples of dual Banach algebras with natural preduals include:

(i) *the measure algebra* $M(G)$ on a locally compact group G where the natural predual is the algebra $\mathcal{C}_0(G)$ of continuous functions vanishing at infinity,
(ii) *the Fourier–Stietjes algebra* $\mathrm{B}(G)$ on G where the natural predual is the *full group* C^**-algebra* $\mathcal{C}^*(G)$, and
(iii) *the reduced Fourier–Stietjes algebra* $\mathrm{B}_{\mathrm{r}}(G)$ on G where the natural predual is *the reduced group* C^**-algebra* $\mathcal{C}^*_{\mathrm{r}}(G)$.

We shall quickly review $\mathrm{B}(G)$ and $\mathrm{B}_{\mathrm{r}}(G)$ for discrete groups G below, but for a general definition we refer the reader to [3] where these objects were introduced. (There are other important classes of dual Banach algebras such as that of the algebras $\mathcal{B}(E)$ of bounded operators on reflexive Banach spaces E where the natural preduals are $E^*\widehat{\otimes}E$, but they are out of the scope of this paper.)

REMARK. We remark that for the above examples, their natural preduals are also isometric. But unlike the case of von Neumann algebras, these Banach algebras (considered as Banach spaces) usually do not have unique isometric preduals.

For example, in the case of measure algebras on compact groups, which is of the most interest to us, we see that for any infinite compact group G and the identity element $1 \in G$ (in fact, any infinite compact space and any of its limit points would do), there is a linear isometry $\kappa : M(G) \to M(G \setminus \{1\})$; this can be seen by decomposing measures into their continuous and discrete parts. Thus, the pair $(\mathcal{C}_0(G \setminus \{1\}), \kappa)$ is an isometric predual of $M(G)$ that is different from the canonical predual $\mathcal{C}(G)$, for otherwise $G \setminus \{1\}$ would be homeomorphic to G by the Banach–Stone theorem.

As dual Banach algebras, the above algebras also do not in general have unique preduals, which is again unlike the case of von Neumann algebras. For a locally compact group G, there are additional conditions that would force $\mathcal{C}_0(G)$ to be the unique predual of the dual Banach algebra $M(G)$: it is proved in Section 3 of [2] that $\mathcal{C}_0(G)$ is the unique predual of $M(G)$ with respect to which a certain natural comultiplication of $M(G)$ is weak*-continuous (in addition to the separate weak*-continuity of the multiplication of $M(G)$). But, without this additional condition, even in the case where $G = \mathbb{Z}$, it is proved in Section 3 of [1] that there are uncountably many isometric preduals of $\ell^1(\mathbb{Z}) = M(\mathbb{Z})$ with respect to which $\ell^1(\mathbb{Z})$ is a dual Banach algebra.

In this paper, we shall prove that the situation is much nicer when G is a compact group: in that case, $\mathcal{C}(G)$ is even the unique predual of $M(G)$ with respect to which the multiplication of $M(G)$ is weak*-continuous on the left. This is a considerable improvement of Section 3 in [2] for compact groups, and provides a sharp contrast to the results of [1]. In fact, this is a special case of our main result, Theorem 1, where the same holds true for any compact quantum group.

A *compact quantum group*, as defined by Woronowicz [8], is a pair (A, Δ) that consists of a *unital* C^*-algebra A and *comultiplication* Δ on A such that the linear spans of $\Delta(A)(1 \otimes A)$ and of $\Delta(A)(A \otimes 1)$ are dense in $A \otimes_{\min} A$. Here, a *comultiplication* Δ on A is a unital $*$-homomorphism $\Delta : A \to A \otimes_{\min} A$ satisfying the *coassociativity*

$$(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta \quad \text{where } \iota \text{ is the identity mapping.}$$

See also the expository paper [6], which will serve as our reference on compact quantum groups.

The most basic examples of compact quantum groups are the ones associated with compact groups G. In that case, the unital C^*-algebra is the algebra $\mathcal{C}(G)$ of continuous functions on G and the comultiplication Δ is defined as

$$\Delta(f)(s,t) = f(st) \quad (s,t \in G),$$

where we identify $\mathcal{C}(G) \otimes_{\min} \mathcal{C}(G)$ naturally as $\mathcal{C}(G \times G)$.

