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COMPACT QUANTUM GROUPS WITH REPRESENTATIONS OF BOUNDED DEGREE

JACEK KRAJCZOK and PIOTR M. SOŁTAN

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ABSTRACT. We show that a compact quantum group whose all irreducible representations have dimension bounded by a fixed constant must be of Kac type, in other words, its Haar measure is a trace. The proof is based on establishing several facts concerning operators related to modular properties of the Haar measure. In particular we study the spectrum of these operators and the dimensions of some of their eigenspaces in relation to the quantum dimension of the corresponding irreducible representation.

KEYWORDS: *Compact quantum group, representation, dimension, quantum dimension.*

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1. INTRODUCTION

Let $\mathbb{G}$ be a compact quantum group. It is known that all irreducible representations of $\mathbb{G}$ are finite dimensional. We will say that $\mathbb{G}$ *has representations of bounded degree* if the dimensions (in algebraic literature called *degrees*) of all irreducible representations of $\mathbb{G}$ are bounded by some fixed constant. This property appeared recently in the paper [3] in connection with property (T) for discrete quantum groups, where $\mathbb{G}$ with representations of bounded degree was termed *low* (let us also mention that in a recent preprint [2] the main results of [3] have been established without the assumption of bounded degree of representations). The authors of [3] remark that compact quantum groups with representations of bounded degree exist and provide some examples ([3], Remark 1.6). In fact examples of such compact quantum groups have been plentiful in non-commutative geometry (see e.g. [5] or [1]).

Classical groups with representations of bounded degree have been studied already in [7]. It was proved by C.C. Moore in [8] that such groups must be *virtually abelian*, i.e. they have an abelian subgroup of finite index. The interest in establishing a quantum analog of this result lead first to a much more mundane question whether a compact quantum group with representations of bounded degree must necessarily be of Kac type (have tracial Haar measure, see Section 5 of [14] or [4]). Quite surprisingly this question turned out to be rather difficult to settle. In this paper we show that indeed a compact quantum group with representations of bounded degree is of Kac type. Compact quantum groups of Kac type are characterized in many ways e.g. in Proposition 1.7.9 of [9] (see also Theorem 3.4 of [4]). The task is carried out by exploiting a number of inequalities between various numerical invariants like the quantum dimension or dimensions of certain eigenspaces of operators naturally associated with representations of quantum groups which are not of Kac type.

All necessary definitions and basic theory of compact quantum groups can be found in the book [9]. We will also follow almost all notational conventions of that book. In particular we refer the reader to Chapter 1 of [9] for the definitions of

(•) contragredient representation U^{c} ([9], Definition 1.3.8);
(•) intertwiners $\mathrm{Mor}(U,V)$ and self-intertwiners $\mathrm{End}(U)$ ([9], Section 1.3);
(•) direct sums and tensor products of representations ([9], Section 1.3);
(•) conjugate representation $\overline{U}$ ([9], Definition 1.4.5).

The paper is organized as follows: in Section 2 we recall certain aspects of the theory of compact quantum groups and introduce some notation needed later on. In particular we fix notation concerning decomposition of a tensor product of representations into direct sum. Section 3 deals with spectral projections of operators ρ_α (see Section 2 and [9]). Theorem 3.3 in that section is an important technical tool for establishing our main result. The longest Section 4 focuses on the proof of our main theorem (Theorem 4.3) and finally in the appendix we briefly mention an algebraic characterization of the property of having representations of bounded degree.

2. NOTATION

Let $\mathbb{G}$ be a compact quantum group. For a finite dimensional unitary representation $U \in \mathrm{B}(\mathscr{H}_U) \otimes \mathrm{C}(\mathbb{G})$ we will use the symbol ρ_U for the unique positive invertible element of $\mathrm{Mor}(U, U^{\mathrm{cc}})$ such that $\mathrm{Tr}(\,\cdot\,\rho_U) = \mathrm{Tr}(\,\cdot\,{\rho_U}^{-1})$ on $\mathrm{End}(U)$ ([9], Proposition 1.4.4). We let $\mathrm{Irr}\,\mathbb{G}$ denote the set of equivalence classes of irreducible representations of $\mathbb{G}$. For each $\alpha \in \mathrm{Irr}\,\mathbb{G}$ we fix a unitary representative $U^\alpha \in \alpha$ on a Hilbert space $\mathscr{H}_\alpha$ of dimension n_α (i.e. $n_\alpha = \dim U^\alpha$). The choice of U^α is made so that defining $\overline{\alpha}$ as the class of $\overline{U}^\alpha$ we have $\overline{U}^\alpha = U^{\overline{\alpha}}$. We write ρ_α for ρ_{U^α} and we fix an orthonormal basis $\{\xi^\alpha_1, \dots, \xi^\alpha_{n_\alpha}\}$ of $\mathscr{H}_\alpha$ in which the matrix of ρ_α is diagonal with descending eigenvalues. We also write $\mathrm{Mor}(\alpha,\beta)$ instead of $\mathrm{Mor}(U^\alpha, U^\beta)$ etc.

Throughout the paper we will use the constant

$$\mathbf{N}_{\mathbb{G}} = \sup\{n_\alpha : \alpha \in \mathrm{Irr}\,\mathbb{G}\} \in \mathbb{N} \cup \{+\infty\}.$$

For $\beta, \gamma \in \operatorname{Irr}\mathbb{G}$ the tensor product, $U^{\beta} \mathbin{\bigcirc\!\!\!\!\top} U^{\gamma}$, is equivalent to a direct sum of $U^{\alpha_1}, \ldots, U^{\alpha_n}$ with $\alpha_1, \ldots, \alpha_n \in \operatorname{Irr}\mathbb{G}$ determined uniquely up to permutation. Given $\alpha \in \operatorname{Irr}\mathbb{G}$ we let $m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma)$ be the multiplicity of α in $\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma$, i.e. the number of times α appears in the sequence $(\alpha_1, \ldots, \alpha_n)$ (this can be zero). Thus we have

$$U^{\beta} \mathbin{\bigcirc\!\!\!\!\top} U^{\gamma} \approx \bigoplus_{\alpha \in \operatorname{Irr}\mathbb{G}} \bigoplus_{i=1}^{m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma)} U^{\alpha}. \tag{2.1}$$

Let

$$V(\beta, \gamma) : \bigoplus_{\alpha \in \operatorname{Irr}\mathbb{G}} \bigoplus_{i=1}^{m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma)} \mathscr{H}_{\alpha} \to \mathscr{H}_{\beta} \otimes \mathscr{H}_{\gamma}$$

be the unitary operator implementing equivalence (2.1). Then

$$V(\beta, \gamma) = \sum_{\alpha \in \operatorname{Irr}\mathbb{G}} \sum_{i=1}^{m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma)} V(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma, i)$$

where $V(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma, i) : \mathscr{H}_{\alpha} \to \mathscr{H}_{\beta} \otimes \mathscr{H}_{\gamma}$ are isometries with orthogonal ranges.

