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A C^*-ALGEBRAIC APPROACH TO THE PRINCIPAL SYMBOL. I

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ABSTRACT. We construct a generalised principal symbol mapping as a morphism of C^*-algebras. Variants of Connes' trace theorem are proved in terms of the principal symbol.

KEYWORDS: *Principal symbol, C^*-algebras, singular traces, Connes Trace Theorem.*

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1. INTRODUCTION

One of the fundamental concepts in the theory of pseudodifferential operators (henceforth: PSDO) is that of the principal symbol. The conventional definition of a PSDO (as discussed in, e.g. [48] and Chapter 2 of [42]) is as follows: first, for $m \in \mathbb{R}$ the symbol class $S^m(\mathbb{R}^d \times \mathbb{R}^d)$ is defined to be the set of $\sigma \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that for all multi-indices α and β there exists a constant $C_{\alpha,\beta}$ such that

$$|\partial_x^\beta \partial_\xi^\alpha \sigma(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}.$$

Then the pseudodifferential operator T_σ defined by $\sigma \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$ is the linear operator on the Schwartz class $\mathcal{S}(\mathbb{R}^d)$ given by the integral,

$$(T_\sigma f)(x) = (2\pi)^{-d/2} \int\limits_{\mathbb{R}^d} \sigma(x,\xi)\widehat{f}(\xi)\mathrm{e}^{\mathrm{i}(x,\xi)}\,\mathrm{d}\xi, \quad x \in \mathbb{R}^d, f \in \mathcal{S}(\mathbb{R}^d)$$

where $\widehat{f}$ denotes the Fourier transform of f. The operator T_σ is a well-defined linear map from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}(\mathbb{R}^d)$ ([42], Theorem 2.1.6). We denote $\Psi^m(\mathbb{R}^d) := \{T_\sigma : \sigma \in S^m(\mathbb{R}^d \times \mathbb{R}^d)\}$, which we call the set of order m PSDO.

Given $T_\sigma \in \Psi^0(\mathbb{R}^d)$, the principal symbol of T_σ is usually defined by the following procedure (as outlined in Section 2.5.3 of [42]). First, one assumes that there is a formal asymptotic expansion

$$\sigma \sim \sum_{k=0}^{\infty} \sigma_{-k}$$

where each σ_{-k} is in $S^{-k}(\mathbb{R}^d \times \mathbb{R}^d)$, and the meaning of the asymptotic expansion is that for all $N \geqslant 0$, the difference $\sigma - \sum_{k=0}^{N} \sigma_{-k}$ is in $S^{-N-1}(\mathbb{R}^d \times \mathbb{R}^d)$. Furthermore, each function σ_{-k} is assumed to be positively homogeneous "away from zero" in the sense that for each k there exists an $r > 0$ such that for all $\lambda > 1$, $x \in \mathbb{R}^d$ and $|\xi| > r$, we have $\sigma_{-k}(x, \lambda\xi) = \lambda^{-k}\sigma_{-k}(x, \xi)$.

If such an asymptotic expansion exists, then the operator T_σ is called a classical pseudodifferential operator of order 0, and the set of all such operators is denoted $\Psi^0_{\mathrm{cl}}(\mathbb{R}^d)$. Then the principal symbol of $T_\sigma \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^d)$ is defined to be σ_0. Since σ_0 is homogenous in the second variable, it is canonically identified with a function on $\mathbb{R}^d \otimes \mathbb{S}^{d-1}$. It can be shown that the mapping $\Psi^0_{\mathrm{cl}}(\mathbb{R}^d) \to C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ given by $T_\sigma \mapsto \sigma_0$ is well-defined, and is in fact an algebra homomorphism in the sense that it is linear and if $T, S \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^d)$, then $T \circ S \in \Psi^0_{\mathrm{cl}}(\mathbb{R}^d)$ and the principal symbol of $T \circ S$ is the product of the principal symbols of T and S (see Theorem 2.5.1 of [42]).

In applications it is often useful to work with a C^*-subalgebra of the algebra of all bounded linear operators on $L_2(\mathbb{R}^d)$ containing $\Psi^0(\mathbb{R}^d)$. The purpose of this paper is twofold: first, to illustrate that if one restricts attention to order zero classical pseudodifferential operators then the principal symbol may be defined in a straightforward algebraic manner, and secondly we define a C^*-subalgebra Π of the set of bounded linear operators on $L_2(\mathbb{R}^d)$ which is general enough that Π contains (possibly up to compact perturbations) all of $\Psi^0(\mathbb{R}^d)$, and in addition it contains operators such as pointwise multiplication by bounded functions which need not be smooth or even continuous. We then show that with this new definition there is a unique extension of the principal symbol as a C^*-algebra homomorphism from Π to $L_\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$.

We illustrate the usefulness of this extension of the principal symbol mapping by providing a variant of Connes' trace formula (Theorem 1.5), and using this extension to give a simple proof of an equality recently established by the authors by an ad hoc method in [29] (see Theorem 1.7 below).

The observation that the principal symbol map of pseudo-differential operators can be extended to C^*-algebras is not new, but mainly it has involved maps to continuous functions ([43], Remark 1.8; [40], p. 134; [33], p. 288; [26], p. 197). Principal symbol maps are well-known as inverses of quantisation [3], [21], [39]. The principal symbol introduced here is associated to Kohn–Nirenberg quantisation.

Here we restrict attention to the study of pseudodifferential operators on $\mathbb{R}^d$. In a subsequent paper [32], we will consider a more general case including an extension to certain noncommutative spaces.

Let $D_k = \frac{\partial}{\mathrm{i}\partial t_k}$, $k = 1, \ldots, d$ be the k-th partial derivative operator on $\mathbb{R}^d$. These operators extend to unbounded self-adjoint operators on $L_2(\mathbb{R}^d)$ within the common invariant core $\mathcal{S}(\mathbb{R}^d)$. In what follows, $\nabla := (D_1, \ldots, D_d)$ and

$\Delta := \sum_{k=1}^{d} \frac{\partial^2}{\partial^2 t_k} = -\sum_{k=1}^{d} D_k^2$ denote the gradient and the Laplacian, respectively. For $1 \leqslant k \leqslant d$, the operator $\frac{D_k}{(-\Delta)^{1/2}}$ is defined by the Borel functional calculus (that is, we apply a bounded function $t \to \frac{t_k}{|t|}$ to the commuting tuple ∇). The operators $\frac{D_k}{(-\Delta)^{1/2}}$, $1 \leqslant k \leqslant d$, form a set of mutually commuting bounded self-adjoint operators whose joint spectrum lies in $\mathbb{S}^{d-1}$. Using the Borel functional calculus (see e.g. Section 6.5 in [7]), given $g \in L_\infty(\mathbb{S}^{d-1})$ we may then define the bounded operator $g\Big(\frac{\nabla}{(-\Delta)^{1/2}}\Big)$ on $L_2(\mathbb{R}^d)$ (which should be viewed a homogeneous Fourier multiplier on $L_2(\mathbb{R}^d)$).

Let $\mathcal{L}(L_2(\mathbb{R}^d))$ denote the algebra of all bounded linear operators on the Hilbert space $L_2(\mathbb{R}^d)$. We work primarily with the following C^*-subalgebra of $\mathcal{L}(L_2(\mathbb{R}^d))$.

DEFINITION 1.1. Let $\pi_1 : L_\infty(\mathbb{R}^d) \to \mathcal{L}(L_2(\mathbb{R}^d))$ and $\pi_2 : L_\infty(\mathbb{S}^{d-1}) \to \mathcal{L}(L_2(\mathbb{R}^d))$ be the unital $*$-representations given by the formulae:

$$\pi_1(f) = M_f, \quad f \in L_\infty(\mathbb{R}^d), \quad \pi_2(g) = g\Big(\frac{\nabla}{(-\Delta)^{1/2}}\Big), \quad g \in L_\infty(\mathbb{S}^{d-1}).$$

Let Π be the C^*-subalgebra in $\mathcal{L}(L_2(\mathbb{R}^d))$ generated by the algebras $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_2(L_\infty(\mathbb{S}^{d-1}))$.

In what follows, we denote by $L_\infty(\mathbb{R}^d)\overline{\otimes}L_\infty(\mathbb{S}^{d-1})$ the closure in the weak operator topology of the algebraic tensor product $L_\infty(\mathbb{R}^d) \otimes L_\infty(\mathbb{S}^{d-1})$. We identify $L_\infty(\mathbb{R}^d)\overline{\otimes}L_\infty(\mathbb{S}^{d-1})$ with the algebra $L_\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$.

We now state the main results of the paper.

THEOREM 1.2. *There exists a unique $*$-homomorphism*

$$\mathrm{symb} : \Pi \to L_\infty(\mathbb{R}^d)\overline{\otimes}L_\infty(\mathbb{S}^{d-1})$$

such that:

$$\mathrm{symb}(\pi_1(f)) = f \otimes 1, \quad f \in L_\infty(\mathbb{R}^d), \quad \mathrm{symb}(\pi_2(g)) = 1 \otimes g, \quad g \in L_\infty(\mathbb{S}^{d-1}).$$

We call symb *the principal symbol mapping.*

We recall that $*$-homomorphisms between C^*-algebras are necessarily continuous and norm-decreasing. Therefore, we have

$$\|\mathrm{symb}(T)\|_\infty \leqslant \|T\|_\infty, \quad T \in \Pi.$$

The C^*-algebraic setting is important in Theorem 1.2. Indeed, we show in Lemma 3.5 that the principal symbol mapping does not extend (as a $*$-homomorphism) to the weak closure of Π.

In Lemma 8.2 below, we show that for a uniform compactly based classical PSDO, our notion of a principal symbol coincides with the one in [45]. In fact, our notion of the principal symbol extends the traditional one.

In Theorem 1.2 above we have restricted attention to homogeneous functions of the gradient. It is tempting to ask if one can obtain a similar result allowing arbitrary bounded functions of the gradient. The following theorem shows that this is not possible in general.

THEOREM 1.3. *Let π_3 be the representation of $L_\infty(\mathbb{R}^d)$ on $L_2(\mathbb{R}^d)$ given by,*

$$\pi_3(g) = g(\nabla).$$

Let $\mathcal{A}$ be the C^*-algebra generated by $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_3(L_\infty(\mathbb{R}^d))$. There is no $*$-homomorphism $\pi : \mathcal{A} \to L_\infty(\mathbb{R}^d \times \mathbb{R}^d)$ such that

$$\pi(\pi_1(f)\pi_3(g)) = f \otimes g, \quad f, g \in L_\infty(\mathbb{R}^d).$$

One application of PSDO theory is A. Connes' result [13] linking the Wodzicki residue of a classical PSDO of order $-d$ on a d-dimensional compact Riemannian manifold with a (Dixmier) trace of the latter operator. In particular, this result allows the "deduction of ordinary differential forms and the natural conformal invariant norm on them from the quantized forms" (p. 674 in [13]). With the principal symbol in Theorem 1.2 we extend two results from [13] with new and direct proofs.

There have been many extension of Connes' results (see e.g. [25], [30], or earlier results in special cases [38], [12], to name a few). This is the first, we believe, to extend the result to bounded operators associated to essentially bounded symbols, and the appearance of the principal symbol homomorphism makes the result below particularly elegant. The novel feature of our approach is a consistent use of Cwikel-type estimates (see e.g. Chapter 4 in [46]). The proof is the most direct we know of, and since it involves continuous traces it uses, besides the Cwikel-type estimates, only the Riesz–Markov theorem. Prior to the versions in [25] and here, other versions of the trace theorem on $\mathbb{R}^d$, e.g., [12] and [38] were limited to a subset of the algebra $\pi_1(L_\infty(\mathbb{R}^d))$.

DEFINITION 1.4. An operator $T \in \Pi$ is *compactly based* if there is a compactly supported $\phi \in L_\infty(\mathbb{R}^d)$ such that $T\pi_1(\phi) = T$.

THEOREM 1.5. *If $T \in \Pi$ is compactly based, then $T(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$. For every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\varphi(T(1-\Delta)^{-d/2}) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(T). \tag{1.1}$$

In particular, for every compactly supported $f \in L_\infty(\mathbb{R}^d)$, we recover the known result (see e.g. [25] and prior versions on p. 34 in [2] and Corollary 7.21 in [22]),

$$\varphi(\pi_1(f)(1-\Delta)^{-d/2}) = c_d \int\limits_{\mathbb{R}^d} f(t)\mathrm{d}t. \tag{1.2}$$

An easy corollary of Theorem 1.5 is the known trace theorem due to Connes (on compact manifolds); see e.g. [13] [25]. Recall P is uniform if $p(t,s)$ and its derivatives are bounded in t; P is compactly based if $P\pi_1(\phi) = P$ for some compactly supported smooth function ϕ. Equivalently, the integral kernel of P is compactly supported in the second variable. In theorem below, the Wodzicki residue, $\mathrm{Res}_{\mathrm{W}}$, may be understood in the following sense:

$$\mathrm{Res}_{\mathrm{W}}(P(1-\Delta)^{-d/2}) = \int\limits_{\mathbb{R}^d\times\mathbb{S}^{d-1}} p(t,s)\mathrm{d}t\mathrm{d}s,$$

where p is the principal symbol of the PSDO P.

THEOREM 1.6. *If P is a uniform compactly based classical pseudo-differential operator of order* 0 *on $\mathbb{R}^d$, then*

$$\varphi(P(1-\Delta)^{-d/2}) = \mathrm{Res}_{\mathrm{W}}(P(1-\Delta)^{-d/2})$$

for every bounded normalised trace φ on $\mathcal{L}_{1,\infty}$.

In this paper, D denotes the Dirac operator (an unbounded self-adjoint operator on the Hilbert space $\mathbb{C}^{m(d)} \otimes L_2(\mathbb{R}^d)$, where $m(d) = 2^{\lfloor d/2\rfloor}$),

$$D = \sum_{k=1}^{d} \gamma_k \otimes D_k$$

for some choice of self-adjoint matrices $\{\gamma_k\}_{k=1}^d \subset M_{m(d)}(\mathbb{C})$ with $\gamma_j\gamma_k + \gamma_k\gamma_j = 2\delta_{j,k}$.

The following assertion is proved in [29] by a somewhat ad hoc method (smoothing was introduced and then eliminated). With the method of this paper non-smooth symbols can be used directly. The assertion is the key to the characterisation of quantum differentiable functions given in [29]. Using Theorem 1.5 we are able to give a simple proof of the main result of [29].

THEOREM 1.7. *For every $f \in L_\infty(\mathbb{R}^d)$ such that $\nabla f \in L_d(\mathbb{R}^d, \mathbb{C}^d)$, we have that $[\mathrm{sgn}(D), 1\otimes\pi_1(f)] \in \mathcal{L}_{d,\infty}$ and*

$$\varphi(|[\mathrm{sgn}(D), 1\otimes\pi_1(f)]|^d) = \int\limits_{\mathbb{R}^d} \|(\nabla f)(t)\|_2^d \mathrm{d}t$$

for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$.

The paper is structured as follows. Section 2 contains preliminary material. Section 3 contains the proof of Theorems 1.2 and 1.3. Section 4 contains the proof of (1.2). Sections 5, 6 and 7 contain the proof of Theorem 1.5. Section 8 derives Theorem 1.6 from Theorem 1.5. Finally, Section 9 contains the proof of Theorem 1.7.

2. PRELIMINARIES

2.1. GENERAL NOTATION. Fix throughout a separable infinite dimensional Hilbert space H. We let $\mathcal{L}(H)$ denote the algebra of all bounded operators on H. For a compact operator T on H, let $\mu(T) := \{\mu(k,T)\}_{k=0}^{\infty}$ denote the sequence of singular values of T, arranged in non-increasing order with multiplicities.

The standard trace on $\mathcal{L}(H)$ is denoted by Tr.

Fix an orthonormal basis in H (the particular choice of a basis is inessential). We identify the algebra l_∞ of bounded sequences with the subalgebra of all diagonal operators with respect to the chosen basis. For a given sequence $\alpha \in l_\infty$, we denote the corresponding diagonal operator by $\mathrm{diag}(\alpha)$.

2.2. SCHATTEN IDEALS $\mathcal{L}_p$. For $p \in [1,\infty)$, the Schatten class $\mathcal{L}_p$ is defined as

$$\mathcal{L}_p = \{T \in \mathcal{L}(H) : \mathrm{Tr}(|T|^p) < \infty\}.$$

As usual, $\mathcal{L}_p$ is equipped with the norm

$$\|T\|_p = (\mathrm{Tr}(|T|^p))^{1/p}, \quad T \in \mathcal{L}_p.$$

For every $p \geqslant 1$, $\|\cdot\|_p$ is a norm and $(\mathcal{L}_p, \|\cdot\|_p)$ is a Banach space.

