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# AROUND THE CLOSURES OF THE SET OF COMMUTATORS AND THE SET OF DIFFERENCES OF IDEMPOTENT ELEMENTS OF $\mathcal{B}(\mathcal{H})$

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ABSTRACT. We describe the norm-closures of the set $\mathfrak{C}_{\mathfrak{E}}$ of commutators of idempotent operators and the set $\mathfrak{E} - \mathfrak{E}$ of differences of idempotent operators acting on a finite-dimensional complex Hilbert space, as well as characterise the intersection of the closures of these sets with the set $\mathcal{K}(\mathcal{H})$ of compact operators acting on an infinite-dimensional complex separable Hilbert space $\mathcal{H}$. Finally, we characterise the closures of the set $\mathfrak{C}_{\mathfrak{P}}$ of commutators of orthogonal projections and the set $\mathfrak{P} - \mathfrak{P}$ of differences of orthogonal projections acting on a complex separable Hilbert space.

KEYWORDS: *Commutators, differences, idempotents, projections, closures.*

MSC (2020): 47B47, 47A58.

## 1. INTRODUCTION

1.1. Let $\mathcal{H}$ be a complex separable Hilbert space. By $\mathcal{B}(\mathcal{H})$, we denote the norm-closed algebra of all bounded linear operators acting on $\mathcal{H}$. There are surprisingly few well-understood classes of continuous linear operators acting on $\mathcal{H}$, but amongst the best understood of these is the class of orthogonal projections. Recall that an operator $P \in \mathcal{B}(\mathcal{H})$ is said to be an *orthogonal projection* if $P = P^* = P^2$.

The three most important notions of equivalence of Hilbert space operators are *similarity, unitary equivalence* and *approximate unitary equivalence*. Given $A, B \in \mathcal{B}(\mathcal{H})$, we shall write $A \sim B$ to indicate that $A$ is similar to $B$; i.e. there exists $S \in \mathcal{B}(\mathcal{H})$ invertible such that $B = S^{-1}AS$. We write $A \simeq B$ to indicate that $A$ is unitarily equivalent to $B$; i.e. there exists a unitary operator $U \in \mathcal{B}(\mathcal{H})$ such that $B = U^*AU$. Finally, we write $A \simeq_{\mathrm{a}} B$ to indicate that $A$ is approximately unitarily equivalent to $B$, meaning that there exists a sequence $(U_n)_n$ of unitary operators in $\mathcal{B}(\mathcal{H})$ such that $B = \lim_n U_n^* A U_n$. This is equivalent to saying that the (norm) closure $\mathrm{CLOS}\,\mathcal{U}(A)$ of the unitary orbit $\mathcal{U}(A) := \{U^*AU : U \in \mathcal{B}(\mathcal{H}) \text{ unitary}\}$ of $A$ coincides with $\mathrm{CLOS}\,\mathcal{U}(B)$.

It is easy to verify that an operator $Q \in \mathcal{B}(\mathcal{H})$ is approximately unitarily equivalent to a projection $P$ if and only if $Q$ is unitarily equivalent to $P$, in which case $Q$ is itself a projection. Furthermore, it is a standard exercise in operator theory to show that an operator $E \in \mathcal{B}(\mathcal{H})$ is similar to *some* projection if and only if $E$ is *idempotent*; that is, $E^2 = E$.

There exists a substantial literature centred around the characterisation of specific linear and/or multiplicative combinations of projections and idempotents in $\mathcal{B}(\mathcal{H})$, and indeed in other $C^*$-algebras [1, 3, 10, 11, 16, 18, 19, 22, 23, 24, 25, 26, 27, 29, 30].

We shall focus on two particular instances of this problem, namely: *commutators* and *differences*. More specifically, our interest will lie in describing the *norm-closures* of the sets described below.

1.2. NOTATION. Given a Hilbert space $\mathcal{H}$, we define $\mathfrak{E} := \{E \in \mathcal{B}(\mathcal{H}) : E = E^2\}$, $\mathfrak{P} := \{P \in \mathcal{B}(\mathcal{H}) : P = P^* = P^2\}$, and we set

$$\mathfrak{E} - \mathfrak{E} := \{E - F : E, F \in \mathfrak{E}\} \quad \text{and} \quad \mathfrak{P} - \mathfrak{P} := \{P - Q : P, Q \in \mathfrak{P}\}.$$

Following [13], we refer to differences of idempotents as DOIs, and in light of this we refer to differences of projections as DOPs.

Given $A, B \in \mathcal{B}(\mathcal{H})$, we denote by $[A, B] := AB - BA$ the commutator of $A$ and $B$. We also define the sets

$$\mathfrak{C}_{\mathfrak{E}} := \{[E, F] : E, F \in \mathfrak{E}\} \quad \text{and} \quad \mathfrak{C}_{\mathfrak{P}} := \{[P, Q] : P, Q \in \mathfrak{P}\}.$$

The elements of $\mathfrak{C}_{\mathfrak{E}}$ are commutators of idempotents (we shall refer to them as COIs), and we shall refer to elements of $\mathfrak{C}_{\mathfrak{P}}$ as COPs, short for *commutators of projections*.

Finally (keeping in mind that an *involution* is an invertible operator $S \in \mathcal{B}(\mathcal{H})$ such that $S = S^{-1}$), we shall write

$$\begin{aligned} \mathrm{NEG}_{\mathrm{S}}(\mathcal{H}) &:= \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is similar to } -T\}; \\ \mathrm{NEG}_{\mathrm{U}}(\mathcal{H}) &:= \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is unitarily equivalent to } -T\}; \quad \text{and} \\ \mathrm{NEG}_{\mathrm{INVS}}(\mathcal{H}) &:= \{T \in \mathcal{B}(\mathcal{H}) : T \text{ is similar by an involution to } -T\}. \end{aligned}$$

Obviously

$$\mathrm{NEG}_{\mathrm{U}}(\mathcal{H}) \subseteq \mathrm{NEG}_{\mathrm{S}}(\mathcal{H}) \quad \text{and} \quad \mathrm{NEG}_{\mathrm{INVS}}(\mathcal{H}) \subseteq \mathrm{NEG}_{\mathrm{S}}(\mathcal{H}).$$

1.3. Our first goal will be to classify the norm-closures of the sets $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{E} - \mathfrak{E}$. We note that in the case where the underlying Hilbert space is finite-dimensional, the (non-closed) sets themselves have been classified.

Indeed, when dim $\mathcal{H}$ is finite, the characterisation of $\mathfrak{C}_{\mathfrak{E}}$ is due to Drnovšek et al. [8, Theorem 8].

