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CONSISTENT OPERATOR SEMIGROUPS AND THEIR INTERPOLATION

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ABSTRACT. Under a mild regularity condition we prove that the generator of the interpolation of two C_0-semigroups is the interpolation of the two generators.

KEYWORDS: *Semigroups, interpolation.*

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1. INTRODUCTION

Interpolation is one of the main tools in parabolic differential equations and in particular in semigroup theory, see [8], Section 1.13 of [40], and Chapter 2 of [29]. Frequently interpolation is done between two L^p-spaces or between a Banach space and the domain of a power of the generator of a semigroup. The aim of this paper is to consider abstractly interpolation of continuous semigroups, from the viewpoint of category theory. In one of the main theorems of this paper, Theorem 3.9, we show that the generator of the interpolation of two C_0-semigroups is the interpolation of the two generators. As a corollary this gives the following theorem for complex interpolation.

THEOREM 1.1. *Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space and let $p_0, p_1 \in [1, \infty)$. Let $S^{(p_0)}$ and $S^{(p_1)}$ be bounded consistent C_0-semigroups in L^{p_0} and L^{p_1} with generators $-A_{p_0}$ and $-A_{p_1}$, respectively. Let $\theta \in [0, 1]$ and let $p \in [1, \infty)$ be such that $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$. Let $S^{(p)}$ be the C_0-semigroup on L^p which is consistent with $S^{(p_0)}$. Let $-A_p$ be the generator of $S^{(p)}$. Then*

$$[D(A_{p_0}), D(A_{p_1})]_\theta = D(A_p).$$

The paper is organised as follows. In Section 2 we characterise consistency of semigroups in terms of their resolvents and we obtain a useful expression for the intersection of the domain of the generators. In Section 3 we consider interpolation functors and prove loosely speaking that semigroup generators and interpolation functors commute. In Section 4 we give examples of our theorem in L^p-spaces and distribution spaces for consistent semigroups. Finally in Section 5 we present an illustration in non-linear parabolic equations, where the appropriate interpolation is not between two L^p-spaces or between a Banach space and a power of a semigroup generator.

2. CONSISTENCY OF OPERATOR SEMIGROUPS

In this section we show that two C_0-semigroups are consistent if and only if the resolvents of the generators are consistent for large $\lambda > 0$. We start with the definition of consistent operators.

DEFINITION 2.1. Let X and Y be two vector spaces. Let $T_0 : D(T_0) \to Y$ and $T_1 : D(T_1) \to Y$ be two (linear) operators with domains $D(T_0) \subset X$ and $D(T_1) \subset X$. Then the operators T_0 and T_1 are called *consistent* if $T_0 x = T_1 x$ for all $x \in D(T_0) \cap D(T_1)$. Let X_0 and X_1 be two Banach spaces which are embedded in a vector space X. Let $S^{(0)}$ and $S^{(1)}$ be semigroups in X_0 and X_1, respectively. Then the semigroups $S^{(0)}$ and $S^{(1)}$ are called *consistent* if $S^{(0)}_t$ and $S^{(1)}_t$ are consistent for all $t > 0$.

The following easy lemma gives a sufficient condition for two bounded operators to be consistent.

LEMMA 2.2. *Let (X_0, X_1) be an interpolation couple of Banach spaces. Let T_0 and T_1 be bounded operators in X_0 and X_1, respectively. Let $D \subset X_0 \cap X_1$ and suppose that D is dense in $X_0 \cap X_1$. Further, suppose that $T_0 x = T_1 x$ for all $x \in D$. Then T_0 and T_1 are consistent.*

The boundedness condition on the semigroups in the sequel is just for convenience.

LEMMA 2.3. *Let (X_0, X_1) be an interpolation couple of Banach spaces. Let $S^{(0)}$ and $S^{(1)}$ be bounded C_0-semigroups in X_0 and X_1 with generators $-A_0$ and $-A_1$, respectively. Then the following are equivalent:*

(i) *the semigroups $S^{(0)}$ and $S^{(1)}$ are consistent;*

(ii) *for all $\lambda > 0$ the resolvent operators $(A_0 + \lambda I)^{-1}$ and $(A_1 + \lambda I)^{-1}$ are consistent.*

Proof. (i) $\Rightarrow$ (ii) Let $\lambda > 0$ and $x \in X_0 \cap X_1$. Then

$$(A_0 + \lambda I)^{-1} x = \int_0^\infty e^{-\lambda t} S^{(0)}_t x \, dt = \int_0^\infty e^{-\lambda t} S^{(1)}_t x \, dt = (A_1 + \lambda I)^{-1} x.$$

(ii) $\Rightarrow$ (i) Let $\lambda > 0$ and $x \in X_0 \cap X_1$. Then it follows by induction to n that $(A_0 + \lambda I)^{-n} x = (A_1 + \lambda I)^{-n} x$ for all $n \in \mathbb{N}$. Now let $t > 0$ and $x \in X_0 \cap X_1$. Then the Euler formula gives

$$S_t^{(0)} x = \lim_{n \to \infty} \Big(A_0 + \frac{t}{n} I \Big)^{-n} x = \lim_{n \to \infty} \Big(A_1 + \frac{t}{n} I \Big)^{-n} x = S_t^{(1)} x,$$

as required. ∎

REMARK 2.4. In Proposition 2.2 of [2] the following is proved: the set $\mathcal{U}$ of all λ for which $(A_0 + \lambda I)^{-1}$ and $(A_1 + \lambda I)^{-1}$ are consistent, is open and closed in $\rho(-A_0) \cap \rho(-A_1)$. From this it easily follows that if $\rho(-A_0) = \rho(-A_1)$ and this set is connected, then the consistency of $(A_0 + \lambda_0 I)^{-1}$ and $(A_1 + \lambda_0 I)^{-1}$ for *only one* λ_0 implies the consistency of all resolvent operators.

If the equivalent conditions in Lemma 2.3 are valid, then it is possible that there exists a $\lambda \in \rho(-A_0) \cap \rho(-A_1)$ such that the resolvents $(A_0 + \lambda I)^{-1}$ and $(A_1 + \lambda I)^{-1}$ are not consistent. An example has been given in Section 3 of [2].

PROPOSITION 2.5. *Let (X_0, X_1) be an interpolation couple of Banach spaces. Let $S^{(0)}$ and $S^{(1)}$ be bounded consistent C_0-semigroups in X_0 and X_1 with generators $-A_0$ and $-A_1$, respectively. Then one has the following:*
(i) *the generators A_0 and A_1 are consistent;*
(ii) $D(A_0) \cap D(A_1) = \{x \in D(A_0) \cap X_1 : A_0 x \in X_1\} = (A_0 + I)^{-1}(X_0 \cap X_1)$.

Proof. (i) Let $x \in D(A_0) \cap D(A_1)$ and $F \in (X_0 + X_1)'$. Then

$$F(A_0 x) = \lim_{t \downarrow 0} \frac{1}{t} F((I - S^{(0)})x) = \lim_{t \downarrow 0} \frac{1}{t} F((I - S^{(1)})x) = F(A_1 x).$$

Hence $A_0 x = A_1 x$.

(ii) Let $x \in D(A_0) \cap D(A_1)$. Then it follows from statement (i) that $A_0 x = A_1 x \in X_1$. So $D(A_0) \cap D(A_1) \subset \{x \in D(A_0) \cap X_1 : A_0 x \in X_1\}$. Conversely, suppose $x \in D(A_0) \cap X_1$ and $A_0 x \in X_1$. Let $t > 0$. Then for all $F \in (X_0 + X_1)'$ one deduces that

$$\begin{aligned} F((I - S_t^{(1)})x) &= F((I - S_t^{(0)})x) = F\Big(\int_0^t S_s^{(0)} A_0 x \, ds \Big) = \int_0^t F(S_s^{(0)} A_0 x) \, ds \\ &= \int_0^t F(S_s^{(1)} A_0 x) \, ds = F\Big(\int_0^t S_s^{(1)} A_0 x \, ds \Big). \end{aligned}$$

So

$$\frac{1}{t}(I - S_t^{(1)})x = \frac{1}{t} \int_0^t S_s^{(1)} A_0 x \, ds$$

in X_1. Hence

$$\lim_{t \downarrow 0} \frac{1}{t}(I - S_t^{(1)})x = A_0 x$$

in X_1. Therefore $x \in D(A_1)$. This proves the first equality in statement (ii).

