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HARNACK PARTS OF ρ-CONTRACTIONS

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ABSTRACT. The purpose of this paper is to describe the Harnack parts for the operators of class C_ρ ($\rho > 0$) on Hilbert spaces which were introduced by B. Sz.-Nagy and C. Foiaş. More precisely, we study Harnack parts of operators with ρ-numerical radius one. The case of operators with ρ-numerical radius strictly less than 1 was described earlier. We obtain a general criterion for compact ρ-contractions to be in the same Harnack part. For classical contractions, this criterion can be simplified into a very useful form. Operators with numerical radius one receive also a particular attention. Moreover, we study many properties of Harnack equivalence in the general case.

KEYWORDS: *ρ-Contractions, Harnack parts, operator kernel, compact operators, operator radii, numerical range.*

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1. INTRODUCTION AND PRELIMINARIES

Let H be a complex Hilbert space and $B(H)$ the set of all bounded linear operators on H. For $\rho > 0$, we say that an operator $T \in B(H)$ admits a *unitary ρ-dilation* if there is a Hilbert space $\mathcal{H}$ containing H as a closed subspace and a unitary operator $U \in B(\mathcal{H})$ such that

$$T^n = \rho P_H U^n|_H, \quad n \in \mathbb{N}^*, \tag{1.1}$$

where P_H denotes the orthogonal projection onto the subspace H in $\mathcal{H}$.

In the sequel, we denote by $C_\rho(H)$, $\rho > 0$, the set of all operators in $B(H)$ which admit unitary ρ-dilations. A famous theorem due to B. Sz.-Nagy [26] asserts that $C_1(H)$ is exactly the class of all contractions, i.e., operators T such that $\|T\| \leqslant 1$. C.A. Berger [5] showed that the class $C_2(H)$ is precisely the class of all operators $T \in B(H)$ whose *numerical radius*

$$w(T) = \sup\{|\langle Tx, x\rangle| : x \in H,\ \|x\| = 1\}$$

is less or equal to one. In particular, the classes $C_\rho(H)$, $\rho > 0$, provide a framework for simultaneous investigation of these two important classes of operators. Any operator $T \in C_\rho(H)$ is *power-bounded*:

$$\|T^n\| \leqslant \rho, \quad n \in \mathbb{N}, \tag{1.2}$$

moreover, its *spectral radius*

$$r(T) = \lim_{n \to +\infty} \|T^n\|^{1/n} \tag{1.3}$$

is at most one. In [27], an example of a power-bounded operator which is not contained in any of the classes $C_\rho(H)$, $\rho > 0$, is given. However, J.A.R. Holbrook [19] and J.P. Williams [30], independently, introduced the *ρ-numerical radius* (or the *operator radii*) of an operator $T \in B(H)$ by setting

$$w_\rho(T) := \inf\left\{\gamma > 0 : \frac{1}{\gamma}T \in C_\rho(H)\right\}. \tag{1.4}$$

Note that $w_1(T) = \|T\|$, $w_2(T) = w(T)$ and $\lim_{\rho \to \infty} w_\rho(T) = r(T)$. Also, $T \in C_\rho(H)$ if and only if $w_\rho(T) \leqslant 1$, hence operators in $C_\rho(H)$ are contractions with respect to the *ρ-numerical radius*, and the elements of $C_\rho(H)$ are called ρ-contractions. Considerable attention has been paid to the study of ρ-contractions, see for instance [1], [3], [5], [7], [9], [10], [11], [12], [13], [14], [15], [16], [20], [24], [27], [28], [29] and the references therein (the list will not be exhaustive).

Some properties of the classes $C_\rho(H)$ become more clear (see for instance, [7], [9], [10], and [11]) due to harmonic analysis methods using the following operatorial ρ-kernel

$$K_z^\rho(T) = (I - \bar{z}T)^{-1} + (I - zT^*)^{-1} + (\rho - 2)I, \quad (z \in \mathbb{D}) \tag{1.5}$$

(of a bounded operator T having its spectrum in the closed unit disc) introduced and first systematically developed in [6], [9], [10].

The ρ-kernels are connected to ρ-contractions by the next result. An operator T is in the class $C_\rho(H)$ if and only if, $\sigma(T) \subseteq \overline{\mathbb{D}}$ and $K_z^\rho(T) \geqslant 0$ for any $z \in \mathbb{D}$ (see [10]).

The extension of Harnack domination to ρ-contrations appears in [11] and is studied in the C_ρ balls of $B(H)^n$ ($n > 1$) in [25].

We say that T_1 is Harnack dominated by T_0, if T_0 and T_1 satisfy one of the following equivalent conditions given in the next theorem.

THEOREM 1.1 ([11], Theorem 3.1). *For $T_0, T_1 \in C_\rho(H)$ and a constant $c \geqslant 1$, the following statements are equivalent:*

(i) $\mathrm{Re}\, p(T_1) \leqslant c^2 \mathrm{Re}\, p(T_0) + (c^2 - 1)(\rho - 1)\mathrm{Re}\, p(O_H)$, *for any polynomial p with* $\mathrm{Re}\, p \geqslant 0$ *on* $\overline{\mathbb{D}}$;

(ii) $\mathrm{Re}\, p(rT_1) \leqslant c^2 \mathrm{Re}\, p(rT_0) + (c^2 - 1)(\rho - 1)\mathrm{Re}\, p(O_H)$, *for any $r \in]0, 1[$ and each polynomial p with* $\mathrm{Re}\, p \geqslant 0$ *on* $\overline{\mathbb{D}}$;

(iii) $K_z^\rho(T_1) \leqslant c^2 K_z^\rho(T_0)$, *for all $z \in \mathbb{D}$;*

(iv) $\varphi_{T_1}(g) \leqslant c^2\varphi_{T_0}(g)$ *for any function* $g \in C(\mathbb{T})$ *such that* $g \geqslant 0$ *on* $\mathbb{T} = \overline{\mathbb{D}} \setminus \mathbb{D}$;

(v) *if* V_i *acting on* $K_i \supseteq H$ *is the minimal isometric* ρ*-dilation of* T_i $(i = 0, 1)$, *then there is an operator* $S \in B(K_0, K_1)$ *such that* $S(H) \subset H$, $S|_H = I$, $SV_0 = V_1S$ *and* $\|S\| \leqslant c$.

When T_1 is Harnack dominated by T_0 in $C_\rho(H)$ for some constant $c \geqslant 1$, we write $T_1 \underset{c}{\overset{H}{\prec}} T_0$, or also $T_1 \overset{H}{\prec} T_0$. The relation $\overset{H}{\prec}$ is a preorder relation in $C_\rho(H)$. The induced equivalent relation is called Harnack equivalence, and the associated classes are called the Harnack parts of $C_\rho(H)$. So, we say that T_1 and T_0 are Harnack equivalent if they belong to the same Harnack part. In this later case, we write $T_1 \overset{H}{\sim} T_0$.

We say that an operator $T \in C_\rho(H)$ is a strict ρ-contraction if $w_\rho(T) < 1$. In [17] C. Foiaş proved that the Harnack part of contractions containing the null operator O_H consists of all strict contractions. More recently, G. Cassier and N. Suciu proved in Theorem 4.4 of [11] that the Harnack part of $C_\rho(H)$ containing the null operator O_H is the set of all strict ρ-contractions. According to this fact the following natural question arises:

If T *is an operator with* ρ*-numerical radius one, what can be said about the Harnack part of* T*?*

Recall that a ρ-contraction is similar to a contraction [28] but many properties are not preserved under similarity (and an operator similar to a contraction is not necessarily a ρ-contraction!), in particular it is true for the numerical range properties. Thus, the study of Harnack parts for ρ-contractions cannot be deduced from the contractions case; see for instance Theorem 2.17, Remark 2.19, Theorem 2.25 and Remark 2.26. Notice also that some properties are of different nature (see for example Theorem 2.1 and Remark 2.7). We find a few answers in the literature of the previous question, essentially in the class of contractions with norm one. In [2], the authors have proved that if T is either isometry or coisometry then the Harnack part of T is trivial (i.e. equal to $\{T\}$), and if T is compact or $r(T) < 1$, or normal and nonunitary, then its Harnack part is not trivial in general. The authors have asked that it seems interesting to give necessary and/or sufficient conditions for a contraction to have a trivial Harnack part. It was proved in [21] that the Harnack part of a contraction T is trivial if and only if T is an isometry or a coisometry (the adjoint of an isometry), this is a response to the question posed by T. Ando and al. in the class of contractions. Recently the authors of [4] proved that maximal elements for the Harnack domination in $C_1(H)$ are precisely the singular unitary operators and the minimal elements are isometries and coisometries.

This paper is a continuation and refinement of the research treatment of the Harnack domination in the general case of the ρ-contractions. Note that this treatment gives certain useful properties and leads to new techniques for studies of the Harnack part of an operator with ρ-numerical radius one. More precisely, we show that two ρ-contractions belonging to the same Harnack parts have the same spectral values in $\mathbb{T}$. This property has several consequences and applications. In particular, it will be shown that if T_0 is a compact operator (i.e. $T_0 \in \mathcal{K}(H)$) with $w_\rho(T_0) = 1$ and whose spectral radius is strictly less than one, then a ρ-contraction $T_1 \in \mathcal{K}(H)$ is Harnack equivalent to T_0 if and only if they satisfy the *null spaces condition*: $\mathcal{N}(K_z^\rho(T_1)) = \mathcal{N}(K_z^\rho(T_0))$ for all $z \in \mathbb{T}$ and an additional *conorms condition* on the ρ-operator kernels of T_0 and T_1. We give an example showing that this *conorms condition* cannot be removed in general (see Remark 2.19). We also study a situation where the *null spaces condition* is sufficient to characterize Harnack equivalence. It is the case for all usual contractions, moreover we show that if T_0 is a compact contraction with $\|T_0\| = 1$, then a contraction $T_1 \in \mathcal{K}(H)$ is Harnack equivalent to T_0 if and only if $I - T_1^* T_1$ and $I - T_0^* T_0$ have the same null space and T_0 and T_1 restricted to the null space of $I - T_0^* T_0$ coincide. A nice application is the description of the Harnack part of the (nilpotent) Jordan block of size n. We also obtain precise results about the relationships between the trace of the closure of the numerical range on the torus and the Harnack domination for every $\rho \in [1, 2]$. The case of $\rho = 2$ plays a crucial role. We characterize the weak stability of a ρ-contraction in terms of its minimal isometric ρ-dilation. The details of these basic facts are explained in Section 2. The last section is devoted to applications in order to describe the Harnack part of some nilpotent matrices with numerical radius one, in three cases: a nilpotent matrix of order two in the two dimensional case, a nilpotent matrix of order two in $\mathbb{C}^n$ and a nilpotent matrix of order three in the three dimensional case. In particular, we show that in the first case the Harnack part is trivial, while in the third case the Harnack part is an orbit associated with the action of a group of unitary diagonal matrices.

2. MAIN RESULTS

2.1. SPECTRAL PROPERTIES AND HARNACK DOMINATION. We denote by $\Gamma(T)$ the set of complex numbers defined by $\Gamma(T) = \sigma(T) \cap \mathbb{T}$, where $\mathbb{T} = \overline{\mathbb{D}} \setminus \mathbb{D}$ is the unidimensional torus. In the following results, we prove that ρ-contractions belonging to the same Harnack parts have the same spectral values in the torus.