3. SPECTRAL PROJECTIONS OF ρ_U OPERATORS

For a finite dimensional unitary representation $U \in B(\mathscr{H}_U) \otimes C(\mathbb{G})$ of $\mathbb{G}$ and a number $t > 0$ let $\rho_U(t)$ denote the spectral projection of ρ_U corresponding to the subset $\{t\}$ of $\mathbb{R}_+$, i.e. $\rho_U(t) = \chi_{\{t\}}(\rho_U)$. Similarly let $\mathscr{H}_U(t)$ denote the range of the projection $\rho_U(t)$. When $U = U^{\alpha}$ for some $\alpha \in \operatorname{Irr}\mathbb{G}$ we will write $\rho_{\alpha}(t)$ and $\mathscr{H}_{\alpha}(t)$ as usual. The properties of these spectral projections are summarized in the next proposition.

PROPOSITION 3.1. *Let $U \in B(\mathscr{H}_U) \otimes C(\mathbb{G})$ and $V \in B(\mathscr{H}_V) \otimes C(\mathbb{G})$ be finite dimensional unitary representations of $\mathbb{G}$. Then for any $t > 0$ we have:*

(i) *if $T \in \operatorname{Mor}(U, V)$ then $T\rho_U(t) = \rho_V(t)T$;*

(ii) *$\rho_{U \oplus V}(t) = \rho_U(t) \oplus \rho_V(t) \in B(\mathscr{H}_U) \oplus B(\mathscr{H}_V) \subset B(\mathscr{H}_U \oplus \mathscr{H}_V)$;*

(iii) *$\rho_{U \mathbin{\bigcirc\!\!\!\!\top} V}(t) = \sum\limits_{t'>0} \rho_U(t') \otimes \rho_V(t/t') \in B(\mathscr{H}_U \otimes \mathscr{H}_V)$;*

(iv) *$\rho_{\overline{U}}(t) = \rho_U(t^{-1})^{\top}$.*

Proof. Let $\{f_z\}_{z \in \mathbb{C}}$ be the family of Woronowicz characters of $\mathbb{G}$ ([14], Theorem 5.6, [9], Definition 1.7.1). Applying $(\mathrm{id} \otimes f_n)$ with $n \in \mathbb{N}$ to both sides of

$$(T \otimes 1)U = V(T \otimes 1)$$

we obtain

$$\sum_{t>0} T t^n \rho_U(t) = T\rho_U^n = \rho_V^n T = \sum_{t>0} t^n \rho_V(t) T, \quad n \in \mathbb{N}$$

which implies

$$T\rho_U(t) = \rho_V(t)T, \quad t > 0.$$

Points (ii), (iii) and (iv) follow from the equalities

$$\rho_{U\oplus V} = \rho_U \oplus \rho_V, \quad \rho_{U \mathbin{\bigcirc\!\!\!\!\top} V} = \rho_U \otimes \rho_V \quad \text{and} \quad \rho_{\overline{U}} = ({\rho_U}^{-1})^\top \tag{3.1}$$

(see Section 1.4 of [9]). ∎

PROPOSITION 3.2. *For any $\alpha, \beta, \gamma \in \operatorname{Irr}\mathbb{G}$ we have*

$$m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma) = m(\beta, \alpha \mathbin{\bigcirc\!\!\!\!\top} \overline{\gamma}) = m(\gamma, \overline{\beta} \mathbin{\bigcirc\!\!\!\!\top} \alpha).$$

Proof. Using Theorem 2.2.6 of [9] we obtain:

$$\begin{aligned} m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma) &= \dim \operatorname{Mor}(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma) = \dim \operatorname{Mor}(\alpha \mathbin{\bigcirc\!\!\!\!\top} \overline{\gamma}, \beta) \\ &= \dim \operatorname{Mor}(\beta, \alpha \mathbin{\bigcirc\!\!\!\!\top} \overline{\gamma}) = m(\beta, \alpha \mathbin{\bigcirc\!\!\!\!\top} \overline{\gamma}), \\ m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma) &= \dim \operatorname{Mor}(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma) = \dim \operatorname{Mor}(\overline{\beta} \mathbin{\bigcirc\!\!\!\!\top} \alpha, \gamma) \\ &= \dim \operatorname{Mor}(\gamma, \overline{\beta} \mathbin{\bigcirc\!\!\!\!\top} \alpha) = m(\gamma, \overline{\beta} \mathbin{\bigcirc\!\!\!\!\top} \alpha). \quad ∎ \end{aligned}$$

The next theorem provides the most important technical tools to be used in the proof of our main result in Section 4.

THEOREM 3.3. *For any $\alpha, \beta \in \operatorname{Irr}\mathbb{G}$ and $s, t > 0$ we have*

$$\sum_{\gamma \in \operatorname{Irr}\mathbb{G}} \sum_{i=1}^{m(\alpha, \gamma \mathbin{\bigcirc\!\!\!\!\top} \beta)} d_\gamma V(\alpha, \gamma \mathbin{\bigcirc\!\!\!\!\top} \beta, i)^* (\rho_\gamma(s) \otimes \rho_\beta(t)) V(\alpha, \gamma \mathbin{\bigcirc\!\!\!\!\top} \beta, i) = \frac{d_\alpha}{t} (\dim \mathscr{H}_\beta(t))\ \rho_\alpha(st), \tag{3.2a}$$

$$\sum_{\gamma \in \operatorname{Irr}\mathbb{G}} \sum_{i=1}^{m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma)} d_\gamma V(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma, i)^* (\rho_\beta(t) \otimes \rho_\gamma(s)) V(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma, i) = d_\alpha t (\dim \mathscr{H}_\beta(t))\ \rho_\alpha(st). \tag{3.2b}$$

REMARK 3.4. When $\mathbb{G}$ is of Kac type equations (3.2a), (3.2b) reduce to

$$\begin{aligned} \delta_{s,1}\delta_{t,1} \sum_{\gamma \in \operatorname{Irr}\mathbb{G}} m(\alpha, \gamma \mathbin{\bigcirc\!\!\!\!\top} \beta) n_\gamma &= \delta_{s,1}\delta_{t,1} n_\alpha n_\beta, \\ \delta_{s,1}\delta_{t,1} \sum_{\gamma \in \operatorname{Irr}\mathbb{G}} m(\alpha, \beta \mathbin{\bigcirc\!\!\!\!\top} \gamma) n_\gamma &= \delta_{s,1}\delta_{t,1} n_\alpha n_\beta, \end{aligned}$$

which can be seen as an obvious equality of dimensions (see Proposition 3.2)

$$\dim \Big(\bigoplus\nolimits_{\gamma \in \operatorname{Irr}\mathbb{G}} m(\gamma, \alpha \mathbin{\bigcirc\!\!\!\!\top} \overline{\beta}) \cdot U^\gamma \Big) = \dim(\alpha \mathbin{\bigcirc\!\!\!\!\top} \beta) = \dim \Big(\bigoplus\nolimits_{\gamma \in \operatorname{Irr}\mathbb{G}} m(\gamma, \overline{\beta} \mathbin{\bigcirc\!\!\!\!\top} \alpha) \cdot U^\gamma \Big).$$