The assertion below is used twice in Section 6. It follows immediately from Lemma 8 in [47].

PROPOSITION 2.1. *If $a, b \in \mathcal{L}(H)$ are self-adjoint operators, $b \geqslant 0$, then*

$$\|b^\lambda a b^{1-\lambda}\|_1 \leqslant \|ab\|_1, \quad 0 < \lambda < 1.$$

2.3. WEAK SCHATTEN IDEALS $\mathcal{L}_{p,\infty}$. Given $p \geqslant 1$, we let $\mathcal{L}_{p,\infty}$ denote the ideal on $\mathcal{L}(H)$ defined as:

$$\mathcal{L}_{p,\infty} = \{T \in \mathcal{L}(H) : \mu(k,T) = O((k+1)^{-1/p})\}.$$

We set

$$\|T\|_{p,\infty} = \sup_{k\geqslant 0}(k+1)^{1/p}\mu(k,T), \quad T \in \mathcal{L}_{p,\infty}.$$

For every $p \geqslant 1$, $\|\cdot\|_{p,\infty}$ is a quasi-norm (in other words, it satisfies the norm axioms, except that the triangle inequality is replaced by $\|x+y\| \leqslant K(\|x\| + \|y\|)$ for some uniform constant $K > 1$) and $(\mathcal{L}_{p,\infty}, \|\cdot\|_{p,\infty})$ is a quasi-Banach space. For $p > 1$, $\|\cdot\|_{p,\infty}$ is equivalent to a (unitarily invariant Banach) norm. For $p = 1$, the space $(\mathcal{L}_{1,\infty}, \|\cdot\|_{1,\infty})$ is not Banach — that is, its quasi-norm is not equivalent to any norm. In [36], the Banach envelope of $\mathcal{L}_{1,\infty}$ was thoroughly investigated.

We have

$$|T|^p \in \mathcal{L}_{1,\infty} \Longleftrightarrow T \in \mathcal{L}_{p,\infty}.$$

The following Hölder property is widely used throughout the paper: for all $p, q, r \geqslant 1$ satisfying $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ there exists a positive constant $c_{q,r}$ such that

$$\|AB\|_{p,\infty} \leqslant c_{q,r}\|A\|_{q,\infty}\|B\|_{r,\infty}. \tag{2.1}$$

2.4. TRACES ON $\mathcal{L}_{1,\infty}$.

DEFINITION 2.2. If $\mathcal{I}$ is an ideal in $\mathcal{L}(H)$, then a unitarily invariant linear functional $\varphi : \mathcal{I} \to \mathbb{C}$ is said to be a trace.

Since $U^{-1}TU - T = [U^{-1}, TU]$ for all $T \in \mathcal{I}$ and for all unitaries $U \in \mathcal{L}(H)$, and since the set of unitary operators spans $\mathcal{L}(H)$, it follows that traces are precisely the linear functionals on $\mathcal{I}$ satisfying the condition

$$\varphi(TS) = \varphi(ST), \quad T \in \mathcal{I}, S \in \mathcal{L}(H).$$

The latter may be reinterpreted as the vanishing of the linear functional φ on the commutator subspace which is denoted $[\mathcal{I}, \mathcal{L}(H)]$ and defined to be the linear span of all commutators $[T, S] : T \in \mathcal{I}, S \in \mathcal{L}(H)$.

It is shown in Lemma 5.2.2 of [30] that if T_1 and T_2 are two positive operators with identical singular value sequences, i.e. $\mu(T_1) = \mu(T_2)$, then for all traces φ we have $\varphi(T_1) = \varphi(T_2)$.

For $p > 1$, the ideal $\mathcal{L}_{p,\infty}$ does not admit a non-zero trace [18], while for $p = 1$, there exists a plethora of traces on $\mathcal{L}_{1,\infty}$ (see e.g. [11], [30] or [44]). A standard example of a set of traces on $\mathcal{L}_{1,\infty}$ is the class of Dixmier traces introduced in [16].

An extensive discussion of traces, and more recent developments in the theory, may be found in [30] including a discussion of the following facts. We refer the reader to an alternative approach to the theory of traces on $\mathcal{L}_{1,\infty}$ suggested in [44] (based on the fundamental paper [35] by Pietsch):

(i) All Dixmier traces on $\mathcal{L}_{1,\infty}$ are positive.
(ii) All positive traces on $\mathcal{L}_{1,\infty}$ are continuous in the quasi-norm topology.
(iii) There exist positive traces on $\mathcal{L}_{1,\infty}$ which are not Dixmier traces (see [44]).
(iv) There exist traces on $\mathcal{L}_{1,\infty}$ which fail to be continuous (see [30]).

We say that a trace φ on $\mathcal{L}_{1,\infty}$ is normalised if $\varphi(A) = 1$ for any (and, hence, for every) positive $A \in \mathcal{L}_{1,\infty}$ such that $\mu(A) = \{\frac{1}{k+1}\}_{k\geqslant 0}$.

2.5. FOURIER TRANSFORM.

We follow the convention that the Fourier transform on $L_2(\mathbb{R}^d)$ is defined by the formula

$$(\mathcal{F}\xi)(t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \xi(u) \mathrm{e}^{-\mathrm{i}\langle t,u\rangle} \mathrm{d}u, \quad t \in \mathbb{R}^d.$$

So the inverse Fourier transform is given by the formula

$$(\mathcal{F}^{-1}\xi)(t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} \xi(u) \mathrm{e}^{\mathrm{i}\langle t,u\rangle} \mathrm{d}u, \quad t \in \mathbb{R}^d.$$

Since the constant coefficients $(2\pi)^{-d/2}$ do not play any role in our paper, we often omit them.

The Fourier transform relates the representations π_1 and π_3 as follows:

$$\pi_1(x) = \mathcal{F}^{-1}\pi_3(x)\mathcal{F}, \quad x \in L_\infty(\mathbb{R}^d).$$

In particular, the operator $\pi_3(x)$ acts by the formula

$$(\pi_3(x)\xi)(t) = \int_{\mathbb{R}^d} (\mathcal{F}^{-1}x)(t-u)\xi(u)\mathrm{d}u, \quad t \in \mathbb{R}^d. \tag{2.2}$$

In the following we require information about the Fourier transform of the function $t \mapsto (1+|t|^2)^{-d/2}$. This Fourier transform may be conveniently expressed in terms of a modified Bessel function of the second kind, which may be defined by the improper integral:

$$K_0(x) := \int_0^\infty \frac{\cos(xt)}{\sqrt{1+t^2}}\,\mathrm{d}t, \quad x > 0. \tag{2.3}$$

(see Formula 9.6.21 of [1]). Then an elementary computation shows that the Fourier transform of $t \mapsto (1+|t|^2)^{-d/2}$ is

$$\mathcal{F}(t \mapsto (1+|t|^2)^{-d/2})(u) = c_d K_0(|u|) \tag{2.4}$$

where c_d is a positive constant depending only on d.

2.6. CWIKEL-TYPE ESTIMATES. The following observation is made in Chapter 4 in [46]: there is a constant $c_d > 0$ such that

$$\|\pi_1(f_1)\pi_3(f_2)\|_2 = c_d\|f_1\|_2\|f_2\|_2, \quad f_1, f_2 \in L_2(\mathbb{R}^d). \tag{2.5}$$

The following result is due to Cwikel [15], and is stated as Theorem 4.2 in [46].

THEOREM 2.3. *If $f_1 \in L_p(\mathbb{R}^d)$ and $f_2 \in L_{p,\infty}(\mathbb{R}^d)$, $p > 2$, then $\pi_1(f_1)\pi_3(f_2) \in \mathcal{L}_{p,\infty}$. Moreover,*

$$\|\pi_1(f_1)\pi_3(f_2)\|_{p,\infty} \leqslant c_{p,d}\|f_1\|_p\|f_2\|_{p,\infty}.$$

Recall that the spaces $(l_p(L_q))(\mathbb{R}^d)$ and $(l_{p,\infty}(L_q))(\mathbb{R}^d)$ are defined as follows. Let $K = [0,1]^d$ be a fixed unit cube. For every locally integrable function f, we set

$$\|f\|_{l_p(L_q)} = \|\{\|f\|_{L_q(m+K)}\}_{m\in\mathbb{Z}^d}\|_p, \quad 1 \leqslant p, q \leqslant \infty,$$
$$\|f\|_{l_{p,\infty}(L_q)} = \|\{\|f\|_{L_q(m+K)}\}_{m\in\mathbb{Z}^d}\|_{p,\infty}, \quad 1 \leqslant p, q \leqslant \infty.$$

Observe that $f \in (l_1(L_2))(\mathbb{R}^d)$ implies that $f \in L_1(\mathbb{R}^d)$.

Theorem 2.4 below is due to Birman and Solomyak [5], [6]. We refer the reader to Theorem 4.5 of [46] for a modern proof of Theorem 2.4.

THEOREM 2.4. *If $f_1 \in (l_1(L_2))(\mathbb{R}^d)$ and $f_2 \in (l_1(L_2))(\mathbb{R}^d)$, then $\pi_1(f_1)\pi_3(f_2) \in \mathcal{L}_1$. Moreover, we have*

$$\|\pi_1(f_1)\pi_3(f_2)\|_1 \leqslant c_d\|f_1\|_{l_1(L_2)}\|f_2\|_{l_1(L_2)}.$$

In particular, we have that $\pi_1(f_1)\pi_3(f_2) \in \mathcal{L}_1$ for all compactly supported $f_1, f_2 \in L_\infty(\mathbb{R}^d)$.

The following Cwikel estimate is proved by Birman, Karadzhov and Solomyak ([4], Assertion 5.8, p. 104). We refer the reader to the forthcoming paper [27] which proves a more general (and much stronger) result.

THEOREM 2.5. *If* $f_1 \in (l_1(L_2))(\mathbb{R}^d)$ *and* $f_2 \in (l_{1,\infty}(L_2))(\mathbb{R}^d)$, *then* $\pi_1(f_1)\pi_3(f_2) \in \mathcal{L}_{1,\infty}$. *Moreover, we have*

$$\|\pi_1(f_1)\pi_3(f_2)\|_{1,\infty} \leqslant c_d \|f_1\|_{l_1(L_2)} \|f_2\|_{l_{1,\infty}(L_2)}.$$

In particular, applying Theorem 2.5 to the function $f_2 : t \to (1+|t|^2)^{-d/2}$, $t \in \mathbb{R}^d$, we obtain

$$\|\pi_1(f)(1-\Delta)^{-d/2}\|_{1,\infty} \leqslant c_d \|f\|_{l_1(L_2)}, \quad f \in (l_1(L_2))(\mathbb{R}^d).$$

2.7. LEBESGUE POINTS OF A LOCALLY INTEGRABLE FUNCTION. Recall that the Lebesgue points of a locally integrable function x on $\mathbb{R}^d$ are those points $t \in \mathbb{R}^d$ such that

$$\lim_{r\to 0} \frac{1}{\mathrm{Vol}(B(t,r))} \int\limits_{B(t,r)} |x(u) - x(t)| \mathrm{d}u = 0.$$

In particular, we have

$$x(t) = \lim_{r\to 0} \frac{1}{\mathrm{Vol}(B(t,r))} \int\limits_{B(t,r)} x(u) \mathrm{d}u.$$

The Lebesgue differentiation theorem (see e.g. Theorem 7.7 of [41]) states the following.

THEOREM 2.6. *If x is a locally integrable function on $\mathbb{R}^d$, then almost every $t \in \mathbb{R}^d$ is a Lebesgue point for x.*

3. THE PRINCIPAL SYMBOL IN THE C^*-ALGEBRAIC SETTING

The main result of this section is Theorem 1.2. The following simple result is well known (see e.g. Proposition 2.4.1 in [8]).

LEMMA 3.1. *Let* $\{A_n\}_{n\geqslant 0}, \{B_n\}_{n\geqslant 0} \subset \mathcal{L}(H)$ *be bounded sequences. If* $A_n \to A$ *and* $B_n \to B$ *in the strong operator topology, then* $A_nB_n \to AB$ *in the strong operator topology.*

For a function x on $\mathbb{R}^d$, the dilation operator σ_n is defined by

$$(\sigma_n x)(s) = x\Big(\frac{s}{n}\Big), \quad s \in \mathbb{R}^d$$

for $n > 0$. Similarly, the translation operator T_t, $t \in \mathbb{R}^d$ is defined by

$$(T_t x)(s) = x(s-t).$$

On $L_2(\mathbb{R}^d)$, we frequently use an alternative expression

$$T_t = \mathrm{e}^{\mathrm{i}(t,\nabla)}.$$

LEMMA 3.2. *If* 0 *is a Lebesgue point for* $x \in L_\infty(\mathbb{R}^d)$, *then as* $n \to \infty$, *we have* $\pi_1(\sigma_n x) \to x(0)1$ *in the strong operator topology.*

Proof. Without loss of generality, $\|x\|_\infty \leqslant 1$. Fix $\xi \in L_2(\mathbb{R}^d)$. We claim that $\pi_1(\sigma_n x)\xi \to x(0)\xi$ in $L_2(\mathbb{R}^d)$.

To see the claim, fix $\varepsilon > 0$ and choose $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\|\xi - \eta\|_{L_2} \leqslant \varepsilon$. Let η be supported in a ball centred at 0 of radius R. We have:

$$\begin{aligned}\|(\sigma_n x - x(0)) \cdot \eta\|^2 &= \int\limits_{|t|\leqslant R} \left| x\left(\frac{t}{n}\right) - x(0)\right|^2 \cdot |\eta(t)|^2 \mathrm{d}t \\ &\leqslant 2\|\eta\|_\infty^2 \cdot n^d \int\limits_{|t|\leqslant R/n} |x(t) - x(0)|\mathrm{d}t.\end{aligned}$$

By the definition of a Lebesgue point, the right hand side tends to 0 as $n \to \infty$. Fix N sufficiently large such that if $n > N$,

$$2\|\eta\|_\infty^2 \cdot n^d \int\limits_{|t|\leqslant R/n} |x(t) - x(0)|\mathrm{d}t \leqslant \varepsilon^2.$$

It follows that for all $n > N$ we have

$$\|(\sigma_n x - x(0)) \cdot \eta\| \leqslant \varepsilon.$$

Thus,

$$\begin{aligned}\|(\sigma_n x - x(0)) \cdot \xi\| &\leqslant \|(\sigma_n x - x(0)) \cdot \eta\| + \|(\sigma_n x - x(0)) \cdot (\xi - \eta)\| \\ &\leqslant \varepsilon + \|\sigma_n x - x(0)\|_\infty \|\xi - \eta\| \leqslant 3\varepsilon,\end{aligned}$$

for all $n > N$. Thus, $\pi_1(\sigma_n x)\xi \to x(0)\xi$ in $L_2(\mathbb{R}^d)$. ∎

The following is a simple consequence of the fact that for each $1 \leqslant k \leqslant d$ the operator $\frac{D_k}{(-\Delta)^{1/2}}$ commutes with translations and dilations.

LEMMA 3.3. *For all* $f \in L_\infty(\mathbb{R}^d)$, $g \in L_\infty(\mathbb{S}^{d-1})$ *and for every* $n \in \mathbb{N}$, *we have*

$$\begin{aligned}&\sigma_n \pi_1(f)\sigma_{1/n} = \pi_1(\sigma_n f), \quad \sigma_n \pi_2(g)\sigma_{1/n} = \pi_2(g), \quad \text{and} \\ &T_t \pi_1(f) T_{-t} = \pi_1(T_t f), \quad T_t \pi_2(g) T_{-t} = \pi_2(g).\end{aligned}$$

Moreover, we have

$$T_t \pi_3(f) = \pi_3(f) T_t.$$

Proof. We have

$$T_t \pi_3(f) = \mathrm{e}^{\mathrm{i}(t,\nabla)} f(\nabla) = f(\nabla)\mathrm{e}^{\mathrm{i}(t,\nabla)} = \pi_3(f) T_t.$$

If $g \in L_\infty(\mathbb{S}^{d-1})$, then set

$$f(t) = g\Big(\frac{t}{|t|}\Big), \quad t \in \mathbb{R}^d.$$

Hence,

$$\pi_2(g)T_t = \pi_3(f)T_t = T_t\pi_3(f) = T_t\pi_2(g).$$

For $\xi \in L_2(\mathbb{R}^d)$, we have

$$(T_t\pi_1(f))(\xi) = T_t(f\xi) = (T_tf)\cdot(T_t\xi) = (\pi_1(T_tf)T_t)(\xi).$$

This proves the last assertions. The proofs of the first equalities are similar and are, therefore, omitted. ∎

It is at this point that the assumption that we work with homogeneous functions of the gradient is used, since Lemma 3.3 is false when π_2 is replaced with π_3. Indeed, the equality $\sigma_n\pi_3(g)\sigma_{1/n}$ is not equal to $\pi_3(g)$ for inhomogeneous functions $g \in L_\infty(\mathbb{R}^d)$.

