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ISOMORPHISMS AND GAP THEOREMS FOR FIGÀ-TALAMANCA–HERZ ALGEBRAS

JEAN ROYDOR

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ABSTRACT. It is an open question whether the Figà-Talamanca–Herz algebra $\mathcal{A}_p(G)$ determines the group G. We consider Figà-Talamanca–Herz algebras equipped with their p-operator space structure and we prove that two locally compact groups G and H are isomorphic if and only if there exists an algebra isomorphism $\Phi : \mathcal{A}_p(G) \to \mathcal{A}_p(H)$ with p-completely bounded norm $\|\Phi\|_{\mathrm{pcb}} < (2^{p-2} + 1/2)^{1/p}$ if $1 < p \leqslant 2$ or $\|\Phi\|_{\mathrm{pcb}} < (2^{1-p} + 1)^{1/p}$ if $2 \leqslant p < \infty$. In our second theorem, we prove an "almost norm one" version of Host's idempotents theorem for uniformly smooth or uniformly convex Banach spaces. As applications, we obtain several gap results: for instance for norms of idempotent p-completely bounded multipliers and amenability constant of Figà-Talamanca–Herz algebras.

KEYWORDS: *Figà-Talamanca–Herz algebra, p-operator space, p-completely bounded map, uniformly smooth and uniformly convex Banach space.*

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1. INTRODUCTION

For $1 \leqslant p \leqslant \infty$ and an abelian group G, the space $\mathcal{A}_p(G)$ has been defined by A. Figà-Talamanca in [17]. Subsequently C.S. Herz [19] extended this definition to general locally compact groups and proved that $\mathcal{A}_p(G)$ is a Banach algebra, now usually called Figà-Talamanca–Herz algebras (see definition in Section 2). In the case $p = 2$, $\mathcal{A}_2(G)$ coincides with Eymard's Fourier algebra [14]. It is a classical result of M.E. Walter [41] that two locally compact groups G and H are isomorphic if and only if $\mathcal{A}_2(G)$ and $\mathcal{A}_2(H)$ are isometrically isomorphic as Banach algebras. Actually, contractivity of the algebra isomorphism is sufficient to identify the underlying groups, see Corollary 5.4 of [35]. For $p = 1$ or ∞, $\mathcal{A}_p(G)$ does not determine the group, however for $1 < p \neq 2 < \infty$, it is still an open question whether the Banach algebra $\mathcal{A}_p(G)$ up to isometric algebra isomorphism determines the group. In the early seventies, N. Lohoué answered positively for abelian groups [32]. Recently M.G. Cowling revisited this question and gave also a positive answer for connected Lie groups [7] and more generally the reader is referred to that paper for a nice summary of the known results related to this question.

We consider Figà-Talamanca–Herz algebras equipped with their p-operator space structure and p-completely bounded morphisms, see Section 2 for preliminaries on these notions. This category has been initiated by G. Pisier in Chapter 8 of [36] and axiomatized by C. Le Merdy in [29]. Then M. Daws applied this theory to define two p-operator space structures on Figà-Talamanca–Herz algebras [9], here we consider the so-called dual p-operator space structure. The first idea in our paper is to recast in this subcategory the open question mentioned above: one can hope that the Figà-Talamanca–Herz algebra $\mathcal{A}_p(G)$ up to p-completely isometric algebra isomorphism determines the group. In this paper, we even prove a more accurate result.

THEOREM A. *Let G and H be two locally compact groups. Let $\Phi : \mathcal{A}_p(G) \to \mathcal{A}_p(H)$ be a surjective algebra isomorphism. If one of the following conditions hold:*

(i) *for $1 < p \leqslant 2$, if $\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\| < (2^{p-2} + 1/2)^{1/p}$,*

(ii) *for $2 \leqslant p < \infty$, if $\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\| < (2^{1-p} + 1)^{1/p}$,*

then there are $t \in G$ and a topological isomorphism $\phi : H \to G$ such that

$$\Phi(u)(h) = u(t\phi(h)),$$

for any $u \in \mathcal{A}_p(G)$, $h \in H$.

Consequently, Φ is actually p-completely isometric (i.e. for every $n \geqslant 1$, $\mathrm{id}_{\mathbb{M}_n} \otimes \Phi$ is isometric).

The proof of Walter's result (mentioned in the first paragraph) is by duality and uses Kadison's description of surjective isometries between von Neumann algebras [28] (as $\mathcal{A}_2(G)^* = \mathcal{VN}(G)$, the group von Neumann algebra of G). In the case $1 < p \neq 2 < \infty$, the difficulty is that no description of surjective isometries or even p-complete isometries between the dual spaces of Figà-Talamanca–Herz algebras is known (hence, a priori, our hypothesis on the p-completely bounded norm of the algebra isomorphism does not seem helpful). To overcome this difficulty, we use a new trick for 2×2 matrices with coefficients in the algebra of bounded operators on an L_p-space. Consequently, we do not require a full "p-complete hypothesis", we just need tensorization by 2×2 matrices. Moreover one should note that our Theorem A reveals a gap phenomenon: an algebra isomorphism between Figà-Talamanca–Herz algebras is either p-completely isometric or has p-completely bounded norm greater than $(2^{p-2} + 1/2)^{1/p}$ if $1 < p \leqslant 2$ or $(2^{1-p} + 1)^{1/p}$ if $2 \leqslant p < \infty$.

Since a gap phenomenon appears in this structural result, intuitively, one can expect gap phenomena to occur on other aspects of Figà-Talamanca–Herz algebras. The second idea of this paper is to prove an "almost norm one" version of Host's idempotents theorem in order to exhibit such phenomena. We show gap results (see Corollaries 4.4, 4.5, 4.6 and 4.10 in Section 4) for: bounds of approximate identities of ideals, p-completely bounded norms of homomorphisms (for amenable groups), p-completely bounded norms of idempotents multipliers and amenability constants of Figà-Talamanca–Herz algebras. These four gap results are direct repercussions of our second theorem (in the next statement, E^* denotes the dual space of E and $\|\alpha\|_\infty = \sup_{g\in G} \|\alpha(g)\|_{E^*}$, idem for β).