Next, let $x \in D(A_0) \cap D(A_1)$. Then $(A_0 + I)x = (A_1 + I)x \in X_0 \cap X_1$ again by statement (i). So $x \in (A_0 + I)^{-1}(X_0 \cap X_1)$. Conversely, let $x \in X_0 \cap X_1$. Then obviously $(A_0 + I)^{-1}u \in D(A_0)$. Since $(A_0 + I)^{-1}$ and $(A_1 + I)^{-1}$ are consistent by Lemma 2.3, it follows that $(A_0 + I)^{-1}x = (A_1 + I)^{-1}x \in D(A_1)$. Therefore $(A_0 + I)^{-1}x \in D(A_0) \cap D(A_1)$. ∎

3. INTERPOLATION OF CONSISTENT OPERATOR SEMIGROUPS

In this section we consider interpolation of semigroups and their generators. In all what follows, we adopt the terminology of Section 1.2 of [40], with minor modifications.

Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces. Recall from Subsection 1.2.2 of [40] that $L((X_0, X_1), (Y_0, Y_1))$ denotes the vector space of all linear maps $T : X_0 + X_1 \to Y_0 + Y_1$ such that $T|_{X_0} \in \mathcal{L}(X_0, Y_0)$ and $T|_{X_1} \in \mathcal{L}(X_1, Y_1)$. If $T \in L((X_0, X_1), (Y_0, Y_1))$, then clearly the operators $T|_{X_0}$ and $T|_{X_1}$ are consistent. There is a converse.

LEMMA 3.1. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces, $T_0 \in \mathcal{L}(X_0, Y_0)$ and $T_1 \in \mathcal{L}(X_1, Y_1)$. Suppose that T_0 and T_1 are consistent. Then there exists a unique $T \in L((X_0, X_1), (Y_0, Y_1))$ such that $T|_{X_0} = T_0$ and $T|_{X_1} = T_1$.*

Moreover, the operator T is continuous from $X_0 + X_1$ into $Y_0 + Y_1$ and

$$\|T\|_{X_0+X_1 \to Y_0+Y_1} \leqslant \|T_0\|_{X_0 \to Y_0} \vee \|T_1\|_{X_1 \to Y_1}.$$

Proof. The first part is easy and the operator $T \in L((X_0, X_1), (Y_0, Y_1))$ is given by $T(x_0 + x_1) = T_0 x_0 + T_1 x_1$ for all $x_0 \in X_0$ and $x_1 \in X_1$. Here we use that T_0 and T_1 are consistent.

Next, let $x \in X_0 + X_1$. Let $x_0 \in X_0$ and $x_1 \in X_1$ be such that $x = x_0 + x_1$. Then

$$\|Tx\|_{X_0+X_1} \leqslant \|T_0 x_0\|_{X_0} + \|T_1 x_1\|_{X_1} \leqslant (\|T_0\|_{X_0 \to Y_0} \vee \|T_1\|_{X_1 \to Y_1})(\|x_0\|_{X_0} + \|x_1\|_{X_1}).$$

So $\|Tx\|_{Y_0+Y_1} \leqslant (\|T_0\|_{X_0 \to Y_0} \vee \|T_1\|_{X_1 \to Y_1}) \|x\|_{X_0+X_1}$. This proves the last assertion. ∎

Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces. We provide $L((X_0, X_1), (Y_0, Y_1))$ with the norm

$$\|T\|_{L((X_0,X_1),(Y_0,Y_1))} = \|T|_{X_0}\|_{X_0 \to Y_0} \vee \|T|_{X_1}\|_{X_1 \to Y_1}.$$

Then $L((X_0, X_1), (Y_0, Y_1))$ is a Banach space. For the concept of interpolation functor we refer to Subsection 1.2.2 of [40]. If $\mathcal{F}$ is an interpolation functor and $T \in L((X_0, X_1), (Y_0, Y_1))$, then we denote by $T^{\mathcal{F}} : \mathcal{F}(X_0, X_1) \to \mathcal{F}(Y_0, Y_1)$ the restriction of T to $\mathcal{F}(X_0, X_1)$. Note that $T^{\mathcal{F}}$ is a bounded operator. Alternatively, since we are interested in consistent operators, we also introduce another notation. Let $T_0 \in \mathcal{L}(X_0, Y_0)$ and $T_1 \in \mathcal{L}(X_1, Y_1)$. Suppose that T_0 and T_1 are consistent. By Lemma 3.1 there exists a unique $T \in L((X_0, X_1), (Y_0, Y_1))$ such that $T|_{X_0} = T_0$ and $T|_{X_1} = T_1$. Then we define

$$\mathcal{F}(T_0, T_1) = T^{\mathcal{F}}.$$

So $\mathcal{F}(T_0, T_1)$ is a bounded operator from $\mathcal{F}(X_0, X_1)$ into $\mathcal{F}(Y_0, Y_1)$. Since T_0, T_1 and $\mathcal{F}(T_0, T_1) = T^{\mathcal{F}}$ are all three restrictions of the same operator T on $X_0 + X_1$, it is obvious that the three operators T_0, T_1 and $\mathcal{F}(T_0, T_1) = T^{\mathcal{F}}$ are pairwise consistent.

LEMMA 3.2. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces and $\mathcal{F}$ an interpolation functor. Then there exists an $M > 0$ such that*

$$\|T^{\mathcal{F}}\|_{\mathcal{F}(X_0,X_1)\to\mathcal{F}(Y_0,Y_1)} \leqslant M \, \|T\|_{L((X_0,X_1),(Y_0,Y_1))}$$

for all $T \in L((X_0, X_1), (Y_0, Y_1))$.

Proof. The operator $T \mapsto T^{\mathcal{F}}$ from the Banach space $L((X_0, X_1), (Y_0, Y_1))$ into the Banach space $\mathcal{L}(\mathcal{F}(X_0, X_1), \mathcal{F}(Y_0, Y_1))$ has a closed graph. ∎

DEFINITION 3.3. We say that an interpolation functor $\mathcal{F}$ has *Property* (d) (for dense) if for every interpolation couple (X_0, X_1) the subspace $X_0 \cap X_1$ is *dense* in the interpolation space $\mathcal{F}(X_0, X_1)$.

EXAMPLE 3.4. The complex interpolation has Property (d). With exception of the limit values also the real interpolation has Property (d). For complex and real interpolation, see Subsections 1.9.3 and 1.6.2 of [40].

EXAMPLE 3.5. The real interpolation with parameters the limit values does not have Property (d), see Remark 1.18.3.5 of [40].

The next lemma is easy to prove.

LEMMA 3.6. *Let (X_0, X_1) and (Y_0, Y_1) be two interpolation couples of Banach spaces and $\mathcal{F}$ an interpolation functor which has Property* (d). *Let $T_0 \in \mathcal{L}(X_0, Y_0)$, $T_1 \in \mathcal{L}(X_1, Y_1)$ and suppose that T_0 and T_1 are consistent. Then $\mathcal{F}(T_0, T_1)$ is the unique extension of the operator $T|_{X_0 \cap X_1} : X_0 \cap X_1 \to Y_0 \cap Y_1$ which is continuous from the space $\mathcal{F}(X_0, X_1)$ into the space $\mathcal{F}(Y_0, Y_1)$.*

Next we consider a functor on consistent semigroups.

PROPOSITION 3.7. *Let $\mathcal{F}$ be an interpolation functor. Let (X_0, X_1) be an interpolation couple of Banach spaces. Let $S^{(0)}$ and $S^{(1)}$ be consistent semigroups in X_0 and X_1, respectively. Then one has the following:*

(i) *The family $(\mathcal{F}(S^{(0)}_t, S^{(1)}_t))_{t>0}$ on $\mathcal{F}(X_0, X_1)$ is a semigroup which is consistent with both $S^{(0)}$ and $S^{(1)}$.*

(ii) *If both* $S^{(0)}$ *and* $S^{(1)}$ *are bounded semigroups, then the semigroup*

$$(\mathcal{F}(S_t^{(0)}, S_t^{(1)}))_{t>0}$$

is also bounded.