THEOREM 1.1 (Drnovšek, Radjavi and Rosenthal). *If $n := \dim \mathcal{H} < \infty$ and $T \in \mathcal{B}(\mathbb{C}^n)$, then $T \in \mathfrak{C}_{\mathfrak{E}}$ if and only if $T \sim -T$ and the Riesz component $T_1$ of $T$ corresponding to $\left\{\frac{1}{2}\mathrm{i}\right\}$ has the property that $T_1^2 + \frac{1}{4}I$ has a square root.*

The corresponding result for $\mathfrak{E} - \mathfrak{E}$ is due to Hartwig and Putcha [13, Theorem 1b].

THEOREM 1.2 (Hartwig and Putcha). *If* $n := \dim \mathcal{H} < \infty$ *and* $T \in \mathcal{B}(\mathbb{C}^n)$, *then* $T \in \mathfrak{E} - \mathfrak{E}$ *if and only if the elementary divisors (see, e.g.* [15]*) of* $T$ *satisfy the following three conditions:*

(i) *there are no restrictions on the elementary divisors* $z^k$*;*

(ii) *the elementary divisors* $(z-\alpha)^k$, $(z+\alpha)^k$ *with* $\alpha \neq 0, \pm 1$ *occur in pairs with the same multiplicities; and*

(iii) *the elementary divisors* $(z-1)^{m_k}$, $(z+1)^{n_k}$, $k = 1, 2, \ldots, r$ *obey* $|m_k - n_k| \leqslant 1$ *when listed in non-increasing order.*

A complete characterisation of the sets $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{E} - \mathfrak{E}$ in the case where $\dim \mathcal{H} = \infty$ is not yet available, though the paper of Wang and Wu [28] has many interesting partial results. The problem of characterising $\mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$ and $\mathrm{CLOS}\,(\mathfrak{E} - \mathfrak{E})$ in this setting seem quite delicate. To wit: although all nilpotent operators of order two are known to lie in $\mathfrak{C}_{\mathfrak{E}}$, it is not known which nilpotent operators of order three are commutators of idempotents.

1.4. Our study of the classes $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{E} - \mathfrak{E}$ will require us to understand the spectrum of an operator which is similar to its own negative. Recall that the *semi-Fredholm domain* of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set

$$\rho_{\mathrm{sF}}(T) := \{\alpha \in \mathbb{C} : \mathrm{ran}\,(T - \alpha I) \text{ is closed}, \min(\mathrm{NUL}(T-\alpha I), \mathrm{NUL}(T-\alpha I)^*) < \infty\}.$$

A standard result (see, e.g. [4]) shows that $\alpha \in \rho_{\mathrm{sF}}(T)$ if and only if $\pi(T - \alpha I)$ is either left- or right-invertible in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, where $\mathcal{K}(\mathcal{H})$ denotes the closed, two-sided ideal of compact operators acting on $\mathcal{H}$, and $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical quotient map. When $\alpha \in \rho_{\mathrm{sF}}(T)$, one defines the *semi-Fredholm index*

$$\mathrm{IND}\,(T - \alpha I) = \mathrm{NUL}\,(T - \alpha I) - \mathrm{NUL}\,(T - \alpha I)^*.$$

Note that if $T \sim -T$, say $-T = R^{-1}TR$ for some invertible operator $R$, then $\alpha \in \rho_{\mathrm{sF}}(T)$ implies that

$$R^{-1}(T - \alpha I)R = (-T - \alpha I) = -(T + \alpha I),$$

so that $T + \alpha I$ is semi-Fredholm and

$$\mathrm{IND}\,(T + \alpha I) = \mathrm{IND}\,(-(T + \alpha I)) = \mathrm{IND}\,(T - \alpha I).$$

In light of the above observations, the following definition will prove useful.

DEFINITION 1.3. Let $T \in \mathcal{B}(\mathcal{H})$. We say that $T$ is *balanced* if

(i) $\sigma(T) = \sigma(-T)$;

(ii) whenever $\Omega_1, \Omega_2 \subseteq \mathbb{C}$ are disjoint open sets such that $\sigma(T) \subseteq \Omega_1 \cup \Omega_2$, then

$$\dim \mathcal{H}(\Omega_1; T) = \dim \mathcal{H}(-\Omega_1; T),$$

where $\mathcal{H}(\Omega_1; T)$ is the generalised eigenspace (i.e. the range of the corresponding Riesz idempotent $E(\Omega_1; T)$) corresponding to $\sigma(T) \cap \Omega_1$;

(iii) If $\alpha \in \mathbb{C}$, then $T - \alpha I$ is semi-Fredholm if and only if $T + \alpha I$ is semi-Fredholm, in which case

$$\text{IND}\,(T - \alpha I) = \text{IND}\,(T + \alpha I).$$

We denote the set of balanced operators by $\text{BAL}(\mathcal{H})$.

## 2. ELEMENTARY AND GENERAL RESULTS

2.1. Our ultimate goal would be to describe the relationships between each of the classes of operators defined above, as well as their norm-closures in $\mathcal{B}(\mathcal{H})$. In the case where $\dim \mathcal{H} = \infty$, our incomplete understanding of the sets $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{E} - \mathfrak{E}$ themselves complicates matters. For this reason, when considering the closures of $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{E} - \mathfrak{E}$, in this paper we shall focus mostly on two cases. First, we shall direct our attention to the case where $\dim \mathcal{H} < \infty$, where the sets $\mathfrak{C}_{\mathfrak{E}}$ and $\mathfrak{E} - \mathfrak{E}$ are fully understood. Next, we turn to a description of the sets $\text{CLOS}(\mathfrak{C}_{\mathfrak{E}}) \cap \mathcal{K}(\mathcal{H})$ and $\text{CLOS}\,(\mathfrak{E} - \mathfrak{E}) \cap \mathcal{K}(\mathcal{H})$.

Because the sets $\mathfrak{C}_{\mathfrak{P}}$ and $\mathfrak{P} - \mathfrak{P}$ are fully understood independently of the dimension of the underlying Hilbert space, in Section 5 we shall be able to characterise their closures.

We begin with a couple of general and elementary observations concerning the classes $\mathfrak{C}_{\mathfrak{E}}$ and $\text{BAL}(\mathcal{H})$ which will be used throughout the paper.

PROPOSITION 2.1. *For any Hilbert space $\mathcal{H}$,*

$$\mathfrak{C}_{\mathfrak{E}} \subseteq \text{NEG}_{\text{INVS}}(\mathcal{H}) \subseteq \text{NEG}_{\text{S}}(\mathcal{H}) \subseteq \text{BAL}(\mathcal{H}).$$

*Proof.* It is routine to verify that all four sets above are invariant under similarity, and this will be used implicitly below.

The proof that $\mathfrak{C}_{\mathfrak{E}} \subseteq \text{NEG}_{\text{INVS}}$ is an immediate consequence of Theorem 1 in the paper [8], while the inclusion $\text{NEG}_{\text{INVS}}(\mathcal{H}) \subseteq \text{NEG}_{\text{S}}(\mathcal{H})$ is trivial.