Proof of Theorem 3.3. We will use the notation and results of Chapter 2 in [9]. For any $\beta \in \operatorname{Irr}\mathbb{G}$ we let $(R_\beta, \overline{R}_\beta)$ be the *standard solutions* of *conjugate equations* ([9], Section 2.2) as given in Example 2.2.13 of [9]. In particular, denoting by $\mathbf{1}$ the trivial representation, we have $R_\beta \in \operatorname{Mor}(\mathbf{1}, \overline{\beta} \mathbin{\bigcirc\!\!\!\!\top} \beta)$ and $\overline{R}_\beta \in \operatorname{Mor}(\mathbf{1}, \beta \mathbin{\bigcirc\!\!\!\!\top} \overline{\beta})$ and

$$({\overline{R}_\beta}^* \otimes 1_\beta)(1_\beta \otimes R_\beta) = 1_\beta, \quad ({R_\beta}^* \otimes 1_{\overline{\beta}})(1_{\overline{\beta}} \otimes \overline{R}_\beta) = 1_{\overline{\beta}},$$

where 1_β and $1_{\overline{\beta}}$ are the identities of $\mathrm{B}(\mathscr{H}_\beta)$ and $\mathrm{B}(\mathscr{H}_{\overline{\beta}})$ (by taking adjoints we can also rewrite e.g. the first equation as $(1_\beta \otimes {R_\beta}^*)(\overline{R}_\beta \otimes 1_\beta) = 1_\beta$).

By Proposition 3.1 we have

$$R_\beta = \rho_{\overline{\beta}\, \textcircled{\scriptsize T}\, \beta}(1) R_\beta = \sum_{t'>0} (\rho_{\overline{\beta}}(t') \otimes \rho_\beta({t'}^{-1})) R_\beta,$$

so for any $t > 0$

$$\begin{aligned}(\rho_{\overline{\beta}}(t) \otimes 1_\beta) R_\beta &= \sum_{t'>0} (\rho_{\overline{\beta}}(t)\rho_{\overline{\beta}}(t') \otimes \rho_\beta({t'}^{-1})) R_\beta = (\rho_{\overline{\beta}}(t) \otimes \rho_\beta(t^{-1})) R_\beta \\ &= \sum_{t'>0} (\rho_{\overline{\beta}}(t') \otimes \rho_\beta(t^{-1})\rho_\beta({t'}^{-1})) R_\beta = (1_{\overline{\beta}} \otimes \rho_\beta(t^{-1})) R_\beta.\end{aligned} \tag{3.3}$$

For any $\alpha, \beta, \gamma \in \operatorname{Irr} \mathbb{G}$ and $T \in \operatorname{Mor}(\alpha, \gamma \textcircled{\scriptsize T} \beta)$ define $\widetilde{T} = (T^* \otimes 1_{\overline{\beta}})(1_\gamma \otimes \overline{R}_\beta)$. Then $\widetilde{T} \in \operatorname{Mor}(\gamma, \alpha \textcircled{\scriptsize T} \overline{\beta})$ (cf. suggested proof of Theorem 2.2.6 of [9]). Moreover, for $S, T \in \operatorname{Mor}(\alpha, \gamma \textcircled{\scriptsize T} \beta)$ we have $\widetilde{S}^* \widetilde{T} \in \operatorname{End}(\gamma)$, so by irreducibility $\widetilde{S}^* \widetilde{T} = \lambda 1_\gamma$ for some $\lambda \in \mathbb{C}$.

Next we will use the so called *categorical traces* ([9], Theorem 2.2.16) which we denote by Tr_ζ for $\zeta \in \operatorname{Irr} \mathbb{G}$ extended to $\mathrm{B}(\mathscr{H}_\zeta)$ (cf. Remark 2.2.17 and Section 1.4 of [9]):

$$\mathrm{Tr}_\zeta(Z) = {\overline{R}_\zeta}^* (Z \otimes 1_{\overline{\zeta}}) \overline{R}_\zeta, \quad Z \in \mathrm{B}(\mathscr{H}_\zeta).$$

We have

$$\widetilde{S}^* \widetilde{T} = (1_\gamma \otimes {\overline{R}_\beta}^*)(ST^* \otimes 1_{\overline{\beta}})(1_\gamma \otimes \overline{R}_\beta) = (\mathrm{id} \otimes \mathrm{Tr}_\beta)(ST^*), \tag{3.4}$$

and hence

$$\lambda \, \mathrm{Tr}_\gamma(1_\gamma) = (\mathrm{Tr}_\gamma \otimes \mathrm{Tr}_\beta)(ST^*) = \mathrm{Tr}_\alpha(T^* S) \tag{3.5}$$

(the second equality follows from the paragraph after Remark 2.2.17 of [9]). It also follows from (3.4) that if the ranges of T and S are orthogonal ($S^* T = 0$) then so are the ranges of $\widetilde{S}$ and $\widetilde{T}$. Furthermore, since $\mathrm{Tr}_\zeta(1_\zeta) = d_\zeta$ for all ζ, if T is an isometry, putting $S = T$ in (3.5) we obtain

$$\lambda = \frac{d_\alpha}{d_\gamma}.$$

In particular $\sqrt{\frac{d_\gamma}{d_\alpha}}\, \widetilde{T}$ is also an isometry.

Applying this to isometries

$$V(\alpha, \gamma \textcircled{\scriptsize T} \beta, i) \in \operatorname{Mor}(\alpha, \gamma \textcircled{\scriptsize T} \beta), \quad i \in \{1, \ldots, m(\alpha, \gamma \textcircled{\scriptsize T} \beta)\}$$

with orthogonal ranges spanning the whole space $\mathscr{H}_{\gamma \textcircled{\scriptsize T} \beta}$ we obtain operators

$$\widetilde{V}(\alpha, \gamma \textcircled{\scriptsize T} \beta, i) \in \operatorname{Mor}(\gamma, \alpha \textcircled{\scriptsize T} \overline{\beta}), \quad i \in \{1, \ldots, m(\alpha, \gamma \textcircled{\scriptsize T} \beta)\}$$

satisfying

$$\sum_{\gamma\in\operatorname{Irr}\mathbb{G}}\sum_{i=1}^{m(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta)} d_\gamma \widetilde{V}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)\widetilde{V}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)^* = d_\alpha 1_\alpha\otimes 1_{\overline{\beta}}.$$