THEOREM B. *Let G be a locally compact group and $S \subset G$ non-empty. Assume that there exist a uniformly smooth Banach space E and mappings $\alpha : G \to E^*$, $\beta : G \to E$ such that the indicator function of S satisfies for any $g, h \in G$, $\chi_S(gh^{-1}) = \langle \alpha(g), \beta(h)\rangle$. Then, there exists $v_E > 1$ such that if $\|\alpha\|_\infty \|\beta\|_\infty < v_E$, then S is a coset.*

It is well-known that a Banach space is uniformly smooth if and only if its dual space is uniformly convex. Moreover, uniform convexity or uniform smoothness implies reflexivity (see e.g. [26] or [30]). Therefore, the assumption of uniform smoothness could be replaced by uniform convexity in the statement of Theorem B. In the proof, we will see that the constant $v_E > 1$ depends only on the modulus of uniform smoothness of E (or equivalently, by duality, on the modulus of uniform convexity of E^*). Our Theorem B is related to Host's theorem on idempotents of Fourier–Stieltjes algebras. For a locally compact group G, Host's theorem [21] states that an indicator function on G is the coefficient of a unitary representation on a Hilbert space if and only if the support of this function belongs to the coset ring of G (i.e. the ring of subsets of G generated by cosets of open subgroups of G), for abelian groups, this characterization is due to P.J. Cohen [6]. In particular, there is this following special case: the support of an idempotent of a Fourier–Stieltjes algebra is exactly one coset if and only if this idempotent has norm one (see Theorem 2.1 [23]). With this special case in mind, our Theorem B must thus be understood as an "almost norm one" version of Host's theorem for uniformly smooth (or equivalently for uniformly convex) Banach spaces. The use of these geometrical properties of Banach spaces has been inspired from M.G. Cowling and G. Fendler paper [8].

2. PRELIMINARIES

2.1. ON p-OPERATOR SPACES AND p-COMPLETELY BOUNDED MAPS.

Let us first recall the category of p-operator spaces, for details the reader is referred to Chapter 8 of [36] and [29]. For Banach spaces E and F, we denote $\mathbb{B}(E,F)$ the space of all bounded operators from E into F equipped with the operator norm and we denote $\mathbb{B}(E) = \mathbb{B}(E,E)$. For $1 < p < \infty$, a concrete p-operator space is a subspace $\mathcal{X} \subset \mathbb{B}(E)$ where E is a subspace of quotient of an L_p-space. Hence $\mathbb{M}_n(\mathcal{X})$, the vector space of all $n \times n$ matrices with coefficients in $\mathcal{X}$, can be normed by the inclusion

$$\mathbb{M}_n(\mathcal{X}) \subset \mathbb{M}_n(\mathbb{B}(E)) = \mathbb{B}(\ell_p^n(E)),$$

where $\ell_p^n(E)$ is the ℓ_p-direct sum of n copies of E. Conversely, suppose that a Banach space $\mathcal{X}$ is equipped with a sequence of norms $\|\cdot\|_n$ defined on $\mathbb{M}_n(\mathcal{X})$ satisfying the following two conditions:

(i) for any $m, n \geqslant 1$, for any $x \in \mathbb{M}_m(\mathcal{X})$, $y \in \mathbb{M}_n(\mathcal{X})$,

$$\left\| \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \right\|_{m+n} = \max\{\|x\|_m, \|y\|_n\};$$

(ii) for any $n \geqslant 1$, for any $x \in \mathbb{M}_n(\mathcal{X})$, $a \in \mathbb{M}_{m,n}$, $b \in \mathbb{M}_{n,m}$

$$\|axb\|_m \leqslant \|a\|_{\mathbb{B}(\ell_p^n,\ell_p^m)} \|x\|_n \|b\|_{\mathbb{B}(\ell_p^m,\ell_p^n)};$$

then there exist E a subspace of quotient of an L_p-space and a linear map $J : \mathcal{X} \to \mathbb{B}(E)$ such that for every $n \geqslant 1$, the map $\mathrm{id}_{\mathbb{M}_n} \otimes J : \mathbb{M}_n(\mathcal{X}) \to \mathbb{B}(\ell_p^n(E))$ is isometric. This abstract characterization of p-operator spaces is due to C. Le Merdy [29] (this is the p-analog of Ruan's classical characterization of operator spaces, corresponding to the case $p=2$, see e.g. [4], [13], [34], or [37]) and the sequence of norms $\|\cdot\|_n$ on $\mathbb{M}_n(\mathcal{X})$ satisfying conditions (i) and (ii) above is called a p-operator space structure on $\mathcal{X}$. Now let $\mathcal{X} \subset \mathbb{B}(E)$, $\mathcal{Y} \subset \mathbb{B}(F)$ be two p-operator spaces and $T : \mathcal{X} \to \mathcal{Y}$ be a linear map, then T is said to be p-completely bounded if

$$\sup_n \|\mathrm{id}_{\mathbb{M}_n} \otimes T\|_{\mathbb{B}(\ell_p^n(E),(\ell_p^n(F))} < \infty$$

and in this case this supremum is called the p-completely bounded norm (or p-cb norm in short) of T and is denoted $\|T\|_{\mathrm{pcb}}$. Also T is said to be p-completely isometric if $\mathrm{id}_{\mathbb{M}_n} \otimes T$ is isometric for every $n \geqslant 1$. We denote $CB_p(\mathcal{X}, \mathcal{Y})$ the vector space of all p-completely bounded maps from $\mathcal{X}$ into $\mathcal{Y}$. Let us review duality for p-operator spaces: if $\mathcal{X}$ is a p-operator space, then its dual space $\mathcal{X}^*$ can be equipped with a p-operator space structure via the identification $\mathbb{M}_n(\mathcal{X}^*) = CB_p(\mathcal{X}, \mathbb{M}_n)$, for every $n \geqslant 1$. Let $T : \mathcal{X} \to \mathcal{Y}$ be a linear map between p-operator spaces, then we always have for every $n \geqslant 1$,

$$\|\mathrm{id}_{\mathbb{M}_n} \otimes T^*\|_{\mathbb{B}(\mathbb{M}_n(\mathcal{Y}^*),\mathbb{M}_n(\mathcal{X}^*))} \leqslant \|\mathrm{id}_{\mathbb{M}_n} \otimes T\|_{\mathbb{B}(\mathbb{M}_n(\mathcal{X}),\mathbb{M}_n(\mathcal{Y}))}.$$

The reverse inequality is also always true in the case $p = 2$, but for $p \neq 2$, the reverse inequality does not hold in general (see Lemma 4.5 [9] for details), this is one important difference with operator space theory.