If $T \in \text{NEG}_{\text{S}}(\mathcal{H})$ and $-T = S^{-1}TS$, then clearly $\sigma(T) = \sigma(S^{-1}TS) = \sigma(-T)$,

$$\dim\,\mathcal{H}(\Omega_1; T) = \dim\,\mathcal{H}(\Omega_1; S^{-1}TS) = \dim\,\mathcal{H}(\Omega_1; -T) = \dim\,\mathcal{H}(-\Omega_1; T),$$

and for $\alpha \in \rho_{\text{sF}}(T)$,

$$\text{IND}\,(T - \alpha I) = \text{IND}\,S^{-1}(T - \alpha I)S = \text{IND}\,(-T - \alpha I) = \text{IND}\,(T + \alpha I),$$

so that $T \in \text{BAL}(\mathcal{H})$. ∎

The following observation can easily be derived from the above result and Lemma 1.1 of the paper of Wang and Wu [28]. We include an alternate proof since it is so short.

PROPOSITION 2.2. *For any Hilbert space $\mathcal{H}$,*

$$\mathfrak{C}_{\mathfrak{E}} \subseteq \mathfrak{E} - \mathfrak{E}.$$

*Proof.* Since both of these sets are invariant under similarity, it suffices to consider $T \in \mathfrak{C}_{\mathfrak{E}}$ of the form $T = [P, F]$, where $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ is an orthogonal projection, and $F = \begin{bmatrix} F_1 & F_2 \\ F_3 & F_4 \end{bmatrix}$ relative to this decomposition.

Then

$$T = [P, F] = \begin{bmatrix} 0 & F_2 \\ -F_3 & 0 \end{bmatrix} = \begin{bmatrix} I & F_2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ F_3 & 0 \end{bmatrix} \in \mathfrak{E} - \mathfrak{E}. \quad ∎$$

2.2. We finish this section with some easy observations:

(i) $\mathfrak{C}_{\mathfrak{E}}$ is self-adjoint: note that $[E, F]^* = (EF - FE)^* = F^*E^* - E^*F^* = [F^*, E^*]$ and $E^*, F^*$ are idempotents when $E, F$ are.

(ii) From this it follows that CLOS $(\mathfrak{C}_{\mathfrak{E}})$ is also self-adjoint.

(iii) If $T \in \mathfrak{C}_{\mathfrak{P}}$, then $H := \mathrm{i}T$ is self-adjoint, so that CLOS $(\mathrm{i}\mathfrak{C}_{\mathfrak{P}})$ is also contained in the set $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$ of self-adjoint operators on $\mathcal{H}$.

(iv) The set $\mathfrak{P} - \mathfrak{P}$ and its closure CLOS $(\mathfrak{P} - \mathfrak{P})$ are contained in $\mathcal{B}(\mathcal{H})_{\mathrm{sa}}$.

## 3. THE CLOSURES OF $\mathfrak{C}_{\mathfrak{E}}$ AND OF $\mathfrak{E} - \mathfrak{E}$ IN THE FINITE-DIMENSIONAL SETTING

3.1. We now turn our attention to the case where $n := \dim \mathcal{H} < \infty$, and concentrate on the problem of describing the closures of the set $\mathfrak{C}_{\mathfrak{E}}$ of commutators of idempotent operators and the set $\mathfrak{E} - \mathfrak{E}$ of differences of idempotent operators in $\mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$.

PROPOSITION 3.1. *If* $\dim \mathcal{H} < \infty$, *then* $\mathrm{NEG}_{\mathrm{U}}(\mathcal{H})$ *is closed.*

*Proof.* Suppose that $(T_m)_m$ is a sequence in $\mathrm{NEG}_{\mathrm{U}}(\mathcal{H})$, and that $T = \lim\limits_m T_m$. Choose $U_m$ unitary such that $-T_m = U_m^* T_m U_m$. Since $(U_m)_m$ is bounded, there exists a subsequence $(U_{m_j})_j$ which converges in norm to a (necessarily unitary operator) $V \in \mathcal{B}(\mathcal{H})$.

Then

$$V^*TV = \lim_j U_{m_j}^* T_{m_j} U_{m_j} = \lim_j -T_{m_j} = -T,$$

so that $T \in \mathrm{NEG}_{\mathrm{U}}(\mathcal{H})$. ∎

NOTATION 3.2. If $\dim \mathcal{H} < \infty$, then given an operator $T \in \mathcal{B}(\mathcal{H})$ and an eigenvalue $\alpha \in \sigma(T)$, we denote by $\mu(\alpha)$ the algebraic multiplicity of $\alpha$.

PROPOSITION 3.3. *If* $\dim \mathcal{H} = n < \infty$, *then*

$$\mathrm{BAL}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \alpha \in \sigma(T) \textit{ implies } -\alpha \in \sigma(T) \textit{ and } \mu(\alpha) = \mu(-\alpha)\},$$

*and thus* $\mathrm{BAL}(\mathcal{H})$ *is closed.*

*Proof.* Let $T \in \text{BAL}(\mathcal{H})$. Since $\sigma(T) = \sigma(-T)$ by definition of $\text{BAL}(\mathcal{H})$, it follows that $\alpha \in \sigma(T)$ implies that $-\alpha \in \sigma(T)$. Also, taking $\Omega_\varepsilon = \{z \in \mathbb{C} : |z - \alpha| < \varepsilon\}$ for sufficiently small $\varepsilon > 0$ (to ensure that $\Omega_\varepsilon \cap \sigma(T) = \{\alpha\}$), condition (ii) from Definition 1.3 implies that $\mu(\alpha) = \mu(-\alpha)$. Thus

$$\text{BAL}(\mathcal{H}) \subseteq \{T \in \mathcal{B}(\mathcal{H}) : \alpha \in \sigma(T) \text{ implies } -\alpha \in \sigma(T) \text{ and } \mu(\alpha) = \mu(-\alpha)\}.$$

Conversely, the condition that $\alpha \in \sigma(T)$ implies $-\alpha \in \sigma(T)$ is equivalent to the statement that $\sigma(T) = \sigma(-T)$, and if $\Omega \subseteq \mathbb{C}$ is any open set which non-trivially intersects $\sigma(T)$, then there exist $1 \leqslant \kappa \leqslant n$ and $\alpha_1, \alpha_2, \ldots, \alpha_\kappa \in \sigma(T)$ such that $\Omega \cap \sigma(T) = \{\alpha_1, \alpha_2, \ldots, \alpha_\kappa\}$. Thus

$$\dim\, \mathcal{H}(\Omega; T) = \sum_{j=1}^{\kappa} \mu(\alpha_j) = \sum_{j=1}^{\kappa} \mu(-\alpha_j) = \dim\, \mathcal{H}(-\Omega; T).$$

Note that condition (iii) from Definition 1.3 always holds in the finite-dimensional setting. Thus

$$\{T \in \mathcal{B}(\mathcal{H}) : \alpha \in \sigma(T) \text{ implies } -\alpha \in \sigma(T) \text{ and } \mu(\alpha) = \mu(-\alpha)\} \subseteq \text{BAL}(\mathcal{H}),$$

so that equality of these two sets holds.