Now for $s>0$ let $X(s)=d_\alpha\rho_{\alpha\mathbin{\bigcirc\!\!\!\!\top}\overline{\beta}}(s)$. Then by Proposition 3.1(i)

$$\begin{aligned}
X(s) &= \Big(\sum_{\gamma\in\operatorname{Irr}\mathbb{G}}\sum_{i=1}^{m(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta)} d_\gamma \widetilde{V}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)\widetilde{V}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)^*\Big)\rho_{\alpha\mathbin{\bigcirc\!\!\!\!\top}\overline{\beta}}(s)\\
&= \sum_{\gamma\in\operatorname{Irr}\mathbb{G}}\sum_{i=1}^{m(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta)} d_\gamma \widetilde{V}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)\rho_\gamma(s)\widetilde{V}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)^*\\
&= \sum_{\gamma\in\operatorname{Irr}\mathbb{G}}\sum_{i=1}^{m(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta)} d_\gamma (V(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)^*\otimes 1_{\overline{\beta}})(\rho_\gamma(s)\otimes \overline{R}_\beta\overline{R}_\beta{}^*)(V(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)\otimes 1_{\overline{\beta}}).
\end{aligned}$$

Let θ_t be the functional on $\mathrm{B}(\mathscr{H}_{\overline{\beta}})$ given by

$$\theta_t(W)=R_\beta{}^*(W\rho_{\overline{\beta}}(t^{-1})\otimes 1_\beta)R_\beta,\quad W\in\mathrm{B}(\mathscr{H}_{\overline{\beta}})$$

and let us compute $(\mathrm{id}\otimes\theta_t)(X(s))$. Using the conjugate equations and (3.3) we obtain the following equality of operators on $\mathscr{H}_\beta$:

$$\begin{aligned}
&(1_\beta\otimes R_\beta{}^*)(\overline{R}_\beta\overline{R}_\beta{}^*\otimes 1_\beta)(1_\beta\otimes\rho_{\overline{\beta}}(t^{-1})\otimes 1_\beta)(1_\beta\otimes R_\beta)\\
&\qquad\qquad = (\overline{R}_\beta{}^*\otimes 1_\beta)(1_\beta\otimes 1_{\overline{\beta}}\otimes\rho_\beta(t))(1_\beta\otimes R_\beta)=\rho_\beta(t)
\end{aligned}$$

which shows that

$$(\mathrm{id}\otimes\theta_t)(X(s))=\sum_{\gamma\in\operatorname{Irr}\mathbb{G}}\sum_{i=1}^{m(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta)} d_\gamma V(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i)^*(\rho_\gamma(s)\otimes\rho_\beta(t))V(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta,i). \tag{3.6}$$

On the other hand by Proposition 3.1(iii)

$$\begin{aligned}
(\mathrm{id}\otimes\theta_t)(X(s)) &= d_\alpha(\mathrm{id}\otimes\theta_t)\sum_{t'>0}\rho_\alpha(t')\otimes\rho_{\overline{\beta}}(s/t')\\
&= d_\alpha(\mathrm{id}\otimes\theta_t)\sum_{u>0}\rho_\alpha(su)\otimes\rho_{\overline{\beta}}(u^{-1})=\mu d_\alpha\rho_\alpha(st),
\end{aligned}\tag{3.7}$$

where $\mu=R_\beta{}^*(\rho_{\overline{\beta}}(t^{-1})\otimes 1_\beta)R_\beta=R_\beta{}^*(1_{\overline{\beta}}\otimes\rho_\beta(t))R_\beta$ (again by (3.3)). This can be computed using the explicit expression

$$R_\beta=\sum_{b=1}^{n_\beta}\overline{\xi}_b^\beta\otimes\rho_\beta^{-1/2}\xi_b^\beta$$

from Example 2.2.3 of [9] to obtain $\mu=t^{-1}\dim\mathscr{H}_\beta$. Combining this with (3.6) and (3.7) we obtain (3.2a).

Formula (3.2b) can be proved analogously with the operation

$$\mathrm{Mor}(\alpha,\gamma\mathbin{\bigcirc\!\!\!\!\top}\beta)\ni T\longmapsto\widetilde{T}\in\mathrm{Mor}(\gamma,\alpha\mathbin{\bigcirc\!\!\!\!\top}\overline{\beta})$$

replaced by

$$\operatorname{Mor}(\alpha,\gamma \mathbin{\bigcirc\!\!\!\!\top} \beta) \ni T \longmapsto \widehat{T} \in \operatorname{Mor}(\beta,\overline{\gamma} \mathbin{\bigcirc\!\!\!\!\top} \alpha)$$

defined by $\widehat{T} = (1_{\overline{\gamma}} \otimes T^*)(R_\gamma \otimes 1_\beta)$. ∎

REMARK 3.5. Theorem 3.3 can also be proved without invoking the theory of C^*-tensor categories (in other words the material of Chapter 2 in [9]). The proof is then based on the following formula for comultiplication on the dual $\widehat{\mathbb{G}}$ of $\mathbb{G}$ ([11], Section 3):

$$\Delta_{\widehat{\mathbb{G}}}(e^\alpha_{a,a'}) = \sum_{\beta,\gamma\in\operatorname{Irr}\mathbb{G}} \sum_{i=1}^{m(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma)} \sum_{b,b',c,c'} V(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma,i)^{b,c}_a (e^\gamma_{c,c'} \otimes e^\beta_{b,b'}) \overline{V(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma,i)^{b',c'}_{a'}},$$

where for each $\alpha \in \operatorname{Irr}\mathbb{G}$ and any $a,a' \in \{1,\dots,n_\alpha\}$ the matrix unit $e^\alpha_{a,a'}$ is given by

$$e^\alpha_{a,a'} = |\xi^\alpha_{a'}\rangle\langle\xi^\alpha_{a'}|$$

and the numbers $V(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma,i)^{b,c}_a$ are matrix elements of the isometric operators $V(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma,i)$:

$$V(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma,i) = \sum_{a,b,c} V(\alpha,\beta \mathbin{\bigcirc\!\!\!\!\top} \gamma,i)^{b,c}_a (|\xi^\beta_b\rangle \otimes |\xi^\gamma_c\rangle)\langle\xi^\alpha_a|.$$

4. BOUNDED DEGREE OF REPRESENTATIONS IMPLIES KAC TYPE

Before proceeding with our main result (Theorem 4.3) let us introduce the following useful notation. For a finite dimensional unitary representation $U \in \mathrm{B}(\mathscr{H}_U) \otimes \mathrm{C}(\mathbb{G})$ we will write $\Gamma(U)$ for the maximal eigenvalue of ρ_U which is also equal to the operator norm of ρ_U, i.e. $\Gamma(U) = \|\rho_U\|$. Let us also denote by D_U the vector space dimension of $\mathscr{H}_U(\Gamma(U))$. Whenever $U = U^\alpha$ for $\alpha \in \operatorname{Irr}\mathbb{G}$ we will write $\Gamma(\alpha)$ and D_α instead of $\Gamma(U^\alpha)$ and D_{U^α} ($\Gamma(U)$ and D_U depend only on equivalence class of U). The following proposition describing some properties of the map $U \mapsto \Gamma(U)$ is an immediate consequence of (3.1).

