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A BEURLING THEOREM FOR NONCOMMUTATIVE HARDY SPACES ASSOCIATED WITH SEMIFINITE VON NEUMANN ALGEBRAS WITH UNITARILY INVARIANT NORMS

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ABSTRACT. We prove a Beurling-type theorem for H^∞-invariant spaces of $L^\alpha(\mathcal{M},\tau)$, where α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ, where $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and H^∞ is an extension of Arveson's noncommutative Hardy space. We use our main result to characterize the H^∞-invariant subspaces of a noncommutative Banach function space $\mathcal{I}(\tau)$ with the norm $\|\cdot\|_E$ on $\mathcal{M}$, the crossed product of a semifinite von Neumann algebra by an action β, and $B(\mathcal{H})$ for a separable Hilbert space $\mathcal{H}$.

KEYWORDS: *Beurling theorem, semifinite von Neumann algebra, crossed products of von Neumann algebras, invariant subspaces, Banach function spaces.*

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1. INTRODUCTION

Suppose that (X,Σ,ν) is a localizable measure space with the finite subset property (i.e. a measure space is localizable if the multiplication algebra is maximal abelian, and has the finite subset property if for every $A\in\Sigma$ such that $\nu(A)>0$, there exists a $B\in\Sigma$ such that $B\subseteq A$, and $0<\nu(B)<\infty$). We let E be a two-sided ideal of the set of complex-valued, Σ-measurable functions on X, such that all functions equal almost everywhere with respect to ν are identified. If E has a norm $\|\cdot\|_E$ such that $(E,\|\cdot\|_E)$ is a Banach lattice, then we call E a Banach function space. (See the work of de Pagter in [23]).

We let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. For every operator $x\in\mathcal{M}$, we define $d_x(\lambda)=\tau(e^{|x|}(\lambda,\infty))$ for every $\lambda\geqslant 0$ (where $e^{|x|}(\lambda,\infty)$ is the spectral projection of $|x|$ on the interval (λ,∞)), and $\mu(x)=\inf\{\lambda\geqslant 0 : d_x(\lambda)\leqslant t\}$ for a given $t\geqslant 0$. Consider the set $\mathcal{I}=\{x\in\mathcal{M} : x \text{ is a finite rank operator in } (\mathcal{M},\tau) \text{ and } \|\mu(x)\|_E<\infty\}$ and let $\|\cdot\|_{\mathcal{I}(\tau)}:\mathcal{I}\to[0,\infty)$ be such that $\|x\|_{\mathcal{I}(\tau)}=\|\mu(x)\|_E$ for all $x\in\mathcal{I}$. It is known that $\|\cdot\|_{\mathcal{I}(\tau)}$ defines a norm on $\mathcal{I}$ (see [23]). Denote by $\mathcal{I}(\tau)$ the closure of $\mathcal{I}$ under $\|\cdot\|_{\mathcal{I}(\tau)}$. We briefly recall an extension of Arveson's noncommutative Hardy space for a semifinite von Neumann algebra. Let H^∞ be a weak* closed unital subalgebra of $\mathcal{M}$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is a von Neumann subalgebra of $\mathcal{M}$. Assume that there also exists a faithful, normal, conditional expectation $\Phi : \mathcal{M} \to \mathcal{D}$. Then H^∞ is called a *semifinite non-commutative Hardy space* if (i) the restriction of τ on $\mathcal{D}$ is semifinite; (ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for every $x, y \in H^\infty$; (iii) $H^\infty + (H^\infty)^*$ is weak* dense in $\mathcal{M}$; and (iv) $\tau(\Phi(x)) = \tau(x)$ for every positive $x \in \mathcal{M}$.

We want to ask the following question about the space $\mathcal{I}(\tau)$.

PROBLEM 1.1. Consider a semifinite subdiagonal subalgebra H^∞ of $\mathcal{M}$ and a closed subspace $\mathcal{K}$ of $\mathcal{I}(\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. How can the subspace $\mathcal{K}$ be characterized?

It can be shown that when $\mathcal{M}$ is diffuse, and $\|\cdot\|_{\mathcal{I}(\tau)}$ is order continuous, the norm $\|\cdot\|_{\mathcal{I}(\tau)}$ on $\mathcal{I}(\tau)$ is in the family of unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms with respect to the tracial weight τ. (See Definition 3.1.)

Our goal for this paper is to prove a Beurling-type theorem for a von Neumann algebra with semifinite, faithful, normal tracial weight τ, and a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ, for example, the Banach function space $\mathcal{I}(\tau)$ with the norm $\|\cdot\|_{\mathcal{I}(\tau)}$.

In 1937, J. von Neumann introduced the unitarily invariant norms on $M_n(\mathbb{C})$ as a way to metrize the matrix spaces [22]. He showed that the class of unitarily invariant norms on $M_n(\mathbb{C})$ is in correspondence with the class of symmetric gauge norms on $\mathbb{C}^n$. Specifically, he proved that for any unitarily invariant norm α, there exists a symmetric gauge norm Ψ on $\mathbb{C}^n$ such that for every finite rank operator A, then $\alpha(A) = \Psi(a_1, a_2, \ldots, a_n)$, where $\{a_i\}_{1\leqslant i\leqslant n}$ is the spectrum of $|A|$.

Since von Neumann's result, these norms have been extended and generalized in different ways. Schatten defined unitarily invariant norms on 2-sided ideals of the continuous operators on a Hilbert space, $B(\mathcal{H})$ (for example, see [29], [30]). Chen, Hadwin and Shen defined a class of unitarily invariant, $\|\cdot\|_1$-dominating, normalized norms on a finite von Neumann algebra [7]. Unitarily invariant norms also play an important role in the study of non-commutative Banach function spaces. For more information and history of unitarily invariant norms see Schatten [29], Hewitt and Ross [14], Goldberg and Krein [10], or Simon [32].

A. Beurling proved his classical theorem for invariant subspaces in 1949 [4]. We recall the classical Beurling theorem. We let $\mathbb{T}$ be the unit circle, and we let μ be the measure on $\mathbb{T}$ such that $\mathrm{d}\mu = \frac{1}{2\pi}\mathrm{d}\theta$. As is standard, we let $L^\infty(\mathbb{T}, \mu)$ be the commutative von Neumann algebra on $\mathbb{T}$. We define $L^2(\mathbb{T}, \mu)$ to be the $\|\cdot\|_2$-norm closure of $L^\infty(\mathbb{T}, \mu)$, which is a Hilbert space with orthonormal basis

$\{z^n : n \in \mathbb{N}\}$. We define the subspace $H^2 = \overline{\text{span}(\{z^n : n \geqslant 0\})}^{\|\cdot\|_2}$ of $L^2(\mathbb{T}, \mu)$, and define $H^\infty = H^2 \cap L^\infty(\mathbb{T}, \mu)$. It is clear that $L^\infty(\mathbb{T}, \mu)$ has a representation onto $B(L^2(\mathbb{T}, \mu))$ given by the map $\phi \to M_\phi$, where M_ϕ is given by $M_\phi(f) = \phi f$ for every $f \in L^2(\mathbb{T}, \mu)$. Hence, $L^\infty(\mathbb{T}, \mu)$ and H^∞ act naturally by left (or right) multiplication on $L^2(\mathbb{T}, \mu)$. The classical Beurling theorem may be stated as follows (for more information, see [4]): *Suppose that $\mathcal{W}$ is a nonzero, closed, H^∞ invariant subspace of H^2 (namely $z\mathcal{W} \subseteq \mathcal{W}$). Then $\mathcal{W} = \phi H^2$ for some $\phi \in H^\infty$ such that $|\phi| = 1$ a.e. (μ).*

The Beurling theorem has been extended in many ways (see [6], [11], [12], [13], [15] and [33], among others). One example is as follows: we define $L^p(\mathbb{T}, \mu)$ to be the closure of $L^\infty(\mathbb{T}, \mu)$ under the $\|\cdot\|_p$-norm. Also define $H^p = \{f \in L^p(\mathbb{T}, \mu) : \int_{\mathbb{T}} f(\mathrm{e}^{\mathrm{i}\theta})\mathrm{e}^{in\theta}\mathrm{d}\mu(\theta) = 0 \ \forall\ n \in \mathcal{N}\}$. The Beurling theorem may be extended to H^∞-invariant subspaces of the Hardy spaces H^p for $1 \leqslant p \leqslant \infty$. Some further extensions of Beurling's theorem can be found in [5] and [7].

Typical examples of noncommutative Banach functional spaces include so called noncommutative L^p-spaces, $L^p(\mathcal{M}, \tau)$, associated with semifinite von Neumann algebras. Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. We consider $\mathcal{I}$, the set of elementary operators on $\mathcal{M}$ (when $\mathcal{M}$ is finite, $\mathcal{M} = \mathcal{I}$). We recall the construction of $L^p(\mathcal{M}, \tau)$. When $0 < p < \infty$ define a mapping $\|\cdot\|_p : \mathcal{I} \to [0, \infty)$ by $\|x\|_p = (\tau(|p|))^{1/p}$ where $|x| = \sqrt{(x^*x)}$ for every $x \in \mathcal{I}$. It is non-trivial to prove that $\|\cdot\|_p$ is a norm, called the *p-norm*, when $1 \leqslant p < \infty$. We define the space $L^p(\mathcal{M}, \tau) = \overline{\mathcal{I}}^{\|\cdot\|_p}$ for $0 < p < \infty$. When $p = \infty$, we set $L^\infty(\mathcal{M}, \tau) = \mathcal{M}$, which acts naturally on $L^p(\mathcal{M}, \tau)$ by right or left multiplication.

In [7], Chen, Hadwin and Shen proved a Beurling-type theorem for unitarily invariant norms on finite von Neumann algebras. A motivation for this paper is to extend the result in [7] to the setting of unitarily invariant norms on *semifinite* von Neumann algebras. We define the family of unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms on the von Neumann algebra $\mathcal{M}$ with respect to the semifinite, faithful, normal tracial weight τ. Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful normal tracial weight τ. We let $\mathcal{I}$ be the set of finite rank operators in $(\mathcal{M}, \tau)$. A norm $\alpha : \mathcal{I} \to [0, \infty)$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ if α is a norm for which the following conditions hold:

(i) for any unitaries $u, v \in \mathcal{M}$ and $x \in \mathcal{I}$, $\alpha(uxv) = \alpha(x)$;

(ii) for every projection $e \in \mathcal{M}$ with $\tau(e) < \infty$ and any $x \in \mathcal{I}$, there exists $0 < c(e) < \infty$ such that $\alpha(exe) \leqslant c(e)\|exe\|_1$;

(iii)

(a) if $\{e_\lambda\}$ is an increasing net of projections in $\mathcal{I}$ such that $\tau(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$;

(b) if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$.

Chen, Hadwin and Shen's family of norms in [7] is a subset of this family of norms. We also show that the norm $\|\cdot\|_{I(\tau)}$ on a Banach function space $\mathcal{I}(\tau)$ is a unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norm.

However, many of the methods used by Chen, Hadwin and Shen no longer apply when $\mathcal{M}$ is a semifinite von Neumann algebra. We use a similar method to extend their theorem as in Sager's work on $L^p(\mathcal{M},\tau)$-spaces extending work of Blecher and Labuschagne (see [26]). We therefore prove a series of density lemmas for the $L^\alpha(\mathcal{M},\tau)$-spaces.

Following these results, we are able to prove a noncommutative Beurling–Chen–Hadwin–Shen theorem for unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norms with respect to τ on a von Neumann algebra $\mathcal{M}$ with a semifinite, faithful, normal tracial weight τ, and we can fully characterize $\mathcal{K}$ in the case when $K \subseteq L^\alpha(\mathcal{M},\tau)$ is $\mathcal{M}$-invariant. Furthermore, when $\mathcal{M}$ is a factor, we can weaken the conditions on α.

Similar to Sager's result in [26] for L^p-spaces, we prove a Beurling–Chen–Hadwin–Shen theorem for the crossed product of a von Neumann algebra $\mathcal{M}$ by a trace-preserving action β with a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous with respect to the trace τ.

We are also able to prove a similar result to Sager's corollary of the Beurling–Blecher–Labuchagne theorem for crossed products, but for any α, a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.

As $B(\mathcal{H})$ is a factor and can be realized as the crossed product, we can also weaken the conditions on α when $\mathcal{M} = B(\mathcal{H})$. Additionally, we can fully characterize the H^∞ invariant subspaces.

Additionally, we prove a result for a Banach function space E with norm $\|\cdot\|_{E(\tau)}$ and provide an answer for Problem 1.1.

We begin in Section 2 by discussing the background definitions and preliminary results. In Section 3, we define the class of unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norms, which we call the class of α-norms. We discuss the non-commutative Banach function space setting and other applications of α-norms. In Section 4, we discuss Arveson's non-commutative Hardy space. We prove our main result, a Beurling–Chen–Hadwin–Shen theorem for α-norms, in Section 5. We finally apply our main result to our examples and crossed products in Section 6.

2. PRELIMINARIES AND NOTATION

In the following section, we give some useful and necessary defintions and results for a von Neumann algebra with a faithful, normal, semifinite tracial weight. We also discuss the space of operators affiliated with a von Neumann algebra with a faithful, normal, semifinite tracial weight.

2.1. WEAK* TOPOLOGY. Let $\mathcal{M}$ be a von Neumann algebra with a predual $\mathcal{M}_{\#}$. We recall that the weak* topology on $\mathcal{M}$, $\sigma(\mathcal{M}, \mathcal{M}_{\#})$, is the topology on $\mathcal{M}$ induced by the predual space $\mathcal{M}_{\#}$. The following result on weak* topology convergence is useful (see, for instance, Theorem 1.7.8 in [28]).

LEMMA 2.1. *Let $\mathcal{M}$ be a von Neumann algebra. If $\{e_\lambda\}_{\lambda\in\Lambda}$ is a net of projections in $\mathcal{M}$ converging to I in the weak* topology, then $e_\lambda x$, xe_λ, and $e_\lambda xe_\lambda$ converge to x in the weak* topology for all x in $\mathcal{M}$.*

2.2. SEMIFINITE VON NEUMANN ALGEBRAS. Let $\mathcal{M}$ be a von Neumann algebra. We let $\mathcal{M}^+$ be the positive part of $\mathcal{M}$. Recall the defintion of a tracial weight τ on $\mathcal{M}$: A mapping $\tau : \mathcal{M}^+ \to [0,\infty]$ is a *tracial weight* on $\mathcal{M}$ if:

(i) $\tau(x+y) = \tau(x) + \tau(y)$ for $x, y \in \mathcal{M}^+$;

(ii) $\tau(ax) = a\tau(x)$ for every $x \in \mathcal{M}^+$ and $a \in [0,\infty]$; and

(iii) $\tau(xx^*) = \tau(x^*x)$ for every $x \in \mathcal{M}$.