(iii) *Suppose in addition that* $S^{(0)}$ *and* $S^{(1)}$ *are* C_0*-semigroups and that the interpolation functor* $\mathcal{F}$ *has Property* (d). *Then the semigroup* $(\mathcal{F}(S_t^{(0)}, S_t^{(1)}))_{t>0}$ *is a* C_0*-semigroup.*

Proof. (i) This is straightforward.

(ii) This follows from Lemmas 3.1 and 3.2.

(iii) Without loss of generality we may assume that both $S^{(0)}$ and $S^{(1)}$ are bounded semigroups. For all $t > 0$ write $S_t^{\mathcal{F}} = \mathcal{F}(S_t^{(0)}, S_t^{(1)})$. Then also $(S_t^{\mathcal{F}})_{t>0}$ is a bounded semigroup by statement (ii).

Because $\mathcal{F}(X_0, X_1)$ is an intermediate space for the interpolation couple (X_0, X_1), there exists a $c > 0$ such that $\|x\|_{\mathcal{F}(X_0,X_1)} \leqslant c\,\|x\|_{X_0\cap X_1}$ for all $x \in X_0 \cap X_1$. Let $x \in X_0 \cap X_1$ and $t > 0$. Then

$$\|S_t^{\mathcal{F}} x - x\|_{\mathcal{F}(X_0,X_1)} \leqslant c\,\|S_t^{\mathcal{F}} x - x\|_{X_0\cap X_1} = c\,(\|S_t^{(0)} x - x\|_{X_0} + \|S_t^{(1)} x - x\|_{X_1}).$$

Hence $\lim_{t\downarrow 0} \|S_t^{\mathcal{F}} x - x\|_{\mathcal{F}(X_0,X_1)} = 0$ and $\lim_{t\downarrow 0} S_t^{\mathcal{F}} x = x$ in $\mathcal{F}(X_0, X_1)$.

Finally, $X_0 \cap X_1$ is dense in $\mathcal{F}(X_0, X_1)$ since the interpolation functor has Property (d). So $\lim_{t\downarrow 0} S_t^{\mathcal{F}} x = x$ in $\mathcal{F}(X_0, X_1)$ for all $x \in \mathcal{F}(X_0, X_1)$. ∎

We wish to determine the generator of the semigroup $S^{\mathcal{F}}$. We need a lemma.

LEMMA 3.8. *Let* $\mathcal{F}$ *be an interpolation functor which has Property* (d). *Next, let* (X_0, X_1) *be an interpolation couple of Banach spaces. Further, let* $S^{(0)}$ *and* $S^{(1)}$ *be consistent* C_0*-semigroups in* X_0 *and* X_1 *with generators* $-A_0$ *and* $-A_1$*, respectively. Let* $S^{\mathcal{F}} = (\mathcal{F}(S_t^{(0)}, S_t^{(1)}))_{t>0}$ *be the* C_0*-semigroup in* $\mathcal{F}(X_0, X_1)$ *as in Proposition* 3.7. *Let* $-B$ *be the generator of* $S^{\mathcal{F}}$. *Then* $D(A_0) \cap D(A_1) \subset D(B)$ *and* $D(A_0) \cap D(A_1)$ *is a core for* B.

Proof. Without loss of generality we may assume that both $S^{(0)}$ and $S^{(1)}$ are bounded semigroups. The resolvent

$$(B + I)^{-1} : \mathcal{F}(X_0, X_1) \to D(B)$$

is a topological isomorphism. Moreover, the resolvent operators $(B + I)^{-1}$ and $(A_0 + I)^{-1}$ are consistent by Proposition 3.7(i) and Lemma 2.3. By Lemma 2.5(iii) the restriction

$$(B + I)^{-1}|_{X_0\cap X_1} = (A_0 + I)^{-1}|_{X_0\cap X_1} : X_0 \cap X_1 \to D(A_0) \cap D(A_1)$$

is a bijection. Because $X_0 \cap X_1 \subset \mathcal{F}(X_0, X_1)$, this implies immediately the assertion $D(A_0) \cap D(A_1) \subset D(B)$. Since $\mathcal{F}$ has Property (d), the space $X_0 \cap X_1$ is dense in $\mathcal{F}(X_0, X_1)$. Hence $D(A_0) \cap D(A_1)$ is dense in $D(B)$. ∎

We provide the domain of a generator with the graph norm. Note that with the notation of the previous lemma, $(D(A_0), D(A_1))$ is an interpolation couple and $A_0 \in \mathcal{L}(D(A_0), X_0)$ and similarly $A_1 \in \mathcal{L}(D(A_1), X_1)$. Now we are able to prove the main theorem of this paper.

THEOREM 3.9. *Let $\mathcal{F}$ be an interpolation functor which has Property* (d). *Let (X_0, X_1) be an interpolation couple of Banach spaces. Further, let $S^{(0)}$ and $S^{(1)}$ be consistent C_0-semigroups in X_0 and X_1 with generators $-A_0$ and $-A_1$, respectively. Then $-\mathcal{F}(A_0, A_1)$ is the generator of the semigroup $(\mathcal{F}(S_t^{(0)}, S_t^{(1)}))_{t>0}$.*

In particular,

$$D(\mathcal{F}(A_0, A_1)) = \mathcal{F}(D(A_0), D(A_1)).$$

Proof. Without loss of generality we may assume that both $S^{(0)}$ and $S^{(1)}$ are bounded semigroups. Write $S_t^{\mathcal{F}} = \mathcal{F}(S_t^{(0)}, S_t^{(1)})$ for all $t > 0$ and let $-B$ be the generator of the C_0-semigroup $S^{\mathcal{F}}$. We know that $D(A_0) \cap D(A_1) \subset D(B)$ by Lemma 3.8. Also $Bx = A_0 x = A^{\mathcal{F}} x$ for all $x \in D(A_0) \cap D(A_1)$ by Proposition 2.5(i), where we set $A^{\mathcal{F}} = \mathcal{F}(A_0, A_1)$. The operator $A^{\mathcal{F}}$ is bounded from $\mathcal{F}(D(A_0), D(A_1))$ into $\mathcal{F}(X_0, X_1)$. Hence there exists a $c > 0$ such that

$$\|A^{\mathcal{F}} x\|_{\mathcal{F}(X_0,X_1)} \leqslant c\, \|x\|_{\mathcal{F}(D(A_0),D(A_1))}$$

for all $x \in \mathcal{F}(D(A_0), D(A_1))$. If $x \in D(A_0) \cap D(A_1)$, then $Bx = A^{\mathcal{F}} x$ and

$$\|Bx\|_{\mathcal{F}(X_0,X_1)} \leqslant c\, \|x\|_{\mathcal{F}(D(A_0),D(A_1))}.$$

Let $x \in \mathcal{F}(D(A_0), D(A_1))$. Since $D(A_0) \cap D(A_1)$ is dense in $\mathcal{F}(D(A_0), D(A_1))$ by Property (d), there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $D(A_0) \cap D(A_1)$ such that $\lim x_n = x$ in $\mathcal{F}(D(A_0), D(A_1))$. Then $(Bx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in the space $\mathcal{F}(X_0, X_1)$ and $\lim x_n = x$ in $\mathcal{F}(X_0, X_1)$. Since B is a closed operator, it follows that $x \in D(B)$ and $Bx = \lim Bx_n = \lim A^{\mathcal{F}} x_n = A^{\mathcal{F}} x$ in $\mathcal{F}(X_0, X_1)$. Hence B is an extension of $A^{\mathcal{F}}$.