Suppose that $(T_m)_m$ is a sequence in

$$\{T \in \mathcal{B}(\mathcal{H}) : \alpha \in \sigma(T) \text{ implies } -\alpha \in \sigma(T) \text{ and } \mu(\alpha) = \mu(-\alpha)\}$$

and that $T = \lim_m T_m$. Since the function $\sigma$ that takes an element $T \in \mathcal{B}(\mathcal{H})$ to its spectrum $\sigma(T) \subseteq \mathbb{C}$ is continuous when $\dim \mathcal{H}$ is finite, we see that $\alpha \in \sigma(T)$ implies that $-\alpha \in \sigma(T)$. Furthermore, if $\Omega$ is any open neighbourhood of $\alpha \in \sigma(T)$ such that $\Omega \cap \sigma(T) = \{\alpha\}$, then for all $m \geqslant 1$,

$$\dim\, \mathcal{H}(\Omega; T_m) = \dim\, \mathcal{H}(-\Omega; T_m)$$

since $T_m$ is balanced, and so

$$\dim\, \mathcal{H}(\Omega; T) = \dim\, \mathcal{H}(-\Omega; T).$$

From this it follows that

$$\{T \in \mathcal{B}(\mathcal{H}) : \alpha \in \sigma(T) \text{ implies } -\alpha \in \sigma(T) \text{ and } \mu(\alpha) = \mu(-\alpha)\}$$

is closed, and thus that $\text{BAL}(\mathcal{H})$ is closed. ∎

It is worth observing that as a consequence of Proposition 3.3, an operator $T \in \mathcal{B}(\mathbb{C}^n)$ is balanced if and only if its characteristic polynomial is either an even function or an odd function.

THEOREM 3.4. *Suppose that* $\dim\, \mathcal{H} < \infty$. *Then*

$$\text{CLOS}\,(\mathfrak{C}_{\mathfrak{C}}) = \text{BAL}(\mathcal{H}).$$

*Proof.* By Proposition 2.1, $\mathfrak{C}_{\mathfrak{E}} \subseteq \mathrm{BAL}(\mathcal{H})$. Thus

$$\mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}) \subseteq \mathrm{CLOS}\,(\mathrm{BAL}(\mathcal{H})) = \mathrm{BAL}(\mathcal{H}).$$

By Proposition 3.3, we have that

$$\mathrm{BAL}(\mathcal{H}) = \{T \in \mathcal{B}(\mathcal{H}) : \alpha \in \sigma(T) \text{ implies } -\alpha \in \sigma(T) \text{ and } \mu(\alpha) = \mu(-\alpha)\}.$$

Now consider the converse, and suppose that $T \in \mathrm{BAL}(\mathcal{H})$. Then we may write the elements of $\sigma(T)$ (repeated according to their algebraic multiplicity) as an $n$-tuple

$$\Sigma_T := (\alpha_1, \alpha_2, \ldots, \alpha_n),$$

where $\alpha_{2k} = -\alpha_{2k-1}$ for all $1 \leqslant k \leqslant \frac{n}{2}$, and $\alpha_n = 0$ if $n$ is odd. Furthermore, we can upper-triangularise $T$ so that if $[T] = [t_{ij}]$, then

(•) $t_{ij} = 0$ if $i > j$, and
(•) $t_{kk} = \alpha_k$ for all $1 \leqslant k \leqslant n$.

Let $\varepsilon > 0$. It is relatively easy to show that we can find $\beta_k \in \mathbb{C}$, $1 \leqslant k \leqslant n$ such that

(•) each $|\beta_k - \alpha_k| < \varepsilon$;
(•) if $i \neq j$, then $\beta_i \neq \beta_j$;
(•) for all $1 \leqslant k \leqslant \frac{n}{2}$, $\beta_{2k} = -\beta_{2k-1}$; and (•) $\beta_n = 0$ if $n$ is odd.

Let $D = \mathrm{DIAG}(\beta_1, \beta_2, \ldots, \beta_n)$. Then $D$ is a normal operator and (by the last two conditions) $D$ is unitarily equivalent to $-D$. Furthermore, all eigenvalues of $D$ are distinct. By considering Jordan forms, any operator $X \in \mathbb{M}_n(\mathbb{C})$ such that $\sigma(X) = \sigma(D)$ is similar to $D$.

Let $X \in \mathbb{M}_n(\mathbb{C})$ be the operator whose matrix is $[x_{ij}]$, where

$$x_{ij} = \begin{cases} t_{ij} & \text{if } i \neq j, \\ \beta_k & \text{if } i = k = j. \end{cases}$$

Then $X$ is similar to $D$.

Now, by Proposition 3 of [8], $D \in \mathfrak{C}_{\mathfrak{E}}$. Since $\mathfrak{C}_{\mathfrak{E}}$ is invariant under similarity, $X \in \mathfrak{C}_{\mathfrak{E}}$. But

$$\begin{aligned} X - T &= \mathrm{DIAG}(x_{11} - t_{11}, x_{22} - t_{22}, \ldots, x_{nn} - t_{nn}) \\ &= \mathrm{DIAG}(\beta_1 - \alpha_1, \beta_2 - \alpha_2, \ldots, \beta_n - \alpha_n), \end{aligned}$$

so $\|X - T\| = \max\limits_{1 \leqslant k \leqslant n} |\beta_k - \alpha_k| < \varepsilon$.

Since $\varepsilon > 0$ is arbitrary, it follows that $T \in \mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. ∎

COROLLARY 3.5. *Suppose that* $\dim \mathcal{H} < \infty$. *Then*

$$\mathrm{CLOS}(\mathfrak{C}_{\mathfrak{E}}) = \mathrm{CLOS}(\mathrm{NEG}_{\mathrm{INVS}}(\mathcal{H})) = \mathrm{CLOS}(\mathrm{NEG}_{\mathrm{S}}(\mathcal{H})) = \mathrm{CLOS}(\mathrm{BAL}(\mathcal{H})) = \mathrm{BAL}(\mathcal{H}).$$

*Proof.* This is an immediate consequence of Proposition 2.1, Proposition 3.3 and Theorem 3.4. ∎

We now turn our attention to the closure of the set $\mathfrak{E} - \mathfrak{E}$ of differences of idempotents.