2.2. ON FIGÀ-TALAMANCA–HERZ ALGEBRAS. Let us review Figà-Talamanca–Herz algebras (for details the reader is referred to [11], [15], [17], or [19]) and their dual p-operator space structure. Let G be a locally compact group and $1 < p < \infty$, we denote $\lambda_p : G \to \mathbb{B}(L_p(G))$ the left regular representation on $L_p(G)$ defined by

$$(\lambda_p(t)f)(s) = f(t^{-1}s), \quad \text{for any } s, t \in G,\ f \in L_p(G).$$

Denote p' the conjugate index of p, we recall that $\mathcal{A}_p(G)$ is the set of all complex-valued functions defined on G admitting a representation of the form

$$u(t) = \sum_n \langle \lambda_p(t) f_n, g_n \rangle, \quad \text{for } t \in G,$$

where $f_n \in L_p(G)$, $g_n \in L_{p'}(G)$ satisfy $\sum_n \|f_n\|_p \|g_n\|_{p'} < \infty$ and the norm of u is defined by

$$\|u\|_{\mathcal{A}_p(G)} = \inf \Big\{ \sum_n \|f_n\|_p \|g_n\|_{p'} \Big\},$$

where the infimum runs over all its representatives. Then C.S. Herz showed that $\mathcal{A}_p(G)$ equipped with this norm and the pointwise operations is a commutative Banach algebra. Denote the p-pseudomeasures $\mathcal{PM}_p(G)$, the w^*-closed subalgebra of $\mathbb{B}(L_p(G))$ generated by $\lambda_p(G)$, then $\mathcal{A}_p(G)^* = \mathcal{PM}_p(G)$ isometrically, via the pairing: for $L \in \mathcal{PM}_p(G)$ and $u \in \mathcal{A}_p(G)$ as above,

$$\langle L, u \rangle = \sum_n \langle L(f_n), g_n \rangle.$$

Since $\mathcal{A}_p(G)^* = \mathcal{PM}_p(G) \subset \mathbb{B}(L_p(G))$, the algebra $\mathcal{A}_p(G)^*$ admits a canonical p-operator space structure. Hence, its dual space $\mathcal{A}_p(G)^{**}$ can be equipped with the p-operator space structure described in the previous paragraph. From the canonical inclusion $\mathcal{A}_p(G) \subset \mathcal{A}_p(G)^{**}$, the Figà-Talamanca–Herz algebra $\mathcal{A}_p(G)$ inherits this p-operator space structure, which is called the dual p-operator space structure of $\mathcal{A}_p(G)$ in Sections 7 and 5.1 of [9]. By Proposition 5.6 of [9], $\mathcal{A}_p(G)^* = \mathcal{PM}_p(G)$ p-completely isometrically. Therefore, with these p-operator space structures considered on Figà-Talamanca–Herz algebras and on the p-pseudomeasures, we have that for any linear map $T : \mathcal{A}_p(G) \to \mathcal{A}_p(H)$, the map $T^* : \mathcal{PM}_p(H) \to \mathcal{PM}_p(G)$ satisfies for every $n \geqslant 1$,

$$\|\mathrm{id}_{\mathbb{M}_n} \otimes T^*\| \leqslant \|\mathrm{id}_{\mathbb{M}_n} \otimes T\|.$$

We will only need this inequality for $n = 2$ in the proof of Theorem A.

3. DESCRIPTION OF p-COMPLETELY ISOMETRIC ISOMORPHISMS OF FIGÀ-TALAMANCA–HERZ ALGEBRAS

Let us recall more precisely Walter's result mentioned in the first paragraph of the first section: let $\Phi : \mathcal{A}_2(G) \to \mathcal{A}_2(H)$ be an isometric surjective algebra isomorphism between Fourier algebras, then there are $t \in G$ and a topological isomorphism or anti-isomorphism $\phi : H \to G$ such that

$$\Phi(u)(h) = u(t\phi(h)),$$

for any $u \in \mathcal{A}_2(G)$, $h \in H$.

Our purpose in this section is to prove an analogous result for Figà-Talamanca–Herz algebra in the category of p-operator spaces.

For $u \in \mathcal{A}_p(G)$, $t \in G$, the left translation of u by t is denoted $t \cdot u$, i.e. $t \cdot u(s) = u(t^{-1}s)$, $s \in G$.

LEMMA 3.1. *Let $t \in G$, then the left translation mapping $u \in \mathcal{A}_p(G) \mapsto t \cdot u \in \mathcal{A}_p(G)$ is p-completely isometric.*

Proof. Let $[u_{ij}]_{i,j\leqslant n} \in \mathbb{M}_n(\mathcal{A}_p(G))$, recall that

$$\mathbb{M}_n(\mathcal{A}_p(G)) \subset \mathbb{M}_n(\mathcal{A}_p(G)^{**}) = CB_p(\mathcal{A}_p(G)^*, \mathbb{M}_n),$$

therefore

$$\begin{aligned}
\|[t \cdot u_{ij}]_{i,j\leqslant n}\|_{\mathbb{M}_n(\mathcal{A}_p(G))} &= \sup_{m\in\mathbb{N}} \|\mathrm{id}_{\mathbb{M}_m} \otimes [t \cdot u_{ij}]\|_{\mathbb{B}(\mathbb{M}_m(\mathcal{A}_p(G)^*),\mathbb{M}_{mn})}\\
&= \sup_{m\in\mathbb{N}} \{\|[\langle t \cdot u_{ij}, x_{kl}\rangle]_{i,j,k,l}\|_{\mathbb{M}_{mn}},\ \|[x_{kl}]_{k,l\leqslant m}\|_{\mathbb{M}_m(\mathcal{A}_p(G)^*)} \leqslant 1\}\\
&= \sup_{m\in\mathbb{N}} \{\|[\langle u_{ij}, \lambda_p(t^{-1})x_{kl}\rangle]_{i,j,k,l}\|_{\mathbb{M}_{mn}},\ \|[x_{kl}]\|_{\mathbb{M}_m(\mathcal{A}_p(G)^*)} \leqslant 1\}\\
&= \sup_{m\in\mathbb{N}} \{\|[\langle u_{ij}, x_{kl}\rangle]_{i,j,k,l}\|_{\mathbb{M}_{mn}},\ \|[x_{kl}]\|_{\mathbb{M}_m(\mathcal{A}_p(G)^*)} \leqslant 1\}\\
&= \sup_{m\in\mathbb{N}} \|\mathrm{id}_{\mathbb{M}_m} \otimes [u_{ij}]\|_{\mathbb{B}(\mathbb{M}_m(\mathcal{A}_p(G)^*),\mathbb{M}_{mn})}\\
&= \|[u_{ij}]_{i,j\leqslant n}\|_{\mathbb{M}_n(\mathcal{A}_p(G))}.
\end{aligned}$$