It remains to show that $D(B) \subset \mathcal{F}(D(A_0), D(A_1))$. The operator $(A_0 + I)^{-1}$ is bounded from X_0 into $D(A_0)$ and the operator $(A_1 + I)^{-1}$ is bounded from X_1 into $D(A_1)$. Moreover, the operators $(A_0 + I)^{-1}$ and $(A_1 + I)^{-1}$ are consistent by Lemma 2.3. So by interpolation one obtains a bounded operator, denoted by C, from $\mathcal{F}(X_0, X_1)$ into $\mathcal{F}(D(A_0), D(A_1))$. Let $c' > 0$ be such that

$$\|Cx\|_{\mathcal{F}(D(A_0),D(A_1))} \leqslant c'\, \|x\|_{\mathcal{F}(X_0,X_1)}$$

for all $x \in \mathcal{F}(X_0, X_1)$. If $x \in X_0 \cap X_1$, then $Cx = (A_0 + I)^{-1} x$. Hence

$$\|(A_0 + I)^{-1} x\|_{\mathcal{F}(D(A_0),D(A_1))} \leqslant c'\, \|x\|_{\mathcal{F}(X_0,X_1)}$$

for all $x \in X_0 \cap X_1$. Using Proposition 2.5(iii) it follows that

$$\|x\|_{\mathcal{F}(D(A_0),D(A_1))} \leqslant c'\, \|(A_0 + I)x\|_{\mathcal{F}(X_0,X_1)} = c'\, \|(B + I)x\|_{\mathcal{F}(X_0,X_1)}$$

for all $x \in D(A_0) \cap D(A_1)$. But $D(A_0) \cap D(A_1)$ is dense in $D(B)$ by Lemma 3.8. Since $\mathcal{F}(D(A_0), D(A_1))$ is complete, it follows that $D(B) \subset \mathcal{F}(D(A_0), D(A_1))$. ∎

A similar statement is valid for the resolvents.

PROPOSITION 3.10. *Let $\mathcal{F}$ be an interpolation functor which has Property* (d). *Let (X_0, X_1) be an interpolation couple of Banach spaces. Further, let $S^{(0)}$ and $S^{(1)}$ be consistent bounded C_0-semigroups in X_0 and X_1 with generators $-A_0$ and $-A_1$, respectively. Then*

$$\mathcal{F}((A_0+I)^{-1},(A_1+I)^{-1}) = (\mathcal{F}(A_0,A_1)+I)^{-1}.$$

Proof. Write $S_t^{\mathcal{F}} = \mathcal{F}(S_t^{(0)}, S_t^{(1)})$ for all $t > 0$. Let $x \in X_0 \cap X_1$. If $F \in (X_0+X_1)'$, then

$$\begin{aligned} F((\mathcal{F}(A_0,A_1)+I)^{-1}x) &= \int_0^\infty e^{-t}\, F(S_t^{\mathcal{F}} x)\, dt = \int_0^\infty e^{-t}\, F(S_t^{(0)} x)\, dt \\ &= F((A_0+I)^{-1}x) = F(\mathcal{F}((A_0+I)^{-1},(A_1+I)^{-1})x). \end{aligned}$$

So

$$(\mathcal{F}(A_0,A_1)+I)^{-1}x = \mathcal{F}((A_0+I)^{-1},(A_1+I)^{-1})x.$$

Moreover, the operator $(\mathcal{F}(A_0,A_1)+I)^{-1}$ is bounded from $\mathcal{F}(X_0,X_1)$ into itself. Hence $\mathcal{F}((A_0+I)^{-1},(A_1+I)^{-1}) = (\mathcal{F}(A_0,A_1)+I)^{-1}$ by Lemma 3.6. ∎

4. EXAMPLE: L^p-SPACES

One of the commonly used theorems states that semigroups on L^2-spaces, which are induced by forms on L^2, extrapolate *consistently* to the whole L^p-scale, provided one knows Gaussian estimates for the L^2-semigroup. We next describe this situation.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain and $\mathcal{D} \subset \partial\Omega$ be closed. We define

$$C_{\mathcal{D}}^\infty(\Omega) = \{\psi|_\Omega : \psi \in C^\infty(\mathbb{R}^d) \text{ and } \operatorname{supp}\psi \cap \mathcal{D} = \emptyset\}.$$

For all $p \in [1,\infty)$ let $W_{\mathcal{D}}^{1,p}(\Omega)$ be the closure of $C_{\mathcal{D}}^\infty(\Omega)$ in the Sobolev space $W^{1,p}(\Omega)$. If $q \in (1,\infty]$, then we denote by $W_{\mathcal{D}}^{-1,q}(\Omega)$ the antidual of the space $W_{\mathcal{D}}^{1,q'}(\Omega)$, where q' is the dual exponent of q. Let μ be a bounded, measurable, elliptic function on Ω which takes its values in the set of real $d \times d$-matrices, that is, there exists a $c > 0$ such that

$$\operatorname{Re} \mu(x)\kappa \cdot \overline{\kappa} \geqslant c\,|\kappa|^2$$

for all $\kappa \in \mathbb{C}^d$ and almost every $x \in \Omega$. Define the sesquilinear form $\mathfrak{t} : W_{\mathcal{D}}^{1,2}(\Omega) \times W_{\mathcal{D}}^{1,2}(\Omega) \to \mathbb{C}$ by

$$\mathfrak{t}[\psi,\varphi] = \int_\Omega \mu \nabla\psi \cdot \overline{\nabla\varphi}.$$

Let A be the operator associated with $\mathfrak{t}$ in $L^2(\Omega)$ and let $\mathcal{A} : W^{1,2}_{\mathcal{D}}(\Omega) \to W^{-1,2}_{\mathcal{D}}(\Omega)$ be defined by $\langle \mathcal{A}\psi, \varphi\rangle = \mathfrak{t}[\psi, \varphi]$ for all $\psi, \varphi \in W^{1,2}_{\mathcal{D}}(\Omega)$. Then $-A$ and $-\mathcal{A}$ generate analytic semigroups $S^{(2)}$ and $\widetilde{S}^{(2)}$ on $L^2(\Omega)$ and $W^{-1,2}_{\mathcal{D}}(\Omega)$, respectively. For all $q \in [2, \infty)$ define the operator $\widetilde{\mathcal{A}}_q$ in $W^{-1,q}_{\mathcal{D}}(\Omega)$ by

$$D(\widetilde{\mathcal{A}}_q) = \{\psi \in W^{-1,q}_{\mathcal{D}}(\Omega) \cap W^{1,2}_{\mathcal{D}}(\Omega) : \mathcal{A}\psi \in W^{-1,q}_{\mathcal{D}}(\Omega)\} \tag{4.1}$$

and $\widetilde{\mathcal{A}}_q = \mathcal{A}|_{D(\widetilde{\mathcal{A}}_q)}$.

In the following we frequently need the following assumption.

ASSUMPTION 4.1. (i) The boundary around any point $x \in \overline{\partial\Omega \setminus \mathcal{D}}$ admits a bi-Lipschitzian boundary chart; i.e. for all $x \in \overline{\partial\Omega \setminus \mathcal{D}}$ there is an open neighbourhood U and a bi-Lipschitz mapping Ψ from U onto the cube $(-1, 1)^d$ such that $\Psi(x) = 0$ and $\Psi(U \cap \Omega)$ equals the lower half cube.

(ii) (•) The set $\mathcal{D}$ is a $(d-1)$-set in the sense of Jonsson–Wallin ([26], Chapter II) and

(•) the set Ω is a d-set in the sense of Jonsson–Wallin, or for almost all $x \in \Omega$ the matrix $\mu(x)$ is symmetric.

THEOREM 4.2. (i) *The semigroups $S^{(2)}$ and $\widetilde{S}^{(2)}$ are consistent.*

(ii) *Adopt Assumption* 4.1(i). *Then the semigroup $S^{(2)}$ has a kernel with Gaussian upper estimates. Moreover, the semigroup $S^{(2)}$ extends consistently to a C_0-semigroup $S^{(p)}$ on $L^p(\Omega)$ for all $p \in [1, \infty)$. The semigroup $S^{(p)}$ is holomorphic.*

Proof. (i) See Subsection 1.4.2 of [34].

(ii) The first assertion is proved in Theorem 3.1 of [15]. The second one follows from the first by the second proof on p. 1160 of [2]. The holomorphy follows from the Gaussian estimates in combination with Theorem 5.4 of [3]. ∎

It is desirable in various contexts to know the consistency of semigroups on spaces like $L^p(\Omega)$ and $W^{-1,q}_{\mathcal{D}}(\Omega)$, as outlined in the beginning of this section. Before we can prove such a result we establish the following lemma.