THEOREM 2.1. *Let $T_0, T_1 \in C_\rho(H)$, ($\rho \geqslant 1$). If $T_1 \overset{H}{\prec} T_0$ then $\Gamma(T_1) \subseteq \Gamma(T_0)$.*

Proof. Let $T_0, T_1 \in C_\rho(H)$ be such that $T_1 \overset{H}{\prec} T_0$. Then there exists $c \geqslant 1$ such that

$$K_z^\rho(T_1) \leqslant c^2 K_z^\rho(T_0), \quad \text{for all } z \in \mathbb{D}, \tag{2.1}$$

so,

$$\begin{aligned} K_z^\rho(T_1) &= (I - zT_1^*)^{-1}[\rho I + 2(1-\rho)\mathrm{Re}\,(\bar{z}T_1) + (\rho-2)|z|^2 T_1^* T_1](I - \bar{z}T_1)^{-1} \\ &\leqslant c^2 K_z^\rho(T_0), \quad \text{for all } z \in \mathbb{D}. \end{aligned}$$

Hence

$$\rho I + 2(1-\rho)\mathrm{Re}\,(\bar{z}T_1) + (\rho-2)|z|^2 T_1^* T_1 \leqslant c^2 (I - zT_1^*) K_z^\rho(T_0)(I - \bar{z}T_1), \tag{2.2}$$

for all $z \in \mathbb{D}$. Now, let $\lambda = \mathrm{e}^{\mathrm{i}\omega} \in \Gamma(T_1)$ which is contained in the approximate point spectrum $\sigma_{\mathrm{ap}}(T_1)$, then there exists a sequence $(x_n)_{n \geqslant 0}$ of unit vectors such that $T_1 x_n - \mathrm{e}^{\mathrm{i}\omega} x_n = y_n$ converge to 0. From the inequality (2.2), we derive

$$\begin{aligned} \rho I + 2(1-\rho)&\mathrm{Re}\,(\bar{z}\langle T_1 x_n, x_n\rangle) + (\rho-2)|z|^2 \|T_1 x_n\|^2 \\ &\leqslant c^2 \langle K_z^\rho(T_0)(I - \bar{z}T_1)x_n, (I - \bar{z}T_1)x_n\rangle \\ &= c^2 \langle K_z^\rho(T_0)[(1 - \bar{z}\mathrm{e}^{\mathrm{i}\omega})x_n - \bar{z}y_n], (1 - \bar{z}\mathrm{e}^{\mathrm{i}\omega})x_n - \bar{z}y_n\rangle \\ &= c^2 |1 - \bar{z}\mathrm{e}^{\mathrm{i}\omega}|^2 \langle K_z^\rho(T_0)x_n, x_n\rangle - c^2 z(1 - \bar{z}\mathrm{e}^{\mathrm{i}\omega})\langle K_z^\rho(T_0)x_n, y_n\rangle \\ &\quad - c^2 \bar{z}(1 - z\mathrm{e}^{-\mathrm{i}\omega})\langle K_z^\rho(T_0)y_n, x_n\rangle + c^2|z|^2\langle K_z^\rho(T_0)y_n, y_n\rangle, \end{aligned}$$

for any $z \in \mathbb{D}$ and all $n \geqslant 0$. The triangular inequality gives

$$|\, \|T_1 x_n - \mathrm{e}^{\mathrm{i}\omega}x_n\| - \|x_n\|\, | \leqslant \|T_1 x_n\| \leqslant \|T_1 x_n - \mathrm{e}^{\mathrm{i}\omega}x_n\| + 1.$$

Letting $n \to +\infty$, from the two previous inequalities we obtain

$$\rho + 2(1-\rho)\mathrm{Re}\,(\bar{z}\mathrm{e}^{\mathrm{i}\omega}) + (\rho-2)|z|^2 \leqslant c^2 |1 - \bar{z}\mathrm{e}^{\mathrm{i}\omega}|^2 \limsup_{n\to+\infty} \langle K_z^\rho(T_0)x_n, x_n\rangle,$$

for any $z \in \mathbb{D}$. Then, if we take $z = (1-t)\mathrm{e}^{\mathrm{i}\omega}$ with $0 < t < 1$, we get

$$\rho + 2(1-\rho)(1-t) + (\rho-2)(1-t)^2 \leqslant c^2 t^2 \limsup_{n\to+\infty} \langle K_{(1-t)\mathrm{e}^{\mathrm{i}\omega}}^\rho(T_0)x_n, x_n\rangle.$$

Assume that $\mathrm{e}^{\mathrm{i}\omega} \notin \Gamma(T_0)$, then $K_{(1-t)\mathrm{e}^{\mathrm{i}\omega}}^\rho(T_0)$ is uniformly bounded in $]0,1[$, then there exists $\gamma > 0$ such that

$$\rho + 2(1-\rho)(1-t) + (\rho-2)(1-t)^2 \leqslant \gamma c^2 t^2,$$

which implies

$$2t \leqslant (\gamma c^2 + 2 - \rho)t^2,$$

for all $t > 0$, and hence

$$2 \leqslant (\gamma c^2 + 2 - \rho)t.$$

Now, we get a contradiction by letting $t \to 0$. Hence $\mathrm{e}^{\mathrm{i}\omega} \in \Gamma(T_0)$. ∎

From Theorem 2.1, we also obtain the following result.

COROLLARY 2.2. *If T_1 and T_0 are Harnack equivalent in $C_\rho(H)$ then $\Gamma(T_1) = \Gamma(T_0)$.*

Let $T \in B(H)$ and E be a closed invariant subspace of T, ($T(E) \subset E$). Then $T \in B(E \oplus E^{\perp})$, has the following form:

$$T = \begin{pmatrix} T_1 & R \\ 0 & T_2 \end{pmatrix},$$

with $T_1 \in B(E)$, $T_2 \in B(E^{\perp})$ and R is a bounded operator from $E^{\perp}$ to E. We denote by $\Gamma_p(T) = \sigma_p(T) \cap \mathbb{T}$ the point spectrum of $T \in B(H)$ in the unidimensional torus and by $\mathcal{N}(T)$ its null space.

THEOREM 2.3. *Let $T_0, T_1 \in C_\rho(H)$ ($\rho \geqslant 1$). If $T_1 \overset{H}{\prec} T_0$ then $\Gamma_p(T_1) \subseteq \Gamma_p(T_0)$ and $\mathcal{N}(T_1 - \lambda I) \subseteq \mathcal{N}(T_0 - \lambda I)$ for all $\lambda \in \Gamma_p(T_1)$.*

For the proof of this theorem we need the following lemma.

LEMMA 2.4. *Let $T \in C_\rho(H)$. Then*

$$\|(I - \overline{\lambda} T) K_z^\rho(T)(I - \lambda T^*)\| \leqslant \rho(1 + 2|1-\rho| + |\rho - 2|\rho)\Big(1 + \rho \frac{|z-\lambda|}{1-|z|}\Big)^2,$$

for all $z \in \mathbb{D}$ and $\lambda \in \overline{\mathbb{D}}$.

Proof. Let $z \in \mathbb{D}$ and $\lambda \in \overline{\mathbb{D}}$, we have

$$(I - zT^*)^{-1}(I - \lambda T^*) = I + (z - \lambda) \sum_{n=0}^{+\infty} z^n T^{*n+1}.$$

Then by (1.2),

$$\|(I - zT^*)^{-1}(I - \lambda T^*)\| \leqslant 1 + \rho \frac{|z-\lambda|}{1-|z|}.$$

Taking into account this inequality and the fact that

$$K_z^\rho(T_1) = (I - \overline{z} T_1)^{-1}[\rho I + 2(1-\rho)\mathrm{Re}\,(\overline{z} T_1) + (\rho - 2)|z|^2 T_1 T_1^*](I - z T_1^*)^{-1},$$

we obtain the desired inequality. ∎

Proof of Theorem 2.3. Let $\lambda \in \Gamma_p(T_1)$. Then the operator $T_1 \in C_\rho(H)$ on $\mathcal{N}(T_1 - \lambda I) \oplus \mathcal{N}(T_1 - \lambda I)^{\perp}$ takes the form

$$T_1 = \begin{pmatrix} \lambda I_1 & C \\ 0 & \widetilde{T}_1 \end{pmatrix}.$$

Since $|\lambda| = 1$, by using Proposition 3 of [12] we can see that $C = 0$. Thus, we have

$$K_z^\rho(T_1) = \begin{pmatrix} \frac{\rho + 2(1-\rho)\mathrm{Re}\,(\overline{\lambda} z) + (\rho-2)|\lambda|^2|z|^2}{|1-\overline{\lambda} z|^2} I_1 & 0 \\ 0 & K_z^\rho(\widetilde{T}_1) \end{pmatrix}.$$

Now, if $T_0 \in C_\rho(H)$ is such that $T_1 \overset{H}{\prec} T_0$, then there exists $c \geqslant 1$ such that

$$K_z^\rho(T_1) \leqslant c^2 K_z^\rho(T_0), \quad \text{for all } z \in \mathbb{D}.$$

Let $x \in \mathcal{N}(T_1 - \lambda I)$ and $y \in \mathcal{R}(T_0^* - \overline{\lambda} I)$. The Cauchy–Schwarz inequality yields

$$|\langle K_z^\rho(T_1)x, y\rangle|^2 \leqslant c^2 \langle K_z^\rho(T_1)x, x\rangle \langle K_z^\rho(T_0)y, y\rangle.$$

We derive

$$\frac{\rho + 2(1-\rho)\mathrm{Re}\,(\overline{\lambda} z) + (\rho - 2)|\lambda|^2|z|^2}{|1 - \overline{\lambda} z|^2} |\langle x, y\rangle|^2 \leqslant c^2 \langle K_z^\rho(T_0)y, y\rangle \|x\|^2.$$

Since $y \in \mathcal{R}(T_0^* - \overline{\lambda} I)$, there exists $u \in H$ such that $y = (I - \lambda T_0^*)u$. By Lemma 2.4, we have

$$\begin{aligned}\langle K_z^\rho(T_0)y, y\rangle &= \langle (I - \overline{\lambda} T_0) K_z^\rho(T_0)(I - \lambda T_0^*)u, u\rangle \\ &\leqslant \rho(1 + 2|1-\rho| + |\rho - 2|\rho)\Big(1 + \rho \frac{|z - \lambda|}{1 - |z|}\Big)^2 \|u\|^2.\end{aligned}$$

Let $z = r\lambda$, with $0 < r < 1$. Then

$$\frac{|\rho + 2(1-\rho)r + (\rho - 2)r^2|\, |\langle x, y\rangle|^2}{(1-r)\rho(1 + 2|1-\rho| + |\rho - 2|\rho)(1+\rho)^2} \leqslant (1-r)c^2\|u\|^2\|x\|^2.$$

By letting r tend to 1, it follows that $\langle x, y\rangle = 0$, and hence $x \in \mathcal{R}(T_0^* - \overline{\lambda} I)^\perp = \mathcal{N}(T_0 - \lambda I)$. So, $\Gamma_p(T_1) \subseteq \Gamma_p(T_0)$ and $\mathcal{N}(T_1 - \lambda I) \subseteq \mathcal{N}(T_0 - \lambda I)$. ∎

REMARK 2.5. By Theorem 2.3, if $I_H \overset{H}{\prec} T$ on $C_\rho(H)$, $(\rho \geqslant 1)$ then $T = I_H$. This means that I_H is a maximal element for the Harnack domination in $C_\rho(H)$ and its Harnack part is trivial, for all $\rho \geqslant 1$.

From Theorem 2.3, we also obtain the following result.

COROLLARY 2.6. *If T_1 and T_0 are Harnack equivalent in $C_\rho(H)$ then $\Gamma_p(T_1) = \Gamma_p(T_0)$ and $\mathcal{N}(T_1 - \lambda I) = \mathcal{N}(T_0 - \lambda I)$ for all $\lambda \in \Gamma_p(T_0)$.*

REMARK 2.7. After the authors have obtained Theorem 2.1, they have learned that C. Badea, D. Timotin and L. Suciu [4] have proved using an other method that, in the case of contractions ($\rho = 1$), the domination suffices for the equality of the point spectrum in the torus. But in the case of $\rho > 1$ the inclusion in Theorem 2.3 may be strict, for instance, we have:

(i) for $\rho > 1$, we have $0_H \overset{H}{\underset{c}{\prec}} I$ in $C_\rho(H)$ with $c = \sqrt{\rho/(\rho - 1)}$.

(ii) for $\rho > 1$, the operator T defined on $\mathbb{C}^2$ by $T = \left(\begin{smallmatrix} 0 & \rho \\ 0 & 0 \end{smallmatrix}\right)$ satisfies $T \overset{H}{\underset{c}{\prec}} I$ in $C_\rho(H)$ with $c = \sqrt{2\rho/(\rho - 1)}$.