The fourth equality is coming from the fact that the mapping $L \in \mathcal{A}_p(G)^* \mapsto \lambda_p(t^{-1})L \in \mathcal{A}_p(G)^*$ is a surjective p-complete isometry. ∎

In the next lemmata, $\mathbb{B}(L_p)$ denotes the algebra of all bounded operators on an L_p-space equipped with the operator norm, whose unit is denoted by 1. We compute the norm of a 2×2 matrix with coefficients in $\mathbb{B}(L_p)$ using the isometric identifications

$$\mathbb{M}_2(\mathbb{B}(L_p)) = \mathbb{B}(\ell_p^2(L_p)) = \mathbb{B}(L_p \oplus_p L_p).$$

For $1 < p < \infty$, we denote p' the conjugate index of p; let us recall the Clarkson's inequalities (see [5] or [3]): let $f, g \in L_p$,

(i) For $1 < p \leqslant 2$: $\|f+g\|_p^{p'} + \|f-g\|_p^{p'} \leqslant 2(\|f\|_p^p + \|g\|_p^p)^{p'-1}$.

(ii) For $2 \leqslant p < \infty$: $\|f+g\|_p^p + \|f-g\|_p^p \leqslant 2^{p-1}(\|f\|_p^p + \|g\|_p^p)$.

Moreover, each of these inequalities is an equality if and only if f and g have disjoint supports.

LEMMA 3.2. *Let $1 < p < \infty$ and $U, V \in \mathbb{B}(L_p)$ be two surjective isometries, then*

$$\left\| \begin{bmatrix} U & UV \\ -1 & V \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} = \left\| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} = 2^{\max(1/p, 1/p')}.$$

Proof. The first equality (right above) follows from the computation

$$\begin{bmatrix} U^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U & UV \\ -1 & V \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

in the Banach algebra $\mathbb{M}_2(\mathbb{B}(L_p))$ and the obvious fact that

$$\left\| \begin{bmatrix} U^{-1} & 0 \\ 0 & 1 \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} = \left\| \begin{bmatrix} 1 & 0 \\ 0 & V^{-1} \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} = 1.$$

Now let us denote

$$T = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

as an element of $\mathbb{B}(L_p \oplus_p L_p)$. In the case $2 \leqslant p < \infty$, the inequality

$$\left\| \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} \leqslant 2^{1/p'}$$

follows from the second Clarkson's inequality recalled above and the equality is achieved by choosing a column vector with coefficients (in L_p) having disjoint supports. The case $1 < p \leqslant 2$ is obtained by duality,

$$T^* = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} : L_{p'} \oplus_{p'} L_{p'} \longrightarrow L_{p'} \oplus_{p'} L_{p'}$$

and from the previous case, we get that $\|T^*\| = 2^{1/p}$, which gives the result. ∎

The next lemma is the key result to recover the group structure from an algebra isomorphism between Figà-Talamanca–Herz algebras.

LEMMA 3.3. *Let $1 < p < \infty$ and $U, V \in \mathbb{B}(L_p)$ be two surjective isometries, $X \in \mathbb{B}(L_p)$ and $c \geqslant 1$.*

(i) *For $1 < p \leqslant 2$: if*

$$\left\| \begin{bmatrix} U & X \\ -1 & V \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} \leqslant c2^{1/p},$$

then $\|X - UV\|_{\mathbb{B}(L_p)} \leqslant (4c^p - 2^p)^{1/p}$.

(ii) *For $2 \leqslant p < \infty$: if*

$$\left\| \begin{bmatrix} U & X \\ -1 & V \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p))} \leqslant c2^{1/p'},$$

then $\|X - UV\|_{\mathbb{B}(L_p)} \leqslant 2(c^p - 1)^{1/p}$.

Proof. Note first that

$$\begin{bmatrix} U^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U & X \\ -1 & V \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & V^{-1} \end{bmatrix} = \begin{bmatrix} 1 & U^{-1}XV^{-1} \\ -1 & 1 \end{bmatrix},$$

hence without loss of generality, we may assume that $U = V = 1$. For the case $2 \leqslant p < \infty$, let $f \in L_p$, then we can evaluate

$$\left\| \begin{bmatrix} 1 & X \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -f \\ f \end{bmatrix} \right\|_{L_p \oplus_p L_p} \leqslant c2^{1/p'} \left\| \begin{bmatrix} -f \\ f \end{bmatrix} \right\|_{L_p \oplus_p L_p}.$$

Raising to the power p, we get $\|X(f)-f\|_p^p + \|2f\|_p^p \leqslant c^p 2^{1+p/p'}\|f\|_p^p$, which gives the result for $2 \leqslant p < \infty$. The case $1 < p \leqslant 2$ is obtained by the same evaluation. ∎

The next result is probably well-known, we give a quick proof.

LEMMA 3.4. *Let G be a locally compact group, and $1 < p < \infty$; denote λ_p its left regular representation on $L_p(G)$. Let $g_j \in G$ be distinct and $c_j \in \mathbb{C}$ for $1 \leqslant j \leqslant n$. Then*

$$\Big\| \sum_{j=1}^{n} c_j \lambda_p(g_j) \Big\|_{\mathbb{B}(L_p(G))} \geqslant \Big(\sum_{j=1}^{n} |c_j|^p \Big)^{1/p}.$$

Proof. Let $U \subset G$ be a neighborhood of identity of finite measure such that $g_j U$, $j = 1, \dots, n$, are pairwise disjoint. Denote χ_U the characteristic function of U. Then $\lambda_{g_j}(\chi_U) = \chi_{g_j U}$ and

$$\begin{aligned} \Big\| \sum c_j \lambda_{g_j} \Big\|_{\mathbb{B}(L_p)}^p &\geqslant \frac{1}{\|\chi_U\|_p^p} \Big\| \sum c_j \lambda_{g_j}(\chi_U) \Big\|_p^p \geqslant \frac{1}{\|\chi_U\|_p^p} \Big\| \sum c_j \chi_{g_j U} \Big\|_p^p \\ &\geqslant \frac{1}{\|\chi_U\|_p^p} \sum |c_j|^p \|\chi_{g_j U}\|_p^p \geqslant \sum |c_j|^p \end{aligned}$$

which gives the result. ∎

In particular, if g and g' are distinct elements of G, then we have $\|\lambda_p(g) - \lambda_p(g')\|_{\mathbb{B}(L_p(G))} \geqslant 2^{1/p}$ and this gap is exactly the reason why we do not need to assume contractivity of the map $\mathrm{id}_{\mathbb{M}_2} \otimes \Phi$ and we can relax the norm condition in the next theorem.