Such a τ is called *normal* if it is weak* topology continuous; *faithful* if, given $a \in \mathcal{M}^+$, $\tau(a^*a) = 0$ implies that $a = 0$; *finite* if $\tau(I) < \infty$; and *semifinite* if for any nonzero $x \in \mathcal{M}^+$, there exists a nonzero $y \in \mathcal{M}^+$ such that $\tau(y) < \infty$, and $y \leqslant x$. A von Neumann algebra $\mathcal{M}$ for which a faithful, normal, semifinite tracial weight τ exists is called *semifinite*.

2.3. OPERATORS AFFILIATED WITH $\mathcal{M}$. Given a von Neumann algebra $\mathcal{M}$ with a semifinite, faithful, normal tracial weight τ acting on a Hilbert space $\mathcal{H}$, a *measure topology* on $\mathcal{M}$ is given by the system of neighborhoods $U_{\delta,\varepsilon} = \{a \in \mathcal{M} : \|ap\| \leqslant \varepsilon$ and $\tau(p^\perp) \leqslant \delta$ for some projection $p \in \mathcal{M}\}$ for any $\varepsilon, \delta > 0$ (for more details see [21]). We say that a_n is *Cauchy in measure* if, given ε and $\delta > 0$, there exists an n_0 such that if $n, m \geqslant n_0$, then $a_n - a_m$ is in $U_{\delta,\varepsilon}$.

DEFINITION 2.2. Let $\widetilde{\mathcal{M}}$ denote the algebra of closed, densely defined (possibly unbounded) operators on $\mathcal{H}$ affiliated with $\mathcal{M}$.

REMARK 2.3. $\widetilde{\mathcal{M}}$ is also the closure of $\mathcal{M}$ in the measure topology (see [21] for more information).

3. UNITARILY INVARIANT NORMS AND EXAMPLES

In this section, we introduce a class of unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norms on semifinite von Neumann algebras. We also introduce interesting examples from this class.

3.1. L^α-SPACES OF SEMIFINITE VON NEUMANN ALGEBRAS. Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial state τ. We then let

$$\mathcal{I} = \operatorname{span}\{xey : x, y \in \mathcal{M}, e \in \mathcal{M}, e = e^2 = e^* \text{ with } \tau(e) < \infty\}$$

be the set of elementary operators of $(\mathcal{M},\tau)$ (see Remark 2.3 in [31]). For each $1 \leqslant p < \infty$, we define the $\|\cdot\|_p$-norm on $\mathcal{I}$ by

$$\|x\|_p = (\tau(|x|^p))^{1/p} \quad \text{for every } x \in \mathcal{I}.$$

It is a non-trivial fact that the mapping $\|\cdot\|_p$ defines a norm on $\mathcal{I}$. We let $L^p(\mathcal{M},\tau)$ denote the completion of $\mathcal{I}$ with respect to the $\|\cdot\|_p$-norm.

DEFINITION 3.1. We call a norm $\alpha : \mathcal{I} \to [0,\infty)$ a *unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to* τ on $\mathcal{I}$ if it satisfies the following characteristics:

(i) α is *unitarily invariant* if for all unitaries u, v in $\mathcal{M}$ and every x in $\mathcal{I}$, $\alpha(uxv) = \alpha(x)$;

(ii) α is *locally* $\|\cdot\|_1$*-dominating* if for every projection e in $\mathcal{M}$ with $\tau(e) < \infty$, there exists $0 < c(e) < \infty$ such that $\alpha(exe) \geqslant c(e)\|exe\|_1$ for every $x \in \mathcal{I}$;

(iii) α is *mutually continuous with respect to* τ; namely:

(a) if $\{e_\lambda\}$ is an increasing net of projections in $\mathcal{I}$ such that $\tau(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$; or, equivalently, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $e_\lambda \to I$ in the weak* topology, then $\alpha(e_\lambda x - x) \to 0$ for every $x \in \mathcal{I}$;

(b) if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$.

DEFINITION 3.2. Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Suppose $\mathcal{I} = \operatorname{span}\{\mathcal{M}e\mathcal{M} : e = e^2 = e^* \in \mathcal{M} \text{ such that } \tau(e) < \infty\}$ is the set of all elementary operators in $\mathcal{M}$. Suppose α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{I}$. We define $L^\alpha(\mathcal{M},\tau)$ to be the completion of $\mathcal{I}$ under α, namely,

$$L^\alpha(\mathcal{M},\tau) = \overline{\mathcal{I}}^\alpha.$$

NOTATION 3.3. We will denote by $[S]_\alpha$ the completion, with respect to the norm α, of a set S in $\mathcal{M}$.

LEMMA 3.4. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ, and let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Then for any $x \in L^\alpha(\mathcal{M},\tau)$, and $a, b \in \mathcal{M}$,*

$$\alpha(axb) \leqslant \|a\|\alpha(x)\|b\|.$$

Proof. The proof is included here for completeness. It suffices to show that for any $x \in \mathcal{I}$, and $a, b \in \mathcal{M}$,

$$\alpha(axb) \leqslant \|a\|\alpha(x)\|b\|.$$

Without loss of generality, we might assume that $\|a\| < 1$. By the Russo–Dye theorem, there exist a positive integer n and unitary elements $u_1, \ldots, u_n$ in

$\mathcal{M}$ such that $a = \frac{u_1 + \cdots + u_n}{n}$. Therefore,

$$\alpha(ax) = \frac{\alpha((u_1 + \cdots + u_n)x)}{n} \leqslant \alpha(x)$$

since α is unitarily invariant. So, $\alpha(ax) \leqslant \|a\|\alpha(x)$ for every $a \in \mathcal{M}$.

It may be proved similarly that $\alpha(xb) \leqslant \alpha(x)\|b\|$ for every $b \in \mathcal{M}$. ∎

3.2. EXAMPLES OF UNITARILY-INVARIANT, LOCALLY $\|\cdot\|_1$-DOMINATING, MUTUALLY CONTINUOUS NORMS.

REMARK 3.5. It is trivial to show that the $\|\cdot\|_p$-norms of $\mathcal{M}$ with $1 \leqslant p < \infty$ for a semifinite von Neumann algebra $\mathcal{M}$ with a faithful, normal, semifinite tracial weight τ are unitarily equivalent, $\|\cdot\|_1$-dominating, mutually continuous norms with respect to τ on $\mathcal{M}$.

REMARK 3.6. It is also trivial to show that a continuous, unitarily invariant, normalized, $\|\cdot\|_1$-dominating norm on a finite von Neumann algebra $\mathcal{M}$ as given in [7] is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$.

PROPOSITION 3.7. *Suppose that $\mathcal{M}$ is a semifinite factor, and $\alpha : \mathcal{I} \to [0, \infty)$ is a unitarily invariant norm satisfying that, if $\{e_\lambda\}$ is a net in $\mathcal{M}$ with $e_\lambda \to I$ in the weak* topology, then $\alpha(e_\lambda x - x) \to 0$ for each $x \in \mathcal{I}$. Then α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.*

Proof. By assumption, α is unitarily invariant.

Let e be a projection in $\mathcal{M}$ such that $\tau(e) < \infty$. Let $x = exe$ be an element in $e\mathcal{M}e$, which we denote by $\mathcal{M}_e$. As $|x| \leqslant \|x\|e$, we have that $\alpha(x) = \alpha(|x|) \leqslant \|x\|\alpha(e)$. Note $\mathcal{M}_e$ is a finite factor with a tracial state τ_e, defined by $\tau_e(y) = \frac{\tau(y)}{\tau(e)}$ for all $y \in \mathcal{M}_e$. By the Dixmier approximation property, for every $\varepsilon > 0$, there exist $c_1, c_2, \ldots, c_n$ in $[0,1]$ with $\sum_{i=1}^{n} c_i = 1$ and unitaries $u_1, u_2, \ldots, u_n$ in $e\mathcal{M}e$ such that $\left\|\tau_e(|x|)e - \sum_{i=1}^{n} c_i u_i x u_i^*\right\| < \varepsilon$. Therefore, $\alpha\Big(\tau_e(|x|)e - \sum_{i=1}^{n} c_i u_i x u_i^*\Big) \leqslant \varepsilon\alpha(e)$. Thus,

$$\begin{aligned}
\|x\|_1 = \tau(|x|) = \tau(e)\tau_e(|x|) &= \frac{\tau(e)}{\alpha(e)}\alpha(\tau_e(|x|)e) \\
&\leqslant \frac{\tau(e)}{\alpha(e)}\Big[\alpha\Big(\tau_e(|x|)e - \sum_{i=1}^{n} c_i u_i x u_i^*\Big) + \alpha\Big(\sum_{i=1}^{n} c_i u_i x u_i^*\Big)\Big] \\
&\leqslant \varepsilon\tau(e) + \frac{\tau(e)}{\alpha(e)}\sum_{i=1}^{n}\alpha(c_i u_i x u_i^*) \leqslant \varepsilon\tau(e) + \frac{\tau(e)}{\alpha(e)}\alpha(x).
\end{aligned}$$

Letting $\varepsilon \to 0$, we find that $\tau(x) \leqslant \frac{\tau(e)}{\alpha(e)}\alpha(x)$ for every x in $\mathcal{M}_e$. Namely,

$$\|exe\|_1 \leqslant c\alpha(exe) \quad \text{for all } x \in \mathcal{I}. \tag{3.1}$$

where $c = \frac{\tau(e)}{\alpha(e)}$. Thus, α is locally $\|\cdot\|_1$-dominating.

We now show that α is mutually continuous with respect to τ. Actually, we need only to show that, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$. Assume, to the contrary, that there exist a positive number $\varepsilon > 0$ and a family $\{e_n\}$ of projections in $\mathcal{I}$ such that $\alpha(e_n) < \frac{1}{n}$ but $\tau(e_n) > \varepsilon$ for each $n \in \mathbb{N}$. As $\mathcal{M}$ is a semifinite factor and α is unitarily invariant, we might assume further that $\{e_n\}_n$ is a decreasing sequence of projections in $\mathcal{I}$. Let $e_0 = \bigwedge_n e_n$. Then $\tau(e_0) \geqslant \varepsilon$ and $\alpha(e_0) = 0$ as $e_0 \leqslant e_n$ implies $\alpha(e_0) \leqslant \alpha(e_n) < \frac{1}{n}$ for each n. This is a contradiction. Therefore, if $\{e_\lambda\}$ is a net of projections in $\mathcal{I}$ such that $\alpha(e_\lambda) \to 0$, then $\tau(e_\lambda) \to 0$. ∎

3.2.1. NON-COMMUTATIVE BANACH FUNCTION SPACES. In this subsection, we follow the notation of de Pagter in [23]. We suppose, as before, that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial state τ. In this case, we have the ideal of the distribution function d_x, where x is a τ-measurable operator in $\mathcal{M}$. We define d_x by

$$d_x(\lambda) = \tau(e^{|x|}(\lambda, \infty)) \quad \text{for every } \lambda \geqslant 0,$$

where $e^{|x|}(\lambda, \infty)$ is the spectral projection of $|x|$ on (λ, ∞). It is easy to see that d_x is decreasing, right-continuous and $d_x(\lambda) \to 0$ as $\lambda \to \infty$. This allows us to define a generalized singular value function

$$\mu(x; t) = \inf\{\lambda \geqslant 0 : d_x(\lambda) \leqslant t\} \quad \text{for a given } t \geqslant 0 \text{ and for every } x \in \mathcal{M}.$$

DEFINITION 3.8. Suppose that (X, Σ, ν) is a localizable measure space with the finite subset property. Let E be a two-sided ideal of the set of all complex-valued, Σ-measurable functions on X with the identification of all functions equal a.e. with respect to ν. If E has a norm $\|\cdot\|_E$ such that $(E, \|\cdot\|_E)$ is a Banach lattice, then E is called a *Banach function space*.

We assume that E is a symmetric Banach function space on $(0, \infty)$ with Lebesgue measure (see Definition 2.6 in [23]).

Following [23], we let $\mathcal{I} = \{x \in \mathcal{M} : x$ is a finite rank operator in $(\mathcal{M}, \tau)$, and $\|\mu(x)\|_E < \infty\}$ and define a Banach function space $\mathcal{I}(\tau)$ equipped with a norm $\|\cdot\|_{\mathcal{I}(\tau)}$ such that

$$\|x\|_{\mathcal{I}(\tau)} = \|\mu(x)\|_E \quad \text{for every } x \in \mathcal{I}.$$

Denote the closure of $\mathcal{I}$ under $\|\cdot\|_{\mathcal{I}(\tau)}$ by $\mathcal{I}(\tau)$. We will use the following lemma to show that the restriction of $\|\cdot\|_{\mathcal{I}(\tau)}$ on $\mathcal{I}$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.