COROLLARY 2.8. *Let $T_0, T_1 \in C_\rho(H)$ $(\rho \geqslant 1)$ such that $\Gamma_p(T_0) = \Gamma_p(T_1)$. Then T_0 and T_1 are Harnack equivalent in $C_\rho(H)$ if and only if $T_0 = U \oplus \widetilde{T}_0$ and $T_1 = U \oplus \widetilde{T}_1$ on $H = E \oplus E^\perp$, where $E = \bigoplus_{\lambda \in \Gamma_p(T_0)} \mathcal{N}(T_0 - \lambda I) = \bigoplus_{\lambda \in \Gamma_p(T_1)} \mathcal{N}(T_1 - \lambda I)$, U is an unitary diagonal operator on E and $\widetilde{T}_0$ and $\widetilde{T}_1$ are Harnack equivalent in $C_\rho(E^\perp)$.*

Proof. First we prove that if $\lambda,\mu \in \Gamma_p(T_0)$, then $\mathcal{N}(T_0-\lambda I)\perp\mathcal{N}(T_0-\mu I)$ for $\lambda \neq \mu$. Let $x \in \mathcal{N}(T_1-\lambda I)$ and $y \in \mathcal{N}(T_0-\mu I)$. Then

$$\begin{aligned}\langle K_z^\rho(T_0)x,y\rangle &= \langle((I-\bar{z}T_0)^{-1}+(I-zT_0^*)^{-1}+(\rho-2)I)x,y\rangle\\ &= \frac{1}{1-\bar{z}\lambda}\langle x,y\rangle+\frac{1}{1-z\overline{\mu}}\langle x,y\rangle+(\rho-2)\langle x,y\rangle.\end{aligned}$$

By Cauchy–Schwarz inequality

$$|\langle K_z^\rho(T_0)x,y\rangle|^2 \leqslant \langle K_z^\rho(T_0)x,x\rangle\langle K_z^\rho(T_0)y,y\rangle.$$

Thus

$$\begin{aligned}&\Big|\frac{1}{1-\bar{z}\lambda}+\frac{1}{1-z\overline{\mu}}+(\rho-2)\Big|^2|\langle x,y\rangle|^2\\ &\leqslant \frac{(\rho+2(1-\rho)\mathrm{Re}\,(\overline{\lambda}z)+(\rho-2)|z|^2)(\rho+2(1-\rho)\mathrm{Re}\,(\overline{\mu}z)+(\rho-2)|z|^2)}{|1-\bar{z}\lambda|^2|1-\overline{\mu}z|^2}\|x\|^2\|y\|^2.\end{aligned}$$

So

$$\begin{aligned}&\Big|1+\frac{1-\bar{z}\lambda}{1-z\overline{\mu}}+(\rho-2)(1-\bar{z}\lambda)|z|^2\Big|^2|\langle x,y\rangle|^2\\ &\leqslant \frac{(\rho+2(1-\rho)\mathrm{Re}\,(\overline{\lambda}z)+(\rho-2)|z|^2)(\rho+2(1-\rho)\mathrm{Re}\,(\overline{\mu}z)+(\rho-2)|z|^2)}{|1-\overline{\mu}z|^2}\|x\|^2\|y\|^2.\end{aligned}$$

By tending z to λ, we get $\langle x,y\rangle = 0$. By Corollary 4 of [12] the subspace E reduces T_0 and T_1 and we can now easily derive the desired result. ∎

EXAMPLE 2.9. Recall that an operator $T \in B(H)$ is called to be quasi-compact (or quasi-strongly completely continuous in the terminology of [31]) if there exists a compact operator K and an integer m such that $\|T^m-K\| < 1$. Since every operator $T \in C_\rho(H)$ ($\rho \geqslant 1$) is power-bounded, by Theorem 4 of [31], if $T \in C_\rho(H)$ ($\rho \geqslant 1$) is a quasi-compact operator then $\Gamma(T) = \Gamma_p(T)$ and contains a finite number of eigenvalues and each of them is of finite multiplicity. Now if we assume that T_0, T_1 are two quasi-compact operators which are Harnack equivalent in $C_\rho(H)$, ($\rho \geqslant 1$), then $T_0 = U \oplus \widetilde{T}_0$ and $T_1 = U \oplus \widetilde{T}_1$ where U is a unitary diagonal operator on $E = \bigoplus_{\lambda\in\Gamma_p(T_0)} \mathcal{N}(T_0-\lambda I)$ and both operators $\widetilde{T}_0$ and $\widetilde{T}_1$ are Harnack equivalent to 0 in $C_\rho(E^\perp)$.

COROLLARY 2.10. *Let $T_0 \in C_\rho(H)$ ($\rho \geqslant 1$) be a compact normal operator with $w_\rho(T_0) = 1$. If the operator $T_1 \in C_\rho(H)$ is Harnack equivalent to T_0, then $T_{1|E} = T_{0|E}$ where $E = \bigoplus_{\lambda\in\Gamma_p(T_0)} \mathcal{N}(T_0-\lambda I)$, E is a reducing subspace for T_1 and $T_{1|E^\perp}$ is Harnack equivalent to* 0, *i.e.* $w_\rho(T_{1|E^\perp}) < 1$.

Proof. By Corollary 2.8, for all $\lambda \in \Gamma_p(T)$, we have $T_0 = U \oplus \widetilde{T}_0$ and $T_1 = U \oplus \widetilde{T}_1$ on $E \oplus E^\perp$, where $E = \bigoplus_{\lambda\in\Gamma_p(T_0)} \mathcal{N}(T_0-\lambda I)$ and $\widetilde{T}_0$ and $\widetilde{T}_1$ are Harnack equivalent in $C_\rho(E^\perp)$. Since $T_0 \in C_\rho(H)$ is a compact normal operator we also have

$$w_\rho(\widetilde{T_0}) = r(\widetilde{T_0}) = \sup\{|\lambda| : \lambda \in \sigma(T_0) \setminus \Gamma_p(T_0)\} < 1.$$

This means that $\widetilde{T_0}$ and $\widetilde{T_1}$ are Harnack equivalent to 0. ∎

In the following proposition, we prove that the ρ-contractions belonging to the same Harnack parts have the same null space for their operatorial ρ-kernels.

PROPOSITION 2.11. *Let $T_0, T_1 \in C_\rho(H)$ with $\Gamma(T_0) = \emptyset$. If T_0 and T_1 are Harnack equivalent in $C_\rho(H)$ then $\mathcal{N}(K_z^\rho(T_0)) = \mathcal{N}(K_z^\rho(T_1))$ for all $z \in \overline{\mathbb{D}}$.*

Proof. Since $T_0 \overset{H}{\sim} T_1$, then by Theorem 1.1 and Corollary 2.2, there exist $c \geqslant 1$ such that

$$\frac{1}{c^2} K_z^\rho(T_0) \leqslant K_z^\rho(T_1) \leqslant c^2 K_z^\rho(T_0), \quad \text{for all } z \in \overline{\mathbb{D}}. \tag{2.3}$$

If $x \in \mathcal{N}(K_z^\rho(T_0))$, then by the right side of the inequality (2.3), we also have

$$0 \leqslant \langle K_z^\rho(T_1)x, x\rangle \leqslant c^2 \langle K_z^\rho(T_0)x, x\rangle = 0.$$

This implies that $\left\| \sqrt{K_z^\rho(T_1)}x \right\| = 0$, so $K_z^\rho(T_1)x = 0$, hence, for all $z \in \overline{\mathbb{D}}$, $\mathcal{N}(K_z^\rho(T_0)) \subseteq \mathcal{N}(K_z^\rho(T_1))$. The converse inclusion holds by the left-side of the inequality (2.3). ∎

PROPOSITION 2.12. *If $w_\rho(T) = 1$ and $\Gamma(T)$ is empty then there exists $z_0 \in \mathbb{T}$ such that $K_{z_0}^\rho(T)$ is not invertible.*

Proof. Assume by absurdum that $K_z^\rho(T)$ is invertible for all $z \in \mathbb{T}$. We already know from Lemma 3 of [12] that $K_z^\rho(T)$ is invertible in $\mathbb{D}$, a continuity argument tells us that there exists a positive real number γ such that $K_z^\rho(T) \geqslant \gamma I$ for every $z \in \overline{\mathbb{D}}$. Since $\Gamma(T)$ is empty, we easily deduce that $K_z^\rho(T)$ is well defined in an open neighbourhood of $\overline{\mathbb{D}}$. Thus we can find $r > 1$ such that $K_z^\rho(rT) = K_{rz}^\rho(T) \geqslant (\gamma/2)I$ for any $z \in \mathbb{D}$. Hence $1 \leqslant w_\rho(T/r) = 1/r$, which is a contradiction. ∎

2.2. NUMERICAL RANGE PROPERTIES AND HARNACK DOMINATION.

Firstly, we give a proposition which is useful in this subsection.

PROPOSITION 2.13. *Let $T_0, T_1 \in C_{\rho_1}(H)$ and $\rho_2 \geqslant \rho_1$. Then we have:*

(i) *if $T_1 \overset{H}{\underset{c}{\prec}} T_0$ in $C_{\rho_1}(H)$, then $T_1 \overset{H}{\underset{c}{\prec}} T_0$ in $C_{\rho_2}(H)$;*

(ii) *if $T_1 \overset{H}{\underset{c}{\sim}} T_0$ in $C_{\rho_1}(H)$, then $T_1 \overset{H}{\underset{c}{\sim}} T_0$ in $C_{\rho_2}(H)$.*

Proof. (i) Since the C_ρ classes increase with ρ, the two operators T_0 and T_1 belong to $C_{\rho_2}(H)$. From Theorem 1.1, we know that there exists $c \geqslant 1$ such that

$K_z^{\rho_1}(T_1) \leqslant c^2 K_z^{\rho_1}(T_0)$ for all $z \in \mathbb{D}$. As $c \geqslant 1$, it yields to

$$K_z^{\rho_2}(T_1) = K_z^{\rho_1}(T_1) + (\rho_2 - \rho_1)I \leqslant c^2[K_z^{\rho_1}(T_0) + (\rho_2 - \rho_1)I] = c^2 K_z^{\rho_2}(T_0).$$

Using again Theorem 1.1, we obtained the desired conclusion.

The assertion (ii) is a direct consequence of (i). ∎

Let $T \in B(H)$, we denote by $W(T)$ the numerical range of T which is the set given by

$$W(T) = \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

The following result give relationships between numerical range and Harnack domination.

THEOREM 2.14. *Let $T_0, T_1 \in C_\rho(H)$ with $1 \leqslant \rho \leqslant 2$, then we have:*

(i) *assume that $\rho = 1$ and $T_1 \overset{H}{\prec} T_0$, then $\overline{W(T_0)} \cap \mathbb{T} = \overline{W(T_1)} \cap \mathbb{T}$;*

(ii) *suppose that $1 < \rho \leqslant 2$, $T_1 \overset{H}{\prec} T_0$ and $\Gamma(T_0) = \emptyset$, then $\overline{W(T_0)} \cap \mathbb{T} \subseteq \overline{W(T_1)} \cap \mathbb{T}$;*

(iii) *if $T_1 \overset{H}{\sim} T_0$, then $\overline{W(T_0)} \cap \mathbb{T} = \overline{W(T_1)} \cap \mathbb{T}$.*

Proof. (i) Let $\lambda = \mathrm{e}^{\mathrm{i}\omega} \in \overline{W(T_0)} \cap \mathbb{T}$, then there exists a sequence (x_n) of unit vectors such that $\langle T_0 x_n, x_n\rangle \to \lambda$. We have for some $c \geqslant 1$, $0 \leqslant K_{r,\theta}(T_1) \leqslant c^2 K_{r,\theta}(T_0)$ for all $z \in \mathbb{D}$. Multiplying these inequalities by the nonnegative function $1 - \mathrm{Re}(\overline{\lambda}\mathrm{e}^{\mathrm{i}\theta})$, integrating with respect to the Haar measure m and letting r tend to 1, we get $0 \leqslant I - \mathrm{Re}(\overline{\lambda} T_1) \leqslant c^2[I - \mathrm{Re}(\overline{\lambda} T_0)]$. We deduce that

$$1 - \mathrm{Re}(\overline{\lambda}\langle T_1 x_n, x_n\rangle) \to 0.$$

Since $\langle T_1 x_n, x_n\rangle$ belongs to the closed unit disc, it forces $\langle T_1 x_n, x_n\rangle \to \lambda$. Hence $\overline{W(T_0)} \cap \mathbb{T} \subseteq \overline{W(T_1)} \cap \mathbb{T}$. Now , let $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$, then there exists a sequence (y_n) of unit vectors such that $\langle T_1 y_n, y_n\rangle \to \lambda$. As T_1 is a contraction, it follows that $1 = \lim |\langle T_1 y_n, y_n\rangle| \leqslant \underline{\lim}\|T_1 y_n\| \leqslant \overline{\lim}\|T_1 y_n\| \leqslant 1$, thus $\|T_1 y_n\| \to 1$. It implies $\|T_1 y_n - \lambda y_n\|^2 = \|T_1 y_n\|^2 - 2\mathrm{Re}(\overline{\lambda}\langle T_1 y_n, y_n\rangle) + 1 \to 0$. Consequently, we have $\lambda \in \Gamma(T_1)$, by using Theorem 2.1 we see that $\lambda \in \Gamma(T_0) \subseteq \overline{W(T_0)} \cap \mathbb{T}$. So we get the desired equality.