We are now ready to prove the main result of this section.

THEOREM A. *Let G and H be two locally compact groups. Let $\Phi : \mathcal{A}_p(G) \to \mathcal{A}_p(H)$ be a surjective algebra isomorphism. If one of the following conditions hold:*

(i) *for $1 < p \leqslant 2$, if $\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\| < (2^{p-2} + 1/2)^{1/p}$,*

(ii) *for $2 \leqslant p < \infty$, if $\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\| < (2^{1-p} + 1)^{1/p}$,*

then there are $t \in G$ and a topological isomorphism $\phi : H \to G$ such that

$$\Phi(u)(h) = u(t\phi(h)),$$

for any $u \in \mathcal{A}_p(G)$, $h \in H$.

Consequently, Φ is actually p-completely isometric (i.e. for every $n \geqslant 1$, $\mathrm{id}_{\mathbb{M}_n} \otimes \Phi$ is isometric).

Proof. In this proof, we identify the elements of a group with their images via the left regular representation inside the p-pseudomeasures (the dual space of the associated Figà-Talamanca–Herz algebra), in other words, we consider

$$G \simeq \lambda_p(G) \subset \mathcal{PM}_p(G) \subset \mathbb{B}(L_p(G)).$$

It is well-known that, under this identification, the group coincides with the spectrum of the Figà-Talamanca–Herz algebra (the set of all non-zero multiplicative linear functionals defined on it), see [20]. Therefore, as Φ is a surjective algebra isomorphism, Φ^* sends the spectrum of $\mathcal{A}_p(H)$ onto the spectrum of $\mathcal{A}_p(G)$, thus the restriction of Φ^* on H induces a homeomorphism between H and G (not necessarily a group morphism). Let us denote $t = \Phi^*(e_H) \in G$ and define a surjective algebra isomorphism $\Psi : \mathcal{A}_p(G) \to \mathcal{A}_p(H)$ by

$$\Psi(u) = \Phi(t \cdot u), \quad u \in \mathcal{A}_p(G),$$

where $t \cdot u$ denotes the left translation, i.e. $(t \cdot u)(s) = u(t^{-1}s)$, for $s \in G$. Then $\Psi^* : \mathcal{PM}_p(H) \to \mathcal{PM}_p(G)$ is a surjective linear isomorphism such that $\Psi^*(1) = 1$,

$$\langle \Psi^*(e_H), u \rangle = \langle e_H, \Psi(u) \rangle = \langle e_H, \Phi(t \cdot u) \rangle = \langle t, t \cdot u \rangle = \langle e_G, u \rangle.$$

The left translation is completely isometric on $\mathcal{A}_p(G)$ by Lemma 3.1, hence

$$\|\mathrm{id}_{\mathbb{M}_2} \otimes \Psi^*\| = \|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi^*\| \leqslant \|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\|$$

(see the end of Section 2 for this last inequality). Since Ψ is a surjective algebra isomorphism, the restriction of Ψ^* to the spectrum of $\mathcal{A}_p(H)$ also induces a homeomorphism from H onto G, denoted $\phi : H \to G$ in the rest of this proof. Let us prove that ϕ is a group morphism. Let $h_1, h_2 \in H \subset \mathcal{PM}_p(H)$. We recall that $\mathbb{M}_2(\mathcal{PM}_p(H)) \subset \mathbb{M}_2(\mathbb{B}(L_p(H)))$ isometrically, idem for G. Therefore,

$$\begin{aligned} \left\| \begin{bmatrix} \phi(h_1) & \phi(h_1h_2) \\ -1 & \phi(h_2) \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p(G)))} &= \left\| \begin{bmatrix} \Psi^*(h_1) & \Psi^*(h_1h_2) \\ \Psi^*(-1) & \Psi^*(h_2) \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p(G)))} \\ &\leqslant \|\mathrm{id}_{\mathbb{M}_2} \otimes \Psi^*\| \left\| \begin{bmatrix} h_1 & h_1h_2 \\ -1 & h_2 \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p(H)))}. \end{aligned}$$

From now on, we suppose $2 \leqslant p < \infty$ and $\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\| < (2^{1-p}+1)^{1/p}$ (the proof of the case $1 < p \leqslant 2$ being similar with the corresponding bound). By Lemma 3.2

$$\left\| \begin{bmatrix} h_1 & h_1h_2 \\ -1 & h_2 \end{bmatrix} \right\|_{\mathbb{M}_2(\mathbb{B}(L_p(H)))} = 2^{1/p'},$$

hence we can apply Lemma 3.3 to obtain the following strict inequality

$$\begin{aligned} \|\phi(h_1h_2) - \phi(h_1)\phi(h_2)\|_{\mathbb{B}(L_p(G))} &\leqslant 2(\|\mathrm{id}_{\mathbb{M}_2} \otimes \Psi^*\|^p - 1)^{1/p} \\ &\leqslant 2(\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\|^p - 1)^{1/p} < 2^{1/p}. \end{aligned}$$

This strict inequality implies that $\phi(h_1h_2) = \phi(h_1)\phi(h_2)$ by Lemma 3.4. Finally ϕ is a group isomorphism and Φ has the announced form.