The following simple lemma contains in fact a crucial piece of the proof of Theorem 1.2.

LEMMA 3.4. *Let $f_k \in L_\infty(\mathbb{R}^d)$ and let $g_k \in L_\infty(\mathbb{S}^{d-1})$, $1 \leqslant k \leqslant m$. If 0 is a Lebesgue point for every f_k, $1 \leqslant k \leqslant m$, then*

$$\sigma_n \cdot \prod_{k=1}^m \pi_1(f_k)\pi_2(g_k)\cdot\sigma_{1/n} \to \prod_{k=1}^m f_k(0)\cdot\prod_{k=1}^m \pi_2(g_k), \quad n\to\infty,$$

in the strong operator topology.

Proof. Clearly,

$$\sigma_n \cdot \prod_{k=1}^m \pi_1(f_k)\pi_2(g_k)\cdot\sigma_{1/n} = \prod_{k=1}^m (\sigma_n\pi_1(f_k)\sigma_{1/n}\cdot\sigma_n\pi_2(g_k)\sigma_{1/n}).$$

By Lemma 3.3, we have

$$\sigma_n \cdot \prod_{k=1}^m \pi_1(f_k)\pi_2(g_k)\cdot\sigma_{1/n} = \prod_{k=1}^m \pi_1(\sigma_nf_k)\pi_2(g_k).$$

By Lemma 3.2, we have $\pi_1(\sigma_nf_k) \to f_k(0)$ strongly. The assertion follows now from Lemma 3.1. ∎

Now, we prove our first main result.

Proof of Theorem 1.2. Let $P \subset \Pi$ be the $*$-subalgebra in $\mathcal{L}(L_2(\mathbb{R}^d))$ generated by $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_2(L_\infty(\mathbb{S}^{d-1}))$. So if $A \in P$, then A can be written as a linear combination of products of elements of $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_2(L_\infty(\mathbb{S}^{d-1}))$:

$$A = \sum_{l=1}^p \prod_{k=1}^m \pi_1(f_{k,l})\pi_2(g_{k,l}), \quad 1 \leqslant p,m < \infty, \tag{3.1}$$

where each $f_{k,l} \in L_\infty(\mathbb{R}^d)$ and each $g_{k,l} \in L_\infty(\mathbb{R}^d)$. Define the mapping symb on the algebra P by setting

$$\text{symb} : \sum_{l=1}^{p} \prod_{k=1}^{m} \pi_1(f_{k,l})\pi_2(g_{k,l}) \mapsto \sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l} \otimes g_{k,l}. \tag{3.2}$$

We claim symb is well-defined, and is continuous in the norm topology.

Applying Lemma 3.3, we get

$$T_t A T_{-t} = \sum_{l=1}^{p} \prod_{k=1}^{m} \pi_1(T_t f_{k,l})\pi_2(g_{k,l}).$$

Choose a point t which is a Lebesgue point for every $f_{k,l}$, $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant p$. We have that $(T_t f_{k,l})(0) = f_{k,l}(t)$ and hence 0 is a Lebesgue point for every $T_t f_{k,l}$, $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant p$. Then applying Lemma 3.4, we have that as $n \to \infty$

$$\sigma_n T_t A T_{-t} \sigma_{1/n} \to \sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l}(t)\pi_2(g_{k,l})$$

in the strong operator topology. By the Fatou property of the operator norm of $\mathcal{L}(L_2(\mathbb{R}^d))$, we have

$$\Big\| \sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l}(t)\pi_2(g_{k,l}) \Big\|_\infty \leqslant \liminf_{n\to\infty} \|\sigma_n T_t A T_{-t} \sigma_{1/n}\|_\infty$$

$$\Longrightarrow \Big\| \pi_2\Big(\sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l}(t) g_{k,l} \Big) \Big\|_\infty \leqslant \|A\|_\infty.$$

We claim that π_2 is an isometry. The claim follows via the spectral representation of the gradient. Precisely, take $g \in L_\infty(\mathbb{S}^{d-1})$ and extend it to a function $h \in L_\infty(\mathbb{R}^d)$ by setting

$$h(t) = g\Big(\frac{t}{|t|}\Big), \quad t \in \mathbb{R}^d.$$

Clearly, this extension preserves the uniform norm. We have

$$\pi_2(g) = \pi_3(h) = \mathcal{F}^{-1}\pi_1(h)\mathcal{F}.$$

Therefore,

$$\|\pi_2(g)\|_\infty = \|\mathcal{F}^{-1}\pi_1(h)\mathcal{F}\|_\infty = \|\pi_1(h)\|_\infty = \|h\|_\infty = \|g\|_\infty.$$

Since π_2 is an isometric embedding of $L_\infty(\mathbb{S}^{d-1})$ into $\mathcal{L}(L_2(\mathbb{R}^d))$ we arrive at

$$\Big\| \sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l}(t) g_{k,l} \Big\|_\infty \leqslant \Big\| \sum_{l=1}^{p} \prod_{k=1}^{m} \pi_1(f_{k,l})\pi_2(g_{k,l}) \Big\|_\infty \tag{3.3}$$

for every $t \in \mathbb{R}^d$ which is a Lebesgue point for every $f_{k,l}$, $1 \leqslant k \leqslant m$, $1 \leqslant l \leqslant p$.

By Theorem 2.6, almost every point of $\mathbb{R}^d$ is a Lebesgue point of each $f_{k,l}$. So (3.3) holds for almost every $t \in \mathbb{R}^d$. Taking the essential supremum over $t \in \mathbb{R}^d$, and using the formula

$$\Big\| \sum_{n=1}^{N} a_n \otimes b_n \Big\|_\infty = \operatorname*{sup\,ess}_{t\in\mathbb{R}^d} \Big\| \sum_{n=1}^{N} a_n(t) b_n \Big\|_\infty$$

valid for all $\{a_n\}_{n=1}^N \subset L_\infty(\mathbb{R}^d)$ and $\{b_n\}_{n=1}^N \subset L_\infty(\mathbb{S}^{d-1})$, we infer

$$\Big\| \sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l} \otimes g_{k,l} \Big\|_\infty \leqslant \Big\| \sum_{l=1}^{p} \prod_{k=1}^{m} \pi_1(f_{k,l})\pi_2(g_{k,l}) \Big\|_\infty = \|A\|_\infty.$$

This now shows that the mapping (3.2) is well defined, since if A has two distinct representations in (3.1) such as

$$A = \sum_{l=1}^{p} \prod_{k=1}^{m} \pi_1(f_{k,l})\pi_2(g_{k,l}) = \sum_{l=1}^{p'} \prod_{k=1}^{m'} \pi_1(f'_{k,l})\pi_2(g'_{k,l})$$

then

$$\Big\| \sum_{l=1}^{p} \prod_{k=1}^{m} f_{k,l} \otimes g_{k,l} - \sum_{l=1}^{p'} \prod_{k=1}^{m'} f'_{k,l} \otimes g'_{k,l} \Big\|_\infty \leqslant \|A - A\|_\infty = 0.$$

Hence, the mapping symb from (3.2) is well defined, and immediately,

$$\|\mathrm{symb}(A)\|_\infty \leqslant \|A\|_\infty, \quad A \in P, \tag{3.4}$$

so symb is also norm-continuous on P.

The fact that symb is a $*$-homomorphism on P follows directly from the definition. Since symb contracts the uniform norm, it extends to the uniform closure of P, that is to Π. That this map is unique follows from the norm-density of P in Π. ∎

Proof of the Theorem 1.3. Assume the contrary. Let $\pi : \mathcal{A} \to L_\infty(\mathbb{R}^{2d})$ be a $*$-homomorphism such that

$$\pi(\pi_1(f_1)\pi_3(f_2)) = f_1 \otimes f_2, \quad f_1, f_2 \in L_\infty(\mathbb{R}^d).$$

Take $f \in (L_2 \cap L_\infty)(\mathbb{R}^d)$. Since π is a $*$-homomorphism of C^*-algebras, it is contractive. We have

$$\|f \otimes f\|_\infty \leqslant \|\pi_1(f)\pi_3(f)\|_\infty.$$

However, $\|f \otimes f\|_\infty = \|f\|_\infty^2$. Since $\|\cdot\|_\infty \leqslant \|\cdot\|_2$ in the setting of Schatten ideals, it follows from (2.5) that

$$\|\pi_1(f)\pi_3(f)\|_\infty \leqslant \|\pi_1(f)\pi_3(f)\|_2 = c_d\|f\|_2^2.$$

Thus, $\|f\|_\infty^2 \leqslant c_d\|f\|_2^2$. The latter inequality is false for general f. This contradiction proves the claim. ∎

Theorem 1.2 shows that one can define a symbol mapping on the operator norm closure of the subalgebra of $\mathcal{L}(L_2(\mathbb{R}^d))$ generated by $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_2(L_\infty(\mathbb{S}^{d-1}))$. It is natural to ask whether the same is true for the closure in the weak operator topology. The following lemma shows that this is impossible.

LEMMA 3.5. *The closure of Π in the weak operator topology is all of $\mathcal{L}(L_2(\mathbb{R}^d))$, and there is no $*$-homomorphism extending* symb *to all of $\mathcal{L}(L_2(\mathbb{R}^d))$.*

Proof. To show that the weak closure of Π, or equivalently the double commutant Π'', is all of $\mathcal{L}(L_2(\mathbb{R}^d))$ it suffices to show that the commutant Π' is trivial.

If $T \in \Pi'$, then T in particular commutes with all of $\pi_1(L_\infty(\mathbb{R}^d))$. However since $\pi_1(L_\infty(\mathbb{R}^d))$ is a maximal abelian subalgebra in $\mathcal{L}(L_2(\mathbb{R}^d))$ it is equal to its commutant, hence $T = \pi_1(f)$ for some $f \in L_\infty(\mathbb{R}^d)$. Since T additionally commutes with every element of $\pi_2(L_\infty(\mathbb{S}^{d-1}))$, we have that for all $1 \leqslant k \leqslant d$,

$$\Big[\frac{D_k}{(-\Delta)^{1/2}}, \pi_1(f)\Big] = 0. \tag{3.5}$$

This implies that f is constant. One way of seeing this is to apply Theorem 1.7. Let $\{h_n\}_{n\geqslant 0}$ be an approximate identity on $\mathbb{R}^d$ (i.e., a sequence of positive Schwartz functions such that $\|h_n\|_1 = 1$ and such that $h_n \to \delta$ (Dirac point measure) in the space of tempered distributions as $n \to \infty$). Setting $f_n = f * h_n$, we have (here, $t \to T_t$ is a group of translations on $\mathbb{R}^d$):

$$\pi_1(f_n) = \int\limits_{\mathbb{R}^d} h_n(t)\pi_1(T_t(f))\mathrm{d}t.$$

Therefore,

$$\Big[\frac{D_k}{(-\Delta)^{1/2}}, \pi_1(f_n)\Big] = \int\limits_{\mathbb{R}^d} h_n(t)\Big[\frac{D_k}{(-\Delta)^{1/2}}, \pi_1(T_t f)\Big]\mathrm{d}t.$$

By Lemma 3.3, we have (recall that translations commute with functions of the gradient):

$$\Big[\frac{D_k}{(-\Delta)^{1/2}}, \pi_1(T_t f)\Big] = \Big[\frac{D_k}{(-\Delta)^{1/2}}, T_t\pi_1(f)T_{-t}\Big] = T_t\Big[\frac{D_k}{(-\Delta)^{1/2}}, \pi_1(f)\Big]T_{-t} = 0.$$

Since

$$\mathrm{sgn}(D) = \sum_{k=1}^{d} \gamma_k \otimes \frac{D_k}{(-\Delta)^{1/2}},$$

it follows that $[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)] = 0$. Taking the absolute value, raising to the d-th power and using Theorem 1.7, we get

$$\int\limits_{\mathbb{R}^d} \|\nabla f_n(t)\|_2^d\,\mathrm{d}t = \varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]|^d) = 0.$$

Hence, f_n is a constant function for every $n \geqslant 0$. Since $f_n \to f$ in the space of tempered distributions, it follows that f is also a constant function.

If $\phi : \mathcal{L}(L_2(\mathbb{R}^d)) \to A$ is a homomorphism to a commutative algebra A, then the kernel of ϕ contains the linear span of the set of all commutators of elements of $\mathcal{L}(L_2(\mathbb{R}^d)$. However every element in $\mathcal{L}(L_2(\mathbb{R}^d))$ can be expressed as a finite sum of commutators [23]. In particular, ϕ must vanish on Π so cannot be an extension of symb. ∎

4. THE LEBESGUE INTEGRAL FROM THE TRACE FORMULA

The main result of this section is Proposition 4.1 below. It is crucially used in Section 5.

PROPOSITION 4.1. *Let $f \in l_1(L_2(\mathbb{R}^d))$. For every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\varphi(\pi_1(f)(1-\Delta)^{-d/2}) = c_d \int_{\mathbb{R}^d} f(t)\mathrm{d}t.$$

That $\pi_1(f)(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ follows directly from Theorem 2.5.

We begin with a compact analogue of Proposition 4.1. Here, $\Delta_{\mathbb{T}}$ (respectively, $\nabla_{\mathbb{T}}$) is the Laplacian (respectively, gradient) on the torus $\mathbb{T}^d$. Given $f \in L_\infty(\mathbb{T}^d)$, the operator M_f denotes the operator of pointwise multiplication by f on $L_2(\mathbb{T})$.

LEMMA 4.2. *There exists a constant $c_d > 0$ such that*

$$\varphi(M_f(1-\Delta_{\mathbb{T}})^{-d/2}) = c_d \int_{\mathbb{T}^d} f(t)\mathrm{d}t, \quad f \in C(\mathbb{T}^d),$$

for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$.

Proof. Fix a continuous trace φ on $\mathcal{L}_{1,\infty}$ and consider a linear functional θ on $C(\mathbb{T}^d)$ defined by the formula

$$\theta(f) = \varphi(M_f(1-\Delta_{\mathbb{T}})^{-d/2}), \quad f \in C(\mathbb{T}^d).$$

This functional is bounded because

$$|\theta(f)| \leqslant \|\varphi\|_{\mathcal{L}^*_{1,\infty}} \|f\|_\infty \Big\| \Big\{ \frac{1}{(1+|n|^2)^{d/2}} \Big\}_{n\in\mathbb{Z}^d} \Big\|_{1,\infty}.$$

Let $t \to U_t$ be the representation of $\mathbb{T}^d$ on $L_2(\mathbb{T}^d)$ by translations, that is $(U_t\xi)(s) = \xi(s+t)$, $\xi \in L_2(\mathbb{T}^d)$. We have that

$$\begin{aligned} U_t(1-\Delta_{\mathbb{T}})^{-d/2} &= \mathrm{e}^{\mathrm{i}\langle t,\nabla_{\mathbb{T}}\rangle} \cdot (1+|\nabla_{\mathbb{T}}|^2)^{-d/2} = (1+|\nabla_{\mathbb{T}}|^2)^{-d/2} \cdot \mathrm{e}^{\mathrm{i}\langle t,\nabla_{\mathbb{T}}\rangle} \\ &= (1-\Delta_{\mathbb{T}})^{-d/2} U_t. \end{aligned}$$

Clearly,

$$U_t M_f U_t^{-1} = M_{U_t f} \quad t \in \mathbb{T}^d.$$

Since φ is unitarily invariant functional, it follows that

$$\begin{aligned}\theta(f) &= \varphi(U_t M_f (1-\Delta_{\mathbb{T}})^{-d/2} U_t^{-1}) = \varphi(U_t M_f U_t^{-1} (1-\Delta_{\mathbb{T}})^{-d/2}) \\ &= \varphi(M_{U_t f}(1-\Delta_{\mathbb{T}})^{-d/2}) = \theta(U_t f).\end{aligned}$$

Thus, θ is a translation invariant bounded linear functional on $C(\mathbb{T}^d)$.