(ii) Taking into account Proposition 2.13, it suffices to treat the case where $\rho = 2$. Let $\lambda = \mathrm{e}^{\mathrm{i}\omega} \in \overline{W(T_0)} \cap \mathbb{T}$, then there exists a sequence (x_n) of unit vectors such that $\langle T_0 x_n, x_n\rangle \to \lambda$. Set $y_n = (I - \mathrm{e}^{-\mathrm{i}\omega} T_0)x_n$, since $\Gamma(T_0) = \emptyset$ we necessarily have $\gamma = \inf\{\|y_n\| : n \geqslant 0\} > 0$. Taking $u_n = y_n/\|y_n\|$, we can see that

$$\begin{aligned}\langle K^2_{\mathrm{e}^{\mathrm{i}\omega}}(T_0)u_n, u_n\rangle &= \frac{2}{\|y_n\|^2}\langle (I - \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega} T_0))x_n, x_n\rangle \\ &\leqslant \frac{2}{\gamma^2}\langle (I - \mathrm{Re}(\mathrm{e}^{-\mathrm{i}\omega} T_0))x_n, x_n\rangle \to 0.\end{aligned}$$

Since $T_1 \overset{H}{\prec} T_0$, there exists $c \geqslant 1$ such that

$$K_z^2(T_1) \leqslant c^2 K_z^2(T_0), \quad \text{for all } z \in \mathbb{D}. \tag{2.4}$$

On one hand, if $\lambda \in \Gamma(T_1)$ we have obviously $\lambda \in \overline{W(T_1)}$. On the other hand, if $\lambda \notin \Gamma(T_1)$ we can extend (2.4) at $z = \lambda$ and we get

$$0 \leqslant \langle K^2_{e^{i\omega}}(T_1)u_n, u_n\rangle \leqslant c^2\langle K^2_{e^{i\omega}}(T_0)u_n, u_n\rangle \to 0,$$

hence $\langle K^2_{e^{i\omega}}(T_1)u_n, u_n\rangle \to 0$. Observe that $\inf\{\|(I - e^{-i\omega}T_1)^{-1}u_n\| : n \geqslant 0\} \geqslant 1/3$. Set $v_n = (1/\|(I - e^{-i\omega}T_1)^{-1}u_n\|)(I - e^{-i\omega}T_1)^{-1}u_n$, we obtain

$$\langle (I - \mathrm{Re}(e^{-i\omega}T_1))v_n, v_n\rangle \leqslant \frac{9}{2}\langle K^2_{e^{i\omega}}(T_1)u_n, u_n\rangle \to 0.$$

We deduce that $\langle \mathrm{Re}(e^{-i\omega}T_1)v_n, v_n\rangle \to 1$. As $T_1 \in C_2(H)$, it yields to:

$$1 \geqslant |\langle T_1 v_n, v_n\rangle|^2 = |\langle \mathrm{Re}(e^{-i\omega}T_1)v_n, v_n\rangle|^2 + |\langle \mathrm{Im}(e^{-i\omega}T_1)v_n, v_n\rangle|^2,$$

and we derive successively that $\langle \mathrm{Im}(e^{-i\omega}T_1)v_n, v_n\rangle \to 0$ and $\langle T_1 v_n, v_n\rangle \to \lambda$. Thus $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$ and it ends the proof of (ii).

(iii) As before, we may suppose that $\rho = 2$. Assume that $T_1 \overset{H}{\sim} T_0$ and $\lambda \in \overline{W(T_0)} \cap \mathbb{T}$. By Corollary 2.2, we have $\Gamma(T_0) = \Gamma(T_1)$. So, if $\lambda \in \Gamma(T_0)$ then $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$. Now, if $\lambda \notin \Gamma(T_0)$, we proceed as in the second item (ii) to prove that $\lambda \in \overline{W(T_1)} \cap \mathbb{T}$. Interchanging the roles of T_0 and T_1 gives the desired equality. ∎

REMARK 2.15. (i) The condition $\Gamma(T_0) = \emptyset$, in (ii), cannot be relaxed. In fact, we have $T_1 = 0_H \overset{H}{\underset{c}{\prec}} I = T_0$ in $C_\rho(H)$ $(1 < \rho \leqslant 2)$ with $c = \sqrt{\rho/(\rho - 1)}$ but $\overline{W(T_0)} \cap \mathbb{T} = \{1\}$ and $\overline{W(T_1)} \cap \mathbb{T} = \emptyset$.

(ii) When T is a contraction, we have $\overline{W(T)} \cap \mathbb{T} = \Gamma(T)$ (see for instance the end of the proof of (i)). So, the assertion (i) of Theorem 2.14 restore, in the case of domination, the equality of the spectral values in the torus obtained by C. Badea, D. Timotin and L. Suciu in [4] in another way.

COROLLARY 2.16. *Let $T_0 \in C_\rho(H)$ with $1 \leqslant \rho \leqslant 2$. If $\overline{W(T_0)} = \overline{\mathbb{D}}$, and satisfies $\Gamma(T_0) = \emptyset$ when $\rho \neq 1$, then $\overline{W(T_1)} = \overline{\mathbb{D}}$ for every $T_1 \in C_\rho(H)$ such that $T_1 \overset{H}{\prec} T_0$. Furthermore, in the case of Harnack equivalence, we have $\overline{W(T_1)} = \overline{\mathbb{D}}$ as soon as $\overline{W(T_0)} = \overline{\mathbb{D}}$.*

Proof. By Theorem 2.14, Proposition 2.13 and the convexity theorem of Toeplitz–Hausdorff, we obtain the desired conclusions. ∎

2.3. HARNACK PARTS IN THE SPACE OF COMPACT OPERATORS. Recall that the conorm $\gamma(A)$ of an operator $A \in B(H)$ is defined by setting

$$\gamma(A) = \inf\{\|Ax\| : x \in (\mathcal{N}(A)^\perp \text{ and } \|x\| = 1\}.$$

The Moore–Penrose inverse of A, denoted by $A^\dagger$, if it exists, is the unique solution of the following equations:

$$AA^\dagger A = A, \quad A^\dagger AA^\dagger = A^\dagger, \quad (AA^\dagger)^* = AA^\dagger \quad \text{and} \quad (A^\dagger A)^* = A^\dagger A.$$

It is well known that A has a Moore–Penrose inverse if and only if the range $\mathcal{R}(A)$ of A is closed. We denote by $\mathcal{K}(H)$ the set of all compact operators. The next result gives a characterization of Harnack equivalence in $C_\rho(H) \cap \mathcal{K}(H)$ for operators with no spectral values in $\mathbb{T}$ (we can reduce the problem to this case; see Remark 2.19 below).

THEOREM 2.17. *Let $T_0 \in C_\rho(H) \cap \mathcal{K}(H)$ with $w_\rho(T_0) = 1$ and $\Gamma_p(T_0)$ is empty. Then $T_1 \in C_\rho(H) \cap \mathcal{K}(H)$ is Harnack equivalent to T_0 if and only if we have:*

(i) *(null spaces condition) $\mathcal{N}(K_z^\rho(T_1)) = \mathcal{N}(K_z^\rho(T_0))$ for all $z \in \mathbb{T}$;*

(ii) *(conorms condition) $\inf_{z\in\mathbb{T}} \gamma\left(\sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0)\right)\sqrt{K_z^\rho(T_1)^\dagger}\Big) > 0$ and $\inf_{z\in\mathbb{T}} \gamma\left(\sqrt{K_z^\rho(T_0)^\dagger} K_z^\rho(T_1)\right)\sqrt{K_z^\rho(T_0)^\dagger}\Big) > 0$.*

Proof. Set $E_{T_0}(z) = \mathcal{N}(K_z^\rho(T_0))$ and $E_{T_1}(z) = \mathcal{N}(K_z^\rho(T_1))$. Let $T_0, T_1 \in C_\rho(H) \cap \mathcal{K}(H)$ such that $T_0 \overset{H}{\sim} T_1$. Since $\Gamma_p(T_0)$ is empty, by Corollary 2.2, the operators T_0 and T_1 admit no eigenvalues in $\mathbb{T}$. Hence, $K_z^\rho(T_0)$ and $K_z^\rho(T_1)$ are uniformly bounded in $\mathbb{D}$ and may be extended to positive operators on $\overline{\mathbb{D}}$. Furthermore, if we proceed as in the proof of Proposition 2.11, we deduce that $E_{T_0}(z) = E_{T_1}(z) := E(z)$ for all $z \in \mathbb{T}$. Let $P(z)$ denote the orthogonal projection on $E(z)$ and $Q(z) = I - P(z)$. Since T_0 is a compact operator with $r(T_0) < 1$, for any $z \in \overline{\mathbb{D}}$ both of the series $\sum_{n=1}^{+\infty} \overline{z}^n T_0^n$ and $\sum_{n=1}^{+\infty} z^n T_0^{*n}$ are convergent to a compact operator in the operator norm, so we can write $K_z^\rho(T_0) = \rho I + R_z(T_0)$ where $R_z(T_0)$ is a compact operator. We derive that $E_{T_0}(z)$ is a finite dimensional space, that the range $\mathcal{R}(K_z^\rho(T_0))$ of $K_z^\rho(T_0)$ is closed and we have $\mathcal{R}\left(\sqrt{K_z^\rho(T_0)}\right) = \mathcal{R}(K_z^\rho(T_0)) = E_{T_0}(z)^\perp$. Of course, analogous properties hold for $K_z^\rho(T_1)$. Therefore, the Moore–Penrose inverses of $K_z^\rho(T_0)$ and $K_z^\rho(T_1)$ are well defined. From Theorem 1.1, we know that there exists $c \geqslant 1$ such that

$$\frac{1}{c^2} K_z^\rho(T_0) \leqslant K_z^\rho(T_1) \leqslant c^2 K_z^\rho(T_0).$$

Firstly, we show that $E_1(z) := \mathcal{N}\left(\sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger}\right) = E(z)$. Let $x \in E_1(z)$ and write $x = x_1 + x_2$ with respect to the orthogonal decomposition $H = E(z) \oplus E(z)^\perp$. We can see that $x_2 = \sqrt{K_z^\rho(T_1)} y_2$ where $y_2 \in E(z)^\perp$ and then

$$\begin{aligned} 0 &= \left\langle K_z^\rho(T_0)\sqrt{K_z^\rho(T_1)^\dagger}x, \sqrt{K_z^\rho(T_1)^\dagger}x \right\rangle = \left\langle K_z^\rho(T_0)\sqrt{K_z^\rho(T_1)^\dagger}x_2, \sqrt{K_z^\rho(T_1)^\dagger}x_2 \right\rangle \\ &= \left\langle K_z^\rho(T_0)\sqrt{K_z^\rho(T_1)^\dagger}\sqrt{K_z^\rho(T_1)}y_2, \sqrt{K_z^\rho(T_1)^\dagger}\sqrt{K_z^\rho(T_1)}y_2 \right\rangle = \langle K_z^\rho(T_0)y_2, y_2\rangle. \end{aligned}$$

Hence $y_2 \in E_{T_0}(z) \cap E_{T_0}(z)^\perp = \{0\}$ which implies $x = x_1 \in E(z)$. The converse inclusion is obvious.

Now, observe that

$$\begin{aligned} Q(z) = K_z^\rho(T_1)^\dagger K_z^\rho(T_1) &= \sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_1) \sqrt{K_z^\rho(T_1)^\dagger} \\ &\leqslant c^2 \sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger}. \end{aligned}$$

Since the null space of the right hand operator is $E(z)$, the previous inequality implies that $\inf\limits_{z\in\mathbb{T}} \gamma\Big(\sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger}\Big) > 1/c^2$. In the same manner, we prove that $\inf\limits_{z\in\mathbb{T}} \gamma\Big(\sqrt{K_z^\rho(T_0)^\dagger} K_z^\rho(T_1) \sqrt{K_z^\rho(T_0)^\dagger}\Big) > 0$.

Conversely, set $\gamma = \inf\limits_{z\in\mathbb{T}} \gamma\Big(\sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger}\Big) > 0$. We have seen that

$$E_1(z) = \mathcal{N}\Big(\sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger}\Big) = E_{T_0}(z)$$

for any $z \in \mathbb{T}$. Hence, since $\sqrt{A^\dagger} = \sqrt{A}^\dagger$ for any positive operator with closed range, we have successively the following inequalities:

$$\gamma Q(z) \leqslant \sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger} \quad \text{and}$$

$$\begin{aligned} \gamma K_z^\rho(T_1) = \gamma Q(z) K_z^\rho(T_1) Q(z) &= \gamma \sqrt{K_z^\rho(T_1)}\, Q(z) \sqrt{K_z^\rho(T_1)} \\ &\leqslant \sqrt{K_z^\rho(T_1)} \sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger} \sqrt{K_z^\rho(T_1)} \\ &= Q(z) K_z^\rho(T_0) Q(z) = K_z^\rho(T_0). \end{aligned}$$

Thus we have shown that $T_1 \overset{H}{\prec} T_0$. In a similar way we can prove that $T_0 \overset{H}{\prec} T_1$ and the proof is ended. ∎

REMARK 2.18. (i) The condition "$\Gamma_p(T_0)$ is empty" can be relaxed. In this case, we can use Corollary 2.8 and apply Theorem 2.17 for $\widetilde{T}_0$ and $\widetilde{T}_1$ as in the decomposition of T_0 and T_1 respectively, given by the Corollary 2.8.