LEMMA 3.6. (i) *Let $\alpha \in \mathbb{C}$. Then $B := \alpha \oplus -\alpha \in \mathcal{B}(\mathbb{C}^2)$ is a difference of idempotents.*

(ii) *If $\dim \mathcal{H} < \infty$ and $N \in \mathcal{B}(\mathcal{H})$ is nilpotent, then $N \in \mathfrak{E} - \mathfrak{E}$.*

(iii) *If $\mathcal{H}_k$ is a Hilbert space and $B_k \in \mathcal{B}(\mathcal{H}_k)$ lies in $\mathfrak{E} - \mathfrak{E}$, $1 \leqslant k \leqslant K$, then $B := \bigoplus_{k=1}^{K} B_k$ is a difference of idempotents in $\mathcal{B}(\bigoplus_{k=1}^{K} \mathcal{H}_k)$.*

*Proof.* (i) If $\alpha = 0$, then $B = 0$ is trivially a difference of idempotents. If $\alpha \neq 0$, then

$$B \simeq \begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ -\alpha & 0 \end{bmatrix} \in \mathfrak{E} - \mathfrak{E}.$$

(ii) By Proposition 6 of [8], $N \in \mathfrak{C}_{\mathfrak{E}} \subseteq \mathfrak{E} - \mathfrak{E}$.

(iii) Setting $B_k = [E_k, F_k]$, $1 \leqslant k \leqslant K$ and $E := \bigoplus_{k=1}^{K} E_k$, $F := \bigoplus_{k=1}^{K} F_k$, we see that $B = [E, F]$. ∎

PROPOSITION 3.7. *Suppose that $2 \leqslant n := \dim \mathcal{H} < \infty$. If $T \in \mathrm{BAL}(\mathcal{H})$, then $T \in \mathrm{CLOS}\,(\mathfrak{E} - \mathfrak{E})$. Nevertheless, there exists $T \in \mathrm{BAL}(\mathcal{H})$ such that $T \notin \mathfrak{E} - \mathfrak{E}$.*

*Proof.* By Theorem 3.4, $\mathrm{BAL}(\mathcal{H}) = \mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$, while Proposition 2.2 shows that $\mathfrak{C}_{\mathfrak{E}} \subseteq \mathfrak{E} - \mathfrak{E}$, whence

$$\mathrm{BAL}(\mathcal{H}) \subseteq \mathrm{CLOS}\,(\mathfrak{E} - \mathfrak{E}).$$

As for the second statement, note that if $X = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ and $Y = \begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}$, then $W = X \oplus Y$ is invertible, as are $W + I_4$ and $W - I_4$. Note that $W \in \mathrm{BAL}(\mathbb{C}^4)$. By Proposition 4 of [13], $W \in \mathfrak{E} - \mathfrak{E}$ if and only if $W$ is similar to $D \oplus -D$ for some invertible operator $D \in \mathcal{B}(\mathbb{C}^2)$. But clearly the Jordan form of $W$ prohibits this from happening. Thus $W \in \mathrm{BAL}(\mathbb{C}^4) \setminus (\mathfrak{E} - \mathfrak{E})$. ∎

PROPOSITION 3.8. *Let $2 \leqslant n := \dim \mathcal{H} < \infty$, and let $\mathfrak{sl}(\mathcal{H}) := \{A \in \mathcal{B}(\mathcal{H}) : \mathrm{TR}(A) = 0\}$. Then*

$$\mathfrak{sl}(\mathcal{H}) \cap \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E}) = \mathrm{BAL}(\mathcal{H}).$$

*Proof.* Clearly $\mathrm{BAL}(\mathcal{H}) \subseteq \mathfrak{sl}(\mathcal{H})$, and by Proposition 3.7, we have that $\mathrm{BAL}(\mathcal{H}) \subseteq \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$, so that

$$\mathrm{BAL}(\mathcal{H}) \subseteq \mathfrak{sl}(\mathcal{H}) \cap \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E}).$$

Now suppose that $T \in \mathfrak{sl}(\mathcal{H}) \cap (\mathfrak{E} - \mathfrak{E})$. We may decompose $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$ in such a way that relative to this decomposition, we have

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ 0 & T_{22} & T_{23} \\ 0 & 0 & T_{33} \end{bmatrix},$$

where $\sigma(T_{11}) = \{0\}$, $\sigma(T_{22}) \subseteq \{-1,1\}$ and $\sigma(T_{33}) \subseteq \mathbb{C} \setminus \{-1,0,1\}$. Since the spectra of $T_{11}, T_{22}$ and $T_{33}$ are all disjoint, we see that $T$ is similar to $T_{11} \oplus T_{22} \oplus T_{33}$, and that we find from Lemma 2 of [13] that $T \in \mathfrak{E} - \mathfrak{E}$ if and only if each of $T_{11}, T_{22}$ and $T_{33}$ is.

Now $\sigma(T_{11}) = \{0\}$, so $T_{11}$ is balanced. Also, by Proposition 4 of [13], $T_{33} \in \mathfrak{E} - \mathfrak{E}$ if and only if $T_{33}$ is similar to $D \oplus -D$ for some invertible operator $D \in \mathcal{B}(\mathcal{H}_3)$, implying that $T_{33}$ is balanced. Thus $\mathrm{TR}(T_{11}) = 0 = \mathrm{TR}(T_{33})$. Since $\mathrm{TR}(T) = 0$, it follows that $\mathrm{TR}(T_{22}) = 0$. But $\sigma(T_{22}) \subseteq \{-1,1\}$, which then implies that $T_{22}$ is balanced. Hence $T \in \mathrm{BAL}(\mathcal{H})$, being the direct sum of balanced operators.

Finally, if $X \in \mathfrak{sl}(\mathcal{H}) \cap \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$, then $X = \lim_n T_n$, where each $T_n \in \mathfrak{E} - \mathfrak{E}$. Since $\mathrm{TR}(\cdot)$ is continuous and integer-valued on $\mathfrak{E} - \mathfrak{E}$, and since $\mathrm{TR}(X) = 0$, it follows that there exists $n_0 \geqslant 1$ such that $T_n \in \mathfrak{sl}(\mathcal{H}) \cap (\mathfrak{E} - \mathfrak{E})$ for all $n \geqslant n_0$. But then $T_n \in \mathrm{BAL}(\mathcal{H})$ for all $n \geqslant n_0$. Since $\mathrm{BAL}(\mathcal{H})$ is closed by Proposition 3.3, $X \in \mathrm{BAL}(\mathcal{H})$. This completes the proof. ∎

Let $n \geqslant 2$ and denote by $\{e_k : 1 \leqslant k \leqslant n\}$ the standard orthonormal basis for $\mathbb{C}^n$. We shall denote by $J_n$ the standard $n \times n$ Jordan cell in $\mathbb{M}_n(\mathbb{C})$; that is, $J_n$ is the unique operator satisfying $J_n e_1 = 0$, while $J_n e_k = e_{k-1}$, $2 \leqslant k \leqslant n$.