As such, given a compact quantum group, we would think of it as representing some "generalised group" object that is denoted by $\mathbb{G}$, and if $\mathbb{G}$ is given by the pair (A, Δ), then in view of the case of a compact group, we shall write $\mathcal{C}(\mathbb{G})$ instead of A and $M(\mathbb{G})$ instead of A^*. As in the case of a compact group, $M(\mathbb{G})$ is a Banach algebra whose multiplication, generalising the convolution product, is defined as

$$\mu\omega := (\mu \otimes \omega) \circ \Delta \quad (\mu, \omega \in M(\mathbb{G})).$$

Since

$$(\mu \otimes \iota)\Delta(A) \subseteq A \quad \text{and} \quad (\iota \otimes \mu)\Delta(A) \subseteq A \quad \text{for every } \mu \in M(\mathbb{G}) = A^*,$$

we see that the multiplication on $M(\mathbb{G})$ is separately weak*-continuous, and so it is even a dual Banach algebra with respect to the predual $A = \mathcal{C}(\mathbb{G})$.

THEOREM 1. *Let $\mathbb{G}$ be a compact quantum group. Then $\mathcal{C}(\mathbb{G})$ is the unique predual of $M(\mathbb{G})$ with respect to which the multiplication on $M(\mathbb{G})$ is weak*-continuous either on the left or on the right.*

Proof. Suppose that $\mathbb{G} = (A, \Delta)$, so that

$$\mathcal{C}(\mathbb{G}) = A \quad \text{and} \quad M(\mathbb{G}) = A^*.$$

Let $\widetilde{\Delta}$ be the extension of Δ to a normal $*$-homomorphism

$$\mathcal{C}(\mathbb{G})^{**} \to \mathcal{C}(\mathbb{G})^{**} \overline{\otimes} \mathcal{C}(\mathbb{G})^{**}.$$

A simple continuity argument shows that

$$\langle \mu\omega, x \rangle = (\mu \otimes \omega)(\widetilde{\Delta}(x)) \quad (\mu, \omega \in M(\mathbb{G}),\ x \in \mathcal{C}(\mathbb{G})^{**} = M(\mathbb{G})^*).$$

In this proof, we shall use $\langle \cdot, \cdot \rangle$ for the duality between a Banach space and its dual, and $\langle \cdot | \cdot \rangle$ for inner product of a Hilbert space.

A theorem of Woronowicz when A is seperable and of Van Daele for the general case (see Theorem 4.4 of [6]) says that $\mathbb{G}$ has a (unique) Haar measure, i.e. a state h on $\mathcal{C}(\mathbb{G})$ such that

$$\mu \mathrm{h} = \mathrm{h}\mu = \mu(1)\mathrm{h} \quad (\mu \in M(\mathbb{G})).$$

Denote by $(\mathfrak{H}, \pi, \xi_0)$ the cyclic GNS representation for h. We shall denote by $\widetilde{\pi}$ the extension of π to a normal $*$-epimorphism from $\mathcal{C}(\mathbb{G})^{**}$ onto $\pi(\mathcal{C}(\mathbb{G}))''$.

Denote by $\mathfrak{K}$ a Hilbert space on which $\mathcal{C}(\mathbb{G})$ acts universally. Thus $\mathcal{C}(\mathbb{G})^{**}$ can be identified with $\mathcal{C}(\mathbb{G})''$ on $\mathfrak{K}$. By Proposition 5.5 of [6] and its proof, the unitary operator U on $\mathfrak{H} \otimes \mathfrak{K}$ defined in Proposition 5.2 of [6] as

$$U(a\xi_0 \otimes \eta) = (\pi \otimes \iota)(\Delta(a))(\xi_0 \otimes \eta) \quad (a \in \mathcal{C}(\mathbb{G}),\ \eta \in \mathfrak{K}),$$

satisfies the following:

$$\mathcal{C}(\mathbb{G}) = [(\mu \otimes \iota)(U) : \mu \in \mathcal{B}(\mathfrak{H})_*], \quad \text{and}$$
$$(\omega_{\pi(a)\xi_0,\, \pi(b)\xi_0} \otimes \iota)(U) = (\mathrm{h} \otimes \iota)((b^* \otimes 1)\Delta(a)) \quad (a, b \in \mathcal{C}(\mathbb{G})).$$