(ii) According to Theorem 1.5 of [23], we can see that the *conorms condition*

$$\begin{aligned} &\inf_{z\in\mathbb{T}} \gamma\Big(\sqrt{K_z^\rho(T_1)^\dagger} K_z^\rho(T_0) \sqrt{K_z^\rho(T_1)^\dagger}\Big) > 0 \quad \text{and} \\ &\inf_{z\in\mathbb{T}} \gamma\Big(\sqrt{K_z^\rho(T_0)^\dagger} K_z^\rho(T_1) \sqrt{K_z^\rho(T_0)^\dagger}\Big) > 0 \end{aligned}$$

are equivalent to the spectral conditions

$$\begin{aligned} &0 < \inf\{t \in \sigma(K_z^\rho(T_1)^\dagger K_z^\rho(T_0)) \setminus \{0\}\} \quad \text{and} \\ &0 < \inf\{t \in \sigma(K_z^\rho(T_0)^\dagger K_z^\rho(T_1)) \setminus \{0\}\}. \end{aligned}$$

REMARK 2.19. The following example shows that the *null spaces condition*

$$\mathcal{N}(K_z^\rho(T_1)) = \mathcal{N}(K_z^\rho(T_0)) \quad \text{for all } z \in \mathbb{T} \tag{2.5}$$

is not sufficient in general to ensure that T_1 is Harnack equivalent to T_0 in Theorem 2.17. We consider the following two operators acting on $\mathbb{C}^2$ which are given by the matrices

$$T_0 = \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix} \text{ and } T_1 = \begin{pmatrix} \frac{1}{2}+\frac{\mathrm{i}}{2} & 1 \\ 0 & \frac{1}{2}-\frac{\mathrm{i}}{2} \end{pmatrix}.$$

For any $z \in \overline{\mathbb{D}}$, we have

$$K_z^2(T_0) = \begin{pmatrix} \frac{2-\mathrm{Re}(z)}{\left|1-\frac{z}{2}\right|^2} & \frac{\overline{z}}{\left(1-\frac{\overline{z}}{2}\right)^2} \\ \frac{z}{\left(1-\frac{z}{2}\right)^2} & \frac{2-\mathrm{Re}(z)}{\left|1-\frac{z}{2}\right|^2} \end{pmatrix} \quad \text{and}$$

$$K_z^2(T_1) = \begin{pmatrix} \frac{2\sqrt{2}\left(\sqrt{2}-\mathrm{Re}(z\mathrm{e}^{-\mathrm{i}(\pi/4)})\right)}{\left|\sqrt{2}-\mathrm{e}^{-\mathrm{i}(\pi/4)}z\right|^2} & \frac{2\overline{z}}{\left(\sqrt{2}-\overline{z}\mathrm{e}^{\mathrm{i}(\pi/4)}\right)\left(\sqrt{2}-\overline{z}\mathrm{e}^{-\mathrm{i}(\pi/4)}\right)} \\ \frac{2z}{\left(\sqrt{2}-z\mathrm{e}^{-\mathrm{i}(\pi/4)}\right)\left(\sqrt{2}-z\mathrm{e}^{\mathrm{i}(\pi/4)}\right)} & \frac{2\sqrt{2}\left(\sqrt{2}-\mathrm{Re}(z\mathrm{e}^{\mathrm{i}(\pi/4)})\right)}{\left|\sqrt{2}-\mathrm{e}^{\mathrm{i}(\pi/4)}z\right|^2} \end{pmatrix}.$$

An easy computation gives

$$\delta_{T_0}(\theta) := \det(K^2_{\mathrm{e}^{\mathrm{i}\theta}}(T_0)) = 16\frac{(1-\cos\theta)(3-\cos\theta)}{|2-\mathrm{e}^{\mathrm{i}\theta}|^4} \quad \text{and}$$
$$\delta_{T_1}(\theta) := \det(K^2_{\mathrm{e}^{\mathrm{i}\theta}}(T_1)) = 8\frac{(1-\cos\theta)^2}{\left|\sqrt{2}-\mathrm{e}^{\mathrm{i}(\theta-(\pi/4))}\right|^2\left|\sqrt{2}-\mathrm{e}^{\mathrm{i}(\theta+(\pi/4))}\right|^2}.$$

Since $\mathrm{Tr}(K^2_{\mathrm{e}^{\mathrm{i}\theta}}(T_0)) > 0$ and $\delta_{T_0}(\theta) \geqslant 0$ for any $\theta \in \mathbb{R}$, from [10] we derive that $T_0 \in C_2(\mathbb{C}^2)$. The same argument is valid for T_1. Moreover, we easily see that $\mathcal{N}(K_1^2(T_1)) = \mathcal{N}(K_1^2(T_0)) = \mathbb{C}(1,-1)$ and $\mathcal{N}(K_z^2(T_1)) = \mathcal{N}(K_z^2(T_0)) = \{0\}$ for any $z \in \mathbb{T}\setminus\{1\}$. Suppose that T_0 is Harnack dominated by T_1, using Theorem 1.1 we see that there exists $c \geqslant 1$ such that $K_z^2(T_0) \leqslant c^2K_z^2(T_1)$ for any $z \in \mathbb{T}$. Hence $\delta_{T_0}(\theta) \leqslant c^4\delta_{T_1}(\theta)$ which implies

$$2\frac{3-\cos\theta}{|2-\mathrm{e}^{\mathrm{i}\theta}|^4} \leqslant c^4\frac{(1-\cos\theta)}{\left|\sqrt{2}-\mathrm{e}^{\mathrm{i}(\theta-(\pi/4))}\right|^2\left|\sqrt{2}-\mathrm{e}^{\mathrm{i}(\theta+(\pi/4))}\right|^2}$$

for any $\theta \in [-\pi,\pi]\setminus\{0\}$. Letting θ going to 0, we get $4 \leqslant 0$, a contradiction. Finally, T_1 and T_0 satisfy the *null spaces condition* (2.5) but are not Harnack equivalent.

In the sequel, we give a situation where the *null spaces condition* (2.5) ensures the Harnack equivalence (see Corollary 2.23).

The next proposition is concerned with the dimension of $E_T(z) = \mathcal{N}(K_z^\rho(T))$ for any compact operator T in $C_\rho(H)$ with $\Gamma_p(T) = \emptyset$. We set $d_T(z) = \dim(E_T(z))$ for any $z \in \mathbb{T}$. Notice that $d_T(z)$ is well defined ($\Gamma_p(T) = \emptyset$) and is always finite since T is assumed to be compact.

PROPOSITION 2.20. *Let $T \in C_\rho(H) \cap \mathcal{K}(H)$ with $\Gamma_p(T)$ be empty. Then, we have*

$$\sup_{z\in\mathbb{T}} d_T(z) < +\infty.$$

Proof. We proceed by absurdum and suppose that $\sup_{z\in\mathbb{T}} d_T(z) = +\infty$. Then there exists a sequence $(z_n) \subseteq \mathbb{T}$, converging to $z \in \mathbb{T}$, such that $d_T(z_n) \uparrow +\infty$. We easily see that we can find a strictly increasing sequence of positive integers (n_p) for which $d_T(z_{n_p}) > p$. Let E be an arbitrary subspace of H of dimension d. Observe that we have $E^\perp \cap E_T(z_{n_p}) \neq \{0\}$ for any $p > d$. Otherwise, we easily see that $E_T(z_{n_p}) = P_{E_T(z_{n_p})}(E)$ (where $P_{E_T(z_{n_p})}$ is the orthogonal projection $E_T(z_{n_p})$), a fact which is impossible to verify for any $p > d$. Thus, we can choose a unit vector u_p in $E^\perp \cap E_T(z_{n_p})$ for every $p > d$. Replacing (n_p) by one of its subsequences if necessary, we can assume that u_p is weakly convergent to a vector $u \in E^\perp$. Since $R_\alpha(T)$ is compact for any $\alpha \in \mathbb{T}$ and the map $\alpha \mapsto R_\alpha(T)$ is norm continuous on $\mathbb{T}$, we derive

$$0 = \langle K^\rho_{z_{n_p}}(T)u_p, u_p\rangle = \rho + \langle R_{z_{n_p}}(T)u_p, u_p\rangle \to \rho + \langle R_z(T)u, u\rangle \geqslant \langle K^\rho_z(T)u, u\rangle \geqslant 0.$$

It implies both that $u \in E_T(z)$ and that $u \neq 0$. We have obtained that $E_T(z) \cap E^\perp \neq \{0\}$ for an arbitrary finite-dimensional subspace E, a contradiction since $E_T(z)$ is of finite dimension. ∎

The following supremum-infimum formula was used in [8]. It will be useful in the sequel. Let E be a subspace of H, we denote by $E(1)$ the unit sphere of E.

LEMMA 2.21. *Let $A \in B(H)$ be an operator with finite dimensional kernel and closed range. Then we have*

$$\gamma(A) = \sup\{\inf\{\langle |A|x, x\rangle : x \in F^\perp(1)\} : F \subseteq H,\ \dim F \leqslant \dim(\mathcal{N}(A))\}.$$

Proof. The right hand side in the previous equality will be denoted by $\gamma'(A)$. From the definition, we easily deduce that $\gamma(A) = \gamma(|A|)$. Recall that the range of A is closed if and only if the range of $|A|$ is closed and we always have $\mathcal{N}(A) = \mathcal{N}(|A|)$. Thus, we are reduced to prove Lemma 2.21 when A is supposed to be a positive operator ($\langle Ax, x\rangle \geqslant 0$ for any $x \in H$). Let $\varepsilon > 0$, then there exists a unit vector u in $\mathcal{N}(A)^\perp$ such that $\langle Au, u\rangle \leqslant \|Au\| \leqslant \gamma(A) + \varepsilon$. Set $L = \mathcal{N}(A) + \mathbb{C}u$ and $d = \dim(\mathcal{N}(A))$, then we have

$$d + 1 - \dim(L \cap F^\perp) = \dim\Big(\frac{L}{L \cap F^\perp}\Big) = \dim\Big(\frac{L + F^\perp}{F^\perp}\Big) \leqslant \dim\Big(\frac{H}{F^\perp}\Big) \leqslant d,$$

for every subspace $F \subseteq H$ such that $\dim(F) \leqslant d$. Therefore, for any such subspace F, we have $\dim(L \cap F^\perp) \geqslant 1$ and we can find a unit vector $y \in L \cap F^\perp$. We write $y = x + \alpha u$ where $x \in \mathcal{N}(A)$ and $\alpha \in \mathbb{C}$. Since $y \in F^\perp(1)$, we have

$$\inf\{\langle |A|x, x\rangle : x \in F^\perp(1)\} \leqslant \langle Ay, y\rangle = |\alpha|^2\langle Au, u\rangle \leqslant \gamma(A) + \varepsilon.$$

Taking the supremum over all subspaces F of dimension less than d, we get $\gamma'(A) \leqslant \gamma(A) + \varepsilon$. As ε is arbitrary, we derive that $\gamma'(A) \leqslant \gamma(A)$. Then, the equality $\gamma'(A) = \gamma'(A)$ is obtained by considering the particular case where $F = \mathcal{N}(A)$. ∎

We also need the next lemma.

LEMMA 2.22. *We have the two following properties:*
(i) *let $A \in B(H)$ be a nonzero operator with closed range, then $\gamma(A^\dagger) = \|A\|^{-1}$;*
(ii) *let A and B be two positive operators, and suppose that $\mathcal{N}(A) = \mathcal{N}(B)$, then*

$$\gamma(\sqrt{B}A\sqrt{B}) \geqslant \gamma(A)\gamma(B).$$

Proof. (i) Since $\mathcal{R}(A)$ is closed if and only if $\mathcal{R}(A^*) = \mathcal{R}(A^\dagger)$ is closed and $(A^\dagger)^\dagger = A$, it follows from Proposition 1.3 of [23] that

$$\gamma(A^\dagger) = \frac{1}{\|(A^\dagger)^\dagger\|} = \frac{1}{\|A\|}.$$

(ii) Firstly we observe that $\mathcal{N}(\sqrt{B}A\sqrt{B}) = \mathcal{N}(A)$. Indeed let $x \in \mathcal{N}(\sqrt{B}A\sqrt{B})$; then we have $0 = \langle \sqrt{B}A\sqrt{B}x, x\rangle = \|\sqrt{A}\sqrt{B}x\|^2$ which implies that $\sqrt{B}x \in \mathcal{N}(\sqrt{A}) = \mathcal{N}(\sqrt{B})$. Hence $x \in \mathcal{N}(B) = \mathcal{N}(A)$. Thus, we have $\mathcal{N}(\sqrt{B}A\sqrt{B}) \subseteq \mathcal{N}(A)$. The converse inclusion is immediate.