LEMMA 3.9. *Let $\mathcal{H}$ be a Hilbert space with $n = \dim \mathcal{H} < \infty$ and $Z \in \mathcal{B}(\mathcal{H}) \simeq \mathbb{M}_n(\mathbb{C})$. Suppose that:*

(i) *$\sigma(Z) \subseteq \{-1,1\}$;*

(ii) *$\mathrm{TR}(Z) = r \in \mathbb{N}$; and*

(iii) *$\mathrm{NUL}\,(Z - I) \geqslant r$.*

*Then $Z \in \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$.*

*Proof.* Suppose first that $\sigma(Z) = \{1\}$. Then, from (ii) and (iii), we know each elementary divisor of $Z$ has degree one. Hence, $Z$ is similar to $I_r$, whence $Z = I_r$. Thus we may assume that $\sigma(Z) = \{-1,1\}$.

Consider the Jordan form of $Z$, namely

$$Z \sim \Big[\bigoplus\nolimits_{j=1}^{\kappa^+}(I_{m_j} + J_{m_j})\Big] \oplus \Big[\bigoplus\nolimits_{j=1}^{\kappa^-}(-I_{n_j} + J_{n_j})\Big].$$

Let $s := \sum_{j=1}^{\kappa^-} n_j$. Observe that:

(•) the fact that $\mathrm{NUL}\,(Z - I) \geqslant r$ implies that $\kappa^+ \geqslant r$; and

(•) the fact that $\mathrm{TR}(Z) = r$ implies that $\sum_{j=1}^{\kappa^+} m_j - s = r$.

Here we agree that $J_1 = 0$.

Since $-1 \in \sigma(Z)$, $s \geqslant 1$. If $s < r$, set $c = r - s$. According to the above facts, it is easy to deduce that there exists a subset $\Lambda \subset \{1, \dots, \kappa^+\}$ with $|\Lambda| = c$, such that $m_j = 1$, when $j \in \Lambda$. By reindexing, we may assume that $\Lambda = \{\kappa^+ - c +$

$1, \dots, \kappa^+\}$. Then

$$Z \sim \left[\bigoplus_{j=1}^{\kappa^+ - c}(I_{m_j} + J_{m_j})\right] \oplus \left[\bigoplus_{j=1}^{\kappa^-}(-I_{n_j} + J_{n_j})\right] \oplus I_c := Z_1 \oplus I_c.$$

It is clear that $Z \in \text{CLOS}(\mathfrak{E} - \mathfrak{E})$ if $Z_1 \in \text{CLOS}(\mathfrak{E} - \mathfrak{E})$. Now let $r_1 = s$, then $\sigma(Z_1) = \{-1, 1\}$, $\text{TR}(Z_1) = r_1 \in \mathbb{N}$, and $\text{NUL}\,(Z_1 - I) \geqslant r_1$. Furthermore, $s_1 := \sum_{j=1}^{\kappa^-} n_j = s = r_1$. Hence, without loss of generality, we may add a further assumption that "$s \geqslant r$" to $Z$.

For each $n \geqslant 1$, we define a diagonal operator

$$D_n = \text{DIAG}(d_1^{(n)}, d_2^{(n)}, \dots, d_s^{(n)})$$

with the properties that:

(•) for any fixed $n \geqslant 1$, all of the diagonal entries $d_j^{(n)} \in (0,1)$, $1 \leqslant j \leqslant s$ are distinct; and

(•) $\lim_n d_j^{(n)} = 1$ for all $1 \leqslant j \leqslant s$.

Partition the set $\{1, 2, \dots, s\}$ into $r+1$ disjoint sets $\Omega_1, \Omega_2, \dots, \Omega_{r+1}$, where $|\Omega_j| = m_j - 1$ for $1 \leqslant j \leqslant r$ and $\Omega_{r+1}$ contains the remaining elements of $\{1, 2, \dots, s\}$. (It is possible that $\Omega_{r+1}$ might be empty.) Define $D_j^{(n)} = \text{DIAG}\{d_\ell^{(n)} : \ell \in \Omega_j\}$, $1 \leqslant j \leqslant r+1$, and note that $D_n \simeq \bigoplus_{j=1}^{r+1} D_j^{(n)}$. Set

$$Y_n^+ = \left[\bigoplus_{j=1}^{r}(1 \oplus D_j^{(n)})\right] \oplus D_{r+1}^{(n)}.$$

(The point is that $Y_n^+$ is a direct sum of $r$ diagonal operators acting on spaces of dimension $m_1, m_2, \dots, m_r$, and each of these diagonal operators has first entry equal to 1, along with another diagonal operator which brings the dimension of the space upon which $Y_n^+$ acts up to $s$. Other than the 1's which appear $r := \text{TR}(Z)$ times, all other diagonal entries of $Y_n^+$ should be distinct.) Because all of the diagonal entries of $(1 \oplus D_j^{(n)})$ are distinct, we see that

$$(1 \oplus D_j^{(n)}) \sim A_j^{(n)} := (1 \oplus D_j^{(n)}) + J_{m_j}, 1 \leqslant j \leqslant r, \quad n \geqslant 1.$$

Also, because all of the diagonal entries of $D_{r+1}^{(n)}$ are distinct (for any $n \geqslant 1$), we see that

$$D_{r+1}^{(n)} \sim A_{r+1}^{(n)} := D_{r+1}^{(n)} + \left[\bigoplus_{j=r+1}^{\kappa^+} J_{m_j}\right].$$

In other words,

$$Y_n^+ \sim B_n := \left[\bigoplus_{j=1}^{r} A_j^{(n)}\right] \oplus A_{r+1}^{(n)}, \quad n \geqslant 1.$$

Observe also that

$$\lim_n B_n = \lim_n \left[\bigoplus_{j=1}^{r} A_j^{(n)}\right] \oplus A_{r+1}^{(n)} = \left[\bigoplus_{j=1}^{\kappa^+}(I_{m_j} + J_{m_j})\right].$$

Next, let $Y_n^- := -D_n$. Since all of the diagonal entries of $D_n$ were distinct (for all $n \geqslant 1$), we see that

$$Y_n^- = -D_n \sim C_n := -D_n + \left[\bigoplus_{j=1}^{\kappa^-} J_{n_j}\right],$$

and thus

$$\lim_n C_n = \left[\bigoplus_{j=1}^{\kappa^-} (-I_{n_j} + J_{n_j})\right].$$

But $Y_n := Y_n^+ \oplus Y_n^- \simeq I_r \oplus D_n \oplus -D_n \in \mathfrak{E} - \mathfrak{E}$ by the Hartwig–Putcha Theorem [13, Theorem 1a], and $Y_n \sim B_n \oplus C_n$, implying that $B_n \oplus C_n \in \mathfrak{E} - \mathfrak{E}$. Finally,

$$\lim_n (B_n \oplus C_n) = Z,$$

and therefore $Z \in \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$. ▮

LEMMA 3.10. *Let $\mathcal{H}$ be a Hilbert space with $n = \dim \mathcal{H} < \infty$ and let $T \in \mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$. Suppose that* $\mathrm{TR}(T) = r \in \mathbb{N}$. *If $T \in \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$, then* $\mathrm{NUL}\,(T - I) \geqslant r$.