PROPOSITION 4.1. *Let $U \in \mathrm{B}(\mathscr{H}_U) \otimes \mathrm{C}(\mathbb{G})$ and $V \in \mathrm{B}(\mathscr{H}_V) \otimes \mathrm{C}(\mathbb{G})$ be finite dimensional unitary representations of $\mathbb{G}$. We have:*
(i) $\Gamma(U \oplus V) = \max\{\Gamma(U),\Gamma(V)\}$;
(ii) $\Gamma(U \mathbin{\bigcirc\!\!\!\!\top} V) = \Gamma(U)\Gamma(V)$.

The next result will be needed in the proof of Theorem 4.3. In what follows, for $\alpha,\beta,\gamma \in \operatorname{Irr}\mathbb{G}$, we will write $\gamma \preccurlyeq \alpha \mathbin{\bigcirc\!\!\!\!\top} \beta$ if $m(\gamma,\alpha \mathbin{\bigcirc\!\!\!\!\top} \beta) \neq 0$.

PROPOSITION 4.2. *Let $\alpha,\beta,\gamma \in \operatorname{Irr}\mathbb{G}$ be such that $\gamma \preccurlyeq \alpha \mathbin{\bigcirc\!\!\!\!\top} \beta$, $\Gamma(\gamma)=\Gamma(\alpha)\Gamma(\beta)$ and*

$$\frac{d_\gamma}{D_\gamma} = \max\Big\{\frac{d_{\gamma'}}{D_{\gamma'}} : \gamma' \in \operatorname{Irr}\mathbb{G},\ \gamma' \preccurlyeq \alpha \mathbin{\bigcirc\!\!\!\!\top} \beta,\ \Gamma(\gamma') = \Gamma(\alpha)\Gamma(\beta)\Big\}.$$

Then

$$1 \leqslant \frac{d_\gamma D_\alpha}{d_\alpha \Gamma(\beta) D_\gamma} = \frac{d_\gamma}{\Gamma(\gamma) D_\gamma} \frac{\Gamma(\alpha) D_\alpha}{d_\alpha}. \tag{4.1}$$

Let us note that γ as in Proposition 4.2 always exists. Indeed, the representation $U^\alpha \operatorname{\bigcirc\!\!\!\!\top} U^\beta$ is equivalent to $U^{\gamma_1} \oplus \cdots \oplus U^{\gamma_n}$ for some $\gamma_1, \ldots, \gamma_n \in \operatorname{Irr}\mathbb{G}$ (possibly with repetitions). By Proposition 4.1 we have

$$\Gamma(\alpha)\Gamma(\beta) = \Gamma(\alpha \operatorname{\bigcirc\!\!\!\!\top} \beta) = \max\{\Gamma(\gamma_1), \ldots, \Gamma(\gamma_n)\},$$

so there must exist $\gamma \in \operatorname{Irr}\mathbb{G}$ such that $\gamma \preccurlyeq \alpha \operatorname{\bigcirc\!\!\!\!\top} \beta$ and $\Gamma(\gamma) = \Gamma(\alpha)\Gamma(\beta)$.

Proof of Proposition 4.2. The first equality of Theorem 3.3 and the fact that $\rho_{\overline{\beta}} = (\rho_\beta{}^{-1})^\top$ imply

$$\begin{aligned}
&\sum_{\gamma' \in \operatorname{Irr}\mathbb{G}} \sum_{i=1}^{m(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta})} d_{\gamma'} V(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta}, i)^* (\rho_{\gamma'}(\Gamma(\alpha)\Gamma(\beta)) \otimes \rho_{\overline{\beta}}(\Gamma(\beta)^{-1})) V(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta}, i) \\
&\qquad = d_\alpha \Gamma(\beta) (\dim \mathscr{H}_{\overline{\beta}}(\Gamma(\beta)^{-1})) \rho_\alpha(\Gamma(\alpha)) \\
&\qquad = d_\alpha \Gamma(\beta) (\dim \mathscr{H}_\beta(\Gamma(\beta))) \rho_\alpha(\Gamma(\alpha)) = d_\alpha \Gamma(\beta) D_\beta\, \rho_\alpha(\Gamma(\alpha)).
\end{aligned} \tag{4.2}$$

Taking norm of both sides of (4.2) and using Propositions 3.1, 3.2, 4.1 we get

$$\begin{aligned}
d_\alpha \Gamma(\beta) D_\beta &= \Big\| \sum_{\gamma' \in \operatorname{Irr}\mathbb{G}} \sum_{i=1}^{m(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta})} d_{\gamma'} V(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta}, i)^* \\
&\qquad\qquad (\rho_{\gamma'}(\Gamma(\alpha)\Gamma(\beta)) \otimes \rho_{\overline{\beta}}(\Gamma(\beta)^{-1})) V(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta}, i) \Big\| \\
&\leqslant \sum_{\gamma' \in \operatorname{Irr}\mathbb{G}} m(\alpha, \gamma' \operatorname{\bigcirc\!\!\!\!\top} \overline{\beta}) d_{\gamma'} \|\rho_{\gamma'}(\Gamma(\alpha)\Gamma(\beta))\| \\
&= \sum_{\gamma' \in \operatorname{Irr}\mathbb{G}} m(\gamma', \alpha \operatorname{\bigcirc\!\!\!\!\top} \beta) d_{\gamma'} \|\rho_{\gamma'}(\Gamma(\alpha)\Gamma(\beta))\| \\
&= \sum_{\substack{\gamma' \in \operatorname{Irr}\mathbb{G}: \\ \Gamma(\gamma') = \Gamma(\alpha)\Gamma(\beta)}} m(\gamma', \alpha \operatorname{\bigcirc\!\!\!\!\top} \beta) d_{\gamma'} = \sum_{\substack{\gamma' \in \operatorname{Irr}\mathbb{G}: \\ \Gamma(\gamma') = \Gamma(\alpha)\Gamma(\beta)}} m(\gamma', \alpha \operatorname{\bigcirc\!\!\!\!\top} \beta) \frac{d_{\gamma'}}{D_{\gamma'}} D_{\gamma'} \\
&\leqslant \frac{d_\gamma}{D_\gamma} \sum_{\substack{\gamma' \in \operatorname{Irr}\mathbb{G}: \\ \Gamma(\gamma') = \Gamma(\alpha)\Gamma(\beta)}} m(\gamma', \alpha \operatorname{\bigcirc\!\!\!\!\top} \beta) D_{\gamma'} = \frac{d_\gamma}{D_\gamma} D_\alpha D_\beta,
\end{aligned}$$

which yields (4.1). ∎

Now we are able to prove the main theorem of the paper.

THEOREM 4.3. *Assume that* $\mathbf{N}_\mathbb{G} < +\infty$. *Then* $\mathbb{G}$ *is of Kac type.*

The remainder of this section (apart from Corollary 4.4) will be devoted to the proof of Theorem 4.3.

Proof. Case $\mathbf{N}_{\mathbb{G}} = 1$ is trivial, hence assume that $\mathbf{N}_{\mathbb{G}} \geqslant 2$. Assume by contradiction that $\mathbb{G}$ is not of Kac type. Then there exists $\alpha \in \operatorname{Irr}\mathbb{G}$ such that $\Gamma(\alpha) > 1$.