Obviously, the mapping $u \in \mathcal{A}_p(G) \mapsto u \circ \phi \in \mathcal{A}_p(H)$ is p-completely isometric, hence Φ is p-completely isometric by Lemma 3.1. ∎

REMARK 3.5. In Walter's result (mentioned at the beginning of this section), the map ϕ is a group isomorphism or anti-isomorphism, but in our theorem, the map ϕ is necessarily a group isomorphism. The fact that tensorization by 2×2

matrices rules out the anti-isomorphic case appears for instance in [27], where a 2-isometries between noncommutative L_p implies the existence of $*$-isomorphism between the underlying von Neumann algebras (whereas one just obtains a Jordan isomorphism in the isometric case).

For Fourier algebras (corresponding to the case $p = 2$), we obtain the following corollary.

COROLLARY 3.6. *Let G and H be locally compact groups. Let $\Phi : \mathcal{A}_2(G) \to \mathcal{A}_2(H)$ be a surjective algebra isomorphism. If $\|\mathrm{id}_{\mathbb{M}_2} \otimes \Phi\| < \sqrt{3/2}$, then there are $t \in G$ and a topological isomorphism $\phi : H \to G$ such that*

$$\Phi(u)(h) = u(t\phi(h)),$$

for any $u \in \mathcal{A}_2(G)$, $h \in H$.

Consequently, Φ is actually completely isometric.

REMARK 3.7. The two main ingredients of the proof of Theorem A are: identifying the spectrum of the Banach algebra with the underlying group and the fact that under this identification elements of the groups (inside the dual space of the algebra) form a uniformly discrete set (here two distinct elements are at distance $2^{1/p}$).

4. OTHER GAP RESULTS FOR FIGÀ-TALAMANCA–HERZ ALGEBRAS

4.1. PROOF OF THEOREM B. We recall here two basic notions of geometry of Banach spaces; for more details the reader is referred to [12], [16] or [30]. As usual, for a Banach space X, we denote its unit sphere $S_X = \{x \in X : \|x\| = 1\}$. The modulus of uniform smoothness is defined for $\tau > 0$ by

$$\rho_X(\tau) = \sup\Big\{\frac{\|x+z\|}{2} + \frac{\|x-z\|}{2} - 1 \,:\, x \in S_X,\, z \in X, \|z\| \leqslant \tau\Big\}$$

and X is said to be uniformly smooth if $\lim\limits_{\tau\to 0} \rho_X(\tau)/\tau = 0$. Recall that a Banach space X is uniformly convex if its modulus of uniform convexity

$$\delta_X(\varepsilon) = \inf\Big\{1 - \Big\|\frac{x+y}{2}\Big\|,\, x, y \in S_X,\, \|x-y\| \geqslant \varepsilon\Big\}$$

is strictly positive for any $\varepsilon \in (0, 2)$. It is well-known that X is uniformly smooth if and only if X^* is uniformly convex and in this case, we have the duality formula

$$\rho_X(\tau) = \sup\Big\{\frac{\tau\varepsilon}{2} - \delta_{X^*}(\varepsilon) \,:\, 0 \leqslant \varepsilon \leqslant 2\Big\}.$$

Moreover, if X is uniformly convex or uniformly smooth, then X is reflexive, hence in Theorem B, uniform smoothness and uniform convexity play symmetric roles. Recall that if a Banach space E is uniformly smooth (actually smoothness is sufficient) then for every $x \in E$, there exists (and we will denote) the unique $x^* \in S_{E^*}$ satisfying $\langle x^*, x\rangle = \|x\|$. Note that for any scalar $\lambda > 0$, $(\lambda x)^* = x^*$.

LEMMA 4.1. *Let E be uniformly smooth Banach space and $x \in E$. If $\varphi \in S_{E^*}$ satisfies $(1-2\delta_{E^*}(\varepsilon))\|x\| < \langle \varphi, x\rangle$, then $\|\varphi - x^*\| < \varepsilon$.*

Proof. Since $\langle \varphi + x^*, x\rangle > 2(1-\delta_{E^*}(\varepsilon))\|x\|$, we get that $1 - \left\|\frac{\varphi + x^*}{2}\right\| < \delta_{E^*}(\varepsilon)$, therefore $\|\varphi - x^*\| < \varepsilon$. ∎

For the next statement, we recall the notation $\|\alpha\|_\infty = \sup_{g\in G} \|\alpha(g)\|_{E^*}$, idem for β.

THEOREM B. *Let G be a locally compact group and $S \subset G$ non-empty. Assume that there exist a uniformly smooth Banach space E and mappings $\alpha : G \to E^*$, $\beta : G \to E$ such that the indicator function of S satisfies for any $g, h \in G$, $\chi_S(gh^{-1}) = \langle \alpha(g), \beta(h)\rangle$. Then, there exists $v_E > 1$ such that if $\|\alpha\|_\infty \|\beta\|_\infty < v_E$, then S is a coset.*

Proof. Let $M = \|\alpha\|_\infty \|\beta\|_\infty$. Note that $1 \leqslant M$, since S is non-empty. Suppose that $M < (1-2\delta_{E^*}(\varepsilon))^{-1}$ for some $\varepsilon \in (0,2)$ (we will determine a suitable ε later). Then we have the following fact: for any $s \in S$,

$$\| \|\alpha(s)\|^{-1}\alpha(s) - \beta(e)^*\| < \varepsilon.$$

The reason is that

$$(1-2\delta_{E^*}(\varepsilon))\| \|\alpha(s)\|\beta(e)\| \leqslant (1-2\delta_{E^*}(\varepsilon))M < \chi_S(s) < \langle \|\alpha(s)\|^{-1}\alpha(s), \|\alpha(s)\|\beta(e)\rangle,$$

hence we can apply the previous lemma to obtain that $\|\alpha(s)\|^{-1}\alpha(s)$ is close to $(\|\alpha(s)\|\beta(e))^*$, which is actually equal to $\beta(e)^*$.