LEMMA 3.9. *Suppose that y_0 is an element of $\mathcal{I}$ such that $y_0 = \sum_{i=1}^{n} \beta_i p_i$ where $\beta_1, \beta_2, \ldots, \beta_n$ are nonnegative and $p_1, \ldots, p_n$ are projections in $\mathcal{M}$ such that $\tau(p_1) =$*

$\tau(p_2) = \cdots = \tau(p_n)$. Then

$$\|y_0\|_{\mathcal{I}(\tau)} \geqslant \frac{\|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)}}{\tau(p_1 + \cdots + p_n)} \|y_0\|_1.$$

Proof. Note that y_0 is an element of $\mathcal{I}$ such that $y_0 = \sum\limits_{i=1}^{n} \beta_i p_i$ where $\tau(p_1) = \tau(p_2) = \cdots = \tau(p_n)$. Now let $\beta_{n+j} = \beta_j$ for all $1 \leqslant j \leqslant n$ and $y_j = \sum\limits_{i=1}^{n} \beta_{i+j} p_i$ for $1 \leqslant j \leqslant n$. Then, by definition, $\sum\limits_{k=1}^{n} y_k = (\beta_1 + \cdots + \beta_n)(p_1 + \cdots + p_n)$, and also $\|y_k\|_{\mathcal{I}(\tau)} = \|y_0\|_{\mathcal{I}(\tau)}$ for all $1 \leqslant k \leqslant n$. Therefore:

$$\begin{aligned}\|y_0\|_{\mathcal{I}(\tau)} &\geqslant \frac{\|\sum_{k=1}^{n} y_k\|_{\mathcal{I}(\tau)}}{n} \geqslant \Big(\frac{\beta_1 + \cdots + \beta_n}{n}\Big) \|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)} \\ &= \frac{\tau(y_0)}{\tau(p_1 + \cdots + p_n)} \|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)} = \|y_0\|_1 \frac{\|p_1 + \cdots + p_n\|_{\mathcal{I}(\tau)}}{\tau(p_1 + \cdots + p_n)}. \quad \blacksquare\end{aligned}$$

PROPOSITION 3.10. *Suppose that $\mathcal{I}(\tau)$ is a Banach function space. Suppose that $\mathcal{M}$ is a diffuse von Neumann algebra with a semifinite, faithful, normal tracial state τ and with an order continuous norm $\|\cdot\|_{\mathcal{I}(\tau)}$. Then the restriction of $\|\cdot\|_{\mathcal{I}(\tau)}$ on $\mathcal{I}$ is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mututally continuous norm with respect to τ.*

Proof. Note $\|\cdot\|_{\mathcal{I}(\tau)} : \mathcal{I} \to [0, \infty)$ is a norm. Now we will verify that $\|\cdot\|_{\mathcal{I}(\tau)}$ satisfies the following conditions:

(i) $\|uxv\|_{\mathcal{I}(\tau)} = \|x\|_{\mathcal{I}(\tau)}$ for all unitaries u, v in $\mathcal{M}$, and every x in $\mathcal{I}$;

(ii) for every projection e in $\mathcal{M}$ with $\tau(e) < \infty$, there exists $c(e) < \infty$ such that $\|exe\|_{\mathcal{I}(\tau)} \geqslant c(e)\|exe\|_1$ for all $x \in \mathcal{M}$;

(iii)

(a) if $\{e_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{M}$ such that $e_\lambda \to I$ in the weak* topology, then $\|e_\lambda x - x\|_{\mathcal{I}(\tau)} \to 0$ for every $x \in \mathcal{I}$;

(b) if $\{e_\lambda\}_{\lambda \in \Lambda}$ is a net in $\mathcal{M}$ such that $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$, then $\tau(e_\lambda) \to 0$.

(i) We begin by showing that $\|uxv\|_{\mathcal{I}(\tau)} = \|x\|_{\mathcal{I}(\tau)}$.

Given any x and y in $\mathcal{I}$, we know that if $\tau(|x|^n) = \tau(|y|^n)$ for every $n \in \mathbb{N}$, then $\|x\|_{\mathcal{I}(\tau)} = \|y\|_{\mathcal{I}(\tau)}$ from Definition 3.4 in [23]. We have that τ is unitarily invariant by defintion, so for all unitaries u and v in $\mathcal{M}$ and x in $\mathcal{I}$,

$$\tau(|uxv|^n) = \tau(v^{-n}|x|^n v^n) = \tau(|x|^n) \quad \text{for every } n \in \mathbb{N}.$$

Hence $\|uxv\|_{\mathcal{I}(\tau)} = \|x\|_{\mathcal{I}(\tau)}$, and $\|\cdot\|_{\mathcal{I}(\tau)}$ is unitarily invariant.

(iii)(a) We show that if $\{e_\lambda\} \subseteq \mathcal{I}$ is an increasing net of projections such that $e_\lambda \to I$ in the weak* topology, then $e_\lambda x \to x$ in $\|\cdot\|_{\mathcal{I}(\tau)}$-norm for each $x \in \mathcal{I}$.

Suppose that $\{e_\lambda\} \subseteq \mathcal{I}$ is an increasing net of projections such that $e_\lambda \to I$ in the weak* topology. By definition, $\|\cdot\|_{\mathcal{I}(\tau)}$ is order continuous. So for every x in $\mathcal{I}$, $\|\sqrt{x^*(I-e_\lambda)x}\|_{\mathcal{I}(\tau)} \to 0$, and $\|(I-e_\lambda)x\|_{\mathcal{I}(\tau)} = \||(I-e_\lambda)x|\|_{\mathcal{I}(\tau)} = \|\sqrt{x^*(I-e_\lambda)x}\|_{\mathcal{I}(\tau)}$ by (i). Therefore, $\|x - e_\lambda x\|_{\mathcal{I}(\tau)} \to 0$ for every x in $\mathcal{I}$, as desired.

(b) We show that if $\{e_\lambda\} \subseteq \mathcal{I}$ is a net of projections such that $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$, then $\tau(e_\lambda) \to 0$.

We suppose that $\{e_\lambda\} \subseteq \mathcal{I}$ is a net of projections such that $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$. Suppose to the contrary, that $\tau(e_\lambda) \nrightarrow 0$. There exist an $\varepsilon_0 > 0$, a subsequence $\{e_{\lambda_n}\}$ of $\{e_\lambda\}_{\lambda\in\Lambda}$ such that for every $n \geqslant 1$, $\tau(e_{\lambda_n}) \geqslant \varepsilon_0$. As $\|e_\lambda\|_{\mathcal{I}(\tau)} \to 0$, $\|e_{\lambda_n}\|_{\mathcal{I}(\tau)} \to 0$. Recall that $\mathcal{M}$ has no minimal projection. By the properties of the norm $\|\cdot\|_{\mathcal{I}(\tau)}$, we might assume that $\{e_{\lambda_n}\}$ is a decreasing sequence of projections in $\mathcal{I}$. Thus there exist an $x = \bigwedge_n e_{\lambda_n}$ in $\mathcal{M}$ such that $0 \leqslant x \leqslant e_{\lambda_n}$ for every n, and $\varepsilon_0 \leqslant \tau(x) \leqslant \tau(e_{\lambda_n})$. Moreover, we have that $\|e_{\lambda_n}\|_{\mathcal{I}(\tau)} \geqslant \|x\|_{\mathcal{I}(\tau)}$ for every n, so therefore, $\|x\|_{\mathcal{I}(\tau)} = 0$. Hence $x = 0$, which contradicts with the fact that $\varepsilon_0 \leqslant \tau(x)$.

(ii) We show that for a projection $e \in \mathcal{M}$ such that $\tau(e) < \infty$ there exists $c(e) = \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)}$ satisfying $\|exe\|_{\mathcal{I}(\tau)} \geqslant c(e)\|exe\|_1$ for all $x \in \mathcal{M}$.

Suppose that $e = e^2 = e^*$ is a projection in $\mathcal{M}$ such that $\tau(e) < \infty$. Let x be a positive element in $\mathcal{M}$. For any $\varepsilon > 0$, there exist nonnegative numbers $\beta_1, \beta_2, \ldots, \beta_n$ and subprojections $p_1, p_2, \ldots, p_n$ of e in $\mathcal{M}$ such that $\left\|exe - \sum_{i=1}^n \beta_i p_i\right\|_{\mathcal{I}(\tau)} \leqslant \left\|e - \sum_{i=1}^n \beta_i p_i\right\| \|e\|_{\mathcal{I}(\tau)} < \varepsilon$ and $\left\|exe - \sum_{i=1}^n \beta_i p_i\right\|_1 \leqslant \left\|e - \sum_{i=1}^n \beta_i p_i\right\| \|e\|_1 < \varepsilon$. We call $\sum_{i=1}^n \beta_i p_i = y_0$. For each $m \in \mathbb{N}$ and $1 \leqslant i \leqslant n$, we partition $p_i = q_{i,1} + q_{i,2} + \cdots + q_{i,k_i} + q_{i,k_i+1}$ where k_i is a positive integer and $q_{i,1}, q_{i,2}, \ldots, q_{i,k_i}$ are projections in $\mathcal{M}$ such that $\tau(q_{i,1}) = \tau(q_{i,2}) = \cdots = \tau(q_{i,k_i}) = \frac{1}{m}$, and $0 \leqslant \tau(q_{i,k_i+1}) < \frac{1}{m}$. We can write $y_0 = \sum_{i=1}^n \beta_i\Big(\sum_{j=1}^{k_i+1} q_{i,j}\Big) = z_1 + z_2$, where $z_1 = \sum_{i=1}^n \beta_i\Big(\sum_{j=1}^{k_i} q_{i,j}\Big)$ and $z_2 = \sum_{i=1}^n \beta_i q_{k_i+1}$.

We let $q = \sum_{i=1}^n \sum_{j=1}^{k_i} q_{i,j}$. Then, by Lemma 3.9,

$$\|y_0\|_{\mathcal{I}(\tau)} \geqslant \|z_1\|_{\mathcal{I}(\tau)} \geqslant \frac{\|q\|_{\mathcal{I}(\tau)}}{\tau(q)}\|z_1\|_1.$$

Also, by the triangle inequality,

$$\|z_1\| \geqslant \|y_0\|_1 - \|z_2\|_1 \geqslant \|y_0\|_1 - \frac{\sum_{i=1}^n \beta_i}{m},$$

which approaches $\|y_0\|_1$ as $m \to \infty$. Furthermore, by (iii) we have

$$\frac{\|q\|_{\mathcal{I}(\tau)}}{\tau(q)} \geqslant \frac{\|e\|_{\mathcal{I}(\tau)} - \sum_{i=1}^n \beta_i \|q_{i,k_i+1}\|_{\mathcal{I}(\tau)}}{\tau(e)} \to \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)} \quad \text{as } m \to \infty.$$

Therefore,

$$\|y_0\|_{\mathcal{I}(\tau)} \geqslant \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)} \|y_0\|_1.$$

By the choice of y_0, we conclude, for all x in $\mathcal{M}$:

$$\|exe\|_{\mathcal{I}(\tau)} \geqslant \frac{\|e\|_{\mathcal{I}(\tau)}}{\tau(e)} \|exe\|_1. \quad ∎$$

3.3. EMBEDDING FROM $L^\alpha(\mathcal{M},\tau)$ INTO $\widetilde{M}$. We would like to show that there is a natural embedding from $L^\alpha(\mathcal{M},\tau)$ into $\widetilde{\mathcal{M}}$.

Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ, and $\mathcal{H}$ is a Hilbert space. Recall

$$\mathcal{I} = \operatorname{span}\{xey : x, y \in \mathcal{M}, e \in \mathcal{M}, e = e^2 = e^* \text{ with } \tau(e) < \infty\}$$

is the set of elementary operators of $\mathcal{M}$. Define $\widetilde{\mathcal{M}}$ to be the algebra of closed, densely defined operators on $\mathcal{H}$ affiliated with $\mathcal{M}$. We recall that the measure topology on $\mathcal{M}$ is given by the family of neighborhoods $U_{\delta,\varepsilon} = \{a \in \mathcal{M} : \|ap\| \leqslant \varepsilon$ and $\tau(p^\perp) \leqslant \delta$ for some projection $p \in \mathcal{M}\}$ for any $\varepsilon, \delta > 0$.

Suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$.

LEMMA 3.11. *Let $\varepsilon > 0$ be given. There exists $\delta_0 > 0$ such that if e is a projection in $\mathcal{I}$ with $\alpha(e) < \delta_0$, then $\tau(e) < \varepsilon$.*

Proof. Suppose, to the contrary, that there exists an $\varepsilon > 0$ such that for every $\delta_0 > 0$, there exists a projection e_{δ_0} in $\mathcal{I}$ such that $\alpha(e_{\delta_0}) < \delta_0$, and $\tau(e_{\delta_0}) \geqslant \varepsilon$. Let $\delta_0 = \frac{1}{n}$ for each $n \in \mathbb{N}$. Then there exists a sequence $\{e_n\}_{n\in\mathbb{N}}$ such that for every $n \in \mathbb{N}$, $\alpha(e_n) < \frac{1}{n}$, and $\tau(e_n) \geqslant \varepsilon$. This is a contradiction, as α is mutually continuous with respect to τ (see Definition 3.1). Therefore, the lemma is proven. ∎

LEMMA 3.12. *Suppose a sequence $\{a_n\}$ in $\mathcal{I}$ is Cauchy with respect to the norm α. Then $\{a_n\}$ is Cauchy in the measure topology.*

Proof. To prove that $\{a_n\} \subseteq \mathcal{I}$ is Cauchy in the measure topology, it suffices to show that for every $\varepsilon, \delta > 0$, there exists an $N \in \mathbb{N}$ such that for $n, m > N$, there exists a projection $p_{m,n}$ satisfying $\||a_m - a_n|p_{m,n}\| < \delta$ and $\tau((p_{m,n})^\perp) < \varepsilon$. By Lemma 3.11, we know that there exists a $\delta_0 > 0$ such that

$$\text{if } e \text{ is a projection in } \mathcal{I} \text{ with } \alpha(e) < \delta_0, \text{ then } \tau(e) < \varepsilon. \tag{3.2}$$

For each $m,n \in \mathbb{N}$, let $\{e_\lambda(m,n)\}$ be the spectral decomposition of $|a_m - a_n|$ in $\mathcal{M}$. By the spectral decomposition theorem, we have $|a_m - a_n| = \int\limits_0^\infty \lambda \mathrm{d}e_\lambda(m,n)$, and $\tau(|a_m - a_n|) = \int\limits_0^\infty \lambda \mathrm{d}\tau(e_\lambda(m,n))$. Let $\lambda_0 = \delta_0$. Hence $\lambda_0 e_{\lambda_0}(m,n)^\perp \leqslant |a_m - a_n| e_{\lambda_0}(m,n)^\perp$. So

$$\alpha(\lambda_0 e_{\lambda_0}(m,n)^\perp) \leqslant \alpha(|a_m - a_n|) \quad \text{for all } m,n \in \mathbb{N}. \tag{3.3}$$

Recall that $\{a_n\}$ is Cauchy in α-norm. For $\varepsilon_1 = \lambda_0\delta_0 > 0$, there exists $N \in \mathbb{N}$ such that for all $m,n > N$, $\alpha(a_m - a_n) < \varepsilon_1$. Combining with (3.3), we have that for every $m,n > N$, $\lambda_0 \alpha(e_{\lambda_0}(m,n)^\perp) < \varepsilon_1$. This implies that

$$\alpha(e_{\lambda_0}(m,n)^\perp) < \frac{\varepsilon_1}{\lambda_0} = \delta_0.$$

Because of (3.2), $\tau(e_{\lambda_0}(m,n)^\perp) < \varepsilon$ for every $m,n > N$. Put $p_{m,n} = e_{\lambda_0}(m,n)$. Then for every $m,n > N$ we have the following and the proof is complete:

$$\||a_m - a_n| p_{m,n}\| \leqslant \lambda_0 = \delta_0, \quad \text{and} \quad \tau((p_{m,n})^\perp) < \varepsilon. \quad \blacksquare$$

Therefore, there is a natural continuous mapping from $L^\alpha(\mathcal{M},\tau)$ into $\widetilde{\mathcal{M}}$.