LEMMA 4.3. *Let $p \in [1, \infty)$ and $q \in (1, \infty)$. Then $C^\infty_c(\Omega)$ is dense in the space $W^{-1,q}_{\mathcal{D}}(\Omega) \cap L^p(\Omega)$.*

Proof. First of all, $W^{-1,q}_{\mathcal{D}}(\Omega) \cap L^p(\Omega)$ is dense in both $W^{-1,q}_{\mathcal{D}}(\Omega)$ and $L^p(\Omega)$, since $C^\infty_c(\Omega) \subset W^{-1,q}_{\mathcal{D}}(\Omega) \cap L^p(\Omega)$. Therefore

$$\big(W^{-1,q}_{\mathcal{D}}(\Omega) \cap L^p(\Omega)\big)' = \big(W^{-1,q}_{\mathcal{D}}(\Omega)\big)' + \big(L^p(\Omega)\big)' = W^{1,q'}_{\mathcal{D}}(\Omega) + L^{p'}(\Omega)$$

by Theorem 2.7.1 of [7]. Let $F \in (W^{-1,q}_{\mathcal{D}}(\Omega) \cap L^p(\Omega))' = W^{1,q'}_{\mathcal{D}}(\Omega) + L^{p'}(\Omega)$ and suppose that $F(\psi) = 0$ for all $\psi \in C^\infty_c(\Omega)$. Since $F \in L^1(\Omega)$, it follows $F = 0$. Then the statement is implied by the Hahn–Banach theorem. ∎

For all $p \in [1,\infty)$ let $S^{(p)}$ be the semigroup on $L^p(\Omega)$ as in Theorem 4.2(ii) (assuming Assumption 4.1(i) is satisfied).

THEOREM 4.4. *Assume Assumption* 4.1(i). *Then the semigroup $\widetilde{S}^{(2)}$ is consistent with the semigroup $S^{(p)}$ for every $p \in [1,\infty)$.*

Proof. Let $t > 0$. If $u \in C_c^\infty(\Omega)$, then $\widetilde{S}_t^{(2)} u = S_t^{(2)} u = S_t^{(p)} u$ by Theorem 4.2. Now the result follows from Lemmas 2.2 and 4.3. ∎

There is also a version for the operator $\widetilde{\mathcal{A}}_q$ in $W_{\mathcal{D}}^{-1,q}(\Omega)$ under slightly more assumptions. Statement (i) of the next theorem is of direct interest in this section. In the next section we need many more results for the operator $\widetilde{\mathcal{A}}_q$ in a non-linear example in the special case $\mathcal{D} = \emptyset$. Since these results are of independent interest for general (mixed) boundary conditions, and we do not wish to change back and forth the boundary conditions in Section 5, we include the statements in this section.

THEOREM 4.5. *Suppose that Assumption* 4.1 *is valid. Let $q \in [2,\infty)$. Then one has the following:*

(i) $-\widetilde{\mathcal{A}}_q$ *generates a holomorphic semigroup on $W_{\mathcal{D}}^{-1,q}(\Omega)$ which is consistent with the semigroup $S^{(p)}$ for all $p \in [1,\infty)$;*

(ii) $\widetilde{\mathcal{A}}_q + I$ *is a positive operator in the sense of Triebel (*[40], *Section* 1.14*);*

(iii) $\widetilde{\mathcal{A}}_q + I$ *admits a bounded H_∞-functional calculus on $W_{\Gamma}^{-1,q}(\Omega)$, in particular, it has bounded imaginary powers;*

(iv) $D\big((\widetilde{\mathcal{A}}_q + I)^{1/2}\big) = L^q(\Omega)$*;*

(v) $W_{\mathcal{D}}^{1,q} \subset D(\widetilde{\mathcal{A}}_q)$*;*

(vi) *if $W_{\mathcal{D}}^{1,q} = D(\widetilde{\mathcal{A}}_q)$, then*

$$\widetilde{\mathcal{A}}_q + I : W_{\mathcal{D}}^{1,q} \to W_{\mathcal{D}}^{-1,q}$$

is a topological isomorphism.

Proof. (i) It follows from Lemma 6.9(c) of [13] that $-\widetilde{\mathcal{A}}_q$ generates a holomorphic semigroup on $W_{\mathcal{D}}^{-1,q}(\Omega)$. Denote this semigroup by $\widetilde{S}^{(q)}$. Then $\widetilde{S}^{(q)}$ is consistent with $\widetilde{S}^{(2)}$ by the paragraph before Lemma 6.9 in [13]. Hence if $t > 0$ and $u \in C_c^\infty(\Omega)$, then $\widetilde{S}_t^{(q)} \psi = \widetilde{S}_t^{(2)} \psi = S_t^{(p)} \psi$. Finally use again Lemmas 2.2 and 4.3.

(ii) This is proved in Theorem 11.5(i) of [4].

(iii) This is proved in Theorem 11.5(ii) of [4].

(iv) For all $p \in [1,\infty)$ let $-A_p$ be the generator of the semigroup $S^{(p)}$. The operator $((A_{q'} + I)^{1/2})' : L^q(\Omega) \to W_{\mathcal{D}}^{-1,q}(\Omega)$ is a topological isomorphism by Theorem 5.1 of [4]. Moreover, $D(A_q^{1/2}) = W_{\mathcal{D}}^{1,q}(\Omega)$ and the operators A_q, $A_{q'}$ and $\widetilde{\mathcal{A}}_q$ are consistent. Therefore $D((\widetilde{\mathcal{A}}_q + I)^{1/2}) = L^q(\Omega)$.

(v) This has been proved at the end of Section 5 in [13].

(vi) Clearly the operator $\widetilde{\mathcal{A}} + I$ is injective, hence also $\widetilde{\mathcal{A}}_q + I$ is injective. It follows from Lemma 6.9(c) of [13] that the operator $\widetilde{\mathcal{A}}_q + I$ is surjective.

Define the operator $\mathcal{A}_q : W^{1,q}_{\mathcal{D}}(\Omega) \to W^{-1,q}_{\mathcal{D}}(\Omega)$ by

$$\langle \mathcal{A}_q \psi, \varphi \rangle_{W^{-1,q}_{\mathcal{D}}(\Omega) \times W^{1,q'}_{\mathcal{D}}(\Omega)} = \int_\Omega \mu \nabla \psi \cdot \overline{\nabla \varphi},$$

where $\psi \in W^{1,q}_{\mathcal{D}}(\Omega)$ and $\varphi \in W^{1,q'}_{\mathcal{D}}(\Omega)$. Then $\widetilde{\mathcal{A}}_q$ is an extension of $\mathcal{A}_q$. Moreover, $\mathcal{A}_q$ is continuous. So if $W^{1,q}_{\mathcal{D}}(\Omega) = D(\widetilde{\mathcal{A}}_q)$, then $\mathcal{A}_q = \widetilde{\mathcal{A}}_q$ and $\widetilde{\mathcal{A}}_q + I$ is a continuous bijection from $W^{1,q}_{\mathcal{D}}(\Omega)$ into $W^{-1,q}_{\mathcal{D}}(\Omega)$. By the open mapping theorem it is then also a topological isomorphism. ∎

REMARK 4.6. Unfortunately, one cannot expect in general for $q > 2$ the equality $W^{1,q}_{\mathcal{D}} = D(\widetilde{\mathcal{A}}_q)$. The possible obstructions against a higher integrability of the gradient are non-smooth domains (see Theorem A of [25]), mixed boundary conditions, i.e. $\mathcal{D} \neq \emptyset \neq \partial\Omega \setminus \mathcal{D}$ (cf. [37]), or the discontinuity of the coefficient function μ (see the classical paper [32] or Section 4 of [14], for a striking example). But in many cases, relevant in the applications, there exists a $q > d$ such that the equality $W^{1,q}_{\mathcal{D}} = D(\widetilde{\mathcal{A}}_q)$ is valid. This is always the case in two space dimensions (see [21]) and was proved in case of three space dimensions in many cases in [11]. This can then be used as the basis for the treatment of non-linear parabolic equations in case of non-smooth data, as is carried out in [22] and [24].

The analysis of a prototypical example in the next section also will rest on this.

We end this section with an invariance property for the equality $W^{1,q}_{\mathcal{D}}(\Omega) = D(\widetilde{\mathcal{A}}_q)$.