Recall that S is a coset if and only if for any $r, s, t \in S$, the product $rs^{-1}t \in S$. Now fix $r, s, t \in S$,

$$\begin{aligned}
&\chi_S(rs^{-1}t)\\
&\quad = \langle \|\alpha(r)\|^{-1}\alpha(r), \|\alpha(r)\|\beta(t^{-1}s)\rangle = \langle \|\alpha(r)\|^{-1}\alpha(r) - \beta(e)^*, \|\alpha(r)\|\beta(t^{-1}s)\rangle\\
&\qquad + \langle \beta(e)^* - \|\alpha(s)\|^{-1}\alpha(s), \|\alpha(r)\|\beta(t^{-1}s)\rangle + \langle \|\alpha(s)\|^{-1}\alpha(s), \|\alpha(r)\|\beta(t^{-1}s)\rangle\\
&\quad = \langle \|\alpha(r)\|^{-1}\alpha(r) - \beta(e)^*, \|\alpha(r)\|\beta(t^{-1}s)\rangle\\
&\qquad + \langle \beta(e)^* - \|\alpha(s)\|^{-1}\alpha(s), \|\alpha(r)\|\beta(t^{-1}s)\rangle + \|\alpha(s)\|^{-1}\|\alpha(r)\|\chi_S(t).
\end{aligned}$$

By the fact mentioned at the beginning of the proof, the norm of each of the first two terms in the sum is smaller or equal to εM. For $i = r, s, t$, we have $1 = \chi_S(i) \leqslant \|\alpha(i)\|\|\beta(e)\| \leqslant M$, hence for the norm of the last term, we have $M^{-1} \leqslant \|\alpha(s)\|^{-1}\|\alpha(r)\| \leqslant M$. From all this together, we get

$$|\chi_S(rs^{-1}t) - \|\alpha(s)\|^{-1}\|\alpha(r)\|| < 2\varepsilon M,$$

and consequently $\chi_S(rs^{-1}t) \neq 0$ if the condition $2\varepsilon M < M^{-1}$ is satisfied. But $M < (1-2\delta_{E^*}(\varepsilon))^{-1}$ and the modulus of uniform convexity of E^* is a non-decreasing positive function on $(0,2)$ which vanishes at zero, hence there exists $\varepsilon_* \in (0,2)$ such that $1 < (1-2\delta_{E^*}(\varepsilon_*))^{-1} < 1/\sqrt{2\varepsilon_*}$. Therefore, the preceding condition can be satisfied and one can take $v_E = (1-2\delta_{E^*}(\varepsilon_*))^{-1}$. ∎

REMARK 4.2. For any complex Banach space X, $\delta_X(\varepsilon) \leqslant \varepsilon^2/4$ (see [33]), hence one can always take $\varepsilon_* > 2/5$ in the proof of the previous theorem. For applications below, we will just use the bound $(1 - 2\delta_{E^*}(2/5))^{-1}$ (which is thus not optimal).

4.2. APPLICATIONS OF THEOREM B. For $\pi : G \to \mathbb{B}(E)$ an isometric representation on a Banach space E, two spaces of coefficients associated to π, denoted $\mathcal{A}_\pi$ and $\mathcal{B}_\pi$, are studied in [8] (the second one can be thought as a space of multipliers of the first one). The next corollary is a straightforward application of our Theorem B and Theorem 2 [8] (we keep their notation).

COROLLARY 4.3. *Let G be a locally compact group, let **C** be one of the class derived from **Lp** or **U** and π an isometric representation of G on E a Banach space in **C**. There exists $v > 1$ such that for $S \subset G$, if $\chi_S \in \mathcal{B}_\pi$ and $\|\chi_S\|_{\mathcal{B}_\pi} < v$, then S is a coset.*

Before giving gap results for Figà-Talamanca–Herz algebras, for $1 < p < \infty$, we give estimation of δ_p the modulus of uniform convexity of L_p-spaces. By Clarkson's inequalities (recalled at the beginning of Section 3), we have

$$\delta_p(\varepsilon) \geqslant 1 - \Big(1 - \frac{\varepsilon^r}{2^r}\Big)^{1/r}, \quad r = \max\Big\{p, \frac{p}{p-1}\Big\}.$$

Actually, this is an equality in the case $2 \leqslant p < \infty$ and the case $1 < p < 2$ can be computed more precisely (see [18]).

For the next two corollaries, set

$$\iota_p = \frac{1}{2(1-5^{-r})^{1/r} - 1}, \quad r = \max\Big\{p, \frac{p}{p-1}\Big\}.$$

Note that $1 < \iota_p \leqslant (1 - 2\delta_p(2/5))^{-1}$.

The next corollary can be compared with Theorem 2.2 of [39] which describes ideals with approximate identity bounded by 1. Here, we prove the following self-improvement phenomenon.

For F a closed subset of G, we denote $I_p(F) = \{u \in \mathcal{A}_p(G) \ : \ u_{|F} = 0\}$.

COROLLARY 4.4. *Let G be a locally compact group, $1 < p < \infty$ and I be an ideal of $\mathcal{A}_p(G)$. The following are equivalent:*

(i) *I has an approximate identity bounded by 1,*

(ii) *I has an approximate identity bounded by ι_p,*

(iii) *there are H an open amenable subgroup of G and $s \in G$ such that $I = I_p(G \backslash sH)$.*

Proof. Following the notation of (i) implies (ii) in Theorem 2.2 of [39], the indicator function $\chi_{G\backslash F}$ can be written

$$\chi_{G\backslash F}(t) = \langle \pi(t)\xi, \eta\rangle,$$

where π is an isometric representation of G_{d} (i.e. G equipped with the discrete topology) on an $L_{p'}$-space, $\xi \in L_{p'}$, $\eta \in L_p$ and $\|\xi\|_{p'}\|\eta\|_p \leqslant \iota_p$. We can apply

Theorem B to obtain that $G\backslash F$ is a coset and the end of the proof follows Theorem 2.2 of [39]. ∎

The description of completely bounded homomorphisms of Fourier algebras of amenable groups in Theorem 3.7 of [23] has been adapted in [22], to Figà-Talamanca–Herz algebra. But it deals only with p-completely contractive homomorphisms (and not all p-completely bounded ones as in the Fourier algebra case); here we can improve the bound. As in the previous corollary, the proof is a modification of Theorem 12 of [22] using our Theorem B; we leave it to the reader.

In the next statement, the map $\phi : Y \to G$ is affine means that for any $x, y, z \in Y$, $\phi(xy^{-1}z) = \phi(x)\phi(y)^{-1}\phi(z)$.

COROLLARY 4.5. *Let G and H be two locally compact amenable groups. Let $\Phi : \mathcal{A}_p(G) \to \mathcal{A}_p(H)$ be a nonzero algebra homomorphism. If $\|\Phi\|_{\mathrm{pcb}} < \iota_p$, then there exist an open coset $Y \subset H$ and a proper affine map $\phi : Y \to G$ such that*

$$\Phi(u) = \begin{cases} u \circ \phi & \text{on } Y, \\ 0 & \text{off } Y, \end{cases}$$

for any $u \in \mathcal{A}_p(G)$.