Let e be a projection in $\mathcal{M}$ such that $\tau(e) < \infty$, and let $\mathcal{M}_e = e\mathcal{M}e$. Define a faithful, normal, tracial state τ_e on $\mathcal{M}_e$ by $\tau_e(x) = \frac{1}{\tau(e)}\tau(x)$ for every x in $\mathcal{M}_e$.

It can be shown that τ_e is a finite, faithful, normal tracial state on $\mathcal{M}_e$. Suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$. Define $\alpha_e = \alpha|_{e\mathcal{M}e}$. We define $\alpha'_e : \mathcal{M}_e \to [0,\infty]$ by $\alpha'_e(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha_e(y) \leqslant 1\}$ for every x in $\mathcal{M}_e$. It may be shown that α'_e is indeed a norm, and we call α'_e the *dual norm* of α_e (see [7] for more information). We define $L^{\alpha'_e}(\mathcal{M}_e,\tau) = \overline{\mathcal{M}_e}^{\alpha'_e}$.

We may also define $\overline{\alpha}_e : L^1(\mathcal{M}_e,\tau) \to [0,\infty]$ by $\overline{\alpha}_e(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha'_e(y) \leqslant 1\}$ for every x in $\mathcal{M}_e$, and $\overline{\alpha}'_e : L^1(\mathcal{M}_e,\tau) \to [0,\infty]$ by $\overline{\alpha}'_e = \sup\{|\tau(xy)| : y \in \mathcal{M}, \alpha_e(y) \leqslant 1\}$ for every x in $\mathcal{M}_e$. $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$ and $L^{\overline{\alpha}'_e}(\mathcal{M}_e,\tau)$ are defined to be $\overline{\mathcal{M}_e}^{\overline{\alpha}_e}$ and $\overline{\mathcal{M}_e}^{\overline{\alpha}'_e}$, respectively.

LEMMA 3.13. *Let α be a unitarily invariant, $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Then α_e, α'_e, $\overline{\alpha}'_e$ and $\overline{\alpha}_e$ are unitarily invariant norms on $L^\alpha(\mathcal{M},\tau)$.*

Proof. Clearly, $\alpha_e(uxv) = \alpha(uxv) = \alpha(x) = \alpha_e(x)$ for unitaries u and v and an element x in $\mathcal{M}_e \subset \mathcal{M}$. Therefore, α_e is a unitarily invariant norm.

Let u and v be unitaries, and x be an element of $L^{\alpha'_e}(\mathcal{M}_e,\tau_e)$. Then

$$\begin{aligned}\alpha'_e(uxv) &= \sup\{|\tau(uxvy)| : y \in \mathcal{M}, \alpha_e(y) \leqslant 1\} = \sup\{|\tau(xuyv)| : y \in \mathcal{M}, \alpha_e(y) \leqslant 1\} \\ &= \sup\{|\tau(xy_0)| : y_0 \in \mathcal{M}, \alpha_e(y_0) \leqslant 1\} = \alpha'_e(x)\end{aligned}$$

for every $x \in L^{\alpha'_e}(\mathcal{M}_e,\tau_e)$. Therefore, α'_e is unitarily invariant.

The proofs that $\overline{\alpha}_e$ and $\overline{\alpha}'_e$ are unitarily invariant are similar. ∎

LEMMA 3.14. *Suppose α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ on $\mathcal{M}$. Then:*
(i) $\|x\|_1 \leqslant \overline{\alpha}_e(x)$ *for every* $x \in L^{\overline{\alpha}_e}(\mathcal{M}_e, \tau)$*; and*
(ii) $\|x\|_1 \leqslant \overline{\alpha}'_e(x)$ *for every* $x \in L^{\overline{\alpha}'_e}(\mathcal{M}_e, \tau)$.

Proof. (i) Suppose that x is in $L^{\overline{\alpha}_e}(\mathcal{M}_e, \tau) \subseteq L^1(\mathcal{M}_e, \tau)$. Let $x = uh$ be the polar decomposition of x in $L^1(\mathcal{M}_e, \tau)$, such that u is a unitary in $\mathcal{M}_e$, and h is positive in $L^1(\mathcal{M}_e, \tau)$. As $\overline{\alpha}_e$ is unitarily invariant (see Lemma 3.13),

$$\overline{\alpha}_e(x) = \overline{\alpha}_e(uh) = \overline{\alpha}_e(h). \tag{3.4}$$

By definition, $\overline{\alpha}_e(h) \geqslant |\tau(h)| = \|x\|_1$. Hence, combining with (3.4),

$$\|x\|_1 \leqslant \overline{\alpha}_e(x).$$

The proof of (ii) is similar. ∎

LEMMA 3.15. *For every $y \in \mathcal{M}_e$ and every $z \in L^1(\mathcal{M}_e, \tau)$, $\alpha'_e(yz) \leqslant \|y\|\overline{\alpha}'_e(z)$.*

Proof. Suppose $y \in \mathcal{M}_e$ such that $\|y\| = 1$, and let $y = \omega|y|$ be the polar decomposition of y in $\mathcal{M}_e$, i.e. $\omega \in \mathcal{M}_e$ is unitary and $|y| \in \mathcal{M}_e$ is positive. Define $v = |y| + \mathrm{i}\sqrt{1-|y|^2}$. Then by construction, v is unitary in $\mathcal{M}_e$, and $|y| = \frac{v+v^*}{2}$. Consider any z in $L^1(\mathcal{M}_e, \tau)$. Then we have that

$$\overline{\alpha}'_e(yz) = \overline{\alpha}'_e(\omega|y|z) = \overline{\alpha}'_e\Big(\frac{vz + v^*z}{2}\Big) \leqslant \frac{\overline{\alpha}'_e(vz) + \overline{\alpha}'_e(v^*z)}{2}$$

for every z in $L^1(\mathcal{M}_e, \tau)$, and y in $\mathcal{M}_e$ such that $\|y\| = 1$. Thus $\overline{\alpha}'_e(yz) \leqslant \|y\|\overline{\alpha}'_e(z)$ for every z in $L^1(\mathcal{M}_e, \tau)$ and y in $\mathcal{M}_e$. ∎

LEMMA 3.16. *For every $x \in \mathcal{M}_e$, $\alpha_e(x) = \overline{\alpha}_e(x)$.*

Proof. First, we show that $\overline{\alpha}_e(x) \leqslant \alpha_e(x)$ for every x in $\mathcal{M}_e$. By definition, $|\tau(xy)| \leqslant \alpha_e(x)\alpha'_e(y)$ for every x and y in $\mathcal{M}_e$. Suppose $\alpha'_e(y) \leqslant 1$. Then $|\tau(xy)| \leqslant \alpha_e(x)\alpha'_e(y) < \alpha_e(x)$ for every x in $\mathcal{M}_e$, and y in $\mathcal{M}_e$ such that $\alpha'_e(y) \leqslant 1$. Hence, by definition,

$$\overline{\alpha}_e(x) = \sup\{|\tau(xy)| : y \in \mathcal{M}_e, \alpha'_e(y) \leqslant 1\} \leqslant \alpha_e(x). \tag{3.5}$$

Next, we show that $\overline{\alpha}_e(x) \geqslant \alpha_e(x)$. Suppose x is in $\mathcal{M}_e$ with $\alpha_e(x) = 1$. Then by the Hahn–Banach theorem, there exists a φ in $L^{\alpha_e}(\mathcal{M}_e, \tau)^{\#}$ such that $\varphi(x) = \alpha_e(x) = 1$, and $\|\varphi\| = 1$. Since φ is in $L^{\alpha_e}(\mathcal{M}_e, \tau)^{\#}$, there exists ξ in $L^{\overline{\alpha}'_e}(\mathcal{M}_e, \tau)$ such that $\varphi(x) = |\tau(x\xi)| = 1$, and $\overline{\alpha}'_e(\xi) = \|\xi\| = 1$. Let $\xi = uh$ be the polar decomposition of ξ in $L^{\overline{\alpha}'_e}(\mathcal{M}_e, \tau)$, where $u \in \mathcal{M}_e$ is unitary and $h \in L^{\overline{\alpha}'_e}(\mathcal{M}_e, \tau)$ is positive.

By Lemma 3.8 in [7], there exists a family $\{e_\lambda\}$ of projections in $\mathcal{M}_e$ such that $\|h - he_\lambda\|_1 \to 0$, and $e_\lambda h = he_\lambda \in \mathcal{M}_e$ for every $0 < \lambda < \infty$. Also, $u \in \mathcal{M}_e$, so $uhe_\lambda \in \mathcal{M}_e$. Thus $\alpha'_e(uhe_\lambda) = \overline{\alpha}'_e(uhe_\lambda) \leqslant \overline{\alpha}'_e(uh)\|e_\lambda\| \leqslant \overline{\alpha}'_e(uh) = \alpha'_e(\xi) = 1$, as $\alpha_e'(x) = \overline{\alpha}_e'(x)$ for every $x \in \mathcal{M}_e$ by Lemma 3.2 in [7]. So, $\alpha_e(x)|\tau(x\xi)| = |\tau(xuh)| = \lim_{\lambda\to\infty}|\tau(xuhe_\lambda)| \leqslant \sup\{|\tau(xy)| : y \in \mathcal{M}_e, \alpha_e'(y) \leqslant 1\} = \overline{\alpha}_e(x)$. Therefore

$$\alpha_e(x) \leqslant \overline{\alpha}_e(x). \tag{3.6}$$

Hence from equations (3.5) and (3.6), $\alpha_e(x) = \overline{\alpha}_e(x)$, and the lemma is proven. ∎

LEMMA 3.17. *$L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau) = \{x \in L^1(\mathcal{M}_e) : \overline{\alpha}_e(x) < \infty\}$ is a complete space in α_e-norm.*

Proof. It suffices to show that for every Cauchy sequence $\{b_n\}$ in $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$, there exists b in $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$ such that $b_n \to b$ in $\overline{\alpha}_e$-norm. Suppose that $\{b_n\}$ is a Cauchy sequence in $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$. There exists $M > 0$ such that $\overline{\alpha}_e(b_n) \leqslant M$ for every n.

By Lemma 3.14,

$$\|b_n - b_m\|_1 \leqslant \overline{\alpha}(b_n - b_m) \quad \text{for all } m, n \geqslant 1.$$

Therefore, $\{b_n\}$ is Cauchy in $L^1(\mathcal{M}_e,\tau)$, which is complete. So there exists a b_0 in $L^1(\mathcal{M}_e,\tau)$ such that $\|b_n - b_0\|_1 \to 0$.

First, we claim that b_0 is in $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$. Let $y \in \mathcal{M}_e$ such that $\alpha_e'(y) \leqslant 1$. We have that $|\tau(b_n y) - \tau(b_0 y)| = |\tau((b_n - b_0)y)| \leqslant \|b_n - b_0\|_1\|y\|_\infty$ by Hölder's inequality. However, $\|b_n - b_0\|_1\|y\|_\infty \to 0$. Also, by the definition of $\overline{\alpha}$, we also have that $|\tau(b_0 y)| = \lim_{n\to\infty}|\tau(b_n y)| \leqslant \limsup_{n\to\infty}\overline{\alpha}_e(b_n)\alpha_e'(y) \leqslant M$. Therefore, $\overline{\alpha}(b_x) \leqslant M$, and $b_0 \in L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$.

Now, we show that $\overline{\alpha}_e(b_n - b_0) \to 0$. We know that $\{b_n\}$ is Cauchy in $L^{\overline{\alpha}}(\mathcal{M}_e,\tau)$, so for every $n \geqslant 1$,

$$\begin{aligned}|\tau((b_n - b_0)y)| &= \lim_{m\to\infty}|\tau((b_m - b_n)y)| \leqslant \limsup_{m\to\infty}\overline{\alpha}_e(b_n - b_m)\alpha_e'(y)\\ &\leqslant \limsup_{m\to\infty}\overline{\alpha}(b_m - b_n).\end{aligned}$$

Therefore, $\overline{\alpha}_e(b_n - b_0) \leqslant \limsup_{m\to\infty}(b_n - b_m)$ for every $n \geqslant 1$, and since $\{b_n\}$ is Cauchy in $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$,

$$\overline{\alpha}_e(b_n - b_0) \to 0 \quad \text{as } n \to \infty,$$

and the lemma is proven. ∎

Therefore $L^{\overline{\alpha}_e}(\mathcal{M}_e,\tau)$ is a Banach space with respect to $\overline{\alpha}_e$-norm.