Now, let x be a unit vector in $\mathcal{N}(A)^\perp = \overline{\mathcal{R}(A)}$. We see that $\sqrt{B}x$ is a nonzero vector in $\overline{\mathcal{R}(\sqrt{B})} = \overline{\mathcal{R}(B)} = \overline{\mathcal{R}(A)}$. Using Theorem 1.5 of [23], we see that $\gamma(T) = \inf\{t : t \in \sigma(T) \setminus \{0\}\} = \inf\{t : t \in W(T_{|\overline{\mathcal{R}(T)}})\}$ for any positive operator T. Then, we obtain

$$\Big\langle A\Big(\frac{\sqrt{B}x}{\|\sqrt{B}x\|}\Big), \frac{\sqrt{B}x}{\|\sqrt{B}x\|}\Big\rangle \geqslant \gamma(A),$$

and hence $\langle \sqrt{B}A\sqrt{B}x, x\rangle \geqslant \gamma(A)\langle Bx, x\rangle \geqslant \gamma(A)\gamma(B)$. Since $\overline{\mathcal{R}(\sqrt{B}A\sqrt{B})} = \overline{\mathcal{R}(A)}$, taking the infimum over all such x gives the desired inequality. ∎

The previous lemmas enable us to give a situation where the *null spaces condition* (2.5) is equivalent to Harnack equivalence in $C_\rho(H) \cap \mathcal{K}(H)$.

COROLLARY 2.23. *Let $T_0 \in C_\rho(H) \cap \mathcal{K}(H)$ with $w_\rho(T_0) = 1$ and $\Gamma_p(T_0)$ is empty. Assume that* $\dim(\mathcal{N}(K_z^\rho(T_0)))$ *is constant over $\mathbb{T}$. Then $T_1 \in C_\rho(H) \cap \mathcal{K}(H)$ is Harnack equivalent to T_0 if and only if we have: $\mathcal{N}(K_z^\rho(T_1)) = \mathcal{N}(K_z^\rho(T_0))$ for all $z \in \mathbb{T}$.*

Proof. The direct implication follows immediately from Theorem 2.17. We now prove the converse implication. For simplicity, we write $\gamma_{T_0}(z) = \gamma(K_z^\rho(T_0))$ (respectively $\gamma_{T_1}(z) = \gamma(K_z^\rho(T_1))$) for every $z \in \mathbb{T}$. Let z_1 and z_2 be two points of the torus and let F be a subspace of dimension less than $d := \dim(\mathcal{N}(K_z^\rho(T_0)))$

(for arbitrary $z \in \mathbb{T}$ by assumption). If y belongs to $F^{\perp}(1)$, we can write

$$\inf_{x \in F^{\perp}(1)} \{\langle K^{\rho}_{z_1}(T_0)x, x\rangle\} \leqslant \langle K^{\rho}_{z_1}(T_0)y, y\rangle$$
$$\leqslant \|K^{\rho}_{z_1}(T_0) - K^{\rho}_{z_2}(T_0)\| + \langle K^{\rho}_{z_2}(T_0)y, y\rangle.$$

We derive

$$\inf_{x \in F^{\perp}(1)} \{\langle K^{\rho}_{z_1}(T_0)x, x\rangle\} \leqslant \|K^{\rho}_{z_1}(T) - K^{\rho}_{z_2}(T)\| + \inf_{x \in F^{\perp}(1)} \{\langle K^{\rho}_{z_2}(T_0)x, x\rangle\}.$$

By taking the supremum over all such F and using Lemma 2.21, we obtain

$$\gamma_{T_0}(z_1) \leqslant \|K^{\rho}_{z_1}(T_0) - K^{\rho}_{z_2}(T_0)\| + \gamma_{T_0}(z_2).$$

Exchanging z_1 and z_2, we get

$$|\gamma_{T_0}(z_1) - \gamma_{T_0}(z_2)| \leqslant \|K^{\rho}_{z_1}(T_0) - K^{\rho}_{z_2}(T_0)\|.$$

Hence, the function γ_{T_0} is continuous on $\mathbb{T}$. Since T_0 is compact, the range of $K^{\rho}_{z}(T_0)$ is closed and consequently $\gamma_{T_0}(z) > 0$ for any $z \in \mathbb{T}$. The same properties are also satisfied by T_1. Thus, we have

$$\gamma_{T_0} := \inf_{z \in \mathbb{T}} \gamma_{T_0}(z) > 0 \quad \text{and} \quad \gamma_{T_1} := \inf_{z \in \mathbb{T}} \gamma_{T_1}(z) > 0.$$

By continuity and compactness, we also have

$$M(T_0) = \sup_{z \in \mathbb{T}} \|K^{\rho}_{z}(T_0)\| < +\infty \quad \text{and} \quad M(T_1) = \sup_{z \in \mathbb{T}} \|K^{\rho}_{z}(T_1)\| < +\infty.$$

Noticing that $\mathcal{N}(K^{\rho}_{z}(T_0)^{\dagger}) = \mathcal{N}(K^{\rho}_{z}(T_0)) = \mathcal{N}(K^{\rho}_{z}(T_1)) = \mathcal{N}(K^{\rho}_{z}(T_1)^{\dagger})$ and applying Lemma 2.22, we get

$$\gamma\Big(\sqrt{K^{\rho}_{z}(T_1)^{\dagger}}K^{\rho}_{z}(T_0)\sqrt{K^{\rho}_{z}(T_1)^{\dagger}}\Big) \geqslant \gamma(K^{\rho}_{z}(T_0))\gamma(K^{\rho}_{z}(T_1)^{\dagger}) = \gamma_{T_0}(z)\|K^{\rho}_{z}(T_1)\|^{-1}.$$

The operators T_0 and T_1 play the same role, thus we finally obtain

$$\inf_{z \in \mathbb{T}} \gamma\Big(\sqrt{K^{\rho}_{z}(T_1)^{\dagger}}K^{\rho}_{z}(T_0)\sqrt{K^{\rho}_{z}(T_1)^{\dagger}}\Big) \geqslant \gamma_{T_0} M(T_1)^{-1} > 0 \quad \text{and}$$
$$\inf_{z \in \mathbb{T}} \gamma\Big(\sqrt{K^{\rho}_{z}(T_0)^{\dagger}}K^{\rho}_{z}(T_1)\sqrt{K^{\rho}_{z}(T_0)^{\dagger}}\Big) \geqslant \gamma_{T_1} M(T_0)^{-1} > 0.$$

We conclude by using again Theorem 2.17. ∎

The following result gives a simple criterion for a unitary conjugate of T to be Harnack equivalent to T in $\in C_{\rho}(H) \cap \mathcal{K}(H)$ when $\dim(\mathcal{N}(K^{\rho}_{z}(T)))$ is constant over $\mathbb{T}$.

COROLLARY 2.24. *Let $T \in C_{\rho}(H) \cap \mathcal{K}(H)$ with $w_{\rho}(T) = 1$, $\Gamma_p(T)$ is empty and such that* $\dim(\mathcal{N}(K^{\rho}_{z}(T)))$ *is constant over $\mathbb{T}$. Let U be a unitary operator, then U^*TU is Harnack equivalent to T if and only if $U(\mathcal{N}(K^{\rho}_{z}(T))) \subseteq \mathcal{N}(K^{\rho}_{z}(T))$ for all $z \in \mathbb{T}$. Moreover, the set of unitary operators $U \in B(H)$ such that $U^*TU \overset{H}{\sim} T$ form a multiplicative group.*

Proof. We have $K_z^\rho(U^*TU) = U^*K_z^\rho(T)U$ and $\mathcal{N}(K_z^\rho(U^*TU)) = \mathcal{N}(K_z^\rho(T)U) = U^*\mathcal{N}(K_z^\rho(T))$. As we have seen in the proof of Theorem 2.17, we can write $K_z^\rho(T) = \rho I + R_z^\rho(T)$, where $R_z^\rho(T)$ is compact. Thus $\mathcal{N}(K_z^\rho(T))$ is finite dimensional. Assume that $U(\mathcal{N}(K_z^\rho(T))) \subseteq \mathcal{N}(K_z^\rho(T))$ for all $z \in \mathbb{T}$. Then, the restriction of U to $\mathcal{N}(K_z^\rho(T))$ is injective, and hence is surjective. Then $U\mathcal{N}(K_z^\rho(T)) = \mathcal{N}(K_z^\rho(T)) = U^*\mathcal{N}(K_z^\rho(T))$ and $\mathcal{N}(K_z^\rho(U^*TU)) = \mathcal{N}(K_z^\rho(T))$ for all $z \in \mathbb{T}$. By Corollary 2.23 we conclude that $U^*TU \overset{H}{\sim} T$. Conversely, assume that $U^*TU \overset{H}{\sim} T$. By Proposition 2.11, we derive that $\mathcal{N}(K_z^\rho(T)) = \mathcal{N}(K_z^\rho(U^*TU)) = \mathcal{N}(K_z^\rho(T)U) = U^*\mathcal{N}(K_z^\rho(T))$, which implies $U(\mathcal{N}(K_z^\rho(T))) \subseteq \mathcal{N}(K_z^\rho(T))$ for all $z \in \mathbb{T}$.

As we have seen before, the condition $U(\mathcal{N}(K_z^\rho(T))) \subseteq \mathcal{N}(K_z^\rho(T))$ for a unitary operator U is equivalent to $U(\mathcal{N}(K_z^\rho(T))) = \mathcal{N}(K_z^\rho(T))$. As a direct consequence we see that the set of unitary operators $U \in B(H)$ such that $U^*TU \overset{H}{\sim} T$ form a multiplicative group. ∎

In the case of contractions, the characterization of Harnack equivalence in $\mathcal{K}(H)$ is simpler.

THEOREM 2.25. *Let $T_0 \in C_1(H) \cap \mathcal{K}(H)$ with $\|T_0\| = 1$. Then $T_1 \in C_1(H) \cap \mathcal{K}(H)$ is Harnack equivalent to T_0 if and only if $E := \mathcal{N}(I - T_0^*T_0) = \mathcal{N}(I - T_1^*T_1)$ and $T_{0|E} = T_{1|E}$.*

Proof. According to Corollary 2.8, we are reduced to prove Theorem 2.25 with the extra assumption $\Gamma_p(T_0)$ is empty. Let $T_0, T_1 \in C_1(H) \cap \mathcal{K}(H)$ such that $T_0 \overset{H}{\sim} T_1$. On one hand, the fact that

$$K_z(T_0) = (I - zT_0^*)^{-1}[I - |z|^2 T_0^*T_0](I - \bar{z}T_0)^{-1},$$

easily implies that

$$\mathcal{N}(K_z(T_0)) = (I - \bar{z}T_0)(\mathcal{N}(I - T_0^*T_0)) \quad \text{for all } z \in \mathbb{T},$$

and similarly

$$\mathcal{N}(K_z(T_1)) = (I - \bar{z}T_1)(\mathcal{N}(I - T_1^*T_1)) \quad \text{for all } z \in \mathbb{T}.$$

From Proposition 2.11, we get

$$(I - \bar{z}T_0)(\mathcal{N}(I - T_0^*T_0)) = (I - \bar{z}T_1)(\mathcal{N}(I - T_1^*T_1)) \quad \text{for all } z \in \mathbb{T}.$$

We put $E = \mathcal{N}(I - T_0^*T_0)$. Let $x \in \mathcal{N}(I - T_1^*T_1)$ and $z \in \mathbb{T}$. Then $(I - zT_1)x = (I - zT_0)y(z)$ with $y(z) \in E$, hence $y(z) = (I - zT_0)^{-1}(I - zT_1)x$ has an analytic extension in a neighbourhood of $\overline{\mathbb{D}}$. It follows that $x = y(0) = \int\limits_0^{2\pi} y(\mathrm{e}^{\mathrm{i}\theta})\mathrm{d}m(\theta) \in E$, since E is closed. This proves $\mathcal{N}(I - T_1^*T_1) \subseteq E$. Now the equality holds by interchanging the roles of T_0 and T_1. Furthermore, for all $x \in E$ we have

$$y(z) = (I - zT_0)^{-1}(I - zT_1)x \in E \quad \text{for all } z \in \overline{\mathbb{D}}.$$

Notice that $y(z) = x + \sum\limits_{n=1}^{+\infty} z^n T_0^{n-1}(T_0 - T_1)x$. On the other hand, we have

$$T_0^{n-1}(T_0 - T_1)x = \int_0^{2\pi} \mathrm{e}^{-\mathrm{i}n\theta} y(\mathrm{e}^{\mathrm{i}\theta}) \mathrm{d}m(\theta) \in E \quad \text{for all } n \geqslant 1, \quad \text{and}$$

$$\langle (I - T_0^* T_0) T_0^{n-1}(T_0 - T_1)x, T_0^{n-1}(T_0 - T_1)x \rangle = 0 \quad \text{for all } n \geqslant 1.$$

Thus $\|T_0^{n-1}(T_0 - T_1)x\|^2 = \|T_0^n(T_0 - T_1)x\|^2$ for all $n \geqslant 1$, So

$$\|(T_0 - T_1)x\| = \|T_0^n(T_0 - T_1)x\|^2 \to 0,$$

because $r(T_0) < 1$. This implies that $T_0 x = T_1 x$ for all $x \in E$.