LEMMA 4.7. *Let $d \in \{2,3\}$ and $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Further, let $\mathcal{D}$ be a closed subset of the boundary and μ a bounded, elliptic coefficient function. Let $q \in [2,6]$ and let $\widetilde{\mathcal{A}}_q$ be the corresponding operator. Suppose that $W^{1,q}_{\mathcal{D}}(\Omega) = D(\widetilde{\mathcal{A}}_q)$. Let $\xi : \Omega \to \mathbb{R}$ be a uniformly continuous function which admits a positive lower bound. Then the coefficient function $\xi\,\mu$ satisfies the ellipticity condition. Let $\underline{\widetilde{\mathcal{A}}_q}$ be the operator corresponding to $\xi\,\mu$. Then $D(\underline{\widetilde{\mathcal{A}}_q}) = W^{1,q}_{\mathcal{D}}(\Omega)$.*

For the proof see Lemma 6.2 of [11].

5. EXAMPLE: A NON-LINEAR PROBLEM

We conclude the paper with an illustration of the abstract setting arising in non-linear parabolic equations, where the appropriate interpolation is not between two L^p-spaces or between a Banach space and the domain of a semigroup generator which is acting on this space.

Consider the quasilinear initial boundary value problem

$$u' - \nabla \cdot \phi(u)\nabla u + u = |\nabla u|^2, \quad \phi(u)\, \nu \cdot \nabla u|_{\partial\Omega} = f \neq 0, \quad u(0) = u_0. \tag{5.1}$$

on a bounded domain $\Omega \subset \mathbb{R}^d$, where $\phi : \mathbb{R} \to (0, \infty)$ is sufficiently regular and $d \in \{2, 3\}$. Moreover, we suppose that the Neumann datum f is *real-valued*. In case of non-smooth domains it is, in view of the quadratic gradient term on the right hand side, a non-trivial problem to find a suitable Banach space X for the treatment of this initial boundary value problem. Highly relevant examples of parabolic equations (not necessarily quasilinear ones) from science which exhibit a quadratic gradient term may be found in for example [5], [6] and [27].

REMARK 5.1. The inhomogeneous Neumann condition has to be considered in a weak form. We come back to this issue in the first remark after the proof of Theorem 5.2.

If Ω is a Lipschitz domain then under a mild condition we use complex interpolation to construct a suitable space X. We assume from now on that Ω is a Lipschitz domain and use the notation as in Section 4. We choose from now on $\mathcal{D} = \emptyset$. Then Assumption 4.1 is valid. We write $W^{1,q} = W^{1,q}_{\mathcal{D}}(\Omega) = W^{1,q}_{\emptyset}(\Omega)$. In order to avoid confusion with the antidual of $W^{1,q'}_0$ (which is usually denoted by $W^{-1,q}$ in the literature) we denote the anti-dual of $W^{1,q'}$ by $W^{-1,q}_{\emptyset}$. If $\mu(x) = I$ for all $x \in \Omega$ then we denote the corresponding operator $\widetilde{\mathcal{A}}_q$ by $-\widetilde{\Delta}_q$ for all $q \in [2, \infty)$.

In the next theorem we assume that there exists a $q \in (d, 6)$ such that $D(\widetilde{\Delta}_q) = W^{1,q}(\Omega)$. This is satisfied if Ω is a Lipschitz graph domain by Corollary 9.3 in [16] or Theorem 1.6 of [41]; and in the case of polyhedral (possibly non-convex) Lipschitz domains in [20]. Then Theorem 4.5(vi) implies that $\widetilde{\Delta}_q + I$ provides an isomorphism from $W^{1,q}(\Omega)$ onto $W^{-1,q}_{\emptyset}(\Omega)$.

THEOREM 5.2. *Let $d \in \{2, 3\}$ and let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $q \in (d, 6)$ and suppose that $D(\widetilde{\Delta}_q) = W^{1,q}(\Omega)$. Provide the boundary $\partial\Omega$ with the $(d-1)$-dimensional Hausdorff measure. Let $f \in L^\infty(\partial\Omega, \mathbb{R})$. Let $\delta \in (0, 1)$ and $\phi : \mathbb{R} \to [\delta, \infty)$ be a C^2-function. Fix $\theta \in (\frac{q-2}{q-1}, 1)$ and define the space $X = [L^{q/2}(\Omega), W^{-1,q}_{\emptyset}(\Omega)]_\theta$. We denote the restriction of $\widetilde{\Delta}_q$ to X by Δ and assume $u_0 \in D(\Delta)$ is real-valued. Then the problem*

$$u' - \nabla \cdot \phi(u)\nabla u + u = |\nabla u|^2, \quad \phi(u)\, \nu \cdot \nabla u|_{\partial\Omega} = f, \quad u(0) = u_0 \tag{5.2}$$

is well-posed on the interpolation space X, that is there exist $T > 0$ and real-valued $u \in C^1((0, T); X) \cap C([0, T]; D_X(\nabla \cdot \phi(u_0)\nabla))$ which fulfills on $(0, T)$ the quasilinear equation (5.2).

For the proof we need some preparation. Adopt the notation and assumptions as in Theorem 5.2. For all $\xi \in W^{1,q}(\Omega)$ it follows from the Sobolev embedding theorem that ξ is bounded and Hölder continuous. Let $\widetilde{\mathcal{A}}_{q,\xi}$ be as in (4.1) with the choice $\mathcal{D} = \emptyset$ and $\mu(x) = \xi(x)\, I$ for all $x \in \Omega$. For all $p \in [1, \infty)$ let $S^{(p,\xi)}$ be the C_0-semigroup on $L^p(\Omega)$ as in Theorem 4.2(ii) and let $-A_{p,\xi}$ denote the generator of the semigroup $S^{(p,\xi)}$. Then the C_0-semigroup generated by $-\widetilde{\mathcal{A}}_{q,\xi}$ is consistent with the C_0-semigroup $S^{(q/2,\xi)}$ by Theorem 4.5(i). Since $q > d$ it follows that $L^{q/2}(\Omega) \subset W^{-1,q}_{\emptyset}(\Omega)$. Hence $L^{q/2}(\Omega) \subset X \subset W^{-1,q}_{\emptyset}(\Omega)$ and the semigroup on $W^{-1,q}_{\emptyset}(\Omega)$ generated by $-\widetilde{\mathcal{A}}_{q,\xi}$ leaves X invariant and the restriction on X is a C_0-semigroup. Let $-\mathcal{A}_\xi$ be the generator of this C_0-semigroup on X, which is nothing else but the restriction of $-\widetilde{\mathcal{A}}_{q,\xi}$ to X, see Theorem 3.9.

In the subsequent lemma we establish that Neumann inhomogeneities $f \in L^\infty(\partial\Omega)$ in (5.2) can be interpreted as elements in X.

LEMMA 5.3. *There is a natural embedding of $L^\infty(\partial\Omega)$ in X.*

Proof. Using Subsection 1.11.3 of [40] and Theorem 3.1 of [18] one deduces that

$$\begin{aligned} X' &= [L^{q/2}(\Omega), W^{-1,q}(\Omega)]'_\theta = [\big(L^{q/2}(\Omega)\big)', \big(W^{-1,q}(\Omega)\big)']_\theta \\ &= [L^{(q/2)'}(\Omega), W^{1,q'}(\Omega)]_\theta = H^{\theta,r}(\Omega), \end{aligned}$$

where $\frac{1}{r} = (1-\theta)(1-\frac{2}{q}) + \theta(1-\frac{1}{q})$. The condition $\theta > \frac{q-2}{q-1}$ implies that $\theta > \frac{1}{r}$ (actually, it is equivalent). Hence the trace map $\mathrm{Tr} : H^{\theta,r}(\Omega) \to L^r(\partial\Omega)$ is well defined and continuous by Section 1.5 of [19]. Therefore the map $w \mapsto \int\limits_{\partial\Omega} f\, \overline{\mathrm{Tr}\, w}$ from X' into $\mathbb{C}$ is continuous and f can be identified in a natural way with an element of X. ∎

Next we collect some properties of the operator $\mathcal{A}_\xi$.