LEMMA 3.18. *Suppose that $e \in \mathcal{M}$ is a projection such that $\tau(e) < \infty$. Suppose $\{ea_n e\} \subseteq \mathcal{I}$ is Cauchy in α-norm, and $ea_n e$ converges in measure to 0. Then:*

(i) *for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, if q is a projection in $\mathcal{M}$ with $\tau(q) < \delta$, $|\tau(ea_n eq)| < \varepsilon$ for every n;*

(ii) *given $\delta > 0$, $\varepsilon > 0$ and $N \in \mathbb{N}$, there exists p_n, a projection in $\mathcal{M}$, such that $\|ea_n ep_n\| \leqslant \varepsilon$, and $\tau(p_n^\perp) < \delta$ for every $n \geqslant N$;*

(iii) *for every projection q in $\mathcal{I}$, $\tau(ea_n eq) \to 0$ as $n \to \infty$; and*

(iv) *for every b in $\mathcal{M}$, $\tau(ea_neb) \to 0$ as $n \to \infty$.*

Proof. (i) Suppose that, as above, $e \in \mathcal{M}$ is a projection such that $\tau(e) < \infty$ and $\{ea_ne\}$ is a Cauchy sequence in α-norm. Let $\varepsilon > 0$ be given. By assumption, α is a locally $\|\cdot\|_1$-dominating norm, so there exists $c(e)$ such that $\alpha(exe) \geqslant c(e)\|exe\|_1$ for every $x \in \mathcal{M}$. Then, given $\frac{\varepsilon}{2}c(e)$, there exists $N_0 \in \mathbb{N}$ such that for all $n, m > N_0$,

$$\alpha(ea_ne - ea_me) \leqslant \frac{\varepsilon}{2}c(e).$$

Let $\delta = \min\limits_{k \leqslant N_0}\{\frac{\varepsilon}{2\|ea_ke\|_\infty}\}$. Suppose q is a projection in $\mathcal{M}$ such that $\tau(q) \leqslant \delta$. Then for every $k \leqslant N_0$, $|\tau(ea_keq)| \leqslant \|ea_ke\|\|q\|_1$ by Hölder's inequality, and $\tau(q) = \|q\|_1 \leqslant \delta$. Hence $|\tau(ea_keq)| \leqslant \|ea_ke\|\delta < \frac{\varepsilon}{2}$ for all $k \leqslant N_0$ by our choice of δ.

For $k > N_0$,

$$\begin{aligned} |\tau(ea_keq)| &\leqslant |\tau((ea_ke - ea_{N_0}e)q)| + |\tau(ea_{N_0}eq)| \\ &\leqslant \|ea_ke - ea_{N_0}e\|_1\|q\| + \|ea_{N_0}e\|\|q\|_1 \quad \text{(by Hölder's inequality)} \\ &\leqslant \frac{1}{c(e)}\alpha(ea_ke - ea_{N_0}e)\|q\| + \|ea_{N_0}e\|\delta \quad \text{(by Definition 3.1)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Hence, (i) is proven.

(ii) Suppose that $\{ea_ne\}$ is a Cauchy sequence in α-norm and $ea_ne \to 0$ in measure. Then, by the definition of convergence in measure, for any $\varepsilon > 0$, $\delta > 0$ and $N \in \mathbb{N}$, there exists p_n in $\mathcal{M}$ such that $\|ea_nep_n\| < \varepsilon$ and $\tau(p_n^\perp) < \delta$ for every $n \geqslant N$.

(iii) Suppose that $\{ea_ne\}$ is a Cauchy sequence in α-norm such that $ea_ne \to 0$ in measure. Then by (i), given $\varepsilon > 0$ and a projection q in $\mathcal{I}$, there exists a $\delta_1 > 0$ such that if $\tau(q') < \delta_1$, then $|\tau(ea_neq')| < \frac{\varepsilon}{2}$. Let $\delta > 0$ and $\varepsilon_1 = \frac{\varepsilon}{2\tau(q)}$. Then by (ii), there exists $N \in \mathbb{N}$ such that $\|ea_nep_n\| < \varepsilon_1$, and $\tau(p_n^\perp) < \delta$ for every $n \geqslant N$. Thus, for $n \geqslant N$ and any projection $q \in \mathcal{I}$,

$$\tau(ea_neq) = \tau(ea_ne(q - q \cap p_n)) + \tau(ea_ne(q \cap p_n)). \tag{3.7}$$

However, $\tau(q - q \cap p_n) = \tau(q \cup p_n - p_n) \leqslant \tau(p_n^\perp) < \delta$. Therefore,

$$|\tau(ea_ne(q - q \cap p_n))| < \frac{\varepsilon}{2}. \tag{3.8}$$

from (i). Also,

$$\begin{aligned} |\tau(ea_ne(q \cap p_n))| &= |\tau(ea_nep_n(q \cap p_n))| \leqslant \|ea_nep_n\|\|q \cap p_n\|_1 \\ &\leqslant \varepsilon_1\tau(q \cap p_n) < \varepsilon_1\tau(q) = \frac{\varepsilon}{2}. \end{aligned} \tag{3.9}$$

Then from equations (3.7), (3.8) and (3.9), $|\tau(ea_neq)| < \varepsilon$ for any given $\varepsilon > 0$. Therefore, $\tau(ea_ne) \to 0$ for every $q \in \mathcal{M}$ such that q is a projection and $\tau(q) < \infty$.

(iv) Suppose that $\{ea_ne\}$ is a Cauchy sequence in α-norm. Then there exists $M > 0$ such that $\tau(ea_ne) \leqslant \frac{\alpha(ea_ne)}{c(e)} < \frac{M}{c(e)}$. By considering ebe instead, we might assume that $b \in \mathcal{I}$. By the spectral decomposition theorem, b can be approximated by a finite linear combination of projections q_i in $\mathcal{M}$, i.e. there exist $q_i \in \mathcal{I}$ such that $\left\| b - \sum_{i=1}^{n} q_i \right\| < \varepsilon \frac{c(e)}{M}$ for any given $\varepsilon > 0$. Therefore we have the following and the lemma is proven:

$$\begin{aligned}\left|\tau(ea_neb) - \tau\Big(ea_ne\sum_{i=1}^{n} q_i\Big)\right| &= \left|\tau\Big(ea_ne\Big(b - \sum_{i=1}^{n} q_i\Big)\Big)\right| \\ &\leqslant \|\tau(ea_ne)\|_1 \left\| b - \sum_{i=1}^{n} q_i \right\| \leqslant \frac{M}{c(e)}\varepsilon\frac{c(e)}{M} < \varepsilon. \qquad \blacksquare\end{aligned}$$

PROPOSITION 3.19. *There exists a natural embedding from $L^\alpha(\mathcal{M},\tau)$ into $\widetilde{\mathcal{M}}$.*

Proof. By Lemma 3.12, there exists a natural mapping from $L^\alpha(\mathcal{M},\tau)$ to $\widetilde{\mathcal{M}}$. It suffices to show that this mapping is an injection. Suppose that $\{a_n\} \subseteq \mathcal{I}$ is a Cauchy sequence in α-norm such that $x_n \to 0$ in measure. As $L^\alpha(\mathcal{M},\tau)$ is complete, there exists $a \in L^\alpha(\mathcal{M},\tau)$ such that $a_n \to a$ in α-norm. Assume that $a \neq 0$. There exists a projection e in $\mathcal{M}$ such that $\tau(e) < \infty$ and $eae \neq 0$. Thus $\{ea_ne\}$ is Cauchy in α_e-norm, $ea_ne \to 0$ in measure and $ea_ne \to eae \neq 0$ in α_e-norm. By Lemma 3.18, $\tau(ea_neb) \to 0$ for any $b \in \mathcal{M}$. As, $|\tau(ea_neb) - \tau(eaeb)| \leqslant \alpha_e(ea_ne - eae)\alpha_e'(b) \to 0$, we have

$$\tau(eaeb) = 0 \quad \text{for all } b \in \mathcal{I}.$$

On the other hand, by Lemma 3.16 and definition of $\overline{\alpha}_e$, since $eae \neq 0$, there exists some $b_0 \in \mathcal{M}_e$ such that $\alpha_e'(b_0) \leqslant 1$ and $\tau(eaeb_0) > \frac{\alpha(eae)}{2}$. This is a contradiction. Therefore, $a = 0$, and the mapping is an embedding. $\blacksquare$

4. ARVESON'S NON-COMMUTATIVE HARDY SPACE

In this section, we will extend Arveson's classical definition of a non-commutative Hardy space to $L^\alpha(\mathcal{M},\tau)$. We assume, as before, that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ, and we assume that $\mathcal{A} \subseteq \mathcal{M}$ is a weak* closed unital subalgebra of $\mathcal{M}$. We let $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$, and assume that $\Phi : \mathcal{M} \to \mathcal{D}$ is a faithful, normal conditional expection. Let

$$\mathcal{I} = \operatorname{span}\{xey : x, y \in \mathcal{M}, e \in \mathcal{M}, e = e^2 = e^* \text{ with } \tau(e) < \infty\}$$

be the set of elementary operators of $\mathcal{M}$.

DEFINITION 4.1. A weak* closed unital subalgebra $\mathcal{A}$ of $\mathcal{M}$ is called a semifinite subdiagonal subalgebra, or a semifinite non-commutative Hardy space with respect to $(\mathcal{M},\tau)$, if:

(i) the restriction $\tau|_{\mathcal{D}}$ of τ to $\mathcal{D} = \mathcal{A} \cap \mathcal{A}^*$ is semifinite;
(ii) $\Phi(xy) = \Phi(x)\Phi(y)$ for every x and y in $\mathcal{A}$;
(iii) $\mathcal{A} + \mathcal{A}^*$ is weak*-dense in $\mathcal{M}$;
(iv) Φ is τ-preserving (i.e. $\tau(\Phi(x)) = \tau(x)$ for every positive operator $x \in \mathcal{M}$).
We will, in this case, denote $\mathcal{A}$ by H^∞.

DEFINITION 4.2. Let $\alpha : \mathcal{I} \to [0, \infty)$ be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. We denote by H^α the closure $[\mathcal{A} \cap L^\alpha(\mathcal{M}, \tau)]_\alpha$ in α-norm.

REMARK 4.3. Considering the conditional expectation $\Phi : \mathcal{M} \to \mathcal{D}$ from Definition 4.1, we have that Φ extends to a projection from $L^1(\mathcal{M}, \tau)$ to $L^1(\mathcal{D}, \tau)$. We still denote such an extension by Φ, and we have that

$$\Phi(axb) = a\Phi(x)b, \quad \forall a, b \in \mathcal{D},\ x \in L^\alpha(\mathcal{M}, \tau).$$

NOTATION 4.4. We denote $\ker(\Phi) \cap H^\infty$ by H_0^∞, and $\ker(\Phi) \cap H^\alpha$ by H_0^α.

LEMMA 4.5. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let $\mathcal{A} = H^\infty$ be a semifinite subdiagonal subalgebra, as described in Definition* 4.1. *Let $e = e^* = e^2 \in \mathcal{D}$ such that $\tau(e) < \infty$. Then $eH^\infty e$, denoted H_e^∞, is a Hardy space of $\mathcal{M}_e$.*

For the proof see Lemma 3.1 of [2].

LEMMA 4.6. *Suppose $\mathcal{M}$ is a semifinite von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$, as described in Definition* 4.1, *namely that the restriction of τ to $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is semifinite. Let $\alpha : \mathcal{I} \to [0, \infty)$ be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ.*

Then for every $x \in L^\alpha(\mathcal{M}, \tau)$ and for every $e \in \mathcal{D}$ such that $\tau(e) < \infty$, there exist $h_1, h_3 \in eH^\infty e = H_e^\infty$ and $h_2, h_4 \in eH^\alpha e = H_e^\alpha$ such that:
(i) *$h_1 h_2 = e = h_2 h_1$ and $h_3 h_4 = e = h_4 h_3$;*
(ii) *$h_1 ex \in \mathcal{M}$, and $exh_3 \in \mathcal{M}$.*

Proof. Let $ex = \sqrt{exx^*e}u = |x^*e|u$ be the polar decomposition of $(ex)^*$ in $L^\alpha(\mathcal{M}, \tau)$ where u is a partial isometry in $\mathcal{M}$ and $|x^*e|$ is a positive operator in $L^\alpha(\mathcal{M}, \tau)$. Note that $|x^*e|$ is in $eL^\alpha(\mathcal{M}, \tau)e = L^\alpha(\mathcal{M}_e, \tau)$. Since $0 < \tau(e) < \infty$, we know that $\mathcal{M}_e$ is a finite von Neumann algebra with a faithful, normal tracial state $\frac{1}{\tau(e)}\tau$. By Lemma 4.5, we have that H_e^∞ is a finite subdiagonal subalgebra of $\mathcal{M}_e$ with $[H_e^\infty]_\alpha = H_e^\alpha$.

We have that $|x^*e| \in L^\alpha(\mathcal{M}_e, \frac{1}{\tau(e)}\tau)$, and $0 < \tau(e) < \infty$. Then $w = (e + |x^*e|)^{-1}$ is an invertible operator in $\mathcal{M}_e$ with $w^{-1} \in L^\alpha\left(\mathcal{M}_e, \frac{1}{\tau(e)}\tau\right)$. We know that $\mathcal{M}_e$ is a finite von Neumann algebra with faithful, normal tracial state $\frac{1}{\tau(e)}\tau$, and α_e on $\mathcal{M}_e$ is a unitarily invariant, ε-$\|\cdot\|_1$-dominating, continuous norm on

$\mathcal{M}_e$. Therefore, from Proposition 5.2 in [7], there exists a unitary v in $\mathcal{M}_e$, $h_1 \in H_e^\infty$, and $h_2 \in H_e^\alpha$ such that:

(i) $h_1 h_2 = e = h_2 h_1$; and
(ii$_\mathrm{a}$) $w = vh_1$.

By (ii$_\mathrm{a}$), we get (ii$_\mathrm{b}$) $h_1|x^*e| = v^*w|x^*e| = v^*(e + |x^*e|)^{-1}|x^*e| \in \mathcal{M}_e \subseteq \mathcal{M}$. Since u_1 is a partial isometry in $\mathcal{M}$, $h_1 ex = h_1|x^*e|u_1 \in \mathcal{M}$. Therefore, (ii) holds.

The proof for h_3 and h_4 is similar. ∎

The following lemma is also helpful.