Conversely, if $E = \mathcal{N}(I - T_0^* T_0) = \mathcal{N}(I - T_1^* T_1)$ and $T_{0|E} = T_{1|E}$, then for all $z \in \mathbb{T}$, we have

$$\mathcal{N}(K_z(T_0)) = (I - zT_0)(\mathcal{N}(I - T_0^* T_0)) = (I - zT_1)(\mathcal{N}(I - T_1^* T_1)) = \mathcal{N}(K_z(T_1)).$$

Since $\Gamma_p(T_0) = \varnothing$, it follows that for any $z \in \mathbb{T}$, we have

$$d_{T_0}(z) = d := \dim(\mathcal{N}(I - T_0^* T_0)).$$

Then, we can apply Corollary 2.23 and we obtain $T_0 \overset{H}{\sim} T_1$. ∎

REMARK 2.26. In the case of classical contractions, observe that the *null spaces condition* (2.5) is always equivalent to the Harnack equivalence.

For each $n \geqslant 1$, let

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \ddots & 1 \\ 0 & 0 & \dots & \dots & 0 \end{pmatrix}$$

denotes the (nilpotent) Jordan block of size n. By Theorem 2.25 and the fact that $\mathcal{N}(I - J_n^* J_n) = \operatorname{span}\{e_2, \dots, e_n\}$, the Harnack part of J_n is given by the following corollary.

COROLLARY 2.27. *The Harnack part of J_n in $C_1(\mathbb{C}^n)$ is precisely the set of all matrices of the following form, where z is in the open unit disc:*

$$M = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & \ddots & 1 \\ z & 0 & \dots & \dots & 0 \end{pmatrix}.$$

In the case of compact operators, we deduce from Theorem 2.14 the next result.

PROPOSITION 2.28. *Let $T_0, T_1 \in C_\rho(H) \cap \mathcal{K}(H)$ with $1 \leqslant \rho \leqslant 2$, then we have:*

(i) *assume that $\rho = 1$ and $T_1 \overset{H}{\prec} T_0$, then $W(T_0) \cap \mathbb{T} = W(T_1) \cap \mathbb{T}$;*

(ii) *suppose that $1 < \rho \leqslant 2$, $T_1 \overset{H}{\prec} T_0$ and $\Gamma(T_0) = \emptyset$, then $W(T_0) \cap \mathbb{T} \subseteq W(T_1) \cap \mathbb{T}$;*

(iii) *if $T_1 \overset{H}{\sim} T_0$, then $W(T_0) \cap \mathbb{T} = W(T_1) \cap \mathbb{T}$.*

Proof. According to Theorem 2.14, it suffices to prove that $\overline{W(T)} \cap \mathbb{T} = W(T) \cap \mathbb{T}$ for each $T \in C_2(H) \cap \mathcal{K}(H)$. Indeed, let $\lambda \in \overline{W(T)} \cap \mathbb{T}$, then λ is a limit of scalar products $\langle Tx_n, x_n \rangle$ for some sequence (x_n) of unit vectors. Therefore, there exists a subsequence $(x_{j(n)})$ of (x_n) such that $x_{j(n)}$ converges to some x in the weak star topology. Since T is a compact operator, we have $Tx_{j(n)} \to Tx$ in the norm topology, this implies that $\lambda = \langle Tx, x \rangle$, and hence $x \neq 0$. Consequently, $\lambda / \|x\|^2 \in W(T) \subseteq \overline{\mathbb{D}}$. So $1/\|x\|^2 \leqslant 1$ and hence $\|x\|^2 \geqslant 1$, but we also have $\|x\|^2 \leqslant 1$, it tells us that $\|x\| = 1$ and $\lambda \in W(T)$. ∎

2.4. WEAK STABILITY AND HARNACK DOMINATION. One says that an operator is weakly stable if $\lim_{n \to +\infty} T^n = 0$ in the weak topology of $B(H)$. Also we have that this is equivalent to T^* is weakly stable. This notion plays an important role in analysis of operators (see for instance [18] and [22]).

We give the following proposition which is useful to study this property.

PROPOSITION 2.29. *Let H be a separable Hilbert space. Then, we have:*

(i) *Let $T \in C_\rho(H)$ and denote by V its minimal isometric ρ-dilation. Then, for every $m \geqslant 1$, we have*

$$\Big\| \sum_{k=1}^{m} V^{*k+1} x_k \Big\| \leqslant \Big\| \sum_{k=1}^{m} T^{*k} x_k \Big\| \leqslant \rho \Big\| \sum_{k=1}^{m} V^{*k} x_k \Big\|$$

for any m-tuple $(x_1, \dots, x_m)$ of vectors of H.

(ii) *Assume that T_1 be Harnack dominated by T_0 in $C_\rho(H)$ for a constant $c \geqslant 1$. If V_i acting on $K_i \supseteq H$ is the minimal isometric ρ-dilation of T_i $(i = 0, 1)$, then we have the following, for any m-tuple $(x_1, \dots, x_m)$ of vectors of H:*

$$\Big\| \sum_{k=1}^{m} V_1^k x_k \Big\| \leqslant c \Big\| \sum_{k=1}^{m} V_0^k x_k \Big\|.$$

Proof. (i) Let $h = \sum_{i=0}^{n} V^i h_i$ with $h_i \in H$, then we have

$$\begin{aligned} \Big\langle \sum_{k=1}^{m} T^{*k} x_k, Vh \Big\rangle &= \sum_{k=1}^{m} \sum_{i=0}^{n} \langle T^{*k} x_k, V^{i+1} h_i \rangle = \frac{1}{\rho} \sum_{k=1}^{m} \sum_{i=0}^{n} \langle T^{*k} x_k, T^{i+1} h_i \rangle \\ &= \sum_{k=1}^{m} \sum_{i=0}^{n} \langle V^{*k+i+1} x_k, h_i \rangle = \Big\langle \sum_{k=1}^{m} V^{*k+1} x_k, h \Big\rangle. \end{aligned}$$

Since the subset of all elements h having the above form is dense in K, we get

$$\Big\|\sum_{k=1}^{m} T^{*k}x_k\Big\| = \sup_{\|h\|=1}\Big|\Big\langle \sum_{k=1}^{m} T^{*k}x_k, h\Big\rangle\Big| \geqslant \sup_{\|h\|=1}\Big|\Big\langle \sum_{k=1}^{m} T^{*k}x_k, Vh\Big\rangle\Big|$$
$$\geqslant \sup_{\|h\|=1}\Big|\Big\langle \sum_{k=1}^{m} V^{*k+1}x_k, h\Big\rangle\Big| = \Big\|\sum_{k=1}^{m} V^{*k+1}x_k\Big\|$$

and the left-hand side inequality is obtained. The right-hand side inequality is obvious.

(ii) Now, suppose that $T_1 \overset{H}{\underset{c}{\prec}} T_0$ in $C_\rho(H)$ and V_i acting on $K_i \supseteq H$ is the minimal isometric ρ-dilation of T_i $(i = 0, 1)$. Using Theorem 1.1, we know that there exists an operator $S \in B(K_0, K_1)$ such that $S(H) \subset H$, $S|_H = I$, $SV_0 = V_1S$ and $\|S\| \leqslant c$. Let $(x_1, \dots, x_m)$ be an m-tuple of vectors of H. Observe that $SV_0^k = V_1^kS$ for any positive integer k, thus we get

$$\Big\|\sum_{k=1}^{m} V_1^k x_k\Big\| = \Big\|\sum_{k=1}^{m} V_1^k S x_k\Big\| = \Big\|S\Big[\sum_{k=1}^{m} V_0^k x_k\Big]\Big\| \leqslant c\Big\|\sum_{k=1}^{m} V_0^k x_k\Big\|. \quad ∎$$

LEMMA 2.30. *A ρ-contraction T is weakly stable if and only if the minimal isometric ρ-dilation of T is weakly stable.*

Proof. Let us assume that T is weakly stable and $[V, K]$ is the minimal isometric ρ-dilation of T. Hence T^* is also weakly stable, i.e. $T^{*n}h \to 0$ in the weak topology. Since T^* has the Blum–Hanson property, for each $h \in H$ and every increasing sequence $(k_n)_{n\geqslant 0}$ of positive integers, we have

$$\frac{1}{N}\sum_{n=0}^{N} T^{*k_n}h \to 0$$

in the norm topology. For each N, set $x_k = h/N$ if there exists an integer n such that $k = k_n$ and $x_k = 0$ otherwise, and use Proposition 2.29(i). We derive

$$\frac{1}{N}\sum_{n=1}^{N} V^{*k_n+1}h \to 0.$$

It is enough to ensure that

$$\frac{1}{N}\sum_{n=0}^{N} V^{*l_n}x \to 0, \tag{2.6}$$

for any increasing sequence $(l_n)_{n\geqslant 0}$ of positive integers and any $x \in H$. Now, let $x = \sum\limits_{i=1}^{m} V^i x_i$ with $x_i \in H$; we easily deduce from (2.6) that

$$\frac{1}{N}\sum_{n=1}^{N} V^{*l_n}x \to 0.$$

Since the subset of all elements x having the above form is dense in K and the sequence of operators $1/N\Big[\sum\limits_{n=1}^{N} V^{*l_n}\Big]$ is a sequence of contractions, we derive that V^* has the Blum–Hanson property. Thus, the sequence $(V^{*n}x)$ weakly converges to 0 for any $x \in K$. Hence V is weakly stable.

Conversely, assume that V is weakly stable. Then for each $(x,y) \in H^2$ and any $n \geqslant 1$, we have $\langle T^n x, y\rangle = \rho\langle V^n x, y\rangle \to 0$. Hence, T is weakly stable. ∎

COROLLARY 2.31. *Let T_0 and T_1 be two operators in $C_\rho(H)$. Then, we have:*

(i) *Assume that T_1 is Harnack dominated by T_0 in $C_\rho(H)$ and that T_0 is weakly stable (respectively stable). Then T_1 is also weakly stable (respectively stable).*

(ii) *Let T_0 and T_1 be Harnack equivalent in $C_\rho(H)$. Then T_0 is weakly stable (respectively stable) if and only if T_1 is weakly stable (respectively stable).*

Proof. (i) Assume that T_0 is weakly stable. Using Lemma 2.30, we see that the minimal isometric ρ-dilation V_0 is weakly stable. Applying Proposition 2.29(ii) and using the Blum–Hanson property as in the proof of Lemma 2.30, we deduce than V_1 is weakly stable. Using again Lemma 2.30, we obtain the weak stability of T_1.

Now, suppose that T_0 is stable. We deduce from Lemma 3.5 of [11] that V_0 is stable. From Proposition 2.29(ii) we derive that V_1 is stable. Then, by Lemma 3.5 of [11] we obtain the stability of T_1.

The assertion (ii) is a direct consequence of (i). ∎

REMARK 2.32. (i) Concerning the stability of two Harnack equivalent ρ-contractions, the assertion (ii) is exactly Corollary 3.6 of [11].

(ii) Since any ρ-contraction T is similar to a contraction and power bounded, by Proposition 8.5 of [22], the residual spectrum $\sigma_{\mathrm{r}}(T)$ of T is included in $\mathbb{D}$. By Proposition 8.4 of [22] it follows that if any ρ-contraction T is weakly stable then $\sigma_{\mathrm{p}}(T) \subseteq \mathbb{D}$. In this case, according to Lemma 2.30, if V is the minimal isometric ρ-dilation of T, then $\Gamma(V) = \sigma_{\mathrm{c}}(V)$. So, if there exists $\lambda \in \sigma_{\mathrm{p}}(T)$ such that $|\lambda| = 1$ then T is not weakly stable and this ρ-contraction is in the Harnack part of an operator with *ρ-numerical radius* one.

3. EXAMPLES OF HARNACK PARTS FOR SOME NILPOTENT MATRICES WITH NUMERICAL RADIUS ONE

In the sequel, we describe the Harnack parts of some nilpotent matrices with numerical radius one. We begin by a nilpotent matrix of order two in the dimension two.