By the Riesz–Markov theorem (see Theorem V.14 in [37]), a bounded linear functional on $C(\mathbb{T}^d)$ corresponds to a Baire measure on $\mathbb{T}^d$. Clearly, a translation invariant functional corresponds to a translation invariant measure. A translation invariant measure on $\mathbb{T}^d$ is uniformly distributed in the sense of Definition 3.3 in [31]. By Theorem 3.4 in [31], every translation invariant measure is a scalar multiple of the usual Lebesgue measure. Hence, there exists a constant $c_{\varphi,d}$ such that

$$\varphi(M_f(1-\Delta_{\mathbb{T}})^{-d/2}) = c_{\varphi,d} \int_{\mathbb{T}^d} f(t)\mathrm{d}t, \quad f \in C(\mathbb{T}^d).$$

We now show that $c_{\varphi,d}$ depends only on d.

By setting $f = 1$, we obtain that

$$c_{\varphi,d} = (2\pi)^{-d}\varphi(a), \quad a = \Big\{\frac{1}{(1+|n|^2)^{d/2}}\Big\}_{n\in\mathbb{Z}^d}.$$

We have

$$\mathrm{Card}(\{n\in\mathbb{Z}^d : a(n) \geqslant (1+r^2)^{-d/2}\}) = \mathrm{Card}(\{n\in\mathbb{Z}^d : |n|\leqslant r\}) = c_d r^d + O(r^{d-1}),$$

where c_d is the volume of the unit ball in $\mathbb{R}^d$. Thus,

$$\mathrm{Card}(\{n\in\mathbb{Z}^d : a(n)\geqslant t\}) = c_d t^{-1} + O(t^{(1/d)-1}).$$

In other words, we have

$$\mu(k,a) = c_d(k+1)^{-1} + O((k+1)^{-1-(1/d)}), \quad k \in \mathbb{Z}_+.$$

Since $\{(k+1)^{-1-(1/d)}\}_{k\geqslant 0} \in l_1$ and since φ vanishes on $\mathcal{L}_1$, it follows that $\varphi(a) = c_d$. ∎

LEMMA 4.3. *The Banach dual of* $(l_1(L_2))(\mathbb{R}^d)$ *is* $(l_\infty(L_2))(\mathbb{R}^d)$. *Specifically, every bounded linear functional on* $(l_1(L_2))(\mathbb{R}^d)$ *is given by the formula*

$$f \to \int_{\mathbb{R}^d} (fF)(u)\mathrm{d}u \quad \text{for some } F \in (l_\infty(L_2))(\mathbb{R}^d). \tag{4.1}$$

Proof. Firstly, let us establish that (4.1) indeed provides a bounded linear functional on $(l_1(L_2))(\mathbb{R}^d)$. Indeed, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^d}(fF)(u)\mathrm{d}u\Big| &\leqslant \sum_{m\in\mathbb{Z}^d}\Big|\int_{m+K}(fF)(u)\mathrm{d}u\Big| \leqslant \sum_{m\in\mathbb{Z}^d}\|f\chi_{m+K}\|_2\|F\chi_{m+K}\|_2 \\ &\leqslant \sup_{m\in\mathbb{Z}^d}\|F\chi_{m+K}\|_2 \cdot \sum_{m\in\mathbb{Z}^d}\|f\chi_{m+K}\|_2 = \|F\|_{l_\infty(L_2)}\|f\|_{l_1(L_2)}.\end{aligned}$$

Let θ be a bounded linear functional on $(l_1(L_2))(\mathbb{R}^d)$. For a fixed $m \in \mathbb{Z}^d$, the Hilbert space $L_2(m+K)$ is linearly and isometrically embedded into $l_1(L_2(\mathbb{R}^d))$ by inclusion. Hence, θ is a bounded linear functional on $L_2(m+K)$. By the Riesz representation theorem for the dual of a Hilbert space, there exists a function $F_m \in L_2(m+K)$ such that

$$\theta(f) = \int\limits_{m+K} (fF_m)(u)\mathrm{d}u, \quad f \in L_2(m+K).$$

By gluing F_m over all $m \in \mathbb{Z}^d$, we obtain a function F on $\mathbb{R}^d$.

We have

$$\begin{aligned}\|\theta\|_{(l_1(L_2))^*} &= \sup_{\substack{f \in l_1(L_2(\mathbb{R}^d)) \\ \|f\|_{l_1(L_2)}=1}} |\theta(f)| \geqslant \sup_{m \geqslant 0} \sup_{\substack{f \in L_2(m+K) \\ \|f\|_2=1}} |\theta(f)| \\ &= \sup_{m \geqslant 0} \sup_{\substack{f \in L_2(m+K) \\ \|f\|_2=1}} \Big| \int\limits_{m+K} (fF)(u)\mathrm{d}u \Big| = \sup_{m \geqslant 0} \|F\chi_{m+K}\|_2 = \|F\|_{l_\infty(L_2)}.\end{aligned}$$

Therefore, $F \in (l_\infty(L_2))(\mathbb{R}^d)$.

Let now θ_0 be a bounded linear functional given by (4.1) (with the function F defined above). For every fixed $m \in \mathbb{Z}^d$ and for every $f \in L_2(m+K)$, we have $\theta(f) = \theta_0(f)$. By linearity, we have $\theta(f) = \theta_0(f)$ for every compactly supported $f \in (l_1(L_2))(\mathbb{R}^d)$. Since the set of compactly supported functions is norm-dense in $(l_1(L_2))(\mathbb{R}^d)$ and since both θ and θ_0 are continuous, it follows that $\theta = \theta_0$. ∎

The following lemma is enough to prove Proposition 4.1 without the requirement that c_d has no dependence on φ.

LEMMA 4.4. *For every continuous trace φ on $\mathcal{L}_{1,\infty}$, there exists a constant $c_{\varphi,d}$ such that*

$$\varphi(\pi_1(f)(1-\Delta)^{-d/2}) = c_{\varphi,d} \int\limits_{\mathbb{R}^d} f(u)\mathrm{d}u, \quad f \in (l_1(L_2))(\mathbb{R}^d).$$

Proof. By Theorem 2.5, we have that $\pi_1(f)(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ for every $f \in (l_1(L_2))(\mathbb{R}^d)$. Fix a continuous trace φ on $\mathcal{L}_{1,\infty}$ and consider a linear functional θ on $(l_1(L_2))(\mathbb{R}^d)$ defined by the formula

$$\theta(f) = \varphi(\pi_1(f)(1-\Delta)^{-d/2}), \quad f \in (l_1(L_2))(\mathbb{R}^d).$$

Due to Theorem 2.5, the functional θ is bounded:

$$|\theta(f)| \leqslant c_d \|\varphi\|_{\mathcal{L}_{1,\infty}^*} \|f\|_{l_1(L_2)} \|t \to \frac{1}{(1+|t|^2)^{d/2}}\|_{l_{1,\infty}(L_2)}.$$

Recall that T_t denotes the unitary action of $t \in \mathbb{R}^d$ on $L_2(\mathbb{R}^d)$ by translations. For all t the operator T_t commutes with $(1-\Delta)^{-d/2}$ by Lemma 3.3.

Since φ is a unitarily invariant functional, it follows that

$$\begin{aligned}\theta(f) &= \varphi(T_t\pi_1(f)(1-\Delta)^{-d/2}T_t^{-1}) = \varphi(T_t\pi_1(f)T_t^{-1}(1-\Delta)^{-d/2}) \\ &= \varphi(\pi_1(T_tf)(1-\Delta)^{-d/2}) = \theta(T_tf).\end{aligned}$$

Thus, θ is a translation invariant functional on $(l_1(L_2))(\mathbb{R}^d)$.

Let $F \in (l_\infty(L_2))(\mathbb{R}^d)$ be the function corresponding to the functional θ as in Lemma 4.3. We claim that F is translation invariant. Indeed, for every $f \in (l_1(L_2))(\mathbb{R}^d)$ and for every $t \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^d} f(u)F(u)\mathrm{d}u = \theta(f) = \theta(T_tf) = \int_{\mathbb{R}^d} f(u+t)F(u)\mathrm{d}u = \int_{\mathbb{R}^d} f(u)F(u-t)\mathrm{d}u.$$

Thus, $F = T_{-t}F$ for every $t \in \mathbb{R}^d$. Since the only translation invariant functions on $\mathbb{R}^d$ are constants, it follows that the only translation invariant bounded linear functional is an integral with respect to the Lebesgue measure (modulo some constant factor $c_{\varphi,d}$). The assertion follows immediately. ∎

We now focus on removing the dependence of the constant $c_{\varphi,d}$ on φ. The following assertion is absolutely crucial for the proofs in this section. Here, $\nabla_{\mathbb{T}}$ is the gradient on the torus. If $a \in l_\infty(\mathbb{Z}^d)$, then $a(\nabla_{\mathbb{T}})$ is defined by the functional calculus. Note that the operators on the left and right hand sides in lemma below leave the subspace $L_2([0,1]^d)$ invariant and, so, the restrictions make sense.

LEMMA 4.5. *Let $f \in L_\infty([0,1]^d)$ and let $\phi \in C^\infty(\mathbb{R}^d)$ be supported in $[-3,3]^d$ such that $\phi|_{[-1,1]^d} = 1$. We have*

$$\pi_1(f)(1-\Delta)^{-d/2}\pi_1(f)|_{L_2([0,1]^d)} = M_f a(\nabla_{\mathbb{T}})M_f|_{L_2([0,1]^d)}.$$

Here, a is the sequence of Fourier coefficients of the function $t \to K_0(|t|)\phi(t)$ on the cube $[-\pi,\pi]^d$ and K_0 is the modified Bessel function from (2.3).

Proof. According to (2.4), the Fourier transform of the mapping $t \to (1+|t|^2)^{-d/2}$, $t \in \mathbb{R}^d$, is given (up to a constant) by the formula $t \to K_0(|t|)$, $t \in \mathbb{R}^d$. Applying (2.2), we obtain

$$((1-\Delta)^{-d/2}\xi)(t) = \int_{\mathbb{R}^d} K_0(|t-u|)\xi(u)\mathrm{d}u, \quad t \in \mathbb{R}^d. \tag{4.2}$$

It clearly follows that

$$(\pi_1(f)(1-\Delta)^{-d/2}\pi_1(f)\xi)(t) = \int_{[0,1]^d} f(t)f(u)K_0(|t-u|)\xi(u)\mathrm{d}u, \quad t \in [0,1]^d.$$

If $t,u \in [0,1]^d$, then $t-u \in [-1,1]^d$. In particular, we have (distance from a point to a set is taken in the usual Euclidean norm):

$$|t-u| = \mathrm{dist}(t-u, 2\pi\mathbb{Z}^d), \quad t,u \in [0,1]^d.$$

Since $\phi = 1$ on $[-1,1]^d$, it follows that

$$K_0(|t-u|) = K_0(\mathrm{dist}(t-u, 2\pi\mathbb{Z}^d))\psi(t-u), \quad t,u \in [0,1]^d,$$

where ψ is the 2π-periodic extension of $\phi|_{[-\pi,\pi]^d}$.

Define an operator S on $L_2([-\pi,\pi]^d)$ by setting

$$(S\xi)(t) = \int_{[-\pi,\pi]^d} f(t)f(u)K_0(\mathrm{dist}(t-u, 2\pi\mathbb{Z}^d))\psi(t-u)\xi(u)\mathrm{d}u, \quad t \in [-\pi,\pi]^d. \tag{4.3}$$

Comparing the right hand sides in (4.2) and (4.3), we obtain

$$\pi_1(f)(1-\Delta)^{d/2}\pi_1(f)|_{L_2([0,1]^d)} = S|_{L_2([0,1]^d)}.$$

Clearly, $S = M_f T M_f$, where

$$(T\xi)(t) = \int_{[-\pi,\pi]^d} K_0(\mathrm{dist}(t-u, 2\pi\mathbb{Z}^d))\psi(t-u)\xi(u)\mathrm{d}u, \quad t \in [-\pi,\pi]^d. \tag{4.4}$$

Writing

$$K_0(\mathrm{dist}(t, 2\pi\mathbb{Z}))\psi(t) = \sum_{n\in\mathbb{Z}^d} a(n)\mathrm{e}^{\mathrm{i}\langle n,t\rangle},$$

we obtain that

$$(T\xi)(t) = \sum_{n\in\mathbb{Z}^d} a(n)\mathrm{e}^{\mathrm{i}\langle n,t\rangle}\langle x, \mathrm{e}^{\mathrm{i}\langle n,\cdot\rangle}\rangle.$$

In other words, $T = a(\nabla_{\mathbb{T}})$. Hence, $S = M_f a(\nabla_{\mathbb{T}}) M_f$ and the assertion follows. ∎

LEMMA 4.6. *If a is as in Lemma 4.5, then there exists a constant c_d such that*

$$a(n) = \frac{c_d}{(1+|n|^2)^{d/2}} + O((1+|n|^2)^{-p/2}), \quad n \in \mathbb{Z}^d,$$

for every $p > d$.

Proof. Since ϕ is supported on $[-\pi,\pi]^d$ (see Lemma 4.5), it follows that

$$a(n) = \int_{[-\pi,\pi]^d} K_0(|t|)\phi(t)\mathrm{e}^{-\mathrm{i}\langle n,t\rangle}\mathrm{d}t = \int_{\mathbb{R}^d} K_0(|t|)\phi(t)\mathrm{e}^{-\mathrm{i}\langle n,t\rangle}\mathrm{d}t, \quad n \in \mathbb{Z}.$$

Define a Schwartz function ψ on $\mathbb{R}^d$ by setting $\psi(t) = K_0(|t|)\cdot(1-\phi(t))$, $t \in \mathbb{R}^d$. It immediately follows that

$$a(n) = \int_{\mathbb{R}^d} K_0(|t|)\mathrm{e}^{-\mathrm{i}\langle n,t\rangle}\mathrm{d}t - \int_{\mathbb{R}^d} \psi(t)\mathrm{e}^{-\mathrm{i}\langle n,t\rangle}\mathrm{d}t, \quad n \in \mathbb{Z}^d.$$

Since the Fourier transform of the mapping $t \to K_0(|t|)$, $t \in \mathbb{R}^d$, is given by the formula $t \to (1+|t|^2)^{-d/2}$, $t \in \mathbb{R}^d$, (see Example 2.4) and since the Fourier transform of a Schwartz function is again a Schwartz function, the assertion follows. ∎

Now we may complete the proof of Proposition 4.1, by removing the dependence on φ of the constant $c_{\varphi,d}$ in Lemma 4.4.

Proof of Proposition 4.1. Suppose first that $f \in C(\mathbb{R}^d)$ is positive and supported on $[0,1]^d$. Let $f = h^2$. Since φ is a trace, we have

$$\begin{aligned}\varphi(\pi_1(f)(1-\Delta)^{-d/2}) &= \varphi(\pi_1(h)(1-\Delta)^{-d/2}\pi_1(h)) \\ &= \varphi(\pi_1(h)(1-\Delta)^{-d/2}\pi_1(h)|_{L_2([-1,1]^d)}).\end{aligned}$$

Using Lemma 4.5, we obtain

$$\varphi(\pi_1(f)(1-\Delta)^{-d/2}) = \varphi(M_h a(\nabla_{\mathbb{T}})M_h|_{L_2([-1,1]^d)}) = \varphi(M_h a(\nabla_{\mathbb{T}})M_h).$$

Lemma 4.6 yields that $a \in l_{1,\infty}$ and that $a = a_1 + a_2$, where

$$a_1(n) = \frac{c_d}{(1+|n|^2)^{d/2}}, \quad a_2(n) = O\Big(\frac{1}{(1+|n|^2)^{(d+1)/2}}\Big), \quad n \in \mathbb{N}.$$

However $M_h a_2(\nabla_{\mathbb{T}})M_h \in \mathcal{L}_1$ and since φ vanishes on $\mathcal{L}_1$, it follows from Lemma 4.2 that

$$\varphi(\pi_1(f)(1-\Delta)^{-d/2}) = c_d\varphi(M_f(1-\Delta_{\mathbb{T}})^{-d/2}) = c_d\int_{\mathbb{R}^d} f(t)\mathrm{d}t. \tag{4.5}$$

This proves the assertion for positive $f \in C(\mathbb{R}^d)$ supported on $[0,1]^d$.

Consider now the general case and fix $f \in (l_1(L_2))(\mathbb{R}^d)$. Without loss of generality,

$$\int_{\mathbb{R}^d} f(t)\mathrm{d}t \geqslant 0$$

(otherwise, consider $-f$ instead). Let $f_1 \in C(\mathbb{R}^d)$ be positive and supported on $[0,1]^d$ and having the same integral as f. Set $f_2 = f - f_1$. It follows from Lemma 4.4 that

$$\varphi(\pi_1(f)(1-\Delta)^{-d/2}) = \varphi(\pi_1(f_1)(1-\Delta)^{-d/2}) + \varphi(\pi_1(f_2)(1-\Delta)^{-d/2}).$$

The first summand does not depend on φ by (4.5). The second summand vanishes by Lemma 4.4. ∎

5. TRACE FORMULA: FIRST ORDER APPROXIMATION

The main result of this section is Proposition 5.1 below. It delivers a minimalist version of Connes' trace formula, which is further used in Proposition 7.1 as a base of induction.