LEMMA 4.7. *Suppose $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let H^∞ be a semifinite, subdiagonal subalgebra with respect to $(\mathcal{M}, \Phi)$, where Φ is a faithful, normal conditional expectation from $\mathcal{M}$ onto $\mathcal{D} = H^\infty \cap (H^\infty)^*$.*

There exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections in $\mathcal{D}$ such that:
(i) *$e_\lambda \to I$ in the weak* topology on $\mathcal{M}$, and $\tau(e_\lambda) < \infty$ for all $\lambda \in \Lambda$;*
(ii) *for every $x \in L^\alpha(\mathcal{M}, \tau)$,*

$$\lim_\lambda \alpha(e_\lambda x - x) = 0; \quad \lim_\lambda \alpha(xe_\lambda - x) = 0; \quad \textit{and} \quad \lim_\lambda \alpha(e_\lambda xe_\lambda - x) = 0.$$

Proof. We know that H^∞ is a semifinite subdiagonal subalgebra of $\mathcal{M}$, therefore the restriction of τ to $\mathcal{D}$ is semifinite. From Lemma 2.2 in [26], there exists a net of projections $\{e_\lambda\}_{\lambda\in\Lambda}$ in $\mathcal{D}$ such that $e_\lambda \to I$ in the weak* topology on $\mathcal{D}$, and $\tau(e_\lambda) < \infty$ for all $\lambda \in \Lambda$. Therefore,

$$\lim_\lambda |\tau(e_\lambda z - z)| = 0 \quad \text{for every } z \in L^1(\mathcal{D}, \tau).$$

Also, for each y in $L^1(\mathcal{M}, \tau)$, we have that

$$\lim_\lambda |\tau(e_\lambda y - y)| = \lim_\lambda |\tau(\Phi(e_\lambda y - y))| = \lim_\lambda |\tau(e_\lambda \Phi(y) - \Phi(y))| = 0.$$

Namely, $e_\lambda \to I$ in the weak* topology on $\mathcal{M}$, and $\tau(e_\lambda) < \infty$ for every $\lambda \in \Lambda$. (i) is satisfied.

Then from (i) and Definition 3.1, we may conclude that (ii) holds. Namely, for every $x \in L^\alpha(\mathcal{M}, \tau)$, we have the following and the lemma is proven:

$$\lim_\lambda \alpha(e_\lambda x - x) = 0; \quad \lim_\lambda \alpha(xe_\lambda - x) = 0; \quad \text{and} \quad \lim_\lambda \alpha(e_\lambda xe_\lambda - x) = 0. \quad ∎$$

Finally, we recall the definition of a row sum of subspaces of $L^\alpha(\mathcal{M}, \tau)$.

DEFINITION 4.8. Let $\mathcal{M}$ be a von Neumann algebra with a semifinite, normal, faithful tracial weight τ. Suppose X is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$, and $\{X_i\}_{i\in\mathcal{I}}$ are closed subspaces of $L^\alpha(\mathcal{M}, \tau)$. If:
(i) $X_j X_i^* = \{0\}$ for every $i, j \in \mathcal{I}$, $i \neq j$; and
(ii) $X = [\mathrm{span}\{X_i : i \in \mathcal{I}\}]_\alpha$,

we call X the internal row sum of $\{X_i\}_{i\in\mathcal{I}}$, and denote it by $X = \bigoplus_{i\in\mathcal{I}}^{\mathrm{row}} X_i$. Also, we denote $\mathrm{span}\{X_i : i \in \mathcal{I}\}$ by $\sum_{i\in\mathcal{I}} X_i$.

5. BEURLING THEOREM FOR SEMIFINITE HARDY SPACES WITH NORM α

THEOREM 5.1. *Let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal semifinite tracial weight τ, and H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$.*

Then, there exist a closed subspace Y of $L^\alpha(\mathcal{M},\tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that:

(i) *$u_\lambda Y^* = 0$ for every $\lambda \in \Lambda$;*

(ii) *$u_\lambda u_\lambda^* \in \mathcal{D}$, and $u_\lambda u_\mu^* = 0$ for every $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;*

(iii) *$Y = [H_0^\infty Y]_\alpha$;*

(iv) *$\mathcal{K} = Y \bigoplus^{\mathrm{row}} (\bigoplus_{\lambda\in\Lambda}^{\mathrm{row}} H^\alpha u_\lambda)$.*

First, we prove some lemmas.

LEMMA 5.2. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and that H^∞ is a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Suppose also that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M},\tau)$ such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then the following hold:*

(i) *$\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{\mathrm{w}^*} \cap L^\alpha(\mathcal{M},\tau)$;*

(ii) *$\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_\alpha$.*

Proof. (i) It is clear that

$$\mathcal{K} \cap \mathcal{M} \subseteq \overline{\mathcal{K} \cap \mathcal{M}}^{\mathrm{w}^*} \cap L^\alpha(\mathcal{M},\tau).$$

We will prove that

$$\mathcal{K} \cap \mathcal{M} = \overline{\mathcal{K} \cap \mathcal{M}}^{\mathrm{w}^*} \cap L^\alpha(\mathcal{M},\tau).$$

Assume, to the contrary, that $\mathcal{K} \cap \mathcal{M} \subsetneqq \overline{\mathcal{K} \cap \mathcal{M}}^{\mathrm{w}^*} \cap L^\alpha(\mathcal{M},\tau)$. Then there exists an $x \in \overline{\mathcal{K} \cap \mathcal{M}}^{\mathrm{w}^*} \cap L^\alpha(\mathcal{M},\tau)$, with $x \notin \mathcal{K} \cap \mathcal{M}$. By the Hahn–Banach theorem, there exists a $\varphi \in L^\alpha(\mathcal{M},\tau)^\#$ such that $\varphi(x) \neq 0$, and $\varphi(y) = 0$ for every $y \in \mathcal{K} \cap \mathcal{M}$.

Since the restriction of τ to $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is semifinite, there exists a family $\{e_\lambda\}$ of projections in $\mathcal{D}$ such that $\tau(e_\lambda) < \infty$ for every λ, and $e_\lambda \to I$ in the weak* topology. This implies that $e_\lambda x \to x$ in the weak* topology and in α-norm by condition (iii)(a) of Definition 3.1.

Thus, there must exist a λ such that $e_\lambda x \notin \mathcal{K} \cap \mathcal{M}$. Also, $e_\lambda x \in e_\lambda L^\alpha(\mathcal{M},\tau)$.

Define $\psi : \mathcal{M} \to \mathbb{C}$ by $\psi(z) = \varphi(e_\lambda z)$ for every $z \in \mathcal{M}$. Then ψ is a bounded linear functional. We will show that ψ is normal, i.e. for an increasing net f_μ of projections in $\mathcal{M}$ such that $f_\mu \to I$ in weak* topology we have $\psi(f_\mu) \to \psi(I)$. By condition (iii)(a) of Defintion 3.1, we get that $\alpha(e_\lambda f_\mu - e_\lambda I) \to 0$, for a fixed λ. Since $\varphi \in L^\alpha(\mathcal{M}, \tau)^\#$, $\varphi(e_\lambda f_\mu) \to \varphi(e_\lambda I)$. However

$$\varphi(e_\lambda f_\mu) = \psi(f_\mu),$$

and $\varphi(e_\lambda I) = \psi(I)$. Thus, $\psi(f_\mu) \to \psi(I)$. Therefore, ψ is a normal, bounded linear functional, namely, $\psi \in L^1(\mathcal{M}, \tau)$.

There exists a $\xi \in L^1(\mathcal{M}, \tau)$ such that $\psi(z) = \tau(z\xi)$ for every $z \in \mathcal{M}$. Note that $\psi(x) = \varphi(e_\lambda x) = \tau(x\xi) \neq 0$. Thus, there exists a projection $e \in \mathcal{D}$ such that $\tau(e) < \infty$ so that $\psi(ex) = \varphi(e_\lambda ex) = \tau(ex\xi) \neq 0$, and $\psi(ey) = \varphi(e_\lambda ey) = \tau(ey\xi) = 0$ for every $y \in \mathcal{K} \cap \mathcal{M}$.

Recall that $x \in \overline{\mathcal{K} \cap \mathcal{M}}^{w^*}$. Therefore, there exists a sequence $\{y_\mu\}$ in $\mathcal{K} \cap \mathcal{M}$ such that $y_\mu \to x$ in the weak* topology. Note that $\xi e \in L^1(\mathcal{M}, \tau)$. Hence,

$$\tau(y_\mu \xi e) \to \tau(x\xi e).$$

However, $\tau(y_\mu \xi e) = 0$, so $\tau(x\xi e) = 0$, which is a contradiction. Therefore (i) is proven.

(ii) Clearly, $\mathcal{K} \cap \mathcal{M} \subseteq \mathcal{K}$, and $\mathcal{K}$ is α-norm closed, so

$$[\mathcal{K} \cap \mathcal{M}]_\alpha \subseteq \mathcal{K}.$$

We will show that

$$\mathcal{K} = [\mathcal{K} \cap \mathcal{M}]_\alpha.$$

Suppose to the contrary, that $[\mathcal{K} \cap \mathcal{M}]_\alpha \subsetneqq \mathcal{K}$. There exists an $x \in \mathcal{K}$ such that $x \notin [\mathcal{K} \cap \mathcal{M}]_\alpha$. We know that $\mathcal{D}$ is semifinite, so there exists a family of projections $\{e_\lambda\}_{\lambda \in \Lambda}$ such that $\tau(e_\lambda) < \infty$, and $e_\lambda \to I$ in the weak* topology. By Definition 3.1, part (iii)(a), $e_\lambda x \to x$ in α-norm. So, there exists λ such that $e_\lambda x \in \mathcal{K}$, since $x \in \mathcal{K}$, and $e_\lambda x \notin [\mathcal{K} \cap \mathcal{M}]_\alpha$, as $x \notin [\mathcal{K} \cap \mathcal{M}]_\alpha$.

By Lemma 4.6, there exist an $h_1 \in e_\lambda H^\infty e_\lambda$ and an $h_2 \in e_\lambda H^\alpha e_\lambda$ such that $h_1 e_\lambda x \in \mathcal{M}$, and $h_1 h_2 = e_\lambda = h_2 h_1$. Thus, $e_\lambda x = h_2 h_1 e_\lambda x$, $h_1 e_\lambda x \in \mathcal{M}$, and $h_1 e_\lambda x \in \mathcal{K}$, since $H^\infty \mathcal{K} \subseteq \mathcal{K}$. Also, $h_2 \in e_\lambda H^\alpha e_\lambda$, so there exists a sequence $\{a_n\}$ in H^∞ such that $a_n \to h_2$ in α-norm. Hence, $e_\lambda x = h_2 h_1 ex$, $a_n h_1 e_\lambda x \in K \cap \mathcal{M}$, and

$$a_n h_1 e_\lambda x \to h_2 h_1 ex$$

in α-norm. Therefore, $e_\lambda x \in [\mathcal{K} \cap \mathcal{M}]_\alpha$, which is a contradiction. Thus, (ii) is proven. ∎

LEMMA 5.3. *Suppose $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ, and suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Assume that $\mathcal{K}$ is a weak* closed subspace of $\mathcal{M}$ such that*

$H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then
$$\mathcal{K} = \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}.$$

Proof. First we must show that
$$\mathcal{K} \subseteq \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}.$$

Let $x \in \mathcal{K} \subseteq \mathcal{M}$. We know that τ restricted to $\mathcal{D}$ is semifinite, so there exists a net of projections $\{e_\lambda\}_{\lambda\in\Lambda}$ such that $\tau(e_\lambda) < \infty$ and $e_\lambda \to I$ in the weak* topology. Also, $e_\lambda x \to x$ in the weak* topology.

To show that
$$x \in \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*},$$
it is sufficient to show that $e_\lambda x \in [\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}$. We have that $e_\lambda x$ is in $\mathcal{K}$, as $x \in \mathcal{K}$ and $\mathcal{K}$ is H^∞-invariant. We also know $\|e_\lambda x\|_\alpha \leqslant \|e_\lambda\|_\alpha \|x\| < \infty$. Therefore, $e_\lambda x \in L^\alpha(\mathcal{M},\tau)$, and $e_\lambda x \in \mathcal{K} \cap L^\alpha(\mathcal{M},\tau) \subseteq [\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha$. Thus, $x \in \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}$. Hence $\mathcal{K} \subseteq \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}$.

Next, we show that
$$\overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*} \subseteq \mathcal{K}.$$
It suffices to show that $[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M} \subseteq \mathcal{K}$ since $\mathcal{K}$ is weak* closed.

Suppose, to the contrary, that $[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M} \subsetneqq \mathcal{K}$. There exists an $x \in [\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}$ such that $x \notin \mathcal{K}$. Since the restriction of τ to $\mathcal{D}$ is semifinite, there exists a net $\{e_\lambda\}_{\lambda\in\Lambda}$ of projections such that $\tau(e_\lambda) \leqslant \infty$ and $e_\lambda x \to x$ in the weak* topology.

As $x \notin \mathcal{K}$, by the Hahn–Banach theorem, there exists a $\varphi \in \mathcal{M}_\#$ such that $\varphi(x) \neq 0$ and $\varphi(y) = 0$ for all y in $\mathcal{K}$. As $x \in [\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}$ and $x \notin \mathcal{K}$, there exists a λ such that $e_\lambda x \in [\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}$ and $e_\lambda x \notin \mathcal{K}$. Since $\varphi \in \mathcal{M}_\#$, there exists a ξ in $L^1(\mathcal{M},\tau)$ such that $\varphi(z) = \tau(z\xi)$ for every $z \in \mathcal{M}$. It follows that there exists a projection $e \in \mathcal{D}$ with $\tau(e) < \infty$ so that $\tau(x\xi e) \neq 0$, and $\tau(y\xi e) = 0$ for every $y \in \mathcal{K}$.

We claim that there exists a $z = \xi e \in \mathcal{M}e$ such that $\tau(xz) \neq 0$ and $\tau(yz) = 0$ for all $y \in \mathcal{K}$.

Note that $\xi e \in L^1(\mathcal{M},\tau)$ since $\xi \in L^1(\mathcal{M},\tau)$ and $\tau(e) < \infty$. By Lemma 4.6, there exist $h_3 \in eH^\infty e$, and $h_4 \in eH^1 e$ such that $h_3h_4 = e = h_4h_3$ and $\xi e h_3 \in \mathcal{M}$. There exists $\{k_n\}$ in H^∞ such that $k_n \to h_4$ in $\|\cdot\|_1$-norm. So,
$$\begin{aligned}\lim_{n\to\infty} |\tau(ex\xi) - \tau(x\xi e h_3 k_n)| &= \lim_{n\to\infty} |\tau(x\xi e h_3 h_4) - \tau(x\xi e h_3 k_n)| \\ &\leqslant \lim_{n\to\infty} \|x\|\|\xi e h_3\|\|h_4 - k_n\|_1 = 0.\end{aligned}$$
There exists an $N \in \mathbb{N}$ such that $\tau(x\xi e h_3 k_N) \neq 0$, since $\tau(x\xi) \neq 0$. We let $z = \xi e h_3 k_N \in \mathcal{M}$. Then, $z = ze \in \mathcal{M}e$ such that $\tau(xz) = \tau(x\xi e h_3 k_N) \neq 0$, and $\tau(yz) = \tau(y\xi e h_3 k_N) = \tau((eh_3k_N)y\xi) = 0$ for every $y \in \mathcal{K}$.