Here $[X]$ is the closed linear span of a subset X (of a Banach space), and ω_{η_1,η_2} is the normal linear functional $x \mapsto \langle x\eta_1 | \eta_2 \rangle$ on $\mathcal{B}(\mathfrak{H})$ for each $\eta_1, \eta_2 \in \mathfrak{H}$. By continuity, the latter formula can be extended to

$$(\omega_{\widetilde{\pi}(a)\xi_0,\, \widetilde{\pi}(b)\xi_0} \otimes \iota)(U) = (\mathrm{h} \otimes \iota)((b^* \otimes 1)\widetilde{\Delta}(a)) \quad (a, b \in \mathcal{C}(\mathbb{G})^{**}).$$

Suppose now that (E, κ) is a predual of $M(\mathbb{G})$ with respect to which the mapping $T_\mu \colon \omega \mapsto \mu\omega$ is weak*-continuous on $M(\mathbb{G})$, for each $\mu \in M(\mathbb{G})$. Then, for each $\mu \in M(\mathbb{G})$, there exists a bounded linear operator $S_\mu : E \to E$ such that

$$\kappa \circ T_\mu = S_\mu^* \circ \kappa.$$

Thus, for $\omega \in M(\mathbb{G})$ and $x \in E$, we see that

$$\langle \omega, \kappa^*(S_\mu(x)) \rangle = \langle \kappa(T_\mu(\omega)), x \rangle = \langle \mu\omega, \kappa^*(x) \rangle = (\mu \otimes \omega)(\widetilde{\Delta}(\kappa^*(x))).$$

This shows

$$(\mu \otimes \iota)\widetilde{\Delta}(\kappa^*(x)) = \kappa^*(S_\mu(x)) \in \kappa^*(E) \quad (\mu \in M(\mathbb{G}),\ x \in E);$$

noting that $\kappa^* : E^{**} \to M(\mathbb{G})^* = \mathcal{C}(\mathbb{G})^{**}$, we identify E naturally with a subspace of E^{**} (as for any Banach space). Also, since $\kappa^*(E)$ is a σ-weakly dense subspace of $\mathcal{C}(\mathbb{G})^{**}$, it is also σ-strongly dense in $\mathcal{C}(\mathbb{G})^{**}$. Therefore, $\widetilde{\pi}(\kappa^*(E))$ is σ-strongly dense in $\pi(\mathcal{C}(\mathbb{G}))''$. Thus, the previous paragraph implies that

$$\begin{aligned}
\mathcal{C}(\mathbb{G}) &= [(\mu \otimes \iota)(U) :\ \mu \in \mathcal{B}(\mathfrak{H})_*] \\
&= [(\omega_{\widetilde{\pi}(a)\xi_0,\, \pi(b)\xi_0} \otimes \iota)(U) : a \in \kappa^*(E),\ b \in \mathcal{C}(\mathbb{G})] \\
&= [(\mathrm{h} \otimes \iota)((b^* \otimes 1)\widetilde{\Delta}(a)) : a \in \kappa^*(E),\ b \in \mathcal{C}(\mathbb{G})] \\
&= [(\mathrm{h} \cdot b^* \otimes \iota)\widetilde{\Delta}(a) : a \in \kappa^*(E),\ b \in \mathcal{C}(\mathbb{G})] \subseteq \kappa^*(E),
\end{aligned}$$

where $(\mathrm{h} \cdot b^*)(x) := \mathrm{h}(b^* x)$ defines an element of $\mathcal{C}(\mathbb{G})^* = M(\mathbb{G})$. Since both (E, κ) and $\mathcal{C}(\mathbb{G})$ are preduals of $M(\mathbb{G})$, it follows that $\kappa^*(E) = \mathcal{C}(\mathbb{G})$.

For the case where the multiplication is weak*-continuous on the left, we just need to compose Δ with the flip mapping. This completes the proof. ∎

COROLLARY 2. *Let $\mathbb{G}$ be a compact quantum group. Then $\mathcal{C}(\mathbb{G})$ is the unique predual of $M(\mathbb{G})$ that makes $M(\mathbb{G})$ a dual Banach algebra.*

Of course, Theorem 1 also has the following corollary as a special case.