*Proof.* Suppose that $T = \lim_{n\to\infty} T_n$, where $T_n \in \mathfrak{E} - \mathfrak{E}$. By the continuity of the spectrum and therefore of the trace, we may assume without loss of generality that $\mathrm{TR}(T_n) = r$ for all $n \in \mathbb{N}$.

By the Hartwig–Putcha Theorem 1.2, the eigenvalues of $T_n$ which belong to $\mathbb{C} \setminus \{-1, 1\}$ come in pairs $\{-\alpha, \alpha\}$, and therefore do not contribute to the trace. From this, and again by the Hartwig–Putcha Theorem, it follows that if we fix the exponents $m_j^{(n)}$ occurring in the elementary divisors of $T_n$ corresponding to 1 and $n_j^{(n)}$ corresponding to $-1$, for any $n \in \mathbb{N}$, we will always have at least $r$ $j$'s for which

$$m_j^{(n)} - n_j^{(n)} = 1.$$

Then $\mathrm{NUL}\,(T_n - I) \geqslant r$. Since the function $\mathrm{NUL}(\cdot)$ is upper-semicontinuous (for the rank function is lower-semicontinuous), $\mathrm{NUL}\,(T - I) \geqslant r$. ▮

The hypothesis in the next theorem that the trace of the operator $T$ should be non-negative is there only to simplify the statement of the result. Note that $T \in \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$ if and only if $-T \in \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$, so by replacing $T$ by $-T$ if necessary, the trace of $T$ may always be assumed to be a non-negative integer.

THEOREM 3.11. *Let $T \in \mathcal{B}(\mathbb{C}^n) \simeq \mathbb{M}_n(\mathbb{C})$ and suppose that* $\mathrm{TR}(T) = r \in \mathbb{N}$. *The following are equivalent:*

(i) $T \in \mathrm{CLOS}(\mathfrak{E} - \mathfrak{E})$*; and*

(ii) $T \sim B \oplus Z$*, where $B$ is balanced, $\sigma(Z) \subseteq \{-1, 1\}$ and* $\mathrm{NUL}\,(Z - I) \geqslant r$.

*Proof.* (i) $\Longrightarrow$ (ii) By using Riesz idempotent theorem, we may assume that $T = B \oplus Z$, where $\sigma(B) \in \mathbb{C} \setminus \{-1, 1\}$, and $\sigma(Z) \subset \{-1, 1\}$. We claim that $B$ is balanced. Indeed, there exists $0 < \delta < 1$ such that $\sigma(B) \subseteq \{z \in \mathbb{C} : |z| < \delta\}$. If

$T = \lim_m T_m$ where $T_m \in \mathfrak{E} - \mathfrak{E}$ for all $m \geqslant 1$, then by the continuity of the map $X \mapsto \sigma(X)$ in the finite-dimensional setting (with eigenvalues counted according to their algebraic multiplicities), we see that $\sigma(T)$ is the limit of $\sigma(T_m)$, and thus $\sigma(B)$ is the limit of $\sigma(T_m) \cap \{z \in \mathbb{C} : |z| < \delta\}$. But $T_m \in \mathfrak{E} - \mathfrak{E}$ implies that $\mu(\alpha) = \mu(-\alpha)$ whenever $\alpha \notin \{-1, 0, 1\}$, from which we deduce that $0 \neq \alpha \in \sigma(B)$ implies that $-\alpha \in \sigma(B)$ and $\mu(\alpha) = \mu(-\alpha)$; in other words, $B$ is balanced.

Since $T \in \textsc{clos}(\mathfrak{E} - \mathfrak{E})$, by Lemma 3.10, $\textsc{nul}\,(Z - I) = \textsc{nul}\,(T - I) \geqslant r$. Since $B$ is balanced, $\textsc{tr}(Z) = \textsc{tr}(T) = r$.

(ii) $\Longrightarrow$ (i) Suppose that $T \sim B \oplus Z$, with $B$ and $Z$ as in the statement of (ii). Since $B$ is balanced, $B \in \textsc{clos}(\mathfrak{E} - \mathfrak{E})$. And by Lemma 3.9, $Z \in \textsc{clos}(\mathfrak{E} - \mathfrak{E})$. Then it is clear that $T \in \textsc{clos}(\mathfrak{E} - \mathfrak{E})$ also. ∎

## 4. COMPACT OPERATORS

4.1. Recall from Proposition 2.1 that

$$\mathfrak{C}_{\mathfrak{E}} \subseteq \mathrm{NEG}_{\mathrm{INVS}}(\mathcal{H}) \subseteq \mathrm{NEG}_{\mathrm{S}}(\mathcal{H}) \subseteq \mathrm{BAL}(\mathcal{H}).$$

When $\mathcal{H}$ is finite-dimensional, the norm-closures of all of these sets coincide and $\mathrm{BAL}(\mathcal{H})$ is closed (see Corollary 3.5). Our goal in this section is to show that the same result holds if we restrict our attention to the set of compact operators acting on an infinite-dimensional complex separable Hilbert space $\mathcal{H}$. That is to say, we wish to prove that

$$\textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}}) \cap \mathcal{K}(\mathcal{H}) = \mathrm{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H}).$$

We emphasise the fact that we do not require the approximants to be compact; that is, we allow for an element $T \in \textsc{clos}\,(\mathfrak{C}_{\mathfrak{E}}) \cap \mathcal{K}(\mathcal{H})$ (respectively $T \in \mathrm{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$) to be expressed as a limit of operators $(T_m)_m$ which are non-compact elements of $\mathfrak{C}_{\mathfrak{E}}$ (respectively non-compact elements of $\mathrm{BAL}(\mathcal{H})$).

Of course, when $K \in \mathcal{K}(\mathcal{H})$ and $\lambda \in \mathbb{C}$, either $\lambda = 0$ and $K - \lambda I = K + \lambda I = K$ is not semi-Fredholm (in which case condition (iii) of Definition 1.3 does not apply), or $\lambda \neq 0$ in which case $K - \lambda I$, $K + \lambda I$ are both Fredholm of index zero, and so (iii) of Definition 1.3 holds automatically.