We now proceed to choose a sequence $(\alpha_k)_{k\in\mathbb{N}}$ of elements of $\operatorname{Irr}\mathbb{G}$ such that $\alpha_1 = \alpha$ (as above),

(i) $\alpha_{k+1} \preccurlyeq \alpha_k \mathbin{\bar{\oplus}} \alpha_k$,
(ii) $\Gamma(\alpha_{k+1}) = \Gamma(\alpha_k)^2$,

and

(iii) $\frac{d_{\alpha_{k+1}}}{D_{\alpha_{k+1}}} = \max\left\{\frac{d_\gamma}{D_\gamma} : \gamma \in \operatorname{Irr}\mathbb{G},\ \gamma \preccurlyeq \alpha_k \mathbin{\bar{\oplus}} \alpha_k,\ \Gamma(\gamma) = \Gamma(\alpha_k)^2\right\}$,

for all $k \in \mathbb{N}$.

Property (ii) implies $\Gamma(\alpha_k) = \Gamma(\alpha)^{2^{(k-1)}}$ for every $k \in \mathbb{N}$.

We will continue to refine our sequence by choosing appropriate subsequences in order to finally arrive at a contradiction. Let us note that by Proposition 4.2, the sequence of real numbers

$$\Big(\Gamma(\alpha_k)\frac{D_{\alpha_k}}{d_{\alpha_k}}\Big)_{k\in\mathbb{N}}$$

is non-increasing. For each k the matrix of the operator ρ_{α_k} in the basis $\{\xi_1^{\alpha_k}, \dots, \xi_{n_{\alpha_k}}^{\alpha_k}\}$ is

$$\rho_{\alpha_k} = \operatorname{diag}(\Gamma(\alpha_k), \Gamma(\alpha_k)^{\theta_2^k}, \dots, \Gamma(\alpha_k)^{\theta_{n_{\alpha_k}}^k})$$

for some numbers $\theta_2^k, \dots, \theta_{n_{\alpha_k}}^k \in]-\infty, 1]$ such that $\theta_2^k \geqslant \dots \geqslant \theta_{n_{\alpha_k}}^k$. For notational convenience we will also put $\theta_1^k = 1$, so that

$$\rho_{\alpha_k} = \operatorname{diag}(\Gamma(\alpha_k)^{\theta_1^k}, \dots, \Gamma(\alpha_k)^{\theta_{n_{\alpha_k}}^k}).$$

Observe that since $\operatorname{Tr}(\rho_{\alpha_k}) = \operatorname{Tr}(\rho_{\alpha_k}^{-1})$, we have $\theta_{n_{\alpha_k}}^k < 0$ and

$$n_{\alpha_k}\Gamma(\alpha)^{2^{(k-1)}} = n_{\alpha_k}\Gamma(\alpha_k) \geqslant \sum_{j=1}^{n_{\alpha_k}}\Gamma(\alpha_k)^{\theta_j^k} = \sum_{j=1}^{n_{\alpha_k}}\Gamma(\alpha_k)^{-\theta_j^k} \geqslant \Gamma(\alpha_k)^{-\theta_{n_{\alpha_k}}^k} = \Gamma(\alpha)^{-2^{(k-1)}\theta_{n_{\alpha_k}}^k},$$

i.e.

$$\log(n_{\alpha_k}) + 2^{(k-1)}\log(\Gamma(\alpha)) \geqslant -2^{(k-1)}\theta_{n_{\alpha_k}}^k \log(\Gamma(\alpha)),$$

and hence

$$\theta_{n_{\alpha_k}}^k \geqslant -1 - \frac{\log(n_{\alpha_k})}{2^{(k-1)}\log(\Gamma(\alpha))} \geqslant -1 - \frac{\log(n_{\alpha_k})}{\log(\Gamma(\alpha))}. \tag{4.3}$$

We will now show that there is a subsequence $(\alpha_{k_n})_{n\in\mathbb{N}}$ of $(\alpha_k)_{k\in\mathbb{N}}$ such that:

(i) $n_{\alpha_{k_n}} = N$ for each $n \in \mathbb{N}$ and some $N \in \{2, \dots, \mathbf{N}_{\mathbb{G}}\}$;
(ii) dimension $D_{\alpha_{k_n}}$ is the same for each $n \in \mathbb{N}$;
(iii) for each $n \in \mathbb{N}$ we have

$$2^{(k_{n+1}-1)} > 2^{(k_n-1)}\Big(2 + \frac{\log(N)}{\log(\Gamma(\alpha))}\Big);$$

(iv) for each $n \in \mathbb{N}$ and $j \in \{1, \ldots, N\}$ we have

$$\Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_j^{k_{n+1}}} \leqslant \Gamma(\alpha)^{2^{(k_{n+1}-1)}-2^{(k_n-1)}}\Gamma(\alpha)^{2^{(k_n-1)}\theta_j^{k_n}}.$$

We note that the inequality in (iv) is equivalent to

$$2^{(k_n-1)}(1-\theta_j^{k_n}) \leqslant 2^{(k_{n+1}-1)}(1-\theta_j^{k_{n+1}}).$$

It is easy to see that there is a subsequence $(\alpha_{k_n^0})_{n\in\mathbb{N}}$ of $(\alpha_k)_{k\in\mathbb{N}}$ satisfying (i), (ii) and (iii). In order to see that we can refine it so that the resulting subsequence also satisfies (iv) we note first that it follows from the construction of the sequence $(\alpha_k)_{k\in\mathbb{N}}$ that we have

$$\mathrm{Sp}(\rho_{\alpha_{k_n^0}}) \subset \underbrace{\mathrm{Sp}(\rho_\alpha)\cdots\mathrm{Sp}(\rho_\alpha)}_{2^{(k_n^0-1)}}$$

for each $n \in \mathbb{N}$. Now for each $j \in \{1, \ldots, N\}$ the number $\Gamma(\alpha)^{2^{(k_n^0-1)}\theta_j^{k_n^0}}$ belongs to the spectrum of $\rho_{\alpha_{k_n^0}}$, so it can be written as a product of eigenvalues $\{\lambda_1, \ldots, \lambda_{\#\mathrm{Sp}(\rho_\alpha)}\}$ in appropriate powers:

$$\Gamma(\alpha)^{2^{(k_n^0-1)}\theta_j^{k_n^0}} = \prod_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)} \lambda_m^{d(k_n^0,m,j)},$$

where the non-negative integers

$$\{d(k_n^0, m, j) : m \in \{1, \ldots, \#\mathrm{Sp}(\rho_\alpha)\},\ j \in \{1, \ldots, N\}\}$$

satisfy

$$\sum_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)} d(k_n^0, m, j) = 2^{(k_n^0-1)}, \quad n \in \mathbb{N}.$$