Consequently Φ is actually p-completely contractive, i.e. $\|\Phi\|_{\mathrm{pcb}} \leqslant 1$.

Now let us show a gap result for multipliers of Figà-Talamanca–Herz algebras. A function $\varphi : G \to \mathbb{C}$ is a multiplier of $\mathcal{A}_p(G)$ if $m_\varphi : u \in \mathcal{A}_p(G) \mapsto \varphi u \in \mathcal{A}_p(G)$ is well-defined. Considering the p-operator space structure on $\mathcal{A}_p(G)$ (recalled in Section 2), we denote $M_{\mathrm{pcb}}\mathcal{A}_p(G)$ the vector space of all multipliers of $\mathcal{A}_p(G)$ which are p-completely bounded, it is equipped with norm defined by $\|\varphi\|_{\mathrm{pcb}} = \|m_\varphi\|_{\mathrm{pcb}}$. By Theorem 8.3 of [9], a p-completely bounded multiplier φ of $\mathcal{A}_p(G)$ is exactly of the form described in Theorem B with continuous maps α, β and E a subspace of a quotient of an L_p-space; moreover $\|\varphi\|_{\mathrm{pcb}} \leqslant C$ if and only if $\|\alpha\|_\infty \|\beta\|_\infty \leqslant C$.

By Chapter 11, Proposition 4 of [2], we have this general fact: if X is a subspace of a quotient of Y, then $\delta_X(\varepsilon) \geqslant \delta_Y(\varepsilon/2)$, for any $\varepsilon \in (0, 2)$. Hence now, for $1 < p < \infty$, denote

$$\mu_p = \frac{1}{2(1-10^{-r})^{1/r} - 1}, \quad r = \max\left\{p, \frac{p}{p-1}\right\}.$$

As above, $1 < \mu_p \leqslant (1 - 2\delta_p(1/5))^{-1}$.

COROLLARY 4.6. *Let $1 < p < \infty$ and a locally compact group G. For any nonzero idempotent multiplier $\chi \in M_{\mathrm{pcb}}\mathcal{A}_p(G)$, either $\|\chi\|_{\mathrm{pcb}} = 1$ or $\|\chi\|_{\mathrm{pcb}} \geqslant \mu_p$.*

Proof. As χ is idempotent, $\|\chi\|_{\mathrm{pcb}} \geqslant 1$. Suppose that $\|\chi\|_{\mathrm{pcb}} < \mu_p$, by Theorem B, we obtain that the support of χ is a coset of G, i.e. it is of the form sH for some fixed $s \in G$ and some subgroup H of G. Considering the cosets space G/H as a discrete space, we can use the associated quasi-left regular representation $\lambda_{p,H} : G \to \mathbb{B}(\ell_p(G/H))$ defined on the canonical basis vectors by $\lambda_p(g)(\delta_{tH}) = \delta_{gtH}$. Then one can check that $\chi(t) = \langle \lambda_p(t)(\delta_H), \delta_{sH} \rangle$, therefore $\|\chi\|_{\mathrm{pcb}} \leqslant 1$ by the description of p-completely bounded multipliers in the previous paragraph. ∎

REMARK 4.7. If G is discrete, then the multipliers can be represented on an L_p-space, instead of a subspace of a quotient of an L_p-space (see Theorems 5.11 and 8.2 in [36]), hence in this case, we can conclude that either $\|\chi\|_{\mathrm{pcb}} = 1$ or $\|\chi\|_{\mathrm{pcb}} \geqslant \iota_p$.

REMARK 4.8. This last corollary can be compared with [31] and Theorem 3.3 in [40]; it is proved that an idempotent completely bounded multiplier of a Fourier algebra has completely bounded norm 1 or greater than $2/\sqrt{3}$.

REMARK 4.9. Note that this gap phenomenon also occurs for idempotent Schur multipliers on Schatten class, see [1].

To finish this section, let us state a gap result concerning the amenability constant of Figà-Talamanca–Herz algebras. Recall that a Banach algebra $\mathcal{A}$ is amenable if it admits an approximate diagonal. An approximate diagonal for $\mathcal{A}$ is a bounded net (m_α) in the projective tensor product $\mathcal{A}\widehat{\otimes}\mathcal{A}$ satisfying both conditions: for any $a \in \mathcal{A}$, $am_\alpha - m_\alpha a \to 0$ and $\pi(m_\alpha)a \to a$ (here $\pi : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ denotes the multiplication map), see B.E. Johnson's paper [24] for details. Note that, by this last condition, necessarily $\sup_\alpha \|m_\alpha\| \geqslant 1$. Let $C \geqslant 1$; a Banach algebra $\mathcal{A}$ is said to be C-amenable if it admits an approximate diagonal (m_α) satisfying $\sup_\alpha \|m_\alpha\| \leqslant C$. The study of amenabilty constant of Fourier algebras started in [25], and then continued in [38].

The proof of the next corollary is a modification of the proof of Theorem 2.9 in [39], where a certain indicator function is represented on a Banach space obtained as the ultrapower of a space of the form $\ell_p(L_p(G, L_{p'}(G)))$. As taking ultrapower preserves the modulus of uniform convexity, the constant γ_p below can be computed with the modulus of uniform convexity of $\ell_p(L_p(G, L_{p'}(G)))$ using [10] or [3].

COROLLARY 4.10. *Let $1 < p < \infty$. There is $\gamma_p > 1$ such that for any locally compact group G, the following are equivalent:*

(i) *G is abelian;*

(ii) *for every $p \in (1,\infty)$, $\mathcal{A}_p(G)$ is 1-amenable;*

(iii) *there is $p \in (1,\infty)$ such that $\mathcal{A}_p(G)$ is 1-amenable;*

(iv) *there is $p \in (1,\infty)$ such that $\mathcal{A}_p(G)$ is γ_p-amenable.*

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JEAN ROYDOR, INSTITUT DE MATHÉMATIQUES DE BORDEAUX, UNIVERSITÉ DE BORDEAUX, 351 COURS DE LA LIBÉRATION, 33405 TALENCE CEDEX, FRANCE
E-mail address: jean.roydor@math.u-bordeaux1.fr

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