Since $x \in [\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}$ there exists $\{x_n\}$ in $\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)$ such that $x_n \to x$ in α norm, and $ex_n \to ex$ in α-norm. Note $ey = \sqrt{eyy^*e}v =$

$e\sqrt{eyy^*ee}v$. Therefore, $ex_n \to ex$ in $\|\cdot\|_1$-norm, as $\|ey\|_1 = \|e\sqrt{eyy^*ee}\|_1$, $\alpha(ey) = \alpha(e\sqrt{eyy^*ee})$, and α is locally $\|\cdot\|_1$-dominating.

We also have that $|\tau(xz - x_n z)| = |\tau((x - x_n)z)| \leqslant \|e(x_n - x)\|_1\|z\|$. Finally, since $\{x_n\}$ is in $\mathcal{K} \cap L^\alpha(\mathcal{M}, \tau) \subseteq \mathcal{K}$, $\tau(x_n z) = 0$. Hence, $\tau(xz) = 0$, which is a contradiction. Therefore, $\overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*} \subseteq \mathcal{K}$. Thus, $\mathcal{K} = \overline{[\mathcal{K} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}$. ∎

LEMMA 5.4. *Suppose $\mathcal{M}$ is a semifinite von Neumann algebra with a faithful, normal tracial weight τ, and suppose that α is a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually-continuous norm with respect to τ. Let H^∞ be a semifinite, subdiagonal subalgebra of $\mathcal{M}$. Assume that S is a subset of $\mathcal{M}$ such that $H^\infty S \subseteq S$. Then*

$$[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha = [\overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha.$$

Proof. Clearly, $S \cap L^\alpha(\mathcal{M},\tau) \subseteq \overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$ so, $[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha \subseteq [\overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha$.

We will show that $\overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau) \subseteq [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha$. Let $x \in \overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$. We know that there exists a net $\{e_\lambda\}$ in $\mathcal{D}$ of projections such that $\tau(e_\lambda) < \infty$, and $e_\lambda \to I$ in the weak* topology. Thus, $e_\lambda x \to x$ in the weak* topology.

We will show that $e_\lambda x \in [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha$ in order to show that $x \in [S \cap L^\alpha(\mathcal{M},\tau]_\alpha$. By Lemma 5.2, we have that

$$[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M} \subseteq \overline{[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha}^{w^*} \cap L^\alpha(\mathcal{M},\tau).$$

Since $x \in \overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)$, there exists a net $\{x_j\}$ in S such that $x_j \to x$ in the weak* topology. Therefore $e_\lambda x_j \to e_\lambda x$ in the weak* topology for every $\lambda \in \Lambda$. We note that $\alpha(e_\lambda x_j) \leqslant \alpha(e_\lambda)\|x_j\|$, and $H^\infty S \subseteq S$. Therefore $e_\lambda x_j \in S \cap L^\alpha(\mathcal{M},\tau)$, and $e_\lambda x_j \in \overline{[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}$. Thus, $e_\lambda x \in \overline{[S \cap L^\alpha[\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}$. It is clear that $e_\lambda x \in L^\alpha(\mathcal{M},\tau)$. By Lemma 5.2, $\overline{[S \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*} \cap L^\alpha(\mathcal{M},\tau) = [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}$. So $e_\lambda x \in [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha$.

Therefore, $x \in [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha$, whence $\overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau \subseteq [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha$. Hence,

$$[\overline{S}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha = [S \cap L^\alpha(\mathcal{M},\tau)]_\alpha. \quad ∎$$

Now, we prove Theorem 5.1.

Proof of Theorem 5.1. Let $\mathcal{K}_1 = \overline{\mathcal{K} \cap \mathcal{M}}^{w^*}$. $\mathcal{K}_1$ is a weak* closed subspace of $\mathcal{M}$ such that $H^\infty \mathcal{K}_1 \subseteq \mathcal{K}_1$. Then by Theorem 4.5 in [26], there exist a weak* closed subspace $Y_1 \subseteq \mathcal{M}$ and a family $\{u_\lambda\}_{\lambda \in \Lambda}$ of partial isometries in $\mathcal{M}$ such that:

(a) $u_\lambda Y_1^* = 0$ for every $\lambda \in \Lambda$;

(b) $u_\lambda u_\lambda^* \in \mathcal{D}$, and $u_\lambda u_\mu^* = 0$ for every $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$;

(c) $Y_1 = \overline{H_0^\infty Y_1}^{w^*}$;

(d) $\mathcal{K}_1 = Y_1 \bigoplus^{\text{row}} (\bigoplus^{\text{row}}_{\lambda\in\Lambda} H^\infty u_\lambda)$.

Let $Y = [Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha$.

(i) We know that there exists $\{a_n\} \subseteq Y_1^*$ such that $a_n \to a$ in α-norm for some $a \in Y_1^*$. From (a), and the definition of Y_1, $a_n u_i \to a u_i$ in α-norm. Thus, we may conclude that $u_\lambda Y^* = 0$ for every $\lambda \in \Lambda$.

(ii) follows directly from (b).

(iii) We will show that $Y = [H_0^\infty Y]_\alpha$. We have that:

$$
\begin{aligned}
Y &= [Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by definition of } Y)\\
&= [\overline{H_0^\infty Y_1}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by (c))}\\
&= [H_0^\infty Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by Lemma 5.4)}\\
&= [H_0^\infty(\overline{[Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}}^{w^*}) \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by Lemma 5.3)}\\
&\subseteq [\overline{H_0^\infty([Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha \cap \mathcal{M})}^{w^*} \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by Theorem 1.7.8 in [28])}\\
&= [H_0^\infty([Y_1 \cap L^p(\mathcal{M},\tau)]_\alpha \cap \mathcal{M}) \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by Lemma 5.4)}\\
&= [H_0^\infty(Y \cap \mathcal{M}) \cap L^\alpha(\mathcal{M},\tau)]_\alpha \quad \text{(by defintion of } Y)\\
&\subseteq [H_0^\infty Y]_\alpha \subseteq Y.
\end{aligned}
$$

Hence, $Y = [H_0^\infty Y]_\alpha$ as desired.

(iv) Finally, we will show that $\mathcal{K} = Y \bigoplus^{\text{row}} (\bigoplus^{\text{row}}_{\lambda\in\Lambda} H^\alpha u_\lambda)$.
Recall that $Y = [Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha$. We claim that $[H_0^\infty Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha \subseteq [H_0^\infty(Y_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha$. Also, by Lemma 5.2, $H^\alpha u_\lambda = [H^\infty u_\lambda \cap L^\alpha(\mathcal{M},\tau)]_\alpha$ for every $\lambda \in \Lambda$. Now:

$$
\begin{aligned}
\mathcal{K} &= [\mathcal{K}_1 \cap L^\alpha(\mathcal{M},\tau)]_\alpha\\
&= \left[\overline{Y_1 + \sum_{\lambda\in\Lambda} H^\infty u_\lambda}^{w^*} \cap L^\alpha(\mathcal{M},\tau)\right]_\alpha \quad \text{(by definition of } \mathcal{K}_1)\\
&= \left[Y_1 + \sum_{\lambda\in\Lambda} H^\infty u_\lambda \cap L^\alpha(\mathcal{M},\tau)\right]_\alpha \quad \text{(by Lemma 5.4)}\\
&= \left[Y_1 \cap L^\alpha(\mathcal{M},\tau) + \sum_{\lambda\in\Lambda} H^\infty u_\lambda \cap L^\alpha(\mathcal{M},\tau)\right]_\alpha \quad \text{(by (a) and (b))}\\
&= \left[Y + \sum_{\lambda\in\Lambda} H^\alpha u_\lambda\right]_\alpha = Y \bigoplus^{\text{row}} \left(\bigoplus^{\text{row}}_{\lambda\in\Lambda} H^\alpha u_\lambda\right)
\end{aligned}
$$

where the last equality comes from Definition 4.8. ∎

COROLLARY 5.5. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a faithful, normal, semifinite tracial weight τ. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ. Let $\mathcal{K}$ be a subset of L^α such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$. Then there exists a projection q with $\mathcal{K} = \mathcal{M}q$.*

Proof. We note that $\mathcal{M}$ can be considered as a semifinite subdiagonal subalgebra of $\mathcal{M}$ itself. Hence, we let $\mathcal{M} = H^\infty$, and it follows that $\mathcal{D} = \mathcal{M}$ and Φ is the identity map on $\mathcal{M}$. Also, $H_0^\infty = \{0\}$ and $H^\alpha = L^\alpha(\mathcal{M}, \tau)$.

Let $\mathcal{K}$ be a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $\mathcal{M}\mathcal{K} \subseteq \mathcal{K}$. From Theorem 5.1,

$$\mathcal{K} = Y \bigoplus^{\text{row}} \Big(\bigoplus^{\text{row}}_{\lambda \in \Lambda} H^\alpha u_\lambda \Big),$$

where $u_\lambda Y^* = 0$ for every $\lambda \in \Lambda$, $u_\lambda u_\lambda^* \in \mathcal{D}$, and $u_\lambda u_m u^* = 0$ for every $\lambda, \mu \in \Lambda$ such that $\lambda \neq \mu$, and $Y = [H_0^\infty Y]_\alpha$.

It is clear that because $H_0^\infty = \{0\}$, $Y = 0$. Also, since $\mathcal{D} = \mathcal{M}$, we have that

$$H^\alpha u_\lambda = L^\alpha(\mathcal{M}, \tau) u_\lambda = L^\alpha(\mathcal{M}, \tau) u_\lambda u_\lambda^* u_\lambda \subseteq L^\alpha(\mathcal{M}, \tau) u_\lambda^* u_\lambda \subseteq L^\alpha(\mathcal{M}, \tau) u_\lambda = H^\alpha u_\lambda.$$

Therefore, $H^\alpha u_\lambda = L^\alpha(\mathcal{M}, \tau u_\lambda^* u_\lambda)$. Specifically, we find that

$$\mathcal{K} = Y \bigoplus^{\text{row}} \Big(\bigoplus^{\text{row}}_{\lambda \in \Lambda} H^\alpha u_\lambda \Big) = \Big(\bigoplus^{\text{row}}_{\lambda \in \Lambda} L^\alpha(\mathcal{M}, \tau) u_\lambda^* u_\lambda \Big),$$
$$L^\alpha(\mathcal{M}, \tau) \Big(\sum_{\lambda \in \Lambda} u_\lambda^* u_\lambda \Big) = L^\alpha(\mathcal{M}, \tau) q,$$

where we let $\sum_{\lambda \in \Lambda} u_\lambda^* u_\lambda = q$, and q is a projection in $\mathcal{M}$. This ends the proof. ∎

6. APPLICATIONS

6.1. INVARIANT SUBSPACES FOR NON-COMMUTATIVE BANACH FUNCTION SPACES.

We briefly recall our discussion of a non-commutative Banach function space. Let E be a symmetric Banach function space on $(0, \infty)$ with Lebesgue measure. As before, we let $\mathcal{M}$ be a von Neumann algebra with a faithful, normal tracial state τ and $\mathcal{I} = \{x \in \mathcal{M} : x \text{ is a finite rank operator in } (\mathcal{M}, \tau) \text{ and } \|\mu(x)\|_E < \infty\}$. We may then define a Banach function space $\mathcal{I}(\tau)$, and a norm $\|\cdot\|_{\mathcal{I}(\tau)}$ by $\|x\|_{\mathcal{I}(\tau)} = \|\mu(x)\|_{E(0,\infty)}$ for every $x \in \mathcal{I}(\tau)$. We let H^∞ be a semifinite subdiagonal subalgebra of $\mathcal{M}$, as described earlier. The following is an easy corollary of Theorem 5.1 and Proposition 3.10.

COROLLARY 6.1. *Suppose that $\mathcal{I}(\tau)$ is a Banach function space on the diffuse von Neumann algebra $\mathcal{M}$ with order continuous norm $\|\cdot\|_{\mathcal{I}(\tau)}$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $\mathcal{I}(\tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then, there exist a closed subspace Y of $\mathcal{I}(\tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that:

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$;
(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$, *and* $u_\lambda u_\mu^*$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$;
(iii) $Y = [H_0^\infty Y]_\alpha$;
(iv) $\mathcal{K} = Y \bigoplus^{\text{row}} (\bigoplus^{\text{row}}_{\lambda \in \Lambda} H^{\mathcal{I}(\tau)} u_\lambda)$.

6.2. INVARIANT SUBSPACES FOR FACTORS. We also have the following corollary from Theorem 5.1 and Proposition 3.7.

COROLLARY 6.2. *Suppose $\mathcal{M}$ is a factor with a faithful, normal tracial weight τ. Let $\alpha : \mathcal{I} \to [0, \infty)$, where $\mathcal{I}$ is the set of elementary operators in $\mathcal{M}$, be a unitarily invariant norm such that any net $\{e_\lambda\}$ in $\mathcal{M}$ with $e_\lambda \uparrow I$ in the weak* topology implies that $\alpha((e_\lambda - I)x) \to 0$. Let H^∞ be a semifinite subdiagonal subalgebra of $L^\alpha(\mathcal{M}, \tau)$. Let $\mathcal{D} = H^\infty \cap (H^\infty)^*$. Assume that $\mathcal{K}$ is a closed subspace of $L^\alpha(\mathcal{M}, \tau)$ such that $H^\infty \mathcal{K} \subseteq \mathcal{K}$.*

Then, there exist a closed subspace Y of $L^\alpha(\mathcal{M}, \tau)$ and a family $\{u_\lambda\}$ of partial isometries in $\mathcal{M}$ such that:

(i) $u_\lambda Y^* = 0$ *for every* $\lambda \in \Lambda$;

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$, *and* $u_\lambda u_\mu^*$ *for every* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$;

(iii) $Y = [H_0^\infty Y]_\alpha$;

(iv) $\mathcal{K} = Y \bigoplus^{\mathrm{row}} (\bigoplus^{\mathrm{row}}_{\lambda \in \Lambda} H^\alpha u_\lambda)$.