LEMMA 5.4. *Let $\xi \in W^{1,q}(\Omega, \mathbb{R})$ with $\xi \geqslant \delta$.*
(i) *The operator $\mathcal{A}_\xi + I$ is positive.*
(ii) *Let $s = (1-\theta)\,\frac{d}{q} + \theta$ and $\alpha = \frac{1+s}{2}$. Then $D((\mathcal{A}_\xi + I)^\alpha) \subset W^{1,q}(\Omega)$.*
(iii) *$D(\mathcal{A}_\xi) = D(\Delta)$.*

Proof. (i) Both $A_{p,\xi}$ and $\widetilde{\mathcal{A}}_{q,\xi}$ are positive by Theorem 11.5(i) of [4], so A_ξ is positive by Proposition 3.10.

(ii) First note that $L^{q/2}(\Omega) \subset H^{-d/q,q}_{\emptyset}(\Omega)$, where the latter space denotes the antidual to the Bessel potential space $H^{d/q,q'}(\Omega)$. So

$$X \subset [H^{-d/q,q}_{\emptyset}(\Omega), W^{-1,q}_{\emptyset}(\Omega)]_\theta = H^{-s,q}_{\emptyset}(\Omega), \tag{5.3}$$

where we used Theorem 3.1 of [18] in the last step.

The function ξ is uniformly continuous by the Sobolev embedding. Hence Lemma 4.7 implies that $D(\widetilde{\mathcal{A}}_{q,\xi}) = W^{1,q}$. Consequently

$$\widetilde{\mathcal{A}}_{q,\xi} + I : W^{1,q} \to W^{-1,q}_{\emptyset} \tag{5.4}$$

is a topological isomorphism by Theorem 4.5(vi).

By Theorem 4.5(iii) the operator $\widetilde{\mathcal{A}}_{q,\xi} + I$ admits bounded imaginary powers. Using Theorem 1.15.3 of [40] and Theorem 4.5(iv) one deduces that

$$D((\widetilde{\mathcal{A}}_{q,\xi} + I)^{(1-s)/2}) = [W_\emptyset^{-1,q}(\Omega), D((\widetilde{\mathcal{A}}_{q,\xi} + I)^{1/2})]_{1-s}$$

$$= [W_\emptyset^{-1,q}(\Omega), L^q(\Omega)]_{1-s} = H_\emptyset^{-s,q}(\Omega),$$

where we used Theorem 3.1 of [18] in the last step. Together with (5.3) this gives $X \subset D((\widetilde{\mathcal{A}}_{q,\xi} + I)^{(1-s)/2})$ or, equivalently, $(\widetilde{\mathcal{A}}_{q,\xi} + I)^{(1-s)/2} \in \mathcal{L}(X, W_\emptyset^{-1,q})$. Combining this with the isomorphism property (5.4), one gets

$$(\widetilde{\mathcal{A}}_{q,\xi} + I)^{-(1+s)/2} = (\widetilde{\mathcal{A}}_{q,\xi} + I)^{-1}(\widetilde{\mathcal{A}}_{q,\xi} + I)^{(1-s)/2} \in \mathcal{L}(X, W^{1,q}).$$

Since $\mathcal{A}_\xi$ is the restriction of $\widetilde{\mathcal{A}}_{q,\xi}$ to X, this implies that there exists a $c > 0$ such that

$$\|(\mathcal{A}_\xi + I)^{-(1+s)/2}\psi\|_{W^{1,q}} \leqslant c\,\|\psi\|_X$$

for all $\psi \in X$. This is equivalent with $D((\mathcal{A}_\xi + I)^{(1+s)/2}) \subset W^{1,q}$.

(iii) This follows from Corollary 6.8 of [22]. ∎

Besides the preparations up to now, the proof of Theorem 5.2 heavily rests on the following central result of the classical and pioneering paper [39] of Sobolevskii. (For other proofs and refinements see also [1] and [28].)

THEOREM 5.5. *Let Y be a Banach space and B an operator in Y with dense, compactly embedded domain $D(B)$. Assume that there is a $c > 0$ such that B admits the resolvent estimate*

$$\|(B + \lambda)^{-1}\|_{\mathcal{L}(Y)} \leqslant \frac{c}{1 + |\lambda|}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geqslant 0$. Fix $\beta \in (0,1)$ and $u_0 \in D(B^\beta)$. Let $\alpha \in (0,\beta)$. Let $\mathfrak{B} : D(B^\alpha) \to \mathcal{L}(D(B), Y)$ and $g : D(B^\alpha) \to Y$ with $\mathfrak{B}(u_0) = B$. Assume that for all $R > 0$ there is a constant $c(R) > 0$ such that

$$\|\mathfrak{B}(v) - \mathfrak{B}(w)\|_{\mathcal{L}(D(B),Y)} \leqslant c(R)\,\|B^\alpha v - B^\alpha w\|_Y \quad \text{and} \tag{5.5}$$

$$\|g(v) - g(w)\|_Y \leqslant c(R)\,\|B^\alpha v - B^\alpha w\|_Y \tag{5.6}$$

for all $v, w \in D(B^\alpha)$ with $\|B^\alpha v\|_Y \leqslant R$ and $\|B^\alpha w\|_Y \leqslant R$. Then there exist $T > 0$ and $u \in C^1((0,T); Y) \cap C((0,T]; D(B))$ which fulfills on $(0,T)$ the quasilinear parabolic equation

$$u' - \mathfrak{B}(u)u = g(u), \quad u(0) = u_0.$$

REMARK 5.6. There is a nasty missprint in formula (2.72) of [39], in which the Q on the right hand side has to be replaced by ρ. See the Russian original formula (2.72) of [38].

We need one more lemma. It can be proved by adapting the proof of Theorem 5.5.3/1 in [35], but we give an independent proof.

LEMMA 5.7. *Let Ω be a Lipschitz domain. Let $\psi \in C^2(\mathbb{R})$ and $q > d$. Then $\psi \circ u \in W^{1,q}(\Omega)$ for all $u \in W^{1,q}(\Omega, \mathbb{R})$. Moreover, for all $R > 0$ there exists a $c_R > 0$ such that $\|\psi \circ u - \psi \circ v\|_{W^{1,q}} \leqslant c_R \|u - v\|_{W^{1,q}}$ for all $u, v \in W^{1,q}(\Omega, \mathbb{R})$ with $\|u\|_{W^{1,q}} \leqslant R$ and $\|v\|_{W^{1,q}} \leqslant R$.*

Proof. By the Sobolev embedding theorem there exists a $c > 0$ such that $\|u\|_{L^\infty} \leqslant c\,\|u\|_{W^{1,q}}$ for all $u \in W^{1,q}(\Omega, \mathbb{R})$. Let $M = \sup\limits_{|t| \leqslant cR} (|\psi'(t)| + |\psi''(t)|)$. Let $u, v \in W^{1,q}(\Omega, \mathbb{R}) \cap C^\infty(\Omega)$ with $\|u\|_{W^{1,q}} \leqslant R$ and $\|v\|_{W^{1,q}} \leqslant R$. If $k \in \{1, \ldots, d\}$, then

$$\begin{aligned} |\partial_k(\psi \circ u - \psi \circ v)| &\leqslant |\psi' \circ u|\,|\partial_k u - \partial_k v| + |\psi' \circ u - \psi' \circ v|\,|\partial_k v| \\ &\leqslant M\,|\partial_k u - \partial_k v| + M\,|u - v|\,|\partial_k v|. \end{aligned}$$

So $\|\psi \circ u - \psi \circ v\|_{W^{1,q}} \leqslant 2M\,d\,\|u - v\|_{W^{1,q}} + M\,d\,\|u - v\|_{L^\infty} R$. By Theorem 1 of [30] the map $w \mapsto \psi \circ w$ is continuous from $W^{1,q}(\Omega, \mathbb{R})$ into $W^{1,q}$. Since Ω is a Lipschitz domain, $W^{1,q}(\Omega, \mathbb{R}) \cap C^\infty(\Omega)$ is dense in $W^{1,q}(\Omega, \mathbb{R})$, and the lemma follows. ∎