Observe that, by Theorem 2.5, $\pi_1(f)(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ for every function $f \in (l_1(L_2))(\mathbb{R}^d)$. Since $\pi_2(g)$ is a bounded operator which commutes with $(1-\Delta)^{-d/2}$, it follows that also $\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}$ for every $f \in (l_1(L_2))(\mathbb{R}^d)$, $g \in L_\infty(\mathbb{S}^{d-1})$.

PROPOSITION 5.1. *Let $f\in L_\infty(\mathbb{R}^d)$ be compactly supported, and let $g\in L_\infty(\mathbb{S}^{d-1})$. Then*

$$\varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}) = \int_{\mathbb{R}^d} f(t)\mathrm{d}t \cdot \int_{\mathbb{S}^{d-1}} g(s)\mathrm{d}s$$

for every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$.

LEMMA 5.2. *Let $f \in (l_1(L_2))(\mathbb{R}^d)$ and let $g \in L_\infty(\mathbb{S}^{d-1})$. If $\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = 0$, then*

$$\varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}) = 0$$

for every continuous trace φ on $\mathcal{L}_{1,\infty}$.

Proof. Fix a continuous trace φ on $\mathcal{L}_{1,\infty}$ and consider a bounded linear functional θ on $(l_1(L_2))(\mathbb{R}^d)$ defined by the formula

$$\theta(f) = \varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}), \quad f \in (l_1(L_2))(\mathbb{R}^d).$$

Repeating the argument in Lemma 4.4, we obtain that θ is translation invariant and is, therefore, an integral (modulo a constant factor). ∎

LEMMA 5.3. *Let $f \in L_\infty(\mathbb{R}^d)$ and $g \in C(\mathbb{S}^{d-1})$. If f is compactly supported and rotation invariant and if $\int_{\mathbb{S}^{d-1}} g(s)\,\mathrm{d}s = 0$, then*

$$\varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}) = 0$$

for every continuous trace φ on $\mathcal{L}_{1,\infty}$.

Proof. Fix a continuous trace φ on $\mathcal{L}_{1,\infty}$ and consider a linear functional θ on $C(\mathbb{S}^{d-1})$ defined by the formula

$$\theta(g) = \varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}), \quad g \in C(\mathbb{S}^{d-1}).$$

This functional is bounded because (due to Theorem 2.5)

$$|\theta(g)| \leqslant c_d \|\varphi\|_{\mathcal{L}_{1,\infty}^*} \|g\|_\infty \|f\|_{l_1(L_2)} \Big\| t \to \frac{1}{(1+|t|^2)^{d/2}} \Big\|_{l_{1,\infty}(L_2)}.$$

For a given $R \in \mathrm{SO}(d)$, let $U : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$ be the unitary operator given by the formula $U\xi = \xi \circ R$ for $\xi \in L_2(\mathbb{R}^d)$. Since by assumption f is rotation invariant, the operator U commutes with $\pi_1(f)$. Also, U commutes with Δ and hence with $(1-\Delta)^{-d/2}$.

Now, set $h(t) = g(\frac{t}{|t|})$, $t \in \mathbb{R}^d$. Since U commutes with the Fourier transform $\mathcal{F}$, it follows that

$$\begin{aligned} U\pi_2(g)U^{-1} &= U\mathcal{F}^{-1}\pi_1(h)\mathcal{F}U^{-1} = \mathcal{F}^{-1}U\pi_1(h)U^{-1}\mathcal{F} \\ &= \mathcal{F}^{-1}\pi_1(h \circ R)\mathcal{F} = \pi_2(g \circ R). \end{aligned}$$

Since φ is unitarily invariant, it follows that

$$\theta(g) = \varphi(U\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}U^{-1}) = \varphi(\pi_1(f)U\pi_2(g)U^{-1}(1-\Delta)^{-d/2})$$
$$= \varphi(\pi_1(f)\pi_2(g\circ R)(1-\Delta)^{-d/2}) = \theta(g\circ R).$$

Since R is arbitrary, it follows that θ is rotation invariant. Hence, the Baire measure on $\mathbb{S}^{d-1}$ defined by θ via the Riesz–Markov theorem is also rotation invariant, and therefore coincides with the usual rotation invariant measure on $\mathbb{S}^{d-1}$.

Recalling that by assumption $\int_{\mathbb{S}^{d-1}} g(s)\,\mathrm{d}s = 0$, so we arrive at the assertion. ∎

The following is a consequence of Fubini's theorem. A complete proof may be found in proof of Theorem 2.f.2 of [28].

FACT 5.4. Let (X_j,ν_j), $j=1,2$ be σ-finite measure spaces and $p\in[1,\infty]$. If $v\in L_p(X_1,\nu_1)$ and $w\in L_{p,\infty}(X_2,\nu_2)$, then $v\otimes w\in L_{p,\infty}(X_1\times X_2,\nu_1\times\nu_2)$ and

$$\|v\otimes w\|_{L_{p,\infty}} \leqslant \|v\|_{L_p}\|w\|_{L_{p,\infty}}.$$

LEMMA 5.5. *Let $p\in(d,\infty)$, and let $v\in L_p(\mathbb{S}^{d-1})$,. Define*

$$h(t) = v\Big(\frac{t}{|t|}\Big)(1+|t|^2)^{-d/2}, \quad t\in\mathbb{R}^d.$$

Then there is a constant $c_{d,p}>0$ such that

$$\|h\|_{l_{1,\infty}(L_2)} \leqslant c_{d,p}\|v\|_p.$$

Proof. Select $q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$, and write h as

$$h(t) = v\Big(\frac{t}{|t|}\Big)w(|t|)y(t), \quad t\in\mathbb{R}^d,$$

where

$$w(r) = (1+r^2)^{-d/(2p)}, \quad r>0 \quad \text{and,} \quad y(t) = (1+|t|^2)^{-d/(2q)}, \quad t\in\mathbb{R}^d.$$

Let $u(t) := v\big(\frac{t}{|t|}\big)w(|t|)$.

The proof proceeds by showing first that

$$\|h\|_{l_{1,\infty}(L_2)} \leqslant c_p\|u\|_{l_{p,\infty}(L_2)}\|y\|_{l_{q,\infty}(L_\infty)}, \tag{5.1}$$

and secondly that

$$\|u\|_{l_{p,\infty}(L_2)} \leqslant c_{d,p}\|v\|_{L_p}\|w\|_{L_{p,\infty}(\mathbb{R}^+,r^{d-1}\mathrm{d}r)}. \tag{5.2}$$

Then the proof is completed by showing that $\|y\|_{l_{q,\infty}(L_\infty)}$ and $\|w\|_{L_{p,\infty}(\mathbb{R}^+,r^{d-1}\mathrm{d}r)}$ are finite.

First we show (5.1).

For every $m\in\mathbb{Z}^d$ we have:

$$\|uy\chi_{m+[0,1]^d}\|_{L_2} \leqslant \|u\chi_{m+[0,1]^d}\|_{L_2}\|y\chi_{m+[0,1]^d}\|_{L_\infty}.$$

Thus,

$$\|uy\|_{l_{1,\infty}(L_2)} \leqslant \|\{\|u\chi_{m+[0,1]^d}\|_{L_2}\}_{m\in\mathbb{Z}^d} \cdot \{\|y\chi_{m+[0,1]^d}\|_{L_\infty}\}_{m\in\mathbb{Z}^d}\|_{l_{1,\infty}}.$$

Now applying (2.1) we get $\|uy\|_{l_{1,\infty}(L_2)} \leqslant c_p\|u\|_{l_{p,\infty}(L_2)}\|y\|_{l_{q,\infty}(L_\infty)}$. Since $h = uy$, this yields (5.1).

Now we prove (5.2). Consider the mapping

$$i : \mathbb{R}^d \to (\mathbb{S}^{d-1}, \mathrm{d}s) \times (\mathbb{R}_+, r^{d-1}\mathrm{d}r)$$

given by the formula $i(t) = \left(\frac{t}{|t|}, |t|\right)$. This mapping is measure preserving. So we have $u \circ i = v \otimes w$. Then using Fact 5.4, it follows that

$$\begin{aligned}\|u\|_{L_{p,\infty}} &= \|u \circ i\|_{L_{p,\infty}(\mathbb{S}^{d-1}\times\mathbb{R}_+,\mathrm{d}s\times r^{d-1}\mathrm{d}r)} \\ &= \|v \otimes w\|_{L_{p,\infty}(\mathbb{S}^{d-1}\times\mathbb{R}_+,\mathrm{d}s\times r^{d-1}\mathrm{d}r)} \leqslant \|v\|_{L_p}\|w\|_{L_{p,\infty}(\mathbb{R}_+,r^{d-1}\mathrm{d}r)}.\end{aligned}$$

However, we also have that for $p \geqslant 2$, $\|u\|_{l_{p,\infty}(L_2)} \leqslant \|u\|_{L_{p,\infty}}$. This can be seen from the pair of inequalities,

$$\|u\|_{l_2(L_2)} \leqslant \|u\|_{L_2}, \quad \|u\|_{l_p(L_2)} \leqslant \|u\|_{L_p}.$$

The first inequality is in fact an equality, and the second follows from the assumption that $p \geqslant 2$. Then applying the method of real interpolation (see e.g. [28]), it follows that for $p > 2$ that $\|u\|_{l_{p,\infty}(L_2)} \leqslant \|u\|_{L_{p,\infty}}$. So now we have shown that

$$\|u\|_{l_{p,\infty}(L_2)} \leqslant \|u\|_{L_{p,\infty}} \leqslant \|v\|_{L_p}\|w\|_{L_{p,\infty}(\mathbb{R}_+,r^{d-1}\mathrm{d}r)},$$

thus proving (5.2). Finally, to show that $\|y\|_{l_{q,\infty}(L_\infty)}$ and $\|w\|_{L_{p,\infty}(\mathbb{R}^+,r^{d-1}\mathrm{d}r)}$ are finite follows easily from their definitions. ∎

LEMMA 5.6. *Let $f \in L_\infty(\mathbb{R}^d)$ and $g \in L_\infty(\mathbb{S}^{d-1})$. If f is compactly supported and rotation invariant and if g is mean zero, then*

$$\varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}) = 0$$

for every continuous trace φ on $\mathcal{L}_{1,\infty}$.

Proof. Select a sequence of mean zero functions $\{g_n\}_{n\geqslant 1} \subset C(\mathbb{S}^{d-1})$ such that $\|g_n - g\|_{2d} \to 0$ as $n \to \infty$. Set

$$h(t) := g\Big(\frac{t}{|t|}\Big)(1+|t|^2)^{-d/2}, \quad h_n(t) := g_n\Big(\frac{t}{|t|}\Big)(1+|t|^2)^{-d/2}, \quad t \in \mathbb{R}^d.$$

By Lemma 5.5, we have that $h_n \to h$ in $(l_{1,\infty}(L_2))(\mathbb{R}^d)$ as $n \to \infty$. Then applying (2.5) for all $n > 0$ we have

$$\|\pi_1(f)\pi_2(g_n - g)(1-\Delta)^{-d/2}\|_{1,\infty} \leqslant c_d\|f\|_{l_1(L_2)}\|h_n - h\|_{l_{1,\infty}(L_2)}$$

and hence as $n \to \infty$ we have $\|\pi_1(f)\pi_2(g_n - g)(1-\Delta)^{-d/2}\|_{L_{1,\infty}} \to 0$. Since φ is continuous, it follows that as $n \to \infty$,

$$\varphi(\pi_1(f)\pi_2(g_n)(1-\Delta)^{-d/2}) \to \varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}).$$

The assertion follows now from Lemma 5.3. ∎

Proof of Proposition 5.1. Select a compactly supported rotation invariant function $f_1 \in L_\infty(\mathbb{R}^d)$ having the same integral as f. Set $f_2 := f - f_1$,

$$g_1 := \int\limits_{\mathbb{S}^{d-1}} g(s)\mathrm{d}s, \quad g_2 := g - g_1.$$

We have

$$\begin{aligned}\varphi(\pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}) &= \varphi(\pi_1(f_2)\pi_2(g)(1-\Delta)^{-d/2}) + \varphi(\pi_1(f_1)\pi_2(g_2)(1-\Delta)^{-d/2}) \\ &\quad + \Big(\int\limits_{\mathbb{S}^{d-1}} g(s)\mathrm{d}s\Big)\cdot\varphi(\pi_1(f_1)(1-\Delta)^{-d/2}).\end{aligned}$$

Examining the right hand side, the first summand vanishes by Lemma 5.2 and the second summand vanishes by Lemma 5.6. It follows from Proposition 4.1 that

$$\varphi(\pi_1(f_1)(1-\Delta)^{-d/2}) = \int\limits_{\mathbb{R}^d} f_1(t)\mathrm{d}t = \int\limits_{\mathbb{R}^d} f(t)\mathrm{d}t.$$

This concludes the proof. ∎

6. COMMUTATOR ESTIMATES IN $\mathcal{L}_1$

In the following lemma, $C^\infty(\mathbb{R}^d)$ denotes the collection of all infinitely differentiable functions f on $\mathbb{R}^d$ such that f and all its derivatives are bounded.

The main result of this section is Proposition 6.1 below. It is used crucially in the proof of Lemma 7.1.

PROPOSITION 6.1. *If $\phi \in C_c^\infty(\mathbb{R}^d)$ and if $f \in C^\infty(\mathbb{R}^d)$ and $g \in C^\infty(\mathbb{S}^{d-1})$, then*

$$[\pi_1(f), \pi_2(g)(1-\Delta)^{-d/2}\pi_1(\phi)] \in \mathcal{L}_1.$$

The argument of the proof of Lemma 6.2 below is somewhat similar to the one in Lemma 2.29 of [10], to which we refer the reader for additional information.

LEMMA 6.2. *If $\phi \in C_c^\infty(\mathbb{R}^d)$, then*

$$[(1-\Delta)^{1/2}, \pi_1(\phi)](1-\Delta)^{-(d+1)/2} \in \mathcal{L}_1. \tag{6.1}$$

Proof. In order to lighten the notation, denote $K := 1-\Delta$. We have the integral formulae:

$$K^{-1/2} = \frac{1}{\pi}\int\limits_0^\infty \frac{1}{\lambda^{1/2}(\lambda+K)}\,\mathrm{d}\lambda \quad \text{and} \tag{6.2}$$

$$K^{-3/2} = \frac{2}{\pi}\int_0^\infty \frac{1}{\lambda^{1/2}(\lambda+K)^2}\,\mathrm{d}\lambda. \tag{6.3}$$

Hence, since $\frac{1}{\pi}\int_0^\infty \frac{K^{1/2}}{\lambda^{1/2}(\lambda+K)}\,\mathrm{d}\lambda = 1$ and $\frac{1}{\pi}\int_0^\infty \frac{K^{3/2}}{\lambda^{1/2}(\lambda+K)^2}\mathrm{d}\lambda = \frac{1}{2}$, we arrive at the useful expression,

$$\frac{1}{\pi}\int_0^\infty \Big(\frac{K^{1/2}}{\lambda+K} - \frac{K^{3/2}}{(\lambda+K)^2}\Big)\frac{\mathrm{d}\lambda}{\lambda^{1/2}} = \frac{1}{2}. \tag{6.4}$$

Multiplying (6.2) by K and taking the commutator with $\pi_1(\phi)$ yields:

$$[K^{1/2},\pi_1(\phi)] = \frac{1}{\pi}\int_0^\infty \Big[\frac{K}{\lambda+K},\pi_1(\phi)\Big]\frac{\mathrm{d}\lambda}{\lambda^{1/2}}.$$