By choosing an appropriate subsequence $(k_n)_{n\in\mathbb{N}}$ of $(k_n^0)_{n\in\mathbb{N}}$ we can arrange that we have $d(k_{n+1}, m, j) \geqslant d(k_n, m, j)$ for all m and a fixed j. It remains to repeat this procedure for all j refining the sequence each time. Having done so, let us keep the notation $(k_n)_{n\in\mathbb{N}}$ for the resulting sequence of natural numbers. We have

$$\begin{aligned}
\Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_j^{k_{n+1}}} &= \prod_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)} \lambda_m^{d(k_{n+1},m,j)} \\
&= \Big(\prod_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)} \lambda_m^{(d(k_{n+1},m,j)-d(k_n,m,j))}\Big)\Big(\prod_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)} \lambda_m^{d(k_n,m,j)}\Big) \\
&\leqslant \Gamma(\alpha)^{\sum_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)}(d(k_{n+1},m,j)-d(k_n,m,j))} \prod_{m=1}^{\#\mathrm{Sp}(\rho_\alpha)} \lambda_m^{d(k_n,m,j)} \\
&= \Gamma(\alpha)^{2^{(k_{n+1}-1)}-2^{(k_n-1)}}\Gamma(\alpha)^{2^{(k_n-1)}\theta_j^{k_n}}
\end{aligned}$$

(the inequality follows from the fact that $\Gamma(\alpha) = \max\{\lambda_1, \ldots, \lambda_{\#\mathrm{Sp}(\rho_\alpha)}\}$). In other words $(\alpha_{k_n})_{n\in\mathbb{N}}$ satisfies conditions (i)–(iv).

Now using the fact that $\frac{\Gamma(\alpha_{k_{n+1}})}{d_{\alpha_{k_{n+1}}}} \leqslant \frac{\Gamma(\alpha_{k_n})}{d_{\alpha_{k_n}}}$ and properties (i)–(iv) of $(\alpha_{k_n})_{n\in\mathbb{N}}$ we will arrive at a contradiction. We have

$$\begin{aligned}1 &\leqslant \frac{d_{\alpha_{k_{n+1}}}}{d_{\alpha_{k_n}}} \frac{\Gamma(\alpha_{k_n})}{\Gamma(\alpha_{k_{n+1}})}\\ &= \frac{\sum_{j=1}^{N} \Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_j^{k_{n+1}}}}{\sum_{j=1}^{N} \Gamma(\alpha)^{2^{(k_n-1)}\theta_j^{k_n}}} \Gamma(\alpha)^{2^{(k_n-1)}-2^{(k_{n+1}-1)}} \\ &= \frac{\sum_{j=1}^{N-1} \Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_j^{k_{n+1}}} + \Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_N^{k_{n+1}}}}{\sum_{j=1}^{N-1} \Gamma(\alpha)^{2^{(k_n-1)}(-1+\theta_j^{k_n})+2^{(k_{n+1}-1)}} + \Gamma(\alpha)^{2^{(k_n-1)}(-1+\theta_N^{k_n})+2^{(k_{n+1}-1)}}}.\end{aligned} \tag{4.4}$$

Thanks to condition (iv) we have

$$\Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_j^{k_{n+1}}} \leqslant \Gamma(\alpha)^{2^{(k_n-1)}(-1+\theta_j^{k_n})+2^{(k_{n+1}-1)}} \tag{4.5}$$

for every $j \in \{1, \dots, N-1\}$. Moreover, condition (iii) and inequality (4.3) (with $n_{\alpha_k} = N$) implies

$$\begin{aligned} &2^{(k_n-1)}(1-\theta_N^{k_n}) \leqslant 2^{(k_n-1)}\Big(2+\frac{\log(N)}{\log(\Gamma(\alpha))}\Big) < 2^{(k_{n+1}-1)} \leqslant 2^{(k_{n+1}-1)}(1-\theta_N^{k_{n+1}}),\\ &2^{(k_{n+1}-1)}\theta_N^{k_{n+1}} < 2^{(k_n-1)}(-1+\theta_N^{k_n}) + 2^{(k_{n+1}-1)}, \\ &\Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_N^{k_{n+1}}} < \Gamma(\alpha)^{2^{(k_n-1)}(-1+\theta_N^{k_n})+2^{(k_{n+1}-1)}}. \end{aligned} \tag{4.6}$$

Comparing appropriate terms in the numerator and denominator of the right hand side of (4.4) and using (4.5) and (4.6) we find that

$$\frac{\sum_{j=1}^{N-1} \Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_j^{k_{n+1}}} + \Gamma(\alpha)^{2^{(k_{n+1}-1)}\theta_N^{k_{n+1}}}}{\sum_{j=1}^{N-1} \Gamma(\alpha)^{2^{(k_n-1)}(-1+\theta_j^{k_n})+2^{(k_{n+1}-1)}} + \Gamma(\alpha)^{2^{(k_n-1)}(-1+\theta_N^{k_n})+2^{(k_{n+1}-1)}}} < 1$$

which contradicts (4.4) and therefore proves Theorem 4.3. ∎

At the end of this section we use Theorem 4.3 to derive a corollary concerning quantum groups which are not of Kac type.

COROLLARY 4.4. *Let $\mathbb{G}$ be a compact quantum group and let $U \in \mathrm{C}(\mathbb{G} \otimes \mathrm{B}(\mathscr{H}_U)$ be a finite dimensional unitary representation such that $\Gamma(U) > 1$. Then*

$$\sup\{n_\beta : \beta \in \operatorname{Irr}\mathbb{G},\ \beta \preccurlyeq U^{\oplus k_1} ⓣ \cdots ⓣ U^{\oplus k_n},\ n \in \mathbb{Z}_+,\ k_1, \dots, k_n \in \mathbb{Z}\} = +\infty,$$

where we have used conventions $U^{ⓣ(-n)} = \overline{U}^{ⓣ(n)}$ for $n \in \mathbb{N}$ and $U^0 = \mathbf{1}$.

Proof. As any unitary representation decomposes into sum of irreducible ones, it is enough to prove this claim for $U = U^\alpha$, where $\alpha \in \operatorname{Irr}\mathbb{G}$. Let $\mathbb{H}$ be the image of $\mathbb{G}$ in the representation U, i.e. $\mathrm{C}(\mathbb{H})$ is the C^*-algebra generated by

$\{U^{\alpha}_{i,j} : i,j \in \{1,\ldots,n_\alpha\}\}$ and $\Delta_{\mathbb{H}} = \Delta_{\mathbb{G}}|_{\mathrm{C}(\mathbb{H})}$ (cf. Remarks 3 and 1 of [10]). It is easily seen that $\cdot \boldsymbol{h}_{\mathbb{G}}|_{\mathrm{C}(\mathbb{H})}$ is a bi-invariant state on $\mathrm{C}(\mathbb{H})$ and consequently, by uniqueness of Haar measure, we have $\boldsymbol{h}_{\mathbb{H}} = \boldsymbol{h}_{\mathbb{G}}|_{\mathrm{C}(\mathbb{H})}$.