COROLLARY 3. *Let G be a discrete group. Then $\mathcal{C}^*(G)$ is the unique predual of $\mathrm{B}(G)$ that makes $\mathrm{B}(G)$ a dual Banach algebra, and $\mathcal{C}^*_{\mathrm{r}}(G)$ is the unique predual of $\mathrm{B}_{\mathrm{r}}(G)$ that makes $\mathrm{B}_{\mathrm{r}}(G)$ a dual Banach algebra.*

This is because any discrete group G also gives rise to two compact quantum groups as follows. Consider the full group C^*-algebra $\mathcal{C}^*(G)$ of G; this unital C^*-algebra is the completion of $\ell^1(G)$ with respect to its largest C^*-norm. The mapping $\delta_s \mapsto \delta_s \otimes \delta_s$ extends to a comultiplication

$$\Delta : \mathcal{C}^*(G) \to \mathcal{C}^*(G) \otimes_{\min} \mathcal{C}^*(G)$$

making $(\mathcal{C}^*(G), \Delta)$ a compact quantum group. The dual of $\mathcal{C}^*(G)$ is the Fourier–Stieltjes agebra $\mathrm{B}(G)$, which is a commutative Banach algebra of functions on G. We could also consider the reduced group C^*-algebra $\mathcal{C}^*_{\mathrm{r}}(G)$ of G, which is the completion of $\ell^1(G)$ with respect to the operator norm on $\ell^2(G)$, where $\ell^1(G)$ acts on $\ell^2(G)$ by convolution on the left. The C^*-algebra $\mathcal{C}^*_{\mathrm{r}}(G)$ is again unital, it is naturally a quotient of $\mathcal{C}^*(G)$, and the comultiplication Δ of the latter induces a commultiplication, again denoted by Δ, on the former, making $(\mathcal{C}^*_{\mathrm{r}}(G), \Delta)$ a compact quantum group. The dual of $\mathcal{C}^*_{\mathrm{r}}(G)$ is the reduced Fourier–Stieltjes agebra $\mathrm{B}_{\mathrm{r}}(G)$, which is a closed subalgebra of $\mathrm{B}(G)$. These two compact quantum groups $(\mathcal{C}^*(G), \Delta)$ and $(\mathcal{C}^*_{\mathrm{r}}(G), \Delta)$ are actually considered as the same object $\widehat{G}$, the dual of G, in two (possibly different) disguises (the universal and the reduced settings). In fact, when G is abelian (and discrete), $(\mathcal{C}^*(G), \Delta) = (\mathcal{C}^*_{\mathrm{r}}(G), \Delta)$ is nothing but the compact quantum group $(\mathcal{C}(\widehat{G}), \Delta)$ associated with the compact (abelian) group $\widehat{G}$, the dual group of G, as considered above.

We note that in general, not much is known about the uniqueness of the preduals of the Fourier–Stieltjes algebras (reduced or not). Apart from the above, it is only proved in Section 4 of [2] that when G is a separable compact group, $\mathcal{C}^*(G) = \mathcal{C}^*_{\mathrm{r}}(G)$ is the unique isometric predual of $\mathrm{A}(G) = \mathrm{B}(G) = \mathrm{B}_{\mathrm{r}}(G)$ with respect to which $\mathrm{A}(G)$ is a dual Banach algebra and a certain natural comultiplication of $\mathrm{A}(G)$ is weak*-continuous. On the other hand, the results of [1] again show that for the case of $G = \mathbb{T}$, there are uncountably many isometric preduals of $\mathrm{A}(\mathbb{T}) = \ell^1(\mathbb{Z})$ with respect to which $\mathrm{A}(\mathbb{T})$ is a dual Banach algebra.

The duality between a general locally compact group G and its dual object $\widehat{G}$ could be made precise elegantly in the theory of *locally compact quantum groups*, introduced in [5] (see also [4] for the link between the universal setting and the reduced setting of a locally compact quantum group). In this context, the construction of $M(\mathbb{G})$ as a dual Banach algebra holds true for any locally compact quantum group $\mathbb{G}$, and we could ask the same question about which condition would force $\mathcal{C}_0(\mathbb{G})$ to be the unique predual of $M(\mathbb{G})$. But in light of what is known about $M(G)$ and $\mathrm{B}(G)$ /or $\mathrm{B_r}(G)$, this question seems to be out of reach, at least for the present.

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