Focusing on the commutator inside the integral,

$$\begin{aligned}
\Big[\frac{K}{\lambda+K},\pi_1(\phi)\Big] &= K[(\lambda+K)^{-1},\pi_1(\phi)] + [K,\pi_1(\phi)](\lambda+K)^{-1}\\
&= -\frac{K}{\lambda+K}[K,\pi_1(\phi)](\lambda+K)^{-1} + [K,\pi_1(\phi)](\lambda+K)^{-1}\\
&= \Big(1-\frac{K}{\lambda+K}\Big)[K,\pi_1(\phi)](\lambda+K)^{-1}\\
&= \Big(1-\frac{K}{\lambda+K}\Big)([[K,\pi_1(\phi)],(\lambda+K)^{-1}] + (\lambda+K)^{-1}[K,\pi_1(\phi)])\\
&= \Big(\frac{1}{\lambda+K} - \frac{K}{(\lambda+K)^2}\Big)[K,\pi_1(\phi)]\\
&\qquad + \Big(\frac{K}{(\lambda+K)^2} - \frac{1}{\lambda+K}\Big)[[K,\pi_1(\phi)],K](\lambda+K)^{-1}\\
&= \Big(\frac{1}{\lambda+K} - \frac{K}{(\lambda+K)^2}\Big)[K,\pi_1(\phi)] + \frac{\lambda}{(\lambda+K)^2}[K,[K,\pi_1(\phi)]](\lambda+K)^{-1}.
\end{aligned}$$

We introduce the operators $L(\phi)$ and $R(\phi)$, defined as:

$$L(\phi) := K^{-1/2}[K,\pi_1(\phi)] \quad \text{and} \quad R(\phi) := K^{-1}[K,[K,\pi_1(\phi)]].$$

So we have

$$\Big[\frac{K}{\lambda+K},\pi_1(\phi)\Big] = \Big(\frac{K^{1/2}}{\lambda+K} - \frac{K^{3/2}}{(\lambda+K)^2}\Big)L(\phi) + \frac{\lambda K}{(\lambda+K)^2}R(\phi)(\lambda+K)^{-1}.$$

So integrating over λ and applying (6.4) we obtain:

$$[K^{1/2},\pi_1(\phi)] = \frac{1}{2}L(\phi) + \frac{1}{\pi}\int_0^\infty \frac{\lambda K}{(\lambda+K)^2}R(\phi)(\lambda+K)^{-1}\,\frac{\mathrm{d}\lambda}{\lambda^{1/2}}.$$

Then multiplying on the right by $K^{-(d+1)/2}$,

$$[K^{1/2},\pi_1(\phi)]K^{-(d+1)/2} = \frac{1}{2}L(\phi)K^{-(d+1)/2} + \frac{1}{\pi}\int_0^\infty \frac{\lambda K}{(\lambda+K)^2}R(\phi)(\lambda+K)^{-1}K^{-(d+1)/2}\,\frac{d\lambda}{\lambda^{1/2}}.$$

We claim that $L(\phi)K^{-(d+1)/2} \in \mathcal{L}_1$. Indeed, from Lemma 2.1,

$$\|L(\phi)K^{-(d+1)/2}\|_1 = \|K^{-1/2}[K,\pi_1(\phi)]K^{-(d+1)/2}\|_1 \leqslant \|[K,\pi_1(\phi)]K^{-(d+2)/2}\|_1.$$

Then expanding out the commutator,

$$\begin{aligned}[K,\pi_1(\phi)]K^{-(d+2)/2} &= [-\Delta,\pi_1(\phi)](1-\Delta)^{-(d+2)/2}\\ &= \sum_{j=1}^d (D_j[D_j,\pi_1(\phi)] + [D_j,\pi_1(\phi)]D_j)(1-\Delta)^{-(d+2)/2}\\ &= \sum_{j=1}^d (-\pi_1(\partial_j^2\phi) - 2i\pi_1(\partial_j\phi)D_j)(1-\Delta)^{-(d+2)/2}.\end{aligned}$$

Since $\phi \in C_c^\infty(\mathbb{R}^d)$, we may apply Lemma 2.4 that each of the above summands is in $\mathcal{L}_1$, hence $L(\phi)K^{-(d+1)/2} \in \mathcal{L}_1$.

Applying Theorem 2.4, the operator $L(\phi)(1-\Delta)^{-(d+1)/2}$ is in $\mathcal{L}_1$, so the proof is completed on showing

$$\int_0^\infty \frac{\lambda K}{(\lambda+K)^2}R(\phi)(\lambda+K)^{-1}K^{-(d+1)/2}\frac{d\lambda}{\lambda^{1/2}} \in \mathcal{L}_1.$$

Again applying Lemma 2.1 and Theorem 2.4, one can show that

$$R(\phi)K^{-(d+1)/2} \in \mathcal{L}_1.$$

Note that for all $\lambda > 0$ we have

$$\Big\|\frac{\lambda K}{(\lambda+K)^2}\Big\|_\infty \leqslant 1 \quad \text{and} \quad \Big\|\frac{1}{\lambda+K}\Big\|_\infty \leqslant \frac{1}{1+\lambda}.$$

So there is a constant C such that

$$\Big\|\frac{\lambda K}{(\lambda+K)^2}R(\phi)K^{-(d+1)/2}(\lambda+K)^{-1}\Big\|_{\mathcal{L}_1} \leqslant C(1+\lambda)^{-1}.$$

Hence,

$$\Big\|\int_0^\infty \frac{\lambda K}{(\lambda+K)^2}R(\phi)(\lambda+K)^{-1}K^{-(d+1)/2}\frac{d\lambda}{\lambda^{1/2}}\Big\|_{\mathcal{L}_1} \leqslant C\int_0^\infty \frac{1}{\lambda^{1/2}(1+\lambda)}\,d\lambda < \infty.$$

Thus, $[K^{1/2},\pi_1(\phi)]K^{-(d+1)/2} \in \mathcal{L}_1$. ∎

LEMMA 6.3. *If* $\phi \in C_c^\infty(\mathbb{R}^d)$, *then*

$$[\pi_1(\phi), (1-\Delta)^{-d/2}] \in \mathcal{L}_1.$$

Proof. Without loss of generality assume that ϕ is real-valued. Using the Leibniz rule we can write

$$\begin{aligned}[\pi_1(\phi), (1-\Delta)^{-d/2}] &= \sum_{k=0}^{d-1} (1-\Delta)^{-k/2}[\pi_1(\phi), (1-\Delta)^{-1/2}](1-\Delta)^{-(d-1-k)/2} \\ &= \sum_{k=0}^{d-1} (1-\Delta)^{-(k+1)/2}[(1-\Delta)^{1/2}, \pi_1(\phi)](1-\Delta)^{-(d-k)/2}.\end{aligned}$$

Since ϕ is real-valued, the operator $\mathrm{i}[(1-\Delta)^{1/2}, \pi_1(\phi)]$ is self-adjoint. Now applying Lemma 2.1 to each summand, we arrive at

$$\begin{aligned}&\|(1-\Delta)^{-(k+1)/2}[(1-\Delta)^{1/2}, \pi_1(\phi)](1-\Delta)^{-(d-k)/2}\|_1 \\ &\qquad\qquad \leqslant \|[(1-\Delta)^{1/2}, \pi_1(\phi)](1-\Delta)^{-(d+1)/2}\|_1.\end{aligned}$$

So by Lemma 6.2 each summand is in $\mathcal{L}_1$. ∎

LEMMA 6.4. *If* $\phi \in C_c^\infty(\mathbb{R}^d)$, *then*

$$\Big[\pi_1(\phi), \frac{D_k}{(-\Delta)^{1/2}}\Big](1-\Delta)^{-d/2} \in \mathcal{L}_1, \quad 1 \leqslant k \leqslant d.$$

Proof. Without loss of generality, assume that ϕ is real-valued. Let

$$g_k(t) := \frac{t_k}{|t|} - \frac{t_k}{(1+|t|^2)^{1/2}} = \frac{t_k}{|t|} \cdot \frac{1}{(1+|t|^2)^{1/2}(|t| + (1+|t|^2)^{1/2})}, \quad t \in \mathbb{R}^d.$$

We decompose the operator $\big[\pi_1(\phi), \frac{D_k}{(-\Delta)^{1/2}}\big](1-\Delta)^{-d/2}$ into four parts as follows:

$$\Big[\pi_1(\phi), \frac{D_k}{(-\Delta)^{1/2}}\Big](1-\Delta)^{-d/2} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV},$$

where

$$\mathrm{I} = \frac{D_k}{(1-\Delta)^{1/2}}[(1-\Delta)^{1/2}, \pi_1(\phi)](1-\Delta)^{-(d+1)/2}, \quad \mathrm{II} = [\pi_1(\phi), D_k](1-\Delta)^{-(d+1)/2},$$

$$\mathrm{III} = \pi_1(\phi) g_k(\nabla)(1-\Delta)^{-d/2}, \quad \mathrm{IV} = -g_k(\nabla)\pi_1(\phi)(1-\Delta)^{-d/2}.$$

It follows from Lemma 6.2 that $\mathrm{I} \in \mathcal{L}_1$. Since $[D_k, \pi_1(\phi)] = \pi_1(D_k\phi)$, it follows from Theorem 2.4 that $\mathrm{II} \in \mathcal{L}_1$. It follows from Theorem 2.4 that

$$\|\mathrm{III}\|_1 \leqslant \|\pi_1(\phi)(1-\Delta)^{-1-(d/2)}\|_1 < \infty.$$

It follows from Theorem 2.4 and Lemma 2.1 that

$$\|\mathrm{IV}\|_1 \leqslant \|(1-\Delta)^{-1}\pi_1(\phi)(1-\Delta)^{-d/2}\|_1 \leqslant \|\pi_1(\phi)(1-\Delta)^{-1-(d/2)}\|_1 < \infty.$$

A combination of all four inclusions completes the proof. ∎

LEMMA 6.5. *If $\phi \in C_c^\infty(\mathbb{R}^d)$ and if $v \in C^\infty(\mathbb{S}^{d-1})$, then*

$$[\pi_1(\phi), \pi_2(v)](1-\Delta)^{-d/2} \in \mathcal{L}_1.$$

Proof. Let $B = \frac{\nabla}{(-\Delta)^{1/2}} = \left\{\frac{D_k}{(-\Delta)^{1/2}}\right\}_{k=1}^d$. We may extend v to a smooth compactly supported function w on $\mathbb{R}^d$. For example, we may set $w(t) = v\big(\frac{t}{|t|}\big)\phi_0(|t|)$, $t \in \mathbb{R}^d$, where ϕ_0 is a Schwartz function on $\mathbb{R}$ which vanishes outside of the interval on $(\frac{1}{2}, \frac{3}{2})$ and such that $\phi_0(1) = 1$.

The Fourier transform of w is also a Schwartz function. By Definition 1.1, we have

$$\pi_2(v) = v(B) = w(B) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (\mathcal{F}w)(t) e^{i\langle B,t\rangle} dt.$$

Therefore,

$$[\pi_1(\phi), \pi_2(v)](1-\Delta)^{-d/2} = (2\pi)^{-d/2} \int_{\mathbb{R}^d} (\mathcal{F}w)(t)[\pi_1(f), e^{i\langle B,t\rangle}](1-\Delta)^{-d/2} dt.$$

An elementary computation yields

$$[x, e^{iy}] = i \int_0^1 e^{isy}[y,x]e^{i(1-s)y} ds$$

for all bounded operators x and self-adjoint bounded operators y. Thus, for every $t \in \mathbb{R}^d$,

$$\|[\pi_1(\phi), e^{i\langle B,t\rangle}](1-\Delta)^{-d/2}\|_1 \leqslant \sum_{k=1}^d |t_k| \cdot \|[\pi_1(\phi), B_k](1-\Delta)^{-d/2}\|_1.$$

Taking the maximum of the $\mathcal{L}_1$-norms and noting that $\sum_{k=1}^d |t_k| \leqslant d^{1/2}|t|$, we obtain

$$\begin{aligned}&\|[\pi_1(\phi), \pi_2(v)](1-\Delta)^{-d/2}\|_1 \\ &\qquad \leqslant d^{1/2}(2\pi)^{-d/2} \int_{\mathbb{R}^d} |(\mathcal{F}w)(t)| \cdot |t| dt \cdot \max_{1\leqslant k\leqslant d} \|[\pi_1(\phi), B_k](1-\Delta)^{-d/2}\|_1.\end{aligned}$$

The assertion follows now from Lemma 6.4. ∎

Proof of Proposition 6.1. We consider the decomposition

$$\begin{aligned}&[\pi_1(f), \pi_2(g)(1-\Delta)^{-d/2}\pi_1(\phi)] \\ &\qquad = \pi_1(f)\pi_2(g)(1-\Delta)^{-d/2}\pi_1(\phi) - \pi_2(g)(1-\Delta)^{-d/2}\pi_1(f\phi) \\ &\qquad = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}\end{aligned}$$

where

$$\mathrm{I} = \pi_1(f)(1-\Delta)^{-d/2}[\pi_2(g), \pi_1(\phi)], \quad \mathrm{II} = \pi_1(f)[(1-\Delta)^{-d/2}, \pi_1(\phi)]\pi_2(g),$$
$$\mathrm{III} = [\pi_1(f\phi), \pi_2(g)](1-\Delta)^{-d/2}, \quad \mathrm{IV} = \pi_2(g)[\pi_1(f\phi), (1-\Delta)^{-d/2}].$$

By Lemma 6.5, we have $\mathrm{I}, \mathrm{III} \in \mathcal{L}_1$. By Lemma 6.3, we also have $\mathrm{II}, \mathrm{IV} \in \mathcal{L}_1$. A combination of all four inclusions completes the proof. ∎

7. CONNES' TRACE THEOREM IN TERMS OF THE PRINCIPAL SYMBOL

In this section, we prove Theorem 1.5 in full generality. The following proposition is a crucial component of the proof of Theorem 1.5.

PROPOSITION 7.1. *For every normalised continuous trace φ on $\mathcal{L}_{1,\infty}$, for every $m \geqslant 1$, and for every $\phi \in C_c^\infty(\mathbb{R}^d)$, we have*

$$\varphi(T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi)) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(T_{m,\mathbf{f},\mathbf{g}}\pi_1(\phi)). \tag{7.1}$$

Here, for $\mathbf{f} = (f_1, \ldots, f_m) \in L_\infty(\mathbb{R}^d)^m$ and $\mathbf{g} = (g_1, \ldots, g_m) \in L_\infty(\mathbb{S}^{d-1})^m$, we write

$$T_{m,\mathbf{f},\mathbf{g}} = \prod_{k=1}^m \pi_1(f_k)\pi_2(g_k).$$

Our proof of Proposition 7.1 is by induction. The following two lemmas are essential ingredients of the induction step.

LEMMA 7.2. *Fix $m \geqslant 1$, and suppose that* (7.1) *is true for all $\mathbf{f} \in L_\infty(\mathbb{R}^d)^m$ and $\mathbf{g} \in L_\infty(\mathbb{S}^{d-1})^m$ with $f_m \in C^\infty(\mathbb{R}^d)$ and $g_m \in C^\infty(\mathbb{S}^{d-1})$. Then* (7.1) *holds for all $\mathbf{f} \in L_\infty(\mathbb{R}^d)^m$ and $\mathbf{g} \in L_\infty(\mathbb{S}^{d-1})^m$ with $g_m \in C^\infty(\mathbb{S}^{d-1})$.*

Proof. Let $\mathbf{f} \in L_\infty(\mathbb{R}^d)^m$ and $\mathbf{g} \in L^\infty(\mathbb{S}^{d-1})^m$. Define for each $1 \leqslant k < m$, $h_k := f_k$, and $h_m = f_m\phi$. Clearly,

$$[\pi_2(g_m)(1-\Delta)^{-d/2}, \pi_1(\phi)]$$
$$= \pi_2(g_m)[(1-\Delta)^{-d/2}, \pi_1(\phi)] + [\pi_2(g_m), \pi_1(\phi)](1-\Delta)^{-d/2}.$$

By Lemmas 6.3 and 6.5, the above hand side of the latter equality belongs to $\mathcal{L}_1$. Hence,

$$T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi) - T_{m,\mathbf{h},\mathbf{g}}(1-\Delta)^{-d/2}$$
$$= T_{m-1,\mathbf{f},\mathbf{g}}\pi_1(f_m)[\pi_2(g_m)(1-\Delta)^{-d/2}, \pi_1(\phi)] \in \mathcal{L}_1.$$

Since φ vanishes on $\mathcal{L}_1$, it follows that

$$\varphi(T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi)) = \varphi(T_{m,\mathbf{h},\mathbf{g}}(1-\Delta)^{-d/2}). \tag{7.2}$$

By the assumption that (7.1) holds for $\mathbf{f} \in L_\infty(\mathbb{R}^d)^m$ and $\mathbf{g} \in L_\infty(\mathbb{S}^{d-1})^m$ with $f_m \in C^\infty(\mathbb{R}^d)$ and $g_m \in C^\infty(\mathbb{S}^{d-1})$ and (7.2), we have

$$\varphi(T_{m,\mathbf{h},\mathbf{g}}(1-\Delta)^{-d/2}) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \operatorname{symb}(T_{m,\mathbf{h},\mathbf{g}}) \tag{7.3}$$

under the same conditions on $\mathbf{f}$ and $\mathbf{g}$.