THEOREM 3.1. *Let $T_0 = \left(\begin{smallmatrix} 0 & 2 \\ 0 & 0 \end{smallmatrix}\right) \in C_2(\mathbb{C}^2)$; then the Harnack part of T_0 is reduced to $\{T_0\}$.*

Proof. Let $T_1 \in C_2(\mathbb{C}^2)$ such that $T_1 \overset{H}{\sim} T_0$, then by Theorem 1.1, there exists $c \geqslant 1$ such that

$$\frac{1}{c^2} K_z^2(T_0) \leqslant K_z^2(T_1) \leqslant c^2 K_z^2(T_0), \quad \text{for all } z \in \mathbb{D}. \tag{3.1}$$

By Corollary 2.2, the operator T_1 admits no eigenvalues in $\mathbb{T}$. Hence, $K_z^2(T_0)$ and $K_z^2(T_1)$ are uniformly bounded in $\mathbb{D}$ and may be extended to a positive operator on $\overline{\mathbb{D}}$.

We have

$$K_z^2(T_0) = 2 \begin{pmatrix} 1 & \overline{z} \\ z & 1 \end{pmatrix},$$

thus $\det(K_z^2(T_0)) = 4(1 - |z|^2)$ and $d_{T_0}(z) = 1$ over $\mathbb{T}$. Let $v(z) = \left(\begin{smallmatrix} 1 \\ -z \end{smallmatrix}\right)$, then $K_z^2(T_0)v(z) = 0$ on $\mathbb{T}$. This implies by (3.1) that

$$0 = K_{1,\theta}^2(T)v(\mathrm{e}^{\mathrm{i}\theta}) = K_{1,\theta}^2(T)e_1 - \mathrm{e}^{\mathrm{i}\theta} K_{1,\theta}^2(T)e_2 = 0 \quad \text{for all } \theta \in \mathbb{R}. \tag{3.2}$$

Multiplying successively (3.2) by 1 and $\mathrm{e}^{-\mathrm{i}\theta}$, and integrating with respect to the Haar measure m on the torus, we obtain: $Te_2 = 2e_1$ and $T^* e_1 = 2e_2$. Thus T takes the form

$$T = \begin{pmatrix} 0 & 2 \\ b & 0 \end{pmatrix},$$

with $b \in \mathbb{C}$. Since $w(T) \leqslant 1$, we have

$$|2x_2\overline{x}_1 + bx_1\overline{x}_2| \leqslant |x_1|^2 + |x_2|^2.$$

If we take $x_1 = \sqrt{2}/2$ and $x_2 = \sqrt{2}/2\mathrm{e}^{\mathrm{i}\theta}$, we get

$$|1 + b\mathrm{e}^{-2\mathrm{i}\theta}| \leqslant 1.$$

In particular, for $\theta = \arg b/2$

$$1 + |b| \leqslant 1$$

This implies that $b = 0$ and $T = T_0$. ∎

In the following result, we describe the Harnack part of a nilpotent matrix of order two in $C_2(\mathbb{C}^n)$, $n \geqslant 3$, with numerical radius one.

THEOREM 3.2. *Let $T_0 \in C_2(\mathbb{C}^n)$, $n \geqslant 3$ such that*

$$T_0 = \begin{pmatrix} 0 & 0 & \dots & a \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

with $|a| = 2$; then the Harnack part of T_0 is the set of all matrices of $C_2(\mathbb{C}^n)$ of the following form, with $B \in C_2(\mathbb{C}^{n-2})$ such that $w(B) < 1$:

$$T_1 = \begin{pmatrix} 0 & 0 & a \\ 0 & B & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Proof. Let $T \in C_2(\mathbb{C}^n)$ such that $T_1 \overset{H}{\sim} T_0$. By Corollary 2.2, the operator T_1 admits no eigenvalues in $\mathbb{T}$. Hence, $K_z^2(T_0)$ and $K_z^2(T_1)$ are uniformly bounded in $\mathbb{D}$ and may be extended to a positive operators on $\overline{\mathbb{D}}$. We have

$$K_z^2(T_0) = \begin{pmatrix} 2 & 0 & a\bar{z} \\ 0 & 2I_{n-2} & 0 \\ \bar{a}z & 0 & 2 \end{pmatrix},$$

where I_{n-2} denotes the identity matrix on the linear space spanned by the vectors $e_2, \dots, e_{n-1}$ of the canonical basis of $\mathbb{C}^n$. Then $\det(K_z^2(T_0)) = 2^{n-2}(4 - |a|^2|z|^2)$. Let $v(z) = -a\bar{z}e_1 + 2e_n$, then $K_z^2(T_0)v(z) = 0$ on $\mathbb{T}$. Thus by Proposition 2.11, $K_z^2(T_1)v(z) = 0$ on $\mathbb{T}$. This implies that

$$-\,a\mathrm{e}^{-\mathrm{i}\theta}K_{1,\theta}^2(T_1)e_1 + 2K_{1,\theta}^2(T_1)e_n = 0 \quad \text{for all } \theta \in \mathbb{R}. \tag{3.4}$$

Multiplying successively (3.4) by 1, $\mathrm{e}^{\mathrm{i}\theta}$, $\mathrm{e}^{-\mathrm{i}\theta}$ and $\mathrm{e}^{2\mathrm{i}\theta}$, and integrating with respect to m, we obtain:

$$T_1^*e_1 = \bar{a}e_n, \quad T_1e_n = ae_1, \quad T_1^*e_n = 0 \quad \text{and} \quad T_1e_1 = 0. \tag{3.5}$$

By (3.5) , the matrix T_1 take the form (3.3). Hence

$$K_z^2(T_1) = \begin{pmatrix} 2 & 0 & a\bar{z} \\ 0 & K_z^2(B) & 0 \\ \bar{a}z & 0 & 2 \end{pmatrix}. \tag{3.6}$$

By Theorem 2.17, we know that $\mathcal{N}(K_z^2(T_1)) = \mathcal{N}(K_z^2(T_0))$ for all $z \in \mathbb{T}$, it forces $\mathcal{N}(K_z^2(B))$ to be equal to $\{0\}$ for every $z \in \mathbb{T}$. It is clear that B belongs to $C_2(\mathbb{C}^{n-2})$ and according to Proposition 2.12 we should have $w(B) < 1$; we derive that B is Harnack equivalent to 0.

Conversely, let $T_1 \in C_2(\mathbb{C}^n)$ given by (3.3); then we can write $K_z^2(T_1)$ under the form given by (3.6). Since $B \in C_2(\mathbb{C}^{n-2})$ with $w(B) < 1$, B is Harnack equivalent to 0 in $C_2(\mathbb{C}^{n-2})$. Then by Theorem 1.1, there exists $c \geqslant 1$ such that

$$2\frac{1}{c^2}I_{n-2} \leqslant K_z^2(B) \leqslant 2c^2 I_{n-2}, \quad \text{for all } z \in \mathbb{D}.$$

Thus

$$\frac{1}{c^2}K_z^2(T_0) \leqslant K_z^2(T_1) \leqslant c^2 K_z^2(T_0), \quad \text{for all } z \in \mathbb{D}.$$

This means that T_1 is Harnack equivalent to T_0. ∎

THEOREM 3.3. *Let* $T_0 = \left(\begin{smallmatrix} 0 & a & 0 \\ 0 & 0 & a \\ 0 & 0 & 0 \end{smallmatrix}\right)$ *such that* $|a| = \sqrt{2}$; *then the Harnack part of* T_0 *is the set of all matrices of* $C_2(\mathbb{C}^3)$ *of the form*

$$T_1 = a\begin{pmatrix} 0 & \mathrm{e}^{-\mathrm{i}\theta} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\theta} \\ 0 & 0 & 0 \end{pmatrix}, \quad \theta \in \mathbb{R}.$$

Proof. Let $T_1 \in C_2(\mathbb{C}^3)$ such that $T_1 \overset{H}{\sim} T_0$. By Corollary 2.2, the operator T_1 admits no eigenvalues in $\mathbb{T}$. Hence, $K_z^2(T_0)$ and $K_z^2(T_1)$ are uniformly bounded in $\mathbb{D}$ and may be extended to positive operators on $\overline{\mathbb{D}}$. Furthermore, by Theorem 5.2 of [7] $T_1^2 \overset{H}{\sim} T_0^2$; then by Theorem 3.2, the operator T_1^2 takes the following form:

$$T_1^2 = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with $|b| < 1$. If $b \neq 0$ then $\mathcal{N}(T_1^2) = \mathbb{C}e_1$ is invariant by T_1, so $T_1 e_1 = x e_1$ but $0 = T_1^2 e_1 = x^2 e_1$, which implies that $x = 0$ and $T_1 e_1 = 0$. Similarly, $\mathbb{C}^3 \neq \mathcal{R}(T_1) \supseteq \mathcal{R}(T_1^2) = \operatorname{span}\{e_1, e_2\}$ which is invariant by T_1, so $T_1 e_2 = u e_1 + v e_2$ for some $u, v \in \mathbb{C}$. On the other hand, we have

$$K_z^2(T_0) = \begin{pmatrix} 2 & a\bar{z} & a^2\bar{z}^2 \\ \bar{a}z & 2 & a\bar{z} \\ \bar{a}^2 z^2 & \bar{a}z & 2 \end{pmatrix},$$

thus $\det(K_z^2(T_0)) = 4(2 - |a|^2|z|^2)$, so $d_{T_0}(z) = 1$ on $\mathbb{T}$. Let $v(z) = -a^2\bar{z}e_1 + 2ze_3$; then $\mathcal{N}(K_z^2(T_0)) = \mathbb{C}v(z)$ on $\mathbb{T}$. Thus by Proposition 2.11, $K_z^2(T)v(z) = 0$ on $\mathbb{T}$. This implies that

$$-a^2 \mathrm{e}^{-\mathrm{i}\theta} K_{1,\theta}^2(T_1)e_1 + 2\mathrm{e}^{\mathrm{i}\theta} K_{1,\theta}^2(T_1)e_3 = 0 \quad \text{for all } \theta \in \mathbb{R}. \tag{3.7}$$

Using (3.7) in a similar way than before, we get

$$2T_1 e_3 = a^2 T_1^* e_1 \quad \text{and} \quad 2T_1^* e_3 = a^2 T_1 e_1. \tag{3.8}$$

By this we deduce that

$$\begin{aligned} \langle T_1 e_3, e_1 \rangle &= \frac{a^2}{2} \langle T_1^* e_1, e_1 \rangle = \frac{a^2}{2} \langle e_1, T_1 e_1 \rangle = 0 \quad \text{and} \\ \langle T_1 e_3, e_3 \rangle &= \frac{a^2}{2} \langle T_1^* e_1, e_3 \rangle = \frac{a^2}{2} \langle e_1, T e_3 \rangle = 0. \end{aligned}$$

The matrix T_1 takes the form

$$T_1 = \begin{pmatrix} 0 & u & 0 \\ 0 & v & w \\ 0 & 0 & 0 \end{pmatrix}.$$

By (3.8), $2w = a^2 T^* e_1 = a^2 \bar{u} e_2$, hence

$$\bar{a} w = a \bar{u}. \tag{3.9}$$

This implies that u and v must not be equal to 0. Now the fact that

$$T_1^2 = \begin{pmatrix} 0 & uv & uw \\ 0 & v^2 & wv \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a^2 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

implies that $v = b = 0$ and

$$uw = a^2. \tag{3.10}$$

By (3.9) and (3.10) we can deduce that $u = a\mathrm{e}^{-\mathrm{i}\theta}$ and $w = a\mathrm{e}^{\mathrm{i}\theta}$, $\theta \in \mathbb{R}$.

Conversely, let $T \in C_2(\mathbb{C}^3)$ given as above; then

$$K_z^2(T_1) = \begin{pmatrix} 2 & u\overline{z} & a^2\overline{z}^2 \\ \overline{u}z & 2 & w\overline{z} \\ \overline{a}^2 z^2 & \overline{w}z & 2 \end{pmatrix}.$$

Observe that

$$T_1 = U_\theta^* T_0 U_\theta \quad \text{with } U_\theta = \begin{pmatrix} \mathrm{e}^{\mathrm{i}\theta} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\theta} \end{pmatrix}, \theta \in \mathbb{R}.$$

We easily verify that U_θ satisfies the hypotheses of Corollary 2.24. Hence T_1 is Harnack equivalent to T_0. ∎

REMARK 3.4. In the last example, observe that the Harnack part of T_0 is exactly the orbit $\{U_\theta^* T_0 U_\theta : \theta \in \mathbb{R}\}$ under the action of the group given in Corollary 2.24.

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