Proof of Theorem 5.2. Let $\alpha \in (0, 1)$ be as in Lemma 5.4(ii). Then $u_0 \in D(\Delta) \subset D((-\Delta + I)^\alpha) \subset W^{1,q}(\Omega)$ by Lemma 5.4(ii). Hence $\phi \circ u_0 \in W^{1,q}(\Omega)$ by Theorem 1 of [30]. Define $B = \mathcal{A}_{\phi \circ u_0} + I$. Then $D(B) = D(\Delta) = D(\mathcal{A}_\xi)$ for all $\xi \in W^{1,q}(\Omega, \mathbb{R})$ with $\xi \geqslant \delta$ by Lemma 5.4(iii). Since $D(B^\alpha) \subset W^{1,q}(\Omega)$ by Lemma 5.4(ii) one can use again Theorem 1 of [30] and define the map $\mathfrak{B} : D(B^\alpha) \to \mathcal{L}(D(B), X)$ by $\mathfrak{B}(v) = \mathcal{A}_{\phi \circ \operatorname{Re} v}$. It follows from Lemma 6.7 of [22] that there exists a $c > 0$ such that

$$\|\mathcal{A}_\xi u - \mathcal{A}_\eta u\|_X \leqslant c\,\|\xi - \eta\|_{W^{1,q}(\Omega)}\,\|u\|_{D(B)}$$

for all $u \in D(B)$ and $\xi, \eta \in W^{1,q}(\Omega, \mathbb{R})$ with $\xi \geqslant \delta$ and $\eta \geqslant \delta$. Hence

$$\begin{aligned} \|(\mathfrak{B}(v) - \mathfrak{B}(w))u\|_X &= \|(\mathcal{A}_{\phi \circ \operatorname{Re} v} - \mathcal{A}_{\phi \circ \operatorname{Re} w})u\|_X \\ &\leqslant c\,\|\phi \circ \operatorname{Re} v - \phi \circ \operatorname{Re} w\|_{W^{1,q}(\Omega)}\,\|u\|_{D(B)} \end{aligned} \tag{5.7}$$

for all $u \in D(B)$ and all $v, w \in D(B^\alpha)$. By Lemma 5.7 for all $R > 0$ there exists a $c_R > 0$ such that

$$\|\phi \circ \operatorname{Re} v - \phi \circ \operatorname{Re} w\|_{W^{1,q}(\Omega)} \leqslant c_R\,\|\operatorname{Re} v - \operatorname{Re} w\|_{W^{1,q}(\Omega)} \leqslant c_R\,\|v - w\|_{W^{1,q}(\Omega)} \tag{5.8}$$

for all $v, w \in W^{1,q}(\Omega)$ with $\|v\|_{W^{1,q}(\Omega)} \leqslant R$ and $\|w\|_{W^{1,q}(\Omega)} \leqslant R$. The estimates (5.7) and (5.8), combined with the embedding $D(B^\alpha) \subset W^{1,q}(\Omega)$, give (5.5).

Next, if $w \in D(B^\alpha)$, then $w \in W^{1,q}$ and $\nabla w \in L^q(\Omega)$. Therefore $|\nabla w|^2 \in L^{q/2} \subset X$. So we can define $g : D(B^\alpha) \to X$ by $g(v) = |\nabla v|^2 + f$, where we use Lemma 5.3 to identify $f \in L^\infty(\partial\Omega)$ with an element in X. The local Lipschitz condition (5.6) follows immediately from the continuity of the map $v \mapsto |\nabla v|$ from $D(B^\alpha)$ into $L^q(\Omega)$ and the continuous embedding $L^{q/2} \subset X$.

Clearly $D(B)$ is dense in X because $-B$ generates a C_0-semigroup in X. Since $W^{1,q}(\Omega)$ is compactly embedded in $L^q(\Omega)$ and $L^q(\Omega) \subset L^{q/2}(\Omega) \subset X$, one deduces that the space $D(B)$ is compactly embedded in X. Next note that $u_0 \in D(\Delta) = D(B) \subset D(B^\beta)$ for all $\beta \in (\alpha, 1)$, where we used Lemma 5.4(iii) in the equality. Then follows from Theorem 5.5 of Sobolevskii that the *modified problem*

$$u' - \nabla \cdot \phi(\mathrm{Re}\, u)\nabla u + u = |\nabla u|^2, \quad \phi(u)\, \nu \cdot \nabla u|_{\partial\Omega} = f, \quad u(0) = u_0 \tag{5.9}$$

admits a solution u with the regularity as asserted Theorem 5.2.

It remains to show that this solution is *real-valued*. Regard the problem

$$v' - \nabla \cdot \phi(\mathrm{Re}\, u)\nabla v + v = |\nabla u|^2, \quad \phi(\mathrm{Re}\, u)\, \nu \cdot \nabla v|_{\partial\Omega} = f, \quad v(0) = u_0. \tag{5.10}$$

Clearly, u is a solution of this. Due to the already known properties of u, this problem is well-posed in the context of non-autonomous parabolic equations and admits a *unique* solution $v \in L^2(J; W^{1,2}) \cap W^{1,2}(J; W^{-1,2}_\emptyset)$ by Section XVIII.3, Remark 9 of [10]. Since the boundary function f and initial value u_0 are real-valued, also $\overline{u}$ is a solution of (5.10). Therefore $u = \overline{u}$ and u is real-valued. Then the solution u of (5.9) is a solution of (5.2). ∎

Let us finish with some further remarks.

(i) Since the space X is embedded in the space $W^{-1,2}_\emptyset$ one obtains that the solution u is also a solution in the space $W^{-1,2}_\emptyset$. Hence one may perform the duality with any element $\psi \in W^{1,2}$ and obtains that u is also a variational solution in the spirit of Section XVIII.3 of [10]. For these variational solutions it is known that the variational formulation of the inhomogeneous Neumann boundary condition is equal to the classical meaning $\phi(u(t))\, \nu \cdot \nabla u(t) = f$ for almost all t and almost all points $x \in \partial\Omega$ with respect to the boundary measure, if the domain Ω is a strong Lipschitz domain.

(ii) If the inhomogeneous Neumann boundary condition in (5.1) is replaced by a homogeneous one (that is $f = 0$) or if one has a homogeneous Dirichlet condition (that is $u|_{\partial\Omega} = 0$), then L^p spaces are an adequate choice for X in order to treat (5.1), see [31] and [24]. But in our case of inhomogeneous Neumann conditions this is no longer true since L^p spaces enforce a homogeneous Neumann condition, see Section XVIII.4 of [10]. See also Section 1.2 of [9] and Subsection II.2.2 of [17] for an instructive explanation in the elliptic case.

(iii) In contrast to the linear case, also the choice $X = W^{-1,2}_\emptyset(\Omega)$ is inadequate for the problem (5.1). Indeed, if $X = W^{-1,2}_\emptyset(\Omega)$ then for each $t > 0$ the function $u(t, \cdot)$ is then, at best, an element of $W^{1,2}(\Omega)$. Consequently $|\nabla u(t, \cdot)|^2$ is then an element of $L^1(\Omega)$ and fails to be an element of X, as required by the differential equation in (5.1).

(iv) One cannot choose $X = W^{-1,q}_\emptyset$ and apply Theorem 5.5 of Sobolevskii, since the domain of the operator $\nabla \cdot \phi(u_0)\nabla$ in the space $W^{-1,q}_\emptyset$ is at best equal to $W^{1,q}$.

Even if one requires that this domain is equal to $W^{1,q}$, then

$$D\big((\mathcal{A}+I)^{\alpha}\big) = [W_{\emptyset}^{-1,q}, W^{1,q}]_{\alpha} = H_{\emptyset}^{2\alpha-1,q}$$

by Subsection 1.15.4 of [40] and [18]. Hence if $\psi \in D\big((\mathcal{A}+I)^{\alpha}\big)$, then $|\nabla\psi|^2$ is in general not well-defined.

(v) We emphasise that if $p \neq q$ then the space $X = [L^p(\Omega), W_{\emptyset}^{-1,q}(\Omega)]_{\theta}$ is not an interpolation space between the Banach space $W_{\emptyset}^{-1,q}(\Omega)$ and the domain of the operator $-\nabla \cdot \phi(u_0)\nabla$ considered on $W_{\emptyset}^{-1,q}(\Omega)$.

(vi) The problem (5.1) is to be seen, although relevant in science, only as a toy problem. In a quite similar way one can also treat parabolic problems where the right hand side does depend on the gradient of the solution in a much more sophisticated way. This has been done recently in [12] when investigating the drift-diffusion equations for semiconductors with avalanche generation included (see also [23], [33] and pages 111–112 of [36]).

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