We now estimate both sides of (7.3). As $\pi_2(g_m)$ commutes with $(1-\Delta)^{-d/2}$, we have

$$T_{m,\mathbf{h},\mathbf{g}}(1-\Delta)^{-d/2} = T_{m-1,\mathbf{f},\mathbf{g}} \cdot \pi_1(f_m\phi)(1-\Delta)^{-d/2} \cdot \pi_2(g_m).$$

Since φ is continuous, it follows that

$$|\varphi(T_{m,\mathbf{h},\mathbf{g}}(1-\Delta)^{-d/2})| \leqslant \prod_{k=1}^{m-1} \|f_k\|_\infty \cdot \prod_{k=1}^{m} \|g_k\|_\infty \cdot \|\pi_1(f_m\phi)(1-\Delta)^{-d/2}\|_{1,\infty}.$$

By Theorem 2.5, we have

$$|\varphi(T_{m,\mathbf{h},\mathbf{g}}(1-\Delta)^{-d/2})| \leqslant c_d \prod_{k=1}^{m-1} \|f_k\|_\infty \cdot \prod_{k=1}^{m} \|g_k\|_\infty \cdot \|f_m\phi\|_{l_1(L_2)}.$$

Now, estimating the right hand side of (7.3) yields

$$\Big| \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \operatorname{symb}(T_{m,\mathbf{h},\mathbf{g}}) \Big| \leqslant \prod_{k=1}^{m-1} \|f_k\|_\infty \cdot \prod_{k=1}^{m} \|g_k\|_\infty \cdot \|f_m\phi\|_{l_1(L_2)}.$$

That is, both sides of (7.3) are continuous functionals of f_m in the norm $f_m \to \|f_m\phi\|_{l_1(L_2)}$. Consider the semi-norm on $L_\infty(\mathbb{R}^d)$ given by the formula $f \to \|f\phi\|_{l_1(L_2)}$ (this semi-norm is well defined because ϕ is finitely supported, so that $f\phi \in l_1(L_2(\mathbb{R}^d))$). Clearly, $C^\infty(\mathbb{R}^d)$ is dense in $L_\infty(\mathbb{R}^d)$ in the latter semi-norm. Thus, (7.3) holds true provided that $f_m \in L_\infty(\mathbb{R}^d)$ and $g_m \in C^\infty(\mathbb{S}^{d-1})$. The assertion follows now from (7.2). ∎

LEMMA 7.3. *If, for a given $m \geqslant 1$,* (7.1) *holds for every $\mathbf{f} \in L_\infty(\mathbb{R}^d)^m$ and for every $\mathbf{g} \in L_\infty(\mathbb{S}^{d-1})^m$ satisfying the condition $g_m \in C^\infty(\mathbb{S}^{d-1})$, then* (7.1) *holds for all $\mathbf{f} \in L_\infty(\mathbb{R}^d)^m$ and for all $\mathbf{g} \in L_\infty(\mathbb{S}^{d-1})^m$.*

Proof. Let $u_m(t) = g_m(\frac{t}{|t|})(1+|t|^2)^{-d/2}$, $t \in \mathbb{R}^d$. Recalling the notation π_3 from Theorem 1.3, we write

$$T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi) = T_{m-1,\mathbf{f},\mathbf{g}}\pi_1(f_m) \cdot \pi_3(u_m)\pi_1(\phi).$$

Since φ is continuous, it follows that

$$|\varphi(T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi))| \leqslant \prod_{k=1}^{m} \|f_k\|_\infty \cdot \prod_{k=1}^{m-1} \|g_k\|_\infty \|\pi_3(u_m)\pi_1(\phi)\|_{1,\infty}.$$

By Theorem 2.5 and Lemma 5.5 (with $p = 2d$), there exists a constant c_d such that

$$\|\pi_3(u_m)\pi_1(\phi)\|_{1,\infty} \leqslant c_d \|\phi\|_{l_1(L_2)} \|g_m\|_{2d}.$$

Therefore, we have

$$|\varphi(T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi))| \leqslant c_d \prod_{k=1}^{m} \|f_k\|_\infty \cdot \prod_{k=1}^{m-1} \|g_k\|_\infty \cdot \|g_m\|_{2d} \|\phi\|_{l_1(L_2)}.$$

That is, the left hand side of (7.1) is a continuous functional of g_m with respect to the norm $\|\cdot\|_{2d}$. Clearly, the right hand side of (7.1)

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \operatorname{symb}(T_{m,\mathbf{f},\mathbf{g}}\pi_1(\phi)) = \int_{\mathbb{R}^d} \phi(t) \prod_{k=1}^{m} f_k(t)\mathrm{d}t \cdot \int_{\mathbb{S}^{d-1}} \prod_{k=1}^{m} g_k(s)\mathrm{d}s$$

is also a continuous functional of g_m with respect to the norm $\|\cdot\|_{2d}$. By the assumption, (7.1) holds for every $g_m \in C^\infty(\mathbb{S}^{d-1})$. Note that $C^\infty(\mathbb{S}^{d-1})$ is dense in $L_\infty(\mathbb{S}^{d-1})$ with respect to the norm $\|\cdot\|_{2d}$. Thus, (7.1) holds for every $g_m \in L_\infty(\mathbb{S}^{d-1})$. ∎

LEMMA 7.4. *For a given $m \geqslant 1$, if* (7.1) *holds for $m-1$, then* (7.1) *holds for m for every $\mathbf{f} \subset L_\infty(\mathbb{R}^d)$ satisfying the condition $f_m \in C^\infty(\mathbb{R}^d)$ and for every $\mathbf{g} \subset L_\infty(\mathbb{S}^{d-1})$ satisfying the condition $g_m \in C^\infty(\mathbb{S}^{d-1})$.*

Proof. Set $u_k = f_k$, $1 \leqslant k < m$, $v_k = g_k$, $1 \leqslant k < m-1$ and $v_{m-1} = g_{m-1}g_m$. By Proposition 6.1, we have

$$\begin{aligned} T_{m,\mathbf{f},\mathbf{g}}&(1-\Delta)^{-d/2}\pi_1(\phi) - T_{m-1,\mathbf{u},\mathbf{v}}(1-\Delta)^{-d/2}\pi_1(f_m\phi) \\ &= T_{m-1,\mathbf{f},\mathbf{g}} \cdot [\pi_1(f_m), \pi_2(g_m)(1-\Delta)^{-d/2}\pi_1(\phi)] \in \mathcal{L}_1. \end{aligned}$$

Since φ vanishes on $\mathcal{L}_1$, it follows that

$$\varphi(T_{m,\mathbf{f},\mathbf{g}}(1-\Delta)^{-d/2}\pi_1(\phi)) = \varphi(T_{m-1,\mathbf{u},\mathbf{v}}(1-\Delta)^{-d/2}\pi_1(f_m\phi)).$$

By the assumption, we have

$$\begin{aligned} \varphi(T_{m-1,\mathbf{u},\mathbf{v}}(1-\Delta)^{-d/2}\pi_1(f_m\phi)) &= \int_{\mathbb{R}^d} (f_m\phi)(t) \prod_{k=1}^{m-1} u_k(t)\mathrm{d}t \cdot \int_{\mathbb{S}^{d-1}} \prod_{k=1}^{m-1} v_k(s)\mathrm{d}s \\ &= \int_{\mathbb{R}^d} \phi(t) \prod_{k=1}^{m} f_k(t)\mathrm{d}t \cdot \int_{\mathbb{S}^{d-1}} \prod_{k=1}^{m} g_k(s)\mathrm{d}s. \end{aligned}$$

A combination of these two equalities concludes the argument. ∎

Proof of Proposition 7.1. We prove the assertion by induction on m. The base of the induction, that is, the assertion for $m = 1$, follows from Proposition 5.1 due to the trace property of φ. We now prove the induction step.

If (7.1) holds for $m-1$, then, by Lemma 7.4, (7.1) holds for m provided that $f_m \in C^\infty(\mathbb{R}^d)$ and $g_m \in C^\infty(\mathbb{S}^{d-1})$. By Lemma 7.2, (7.1) holds for m provided that $g_m \in C^\infty(\mathbb{S}^{d-1})$. By Lemma 7.3, (7.1) holds for m in full generality. This completes the proof. ∎

Proof of Theorem 1.5. Let $P \subset \Pi$ be the $*$-subalgebra in $\mathcal{L}(L_2(\mathbb{R}^d))$ generated by $\pi_1(L_\infty(\mathbb{R}^d))$ and $\pi_2(L_\infty(\mathbb{S}^{d-1}))$. If

$$S = \prod_{k=1}^m \pi_1(f_k)\pi_2(g_k), \quad \text{then, by Theorem 1.2,} \quad \mathrm{symb}(S) = \prod_{k=1}^m f_k \otimes \prod_{k=1}^m g_k.$$

By Proposition 7.1, we have

$$\varphi(S(1-\Delta)^{-d/2}\pi_1(\phi)) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathrm{symb}(S\pi_1(\phi)). \tag{7.4}$$

By linearity, equality (7.4) holds true for every $S \in P$.

Let $T \in \Pi$ and let ϕ be in $C_c^\infty(\mathbb{R}^d)$ be such that $T\pi_1(\phi) = T$. Fix $S_n \in P$ such that $\|S_n - T\|_\infty \to 0$ as $n \to \infty$. By Lemma 6.3, we have

$$[\pi_1(\phi), (1-\Delta)^{-d/2}] \in \mathcal{L}_1.$$

Since φ vanishes on $\mathcal{L}_1$, it follows that

$$\varphi(S_n\pi_1(\phi)(1-\Delta)^{-d/2}) = \varphi(S_n(1-\Delta)^{-d/2}\pi_1(\phi)).$$

Thus,

$$\varphi(T(1-\Delta)^{-d/2}) - \varphi(S_n(1-\Delta)^{-d/2}\pi_1(\phi)) = \varphi((T-S_n)\pi_1(\phi)(1-\Delta)^{-d/2}).$$

Hence,

$$|\varphi(T(1-\Delta)^{-d/2}) - \varphi(S_n(1-\Delta)^{-d/2}\pi_1(\phi))| \leqslant \|S_n - T\|_\infty \|\pi_1(\phi)(1-\Delta)^{-d/2}\|_{1,\infty} \to 0$$

as $n \to \infty$. Since the principal symbol mapping is continuous (see Theorem 1.2), it follows that

$$\mathrm{symb}(S_n\pi_1(\phi)) \to \mathrm{symb}(T\pi_1(\phi)), \quad n \to \infty,$$

in the uniform norm. However, all these functions are supported in the compact set $\mathrm{supp}(\phi) \times \mathbb{S}^{d-1}$. Since the latter set has finite measure, it follows that the convergence holds also in L_1-norm. Thus, we also have

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathrm{symb}(S_n\pi_1(\phi)) \to \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathrm{symb}(T\pi_1(\phi)) = \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathrm{symb}(T).$$

Therefore, using (7.4) for $S_n \in P$, we conclude that

$$\varphi(T(1-\Delta)^{-d/2}) = \lim_{n\to\infty} \varphi(S_n(1-\Delta)^{-d/2}\pi_1(\phi)) = \lim_{n\to\infty} \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathrm{symb}(S_n\pi_1(\phi))$$

$$= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \mathrm{symb}(T).$$

This completes the proof. ∎

8. APPLICATIONS TO PSEUDO-DIFFERENTIAL OPERATORS

In the following lemma, $C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ denotes the collection of all infinitely differentiable functions f on $\mathbb{R}^d \times \mathbb{S}^{d-1}$ such that f and all its derivatives are bounded.

LEMMA 8.1. *Let $p \in C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and let the operator T be initially defined on Schwartz functions on $\mathbb{R}^d$ by the formula*

$$(Tx)(t) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{i\langle t,u\rangle} p\Big(t, \frac{u}{|u|}\Big)(\mathcal{F}x)(u)du, \quad x \in \mathcal{S}(\mathbb{R}^d)$$

that admits a bounded extension $T : L_2(\mathbb{R}^d) \to L_2(\mathbb{R}^d)$. Moreover, we have $T \in \Pi$ and $\mathrm{symb}(T) = p$.

Proof. For every fixed $t \in \mathbb{R}^d$, we write $p(t, \cdot)$ as a Fourier series with respect to the spherical harmonics on the sphere $\mathbb{S}^{d-1}$ [19]. For every $n \geqslant 0$, let H_n be the (finite-dimensional) eigenspace of the spherical Laplacian $\Delta_{\mathbb{S}^{d-1}}$ [19]

$$H_n = \{f \in L_2(\mathbb{S}^{d-1}) : -\Delta_{\mathbb{S}^{d-1}} f = n(n + d - 2)f\}.$$

It is well-known (see Theorem 4.6 in [19]) that H_n is orthogonal to H_m for $n \neq m$. Let $\{Y_{n,j}\}_{j=1}^{\dim(H_n)}$ be some orthonormal basis in H_n. By Theorem 4.27 in [19], the double sequence

$$\{Y_{n,j} : n \geqslant 0,\ 1 \leqslant j \leqslant \dim(H_n)\},$$

is an orthonormal basis in $L_2(\mathbb{S}^{d-1})$. The key estimate given in Proposition 4.16 in [19] states that

$$\|Y_{n,j}\|_\infty \leqslant (\dim(H_n))^{1/2}. \tag{8.1}$$

For every $s \in \mathbb{S}^{d-1}$, we have (the convergence below is taken with respect to L_2-norm on $\mathbb{S}^{d-1}$):

$$p(t, \cdot) = \sum_{n=0}^{\infty} \sum_{j=1}^{\dim(H_n)} p_{n,j}(t) Y_{n,j}(\cdot).$$

Here, $p_{n,j}(t)$ are the Fourier coefficients of the function $p(t, \cdot)$ with respect to the spherical harmonics. Thus, the definition of T can be formally rewritten as follows:

$$(Tx)(t) = \sum_{n=0}^{\infty} \sum_{j=1}^{\dim(H_n)} p_{n,j}(t) \int_{\mathbb{R}^d} e^{i\langle t,u\rangle} Y_{n,j}\Big(\frac{u}{|u|}\Big)(\mathcal{F}x)(u)du, \quad x \in \mathcal{S}(\mathbb{R}^d).$$

That is, we formally have

$$T = \sum_{n=0}^{\infty} \sum_{j=1}^{\dim(H_n)} \pi_1(p_{n,j})\pi_2(Y_{n,j}). \tag{8.2}$$

We claim that the series in (8.2) actually converges in the uniform norm.

Indeed,

$$p_{n,j}(t) = \langle p(t,\cdot), Y_{n,j}(\cdot)\rangle = (n(n+d-2))^{-d}\langle p(t,\cdot), (-\Delta_{\mathbb{S}^{d-1}})^d Y_{n,j}(\cdot)\rangle$$
$$= (n(n+d-2))^{-d}\langle (-\Delta_{\mathbb{S}^{d-1}})^d p(t,\cdot), Y_{n,j}(\cdot)\rangle.$$

Thus,

$$|p_{n,j}(t)| \leqslant (n(n+d-2))^{-d}\|(-\Delta_{\mathbb{S}^{d-1}})^d p(t,\cdot)\|_\infty \|Y_{n,j}\|_\infty,$$
$$\|p_{n,j}\|_\infty \leqslant (n(n+d-2))^{-d}\|(1\otimes -\Delta_{\mathbb{S}^{d-1}})^d p\|_{L_\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\|Y_{n,j}\|_\infty.$$

Here, $(1\otimes -\Delta_{\mathbb{S}^{d-1}})^d p$ is bounded by the assumption.

We can then bound,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\dim(H_n)} \|p_{n,j}\|_\infty \|Y_{n,j}\|_\infty$$
$$\leqslant \|(1\otimes -\Delta_{\mathbb{S}^{d-1}})^d p\|_{L_\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})} \times \sum_{n=1}^{\infty}(n(n+d-2))^{-d}\sum_{j=1}^{\dim(H_n)} \|Y_{n,j}\|_\infty^2.$$

Using (8.1) and Theorem 4.4 in [19], we infer that

$$\sum_{n=1}^{\infty}(n(n+d-2))^{-d}\sum_{j=1}^{\dim(H_n)} \|Y_{n,j}\|_\infty^2 \leqslant \sum_{n=1}^{\infty}(n(n+d-2))^{-d}\dim^2(H_n) \stackrel{\text{def}}{=} c_d < \infty.$$

Thus,

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\dim(H_n)} \|p_{n,j}\|_\infty\|Y_{n,j}\|_\infty \leqslant c_d\|(1\otimes -\Delta_{\mathbb{S}^{d-1}})^d p\|_{L_\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}.$$

Hence, the series in (8.2) converges in the uniform norm.