6.3. INVARIANT SUBSPACES OF ANALYTIC CROSSED PRODUCTS. Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful normal tracial state τ. We let β be a $*$-automorphism of $\mathcal{M}$ such that $\tau(\beta(x)) = \tau(x)$ for every $x \in \mathcal{M}^+$ (i.e. β is trace-preserving).

Let $l^2(\mathbb{Z})$ denote the Hilbert space which consists of the complex-valued functions f on $\mathbb{Z}$ which satisfy $\sum\limits_{m \in \mathbb{Z}} |f(m)|^2 < \infty$. Let $\{e_n\}_{n \in \mathcal{Z}}$ be the orthonormal basis of $l^2(\mathbb{Z})$ such that $e_n(m) = \delta(n, m)$. We also denote the left regular representation of $\mathcal{Z}$ on $l^2(\mathbb{Z})$ by $\lambda : \mathbb{Z} \to B(l^2(\mathbb{Z}))$, where $\lambda(n)(e_m) = e_{m+n}$.

We let $\mathcal{H} = L^2(\mathcal{M}, \tau) \otimes l^2(\mathbb{Z})$, or equivalently, $H = \bigoplus\limits_{m \in \mathbb{Z}} L^2(\mathcal{M}, \tau) \otimes e_m$. The representations Ψ of $\mathcal{M}$ and Λ of $\mathbb{Z}$ may be defined by:

$$\Psi(x)(\xi \otimes e_m) = (\beta^{-m}\xi) \otimes e_m \quad \text{for all } x \in \mathcal{M}, \xi \in L^2(\mathcal{M}, \tau) \text{ and } m \in \mathbb{Z},$$
$$\Lambda(n)(\xi \otimes e_m) = \xi \times (\lambda(n) e_m) \quad \text{for all } n, m \in \mathbb{Z}.$$

It is not hard to verify that

$$\Lambda(n)\Psi(x)\Lambda(-n) = \Psi(\beta^n(x)) \quad \text{for all } x \in \mathcal{M} \text{ and } n \in \mathbb{Z}.$$

We may define the crossed product of $\mathcal{M}$ by an action β, which we denote by $\mathcal{M} \rtimes_\beta \mathbb{Z}$, to be the von Neumann algebra generated by $\Psi(\mathcal{M})$ and $\Lambda(\mathbb{Z})$ in $B(\mathcal{H})$. When there is no possibility of confusion, we will identify $\mathcal{M}$ with its image $\Psi(\mathcal{M})$ under Ψ in $\mathcal{M} \rtimes_\beta \mathbb{Z}$.

In Chapter 13 of [16], amongst others, it is shown that there exists a faithful, normal conditional expectation, Φ, taking $\mathcal{M} \rtimes_\beta \mathbb{Z}$ onto $\mathcal{M}$ such that

$$\Phi\Big(\sum_{n=-N}^{N} \Lambda(n)\Psi(x_n)\Big) = x_0 \quad \text{where } x_n \in \mathcal{M} \text{ for every } -N \leqslant n \leqslant N.$$

There also exists a semifinite, normal, extended tracial weight on $\mathcal{M} \rtimes_\beta \mathbb{Z}$, which we still denote by τ, and which satisfies

$$\tau(y) = \tau(\Phi(y)), \quad \text{for every postive } y \in \mathcal{M} \rtimes_\beta \mathbb{Z}.$$

EXAMPLE 6.3. Let $\mathcal{M} = l^\infty(\mathbb{Z})$. Then $\mathcal{M}$ is an abelian von Neumann algebra with a semifinite, faithful, normal tracial weight, τ which is given by

$$\tau(f) = \sum_{m \in \mathcal{Z}} f(m), \quad \text{for every positive } f \in l^\infty(\mathbb{Z}).$$

We let β be an action on $l^\infty(\mathcal{Z})$, which we define by

$$\beta(f)(m) = f(m-1), \quad \text{for every } f \in l^\infty(\mathbb{Z}) \text{ and } m \in \mathbb{Z}.$$

It is known (see, for example Proposition 8.6.4 of [16]) that $l^\infty(\mathbb{Z}) \rtimes_\beta \mathbb{Z}$ is a type I_∞ factor. Therefore, for some separable Hilbert space $\mathcal{H}$, $l^\infty(\mathbb{Z}) \rtimes_\beta \mathbb{Z} \simeq B(\mathcal{H})$.

The next result follows from our construction of crossed products. (See also section 3 of [1].)

LEMMA 6.4. *Consider the weak* closed, non-self-adjoint subalgebra $\mathcal{M} \rtimes_\beta \mathbb{Z}_+$ of $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which is generated by*

$$\{\Lambda(n)\Psi(x) : x \in \mathcal{M}, n \geqslant 0\}.$$

Then the following hold:

(i) *$\mathcal{M} \rtimes_\beta \mathbb{Z}_+$ is a semifinite subdiagonal subalgebra with respect to $(\mathcal{M} \rtimes_\beta \mathbb{Z}, \Phi)$. We will denote such a semifinite subdiagonal subalgebra by H^∞ and call H^∞ an analytic crossed product.*

(ii) *We denote by H_0^∞ the space* $\ker(\Phi) \cap H^\infty$. *Then H_0^∞ is a weak* closed nonself-adjoint subalgebra which is generated in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ by*

$$\{\Lambda(n)\Phi(x) : x \in \mathcal{M}, n \geqslant 0\}$$

and satisfies

$$H_0^\infty = \Lambda(1)H^\infty.$$

(iii) *$H^\infty \cap (H^\infty)^* = \mathcal{M}$.*

We are able to characterize the invariant subspaces of a crossed product of a semifinite von Neumann algebra $\mathcal{M}$ by a trace-preserving action β.

COROLLARY 6.5. *Suppose that $\mathcal{M}$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ. Let α be a unitarily invariant, locally $\|\cdot\|_1$-dominating, mutually continuous norm with respect to τ, and β be a trace-preserving, $*$-automorphism of $\mathcal{M}$. Consider the crossed product of $\mathcal{M}$ by an action β, $\mathcal{M} \rtimes_\beta \mathbb{Z}$. Still denote the semifinite, faithful, normal, extended tracial weight on $\mathcal{M} \rtimes_\beta \mathbb{Z}$ by τ.*

Denote by H^∞ the weak closed nonself-adjoint subalgebra in $\mathcal{M} \rtimes_\beta \mathbb{Z}$ which is generated by $\{\Lambda(n)\Psi(x) : x \in \mathcal{M}, n \geqslant 0\}$. Then H^∞ is a semifinite subdiagonal sublagebra of $\rtimes_\beta \mathbb{Z}$.*

Let $\mathcal{K}$ be a closed subspace of $L^{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z}, \tau)$ such that $H^{\infty}\mathcal{K} \subseteq \mathcal{K}$. Then there exist a projection q in $\mathcal{M}$ and a family $\{u_{\lambda}\}_{\lambda\in\Lambda}$ of partial isometries in $\mathcal{M} \rtimes_{\beta} \mathbb{Z}$ which satisfy:

(i) $u_{\lambda}q = 0$ *for all* $\lambda \in \Lambda$;

(ii) $u_{\lambda}u_{\lambda}^{*} \in \mathcal{M}$ *and* $u_{\lambda}u_{\mu}^{*} = 0$ *for all* $\lambda, \mu \in \Lambda$ *with* $\lambda \neq \mu$;

(iii) $\mathcal{K} = (L_{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z})q) \bigotimes^{\text{row}} (\bigotimes_{\lambda\in\Lambda}^{\text{row}} H^{\alpha}u_{\lambda})$.

Proof. From Theorem 5.1, we know that

$$K = Y \bigoplus^{\text{row}} \left(\bigoplus_{\lambda\in\Lambda}^{\text{row}} H^{\alpha}u_{\lambda} \right)$$

such that Y is a closed subspace of $\mathcal{M} \rtimes_{\beta} \mathbb{Z}$ and a family of partial isometries, $\{u_{\lambda}\}$, in $\mathcal{M} \rtimes_{\beta} \mathbb{Z}$ which satisfy:

(a) $u_{\lambda}Y^{*} = 0$ for all $\lambda \in \Lambda$;

(b) $u_{\lambda}u_{\lambda}^{*} \in \mathcal{M}$ and $u_{\lambda}u_{\mu}^{*} = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(c) $Y = [H_0^{\infty}Y]_{\alpha}$.

By Lemma 6.4 and (c), it is clear that

$$Y = [H_0^{\infty}Y]_{\alpha} = [\Lambda(1)H^{\infty}Y]_{\alpha} \subseteq \Lambda(1)Y.$$

We can show, by induction, that $\Lambda(-n)Y \subseteq Y$ for any n in $\mathbb{N}$. From the defintion of H^{∞}, we know that $\Lambda(n)Y \subset Y$ for every $n \geqslant 0$, and $\psi(x)Y \subseteq Y$ for every $x \in \mathcal{M}$. Therefore, $Y \subseteq L^{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z})$ is left $\mathcal{M} \rtimes_{\beta} \mathbb{Z}$-invariant, and from Corollary 5.5, there exists a projection $q \in \mathcal{M}$ with $Y = L^{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z}, \tau)q$. Therefore,

(i) $u_{\lambda}q = 0$ for all $\lambda \in \Lambda$;

(ii) $u_{\lambda}u_{\lambda}^{*} \in \mathcal{M}$ and $u_{\lambda}u_{\mu}^{*} = 0$ for all $\lambda, \mu \in \Lambda$ with $\lambda \neq \mu$;

(iii) $\mathcal{K} = (L_{\alpha}(\mathcal{M} \rtimes_{\beta} \mathbb{Z})q) \bigotimes^{\text{row}} (\bigotimes_{\lambda\in\Lambda}^{\text{row}} H^{\alpha}u_{\lambda})$

hold, and the corollary is proven. ∎

6.4. INVARIANT SUBSPACES FOR $B(\mathcal{H})$. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space with orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$. We let τ =Tr be the usual trace on $B(\mathcal{H})$, namely

$$\tau(x) = \sum_{m\in\mathcal{Z}} \langle xe_m, e_m \rangle \quad \text{for every } x \in B(\mathcal{H}) \text{ with } x > 0.$$

With this τ, $B(\mathcal{H})$ is a von Neumann algebra with a semifinite, faithful, normal tracial weight τ.

We let

$$\mathcal{A} = \{x \in B(\mathcal{H}) : \langle xe_m, e_n \rangle = 0 \ \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$.

Recall from Example 6.3 that the crossed product of $l^{\infty}(\mathbb{Z})$ by an action β, denoted $l^{\infty}(\mathbb{Z}) \rtimes_{\beta} \mathbb{Z}$, where the action β is determined by

$$\beta(f)(m) = f(m-1) \quad \text{for every } f \in l^{\infty}(\mathbb{Z}), m \in \mathbb{Z}$$

is another way to realize $B(\mathcal{H})$.

It is easy to see that $\mathcal{A}$ is $l^\infty(\mathbb{Z}) \rtimes_\beta \mathbb{Z}_+$, a semifinite subdiagonal subalgebra of $l^\infty(\mathbb{Z} \rtimes_\beta \mathbb{Z})$ (see Lemma 6.4).

The following corollary follows from (6.5).

COROLLARY 6.6. *Suppose $\mathcal{H}$ is a separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$, and let*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0,\ \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is the diagonal subalgebra of $B(\mathcal{H})$. Suppose $\alpha : \mathcal{I} \to [0,\infty)$, where $\mathcal{I}$ is the set of elementary operators in $\mathcal{M}$, is an unitarily invariant norm such that any net $\{e_\lambda\}$ in $\mathcal{M}$ with $e_\lambda \uparrow I$ in the weak topology implies that $\alpha((e_\lambda - I)x) \to 0$.*

Assume that $\mathcal{K}$ is a closed subspace of H^α such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then there exists a projection q in $\mathcal{D}$ and $\{u_\lambda\}_{\lambda\in\Lambda}$, a family of partial isometries in H^∞ which satisfy:

(i) $u_\lambda q = 0$ *for every* $\lambda \in \Lambda$;

(ii) $u_\lambda u_\lambda^* \in \mathcal{D}$, *and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \mathcal{D}$ *with* $\lambda \neq \mu$;

(iii) $\mathcal{K} = (B(\mathcal{H})q) \bigoplus^{\mathrm{row}} (\bigoplus^{\mathrm{row}}_{\lambda\in\Lambda} H^\alpha u_\lambda)$.

The following is a corollary of Theorem 6.5 and Proposition 3.7.

COROLLARY 6.7. *Suppose $\mathcal{H}$ is a separable Hilbert space with an orthonormal base $\{e_m\}_{m\in\mathbb{Z}}$, and let*

$$H^\infty = \{x \in B(\mathcal{H}) : \langle xe_m, e_n\rangle = 0,\ \forall n < m\}$$

be the lower triangular subalgebra of $B(\mathcal{H})$. Then $\mathcal{D} = H^\infty \cap (H^\infty)^*$ is the diagonal subalgebra of $B(\mathcal{H})$. Suppose $\alpha : \mathcal{I} \to [0,\infty)$, where $\mathcal{I}$ is the set of elementary operators in $\mathcal{M}$, is an unitarily invariant norm such that any net $\{e_\lambda\}$ in $\mathcal{M}$ with $e_\lambda \uparrow I$ in the weak topology implies that $\alpha((e_\lambda - I)x) \to 0$.*

Assume that $\mathcal{K}$ is a closed subspace of H^α such that $H^\infty\mathcal{K} \subseteq \mathcal{K}$. Then there exists $\{u_\lambda\}_{\lambda\in\Lambda}$, a family of partial isometries in H^∞ which satisfy:

(i) $u_\lambda u_\lambda^* \in \mathcal{D}$ *and* $u_\lambda u_\mu^* = 0$ *for every* $\lambda, \mu \in \Lambda$ *such that* $\lambda \neq \mu$;

(ii) $\mathcal{K} = \bigoplus\limits_{\lambda\in\Lambda}{}^{\mathrm{row}} H^\alpha u_\lambda$.

REMARK 6.8. The result is similar when H^∞ is instead the upper triangular subalgebra of $B(\mathcal{H})$.

REMARK 6.9. Recall that any unitarily invariant norm α gives rise to a symmetric gauge norm Ψ on the spectrum of $|A|$, $\{a_n\}_{1\leqslant n\leqslant N}$, where A is a finite rank operator. Then Corollary 6.7 holds for Ψ.

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