For some $n \in \mathbb{Z}_+$, and $k_1,\ldots,k_n \in \mathbb{Z}$ let $\beta \preccurlyeq \alpha^{\textcircled{\scriptsize T}k_1} \textcircled{\scriptsize T} \cdots \textcircled{\scriptsize T} \alpha^{\textcircled{\scriptsize T}k_n}$ be a (class of) an irreducible representation of $\mathbb{G}$. Then U^β is also irreducible as a representation of $\mathbb{H}$ since

$$\boldsymbol{h}_{\mathbb{H}}(\chi_\beta{}^*\chi_\beta) = \boldsymbol{h}_{\mathbb{G}}(\chi_\beta{}^*\chi_\beta) = 1$$

(where χ_β is the character of U^β, cf. Corollary 5.10 of [14]). Since matrix elements of such representations span a dense subspace in $\mathrm{C}(\mathbb{H})$ we have

$$\operatorname{Irr}\mathbb{H} = \{\beta \in \operatorname{Irr}\mathbb{G} : \beta \in \operatorname{Irr}\mathbb{G},\ \beta \preccurlyeq U^{\textcircled{\scriptsize T}k_1} \textcircled{\scriptsize T} \cdots \textcircled{\scriptsize T} U^{\textcircled{\scriptsize T}k_n},\ n \in \mathbb{Z}_+,\ k_1,\ldots,k_n \in \mathbb{Z}\}.$$

Since $\Gamma(\alpha) > 1$ (where α is considered as a class of representation of $\mathbb{H}$) we must have

$$\sup\{n_\beta : \beta \in \operatorname{Irr}\mathbb{G},\ \beta \preccurlyeq U^{\textcircled{\scriptsize T}k_1} \textcircled{\scriptsize T} \cdots \textcircled{\scriptsize T} U^{\textcircled{\scriptsize T}k_n},\ n \in \mathbb{Z}_+,\ k_1,\ldots,k_n \in \mathbb{Z}\} = +\infty$$

due to Theorem 4.3. ∎

Appendix A. ALGEBRAIC CHARACTERIZATION OF GROUPS WITH REPRESENTATIONS OF BOUNDED DEGREE

In this section we note a characterization of the property of $\mathbb{G}$ having irreducible representations of bounded degree in terms of the comultiplication on $\mathrm{Pol}(\mathbb{G})$. The reasoning is based on the fact that the algebra of $n \times n$ matrices (over a field of characteristic 0) satisfies a polynomial identity of degree $2n$ and not lower (cf. [6]).

PROPOSITION A.1. *Let $\mathbb{G}$ be a compact quantum group. Then $\mathbf{N}_{\mathbb{G}} < +\infty$ if and only if there exists $r \geqslant 2$ such that*

$$\sum_{\pi \in \mathsf{S}_r} \operatorname{sgn}(\pi) x_{\pi(1)} \cdots x_{\pi(r)} = 0, \quad x_1,\ldots,x_r \in \mathrm{c}_{00}(\widehat{\mathbb{G}}). \tag{A.1}$$

Proof. By the Amitsur–Levitzki theorem ([6], Section 4) for any $n \geqslant 2$ we have

$$\sum_{\pi \in \mathsf{S}_{2n}} \operatorname{sgn}(\pi) m_{\pi(1)} \cdots m_{\pi(2n)} = 0, \quad m_1,\ldots,m_{2n} \in \mathsf{M}_n(\mathbb{C}).$$

Moreover $\mathsf{M}_n(\mathbb{C})$ does not have a proper polynomial identity of degree strictly smaller than $2n$ ([6], Section 3, Lemma 2). Since $\mathrm{c}_{00}(\widehat{\mathbb{G}})$ is the algebraic direct sum of matrix algebras of sizes equal to the dimensions of irreducible representations of $\mathbb{G}$, we see that $\mathbf{N}_{\mathbb{G}}$ is finite if and only if (A.1) is satisfied for some r (namely $r = 2\mathbf{N}_{\mathbb{G}}$ or larger). ∎

Condition (A.1) from Proposition A.1 can be rewritten in the following way: for $\pi \in \mathsf{S}_r$ let $\widetilde{\pi}$ be the operator on $c_{00}(\widehat{\mathbb{G}})^{\otimes r}$ permuting the tensor factors and let $\boldsymbol{\mu}$ be the multiplication map $c_{00}(\widehat{\mathbb{G}}) \otimes c_{00}(\widehat{\mathbb{G}}) \to c_{00}(\widehat{\mathbb{G}})$. Further let $(\boldsymbol{\mu}^{(k)})_{k\in\mathbb{N}}$ be the obvious extensions of multiplication to higher tensor powers of $c_{00}(\widehat{\mathbb{G}})$:

$$\boldsymbol{\mu}^{(k)} : c_{00}(\widehat{\mathbb{G}})^{\otimes(k+1)} \to c_{00}(\widehat{\mathbb{G}}), \quad k \in \mathbb{N}.$$

Then (A.1) means simply

$$\sum_{\pi\in\mathsf{S}_r} \operatorname{sgn}(\pi) \cdot \boldsymbol{\mu}^{(r-1)} \circ \widetilde{\pi} = 0.$$

Now recall that $\operatorname{Pol}(\mathbb{G})$ is the (multiplier) Hopf algebra dual to $c_{00}(\widehat{\mathbb{G}})$ ([13]). In particular, for each $k \geqslant 2$ the map $\boldsymbol{\mu}^{(k)}$ is dual to

$$\Delta_{\mathbb{G}}^{(k)} : c_{00}(\widehat{\mathbb{G}}) \to \mathrm{M}(c_{00}(\widehat{\mathbb{G}})^{\otimes(k+1)}),$$

where $\mathrm{M}(\,\cdot\,)$ denotes the multiplier functor ([12], [13]). Thus the condition of having irreducible representations of bounded degree can be expressed in terms of the coalgebra structure of $\operatorname{Pol}(\mathbb{G})$.

COROLLARY A.2. *Let $\mathbb{G}$ be a compact quantum group. Then $\mathbf{N}_{\mathbb{G}} < +\infty$ if and only if there exists $r \geqslant 2$ such that*

$$\sum_{\pi\in\mathsf{S}_r} \operatorname{sgn}(\pi) \cdot \widetilde{\pi} \circ \Delta_{\mathbb{G}}^{(r-1)} = 0. \tag{A.2}$$

REMARK A.3. Let us note that it can be shown that for a *classical* group $\mathbb{G} = G$ condition (A.2) is equivalent to condition P_r considered by Kaplansky ([7], Section 3), which for connected G is further equivalent to commutativity of the group.

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JACEK KRAJCZOK, DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS, FACULTY OF PHYSICS, UNIVERSITY OF WARSAW, POLAND

E-mail address: jk347906@okwf.fuw.edu.pl

PIOTR M. SOŁTAN, DEPARTMENT OF MATHEMATICAL METHODS IN PHYSICS, FACULTY OF PHYSICS, UNIVERSITY OF WARSAW, POLAND

E-mail address: piotr.soltan@fuw.edu.pl

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