Since every summand in (8.2) belongs to the algebra Π, it follows that also $T\in\Pi$. Since the symbol mapping is continuous in the uniform norm, it follows that

$$\mathrm{symb}(T) = \sum_{n=0}^{\infty}\sum_{j=1}^{\dim(H_n)} \mathrm{symb}(\pi_1(p_{n,j})\pi_2(Y_{n,j})) = \sum_{n=0}^{\infty}\sum_{j=1}^{\dim(H_n)} p_{n,j}\otimes Y_{n,j},$$

where the convergence is with respect to the uniform norm. Thus, $\mathrm{symb}(T) = p$. ∎

LEMMA 8.2. *Let P be a uniform classical compactly based pseudo-differential operator of order* 0. *Then there exists $S\in\Pi$ such that:*

(i) $\mathrm{symb}(S)$ *coincides with the principal symbol of the operator P in the sense of pseudo-differential operator theory;*

(ii) *S is also compactly based in the sense of Definition* 1.4*;*

(iii) $P - S \in \mathcal{K}(L_2(\mathbb{R}^d))$.

Proof. Choose a positive $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $P = P\pi_1(\phi)$. By the definition of a classical pseudo-differential operator (see [45]), one can write $P = P_1 + P_2$, where P_1 is a uniform pseudo-differential operator of order 0 such that

$$p_{P_1}(t, \lambda u) = p_{P_1}(t, u), \quad \lambda > 1,\ t, u \in \mathbb{R}^d,\ |u| \geqslant 1,$$

and where P_2 is a uniform pseudo-differential operator of order -1. Let p be the principal symbol of the operator P in the sense [45] of pseudo-differential operator theory, that is,

$$p(t,s) = p_{P_1}(t,s), \quad t \in \mathbb{R}^d, s \in \mathbb{S}^{d-1},$$

and let T be the operator defined in Lemma 8.1 for this p. Let ψ be a smooth function on $\mathbb{R}^d$ such that $\psi(u) = 1$ for $|u| > 2$ and $\psi(u) = 0$ for $|u| < 1$. We have

$$p\Big(t, \frac{u}{|u|}\Big)\psi(u) = p_{P_1}(t,u)\psi(u), \quad t, u \in \mathbb{R}^d.$$

In other words, the integral kernels of the operators $P_1\psi(\nabla)$ and $T\psi(\nabla)$ coincide. It follows that $P_1\psi(\nabla) = T\psi(\nabla)$.

Thus,

$$P = T + P_1(1 - \psi(\nabla)) - T(1 - \psi(\nabla)) + P_2$$

and therefore

$$P = P\pi_1(\phi) = T\pi_1(\phi) + P_1 \cdot (1 - \psi(\nabla))\pi_1(\phi) - T \cdot (1 - \psi(\nabla))\pi_1(\phi) + P_2\pi_1(\phi).$$

By Lemma 8.1, T is bounded. Since both ϕ and $1 - \psi$ are smooth and compactly supported functions, it follows from Theorem 2.4 that $(1 - \psi(\nabla))\pi_1(\phi) \in \mathcal{L}_1$. Thus,

$$T \cdot (1 - \psi(\nabla))\pi_1(\phi) \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

By Theorem 2.4.2 in [42], the uniform pseudo-differential operator P_1 of order 0 is bounded. Thus,

$$P_1 \cdot (1 - \psi(\nabla))\pi_1(\phi) \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

The operator $P_2(1 - \Delta)^{1/2}$ is a uniform pseudo-differential operator of order 0 and is, therefore, bounded. By Theorem 2.3, we have

$$(1 - \Delta)^{-1/2}\pi_1(\phi) \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

Therefore,

$$P_2\pi_1(\phi) = P_2(1 - \Delta)^{1/2} \cdot (1 - \Delta)^{-1/2}\pi_1(\phi) \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

Finally, we get

$$P - T\pi_1(\phi) \in \mathcal{K}(L_2(\mathbb{R}^d)).$$

Setting $S = T\pi_1(\phi)$, we see that $S \in \Pi$ (see Definition 1.1) and this completes the proof. ∎

The following simple fact is well-known. We include it here for convenience of the reader.

LEMMA 8.3. *If $V \in \mathcal{L}_{1,\infty}$ and if A is a compact operator, then*

$$\varphi(AV) = 0$$

for every continuous trace on $\mathcal{L}_{1,\infty}$.

Proof. As discussed in Subsection 2.4, all continuous traces on $\mathcal{L}_{1,\infty}$ are singular. That is, if φ is a continuous trace on $\mathcal{L}_{1,\infty}$ then $\varphi(T) = 0$ for all finite rank operators T (see Corollary 5.7.7 of [30] for a proof of this fact).

Let $\{A_n\}_{n=0}^{\infty}$ be a sequence of finite rank operators converging to A in the operator norm. Then $\|A_nV - AV\| \leqslant \|A_n - A\|_\infty \|V\|_{1,\infty}$, so $A_nV \to AV$ in the $\mathcal{L}_{1,\infty}$ quasi-norm. Since each A_n is finite rank, each operator A_nV is also finite rank. Since φ is continuous and singular,

$$\varphi(AV) = \lim_{n\to\infty} \varphi(A_nV) = 0. \qquad \blacksquare$$

Proof of Theorem 1.6. Let P be a uniform classical compactly based pseudo-differential operator of order 0. Choose a positive $\phi_1 \in C_c^\infty(\mathbb{R}^d)$ such that $P = P\pi_1(\phi_1)$. Let $S \in \Pi$ be the operator defined in Lemma 8.2. Choose a positive $\phi_2 \in C_c^\infty(\mathbb{R}^d)$ such that $S = S\pi_1(\phi_2)$. Choose a positive $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi_k = \phi \cdot \phi_k$, $k = 1, 2$. We then have $P = P\pi_1(\phi)$ and $S = S\pi_1(\phi)$.

By Theorem 2.5, we have

$$\pi_1(\phi)(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

Therefore,

$$S(1-\Delta)^{-d/2} = S \cdot \pi_1(\phi)(1-\Delta)^{-d/2} \in \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

Also, using Lemma 8.2, we obtain

$$(P-S)(1-\Delta)^{-d/2} = (P-S)\cdot \pi_1(\phi)(1-\Delta)^{-d/2} \in \mathcal{K}(L_2(\mathbb{R}^d)) \cdot \mathcal{L}_{1,\infty}(L_2(\mathbb{R}^d)).$$

By Lemma 8.3, we have

$$\varphi(P(1-\Delta)^{-d/2}) = \varphi(S(1-\Delta)^{-d/2}).$$

It follows from Theorem 1.5 that

$$\varphi(S(1-\Delta)^{-d/2}) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(S).$$

By Lemma 8.2, $\mathrm{symb}(S)$ coincides with the principal symbol of the operator P in the sense of pseudo-differential operator theory. Thus,

$$\varphi(P(1-\Delta)^{-d/2}) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(S) = \mathrm{Res}_W(P). \qquad \blacksquare$$

9. APPLICATION TO QUANTISED DIFFERENTIALS

In this section, we prove Theorem 1.7. It computes the trace of a d-th power of the absolute value of a quantised derivative

$$[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]$$

of the function f on $\mathbb{R}^d$ from a certain homogeneous Sobolev space.

We start with the following simple extension of Theorem 1.5. Here, $T \in M_n(\mathbb{C}) \otimes \Pi$ is called *compactly based* if there is a compactly supported $\phi \in L_\infty(\mathbb{R}^d)$ such that $T \cdot (1 \otimes \pi_1(\phi)) = T$.

LEMMA 9.1. *If $T \in M_n(\mathbb{C}) \otimes \Pi$ is compactly based, then $T(1+D^2)^{-d/2} \in \mathcal{L}_{1,\infty}$. For every continuous normalised trace φ on $\mathcal{L}_{1,\infty}$, we have*

$$\varphi(T(1+D^2)^{-d/2}) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{Tr}((\mathrm{id} \otimes \mathrm{symb})(T)).$$

Proof. Let $e_{k,l}$, $1 \leqslant k, l \leqslant n$, be the basis of the set of $n \times n$ matrices $M_n(\mathbb{C})$ of matrices with $(k,l)^{\mathrm{th}}$ entry equal to 1 and 0 elsewhere. By linearity, it suffices to prove the assertion for the case $T = e_{kl} \otimes S$, where $S \in \Pi$. Note that

$$T(1+D^2)^{-d/2} = e_{kl} \otimes S(1-\Delta)^{-d/2}.$$

That is, we want to prove

$$\varphi(e_{kl} \otimes S(1-\Delta)^{-d/2}) = \delta_{kl} \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(S).$$

If $k \neq l$, then $e_{kl} = [e_{kl}, e_{ll}]$ and, therefore,

$$e_{kl} \otimes S(1-\Delta)^{-d/2} = [e_{kl} \otimes S(1-\Delta)^{-d/2}, e_{ll} \otimes 1].$$

Since φ vanishes on the commutators, it follows that

$$\varphi(e_{kl} \otimes S(1-\Delta)^{-d/2}) = 0.$$

If $k = l$, then

$$e_{mm} \otimes S(1-\Delta)^{-d/2} = (U_{mk} \otimes 1) \cdot (e_{kk} \otimes S(1-\Delta)^{-d/2}) \cdot (U_{mk}^{-1} \otimes 1), \quad 1 \leqslant m \leqslant n,$$

for some permutation matrix $U_{mk} \in M_n(\mathbb{C})$. Thus,

$$\varphi(e_{kk} \otimes S(1-\Delta)^{-d/2}) = \varphi(e_{mm} \otimes S(1-\Delta)^{-d/2}), \quad 1 \leqslant m \leqslant n.$$

Summation over $k \in [1, n]$ yields

$$\varphi(e_{kk} \otimes S(1-\Delta)^{-d/2}) = \frac{1}{n}\varphi(1 \otimes S(1-\Delta)^{-d/2}) = \frac{1}{n} \cdot \mathrm{Tr}(1) \cdot \varphi(S(1-\Delta)^{-d/2}).$$

By Theorem 1.5, we conclude

$$\varphi(e_{kk} \otimes S(1-\Delta)^{-d/2}) = \int\limits_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{symb}(S).$$

We have covered both the $k = l$ and $k \neq l$ cases, so this completes the proof. ∎

LEMMA 9.2. *If $f \in C_c^\infty(\mathbb{R}^d)$, then $[\mathrm{sgn}(D), 1 \otimes \pi_1(f)] \in \mathcal{L}_{d,\infty}$ and*

$$\varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|^d) = \int_{\mathbb{R}^d} \|(\nabla f)(t)\|_2^d \mathrm{d}t.$$

Proof. Fix $f \in C_c^\infty(\mathbb{R}^d)$ and choose $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $f\phi = f$. For all $1 \leqslant k \leqslant d$, define an operator $A_k \in \Pi$ by setting

$$A_k = \pi_1(f_k) - \frac{1}{2}\sum_{j=1}^d (\pi_1(f_j)\pi_2(s_k s_j)\pi_1(\phi) + \pi_1(\phi)\pi_2(s_k s_j)\pi_1(f_j)).$$

Here, $f_k = \mathrm{i}D_k f$ and the mapping $s \to s_k s_j$, $s \in \mathbb{S}^{d-1}$, is also denoted by $s_k s_j$ to save the space. Define a (compactly based) operator $A \in M_{m(d)}(\mathbb{C}) \otimes \Pi$ by setting

$$A = \sum_{k=1}^d \gamma_k \otimes A_k.$$

An argument similar to that of Lemma 15 in [29] shows that

$$|[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|^d \in |A|^d (1 + D^2)^{-d/2} + \mathcal{L}_1.$$

Since φ vanishes on $\mathcal{L}_1$, it follows from Lemma 9.1 that

$$\varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|^d) = \varphi(|A|^d (1 + D^2)^{-d/2}) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \mathrm{Tr}((\mathrm{id} \otimes \mathrm{symb})(|A|^d)).$$

Clearly,

$$(\mathrm{id} \otimes \mathrm{symb})(A) = \sum_{k=1}^d \gamma_k \otimes \mathrm{symb}(A_k).$$

By Theorem 1.2, we have

$$(\mathrm{id} \otimes \mathrm{symb})(|A|^d) = 1 \otimes \Big(\sum_{k=1}^d |\mathrm{symb}(A_k)|^2\Big)^{d/2}.$$

It is immediate that

$$(\mathrm{symb}(A_k))(t, s) = f_k(t) - \sum_{j=1}^d f_j(t) s_k s_j, \quad t \in \mathbb{R}^d,\ s \in \mathbb{S}^{d-1}.$$

Thus,

$$\varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|^d) = \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} |(\nabla(f))(t) - s\langle(\nabla(f))(t), s\rangle|^d \mathrm{d}s\mathrm{d}t. \tag{9.1}$$

For an arbitrary $a \in \mathbb{R}^d$, we have

$$\int_{\mathbb{S}^{d-1}} |a - s\langle a, s\rangle|^d \mathrm{d}s = c_d |a|^d.$$

Indeed, for every rotation $R \in \mathrm{SO}(d)$, we have

$$|a - s\langle a, s\rangle| = |Ra - Rs\langle Ra, Rs\rangle|, \quad s \in \mathbb{S}^{d-1}.$$

Taking R such that $a = |a| \cdot Re_1$, we obtain

$$\int_{\mathbb{S}^{d-1}} |a - s\langle a, s\rangle|^d \mathrm{d}s = |a|^d \cdot \int_{\mathbb{S}^{d-1}} |e_1 - s\langle e_1, s\rangle|^d \mathrm{d}s = c_d |a|^d.$$

For every $t \in \mathbb{R}^d$, the preceding paragraph yields

$$\int_{\mathbb{S}^{d-1}} |(\nabla(f))(t) - s\langle(\nabla(f))(t), s\rangle|^d \mathrm{d}s = c_d |(\nabla f)(t)|^d.$$

This concludes the argument. ∎

Proof of Theorem 1.7. Let $f \in L_\infty(\mathbb{R}^d)$ be such that $\nabla f \in L_d(\mathbb{R}^d, \mathbb{C}^d)$. Choose a sequence $\{f_n\}_{n \geqslant 1} \subset C_c^\infty(\mathbb{R}^d)$ such that $f_n \to f$ almost everywhere and such that $\nabla(f_n - f) \to 0$ in $L_d(\mathbb{R}^d)$ (see Theorem 2.1 in [34]). It is proved in [29] (see the proof of Theorem 11 on p.19 there) that

$$\|[\mathrm{sgn}(D), 1 \otimes \pi_1(f)] - [\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]\|_{d,\infty} \leqslant \|\nabla(f_n - f)\|_d \to 0, \quad n \to \infty.$$

Since the absolute value mapping is Lipschitz in $\mathcal{L}_{d,\infty}$ (see Theorem 3.4 in [17]), it follows that

$$\|\, |[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]| - |[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]|\, \|_{d,\infty} \to 0, \quad n \to \infty.$$

For bounded operators A, B, we have

$$A^d - B^d = \sum_{k=0}^{d-1} A^k (A - B) B^{d-1-k}.$$

Hence, for $A, B \in \mathcal{L}_{d,\infty}$, it follows from Hölder inequality that

$$\begin{aligned} \|A^d - B^d\|_{1,\infty} &\leqslant c_d \sum_{k=0}^{d-1} \|A\|_{d,\infty}^k \|A - B\|_{d,\infty} \|B\|_{d,\infty}^{d-1-k} \\ &\leqslant c_d \|A - B\|_{d,\infty} (\|A\|_{d,\infty} + \|B\|_{d,\infty})^{d-1}. \end{aligned}$$

Applying the latter inequality to

$$A = |[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|, \quad B = |[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]|,$$

we obtain

$$\|\, |[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|^d - |[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]|^d \|_{1,\infty} \to 0, \quad n \to \infty.$$

Since φ is continuous, it follows that

$$\varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]|^d) \to \varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f)]|^d), \quad n \to \infty.$$

By Lemma 9.2, we have

$$\varphi(|[\mathrm{sgn}(D), 1 \otimes \pi_1(f_n)]|^d) = \int_{\mathbb{R}^d} \|(\nabla f_n)(t)\|_2^d \mathrm{d}t.$$

Since $\nabla(f_n - f) \to 0$ in $L_d(\mathbb{R}^d)$, it follows that

$$\int_{\mathbb{R}^d} \|(\nabla f_n)(t)\|_2^d \mathrm{d}t \to \int_{\mathbb{R}^d} \|(\nabla f)(t)\|_2^d \mathrm{d}t, \quad n \to \infty.$$

A combination of the last three equalities completes the proof. ∎

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