The fact that every quasinilpotent operator $Q \in \mathcal{B}(\mathcal{H})$ is a limit of nilpotent operators is a deep result due to Apostol and Voiculescu [2]. When $Q$ is both quasinilpotent and *compact*, the fact that the approximating nilpotent operators may be chosen to be of finite rank is much simpler. As we have been unable to locate a specific reference for this result, we have decided to include the outline of its proof. Let $\varepsilon > 0$ and choose a finite-rank operator $F$ such that $\|Q - F\| < \varepsilon$ and $\sigma(F) \subseteq \{z \in \mathbb{C} : |z| < \varepsilon\}$. That this is possible is a consequence of the fact that $Q$ is compact, combined with the upper semicontinuity of the spectrum. Write $F \simeq F_0 \oplus 0$, where $F_0 \in \mathbb{M}_n(\mathbb{C})$ for appropriate $n \geqslant 1$, and upper-triangularising $F_0$, observe that all diagonal entries have magnitude less than $\varepsilon$. Thus a diagonal perturbation $D_0 + F_0$ of $F_0$ of norm at most $\varepsilon$ ($D_0$ simply represents the negative of the diagonal of $F_0$) results in a finite-rank nilpotent operator $N \simeq (D_0 + F_0) \oplus 0$ which approximates $Q$ to within $2\varepsilon$.

PROPOSITION 4.1. *Let $Q \in \mathcal{K}(\mathcal{H})$ be quasinilpotent. Then $Q \in \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$. Moreover, we can choose the approximants in $\mathfrak{C}_{\mathfrak{E}}$ to be nilpotent themselves.*

*Proof.* Let $\varepsilon > 0$. We have just seen that every compact quasinilpotent operator is a limit of finite-rank nilpotent operators, and as such, there exists a finite-rank nilpotent operator $L$ such that $\|Q - L\| < \varepsilon$. Let $\mathcal{R} := \operatorname{span}\{\operatorname{ran} L, \operatorname{ran} L^*\}$. Then $\mathcal{R}$ is finite-dimensional, and relative to the decomposition $\mathcal{H} = \mathcal{R} \oplus \mathcal{R}^{\perp}$, we may write

$$L = \begin{bmatrix} L_0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $L$ is nilpotent, so is $L_0$. By Proposition 6 of [8], $L_0$ is a commutator of two finite-rank idempotents $E_0, F_0 \in \mathcal{B}(\mathcal{R})$. Let $E := \begin{bmatrix} E_0 & 0 \\ 0 & 0 \end{bmatrix}$ and $F := \begin{bmatrix} F_0 & 0 \\ 0 & 0 \end{bmatrix}$. Then $E$ and $F$ are idempotents and

$$L = [E, F] \in \mathfrak{C}_{\mathfrak{E}}.$$

Since $\varepsilon > 0$ was arbitrary, $Q \in \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$. ∎

EXAMPLE 4.2. We temporarily digress to show that $\mathfrak{C}_{\mathfrak{E}} \cap \mathcal{K}(\mathcal{H})$ is not closed. Indeed, let $V \in \mathcal{B}(L^2[0,1], \mathrm{d}x)$ be the classical Volterra operator defined by

$$(Vf)(x) = \int_0^x f(t)\mathrm{d}t, \quad f \in L^2[0,1].$$

It is well known that $V$ is compact and quasinilpotent. By Proposition 4.1, we know $V \in \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$.

A result of Kalisch (see, e.g. [17, Theorem 2] or [9, Proposition 1]) shows that if $\alpha \in \mathbb{C}$, then $V$ and $\alpha V$ are similar if and only if $\alpha = 1$. In particular, $V$ is not similar to $-V$, and thus $V \notin \mathfrak{C}_{\mathfrak{E}}$ by Proposition 2.1.

We now return to the task of extending Corollary 3.5 to the setting of compact operators.

PROPOSITION 4.3. $\operatorname{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H}) \subseteq \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$.

*Proof.* After a moment's thought, and keeping in mind that every quasinilpotent, compact operator lies in $\operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$, without loss of generality, we may assume $K \in \operatorname{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$ is not quasinilpotent and 0 is a cluster point of $\sigma(K)$.

Note that we may denote the sequence of non-zero eigenvalues of $K$ as $(\alpha_n)_n$, where $\alpha_{2k} = -\alpha_{2k-1}$, $k \geqslant 1$. Since $K \in \operatorname{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$, it follows that

$$\dim \mathcal{H}(\{\alpha_{2k}\}; K) = \dim \mathcal{H}(\{\alpha_{2k-1}\}; K) \quad \text{for all } k \geqslant 1.$$

Define $\mathcal{M}_n := \operatorname{span}\{\mathcal{H}(\{\alpha_k\}; K)\}_{k=1}^n$, set $\mathcal{H}_\infty = \bigoplus_n (\mathcal{M}_n \ominus \mathcal{M}_{n-1})$ and $\mathcal{H}_0 = \mathcal{H} \ominus \mathcal{H}_\infty$. Relative to the decomposition $\mathcal{H} = \mathcal{H}_\infty \oplus \mathcal{H}_0$, $K$ admits an upper triangular form

$$K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & & K_{1,0} \\ 0 & K_{22} & K_{23} & \cdots & & K_{2,0} \\ 0 & 0 & K_{33} & \cdots & & K_{3,0} \\ 0 & 0 & 0 & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & K_{00} \end{bmatrix}.$$

Given $n \geqslant 1$, let $P_n$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}_n$, let $P_0$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$ and $P_\infty$ denote the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_\infty$.

Observe that $(P_n + P_0)_n$ is an increasing sequence of projections tending strongly to the identity operator. Since $K \in \mathcal{K}(\mathcal{H})$, it follows that

$$K = \lim_n (P_n + P_0) K (P_n + P_0).$$

Let

$$L_{2n} := P_{2n} K P_{2n} + P_0 K P_0 = \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1,2n} & 0 & \cdots & 0 \\ 0 & K_{22} & K_{23} & \cdots & K_{2,2n} & 0 & \cdots & 0 \\ 0 & 0 & \ddots & \cdots & & 0 & \cdots & 0 \\ 0 & 0 & 0 & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & 0 & 0 & K_{2n,2n} & 0 & \cdots & 0 \\ \vdots & \vdots & & & & & & \\ 0 & \cdots & & & & 0 & \cdots & K_{00} \end{bmatrix}.$$

We may think of this as $L_{2n} = P_{2n} K P_{2n} \oplus P_0 K P_0$. Now, $P_{2n} K P_{2n} \in \mathcal{B}(\mathcal{H}_\infty)$ is a balanced, finite-rank operator. Thus there exist finite-rank idempotents $E_{2n,\infty}$, $F_{2n,\infty} \in \mathcal{B}(\mathcal{H}_\infty)$ such that $P_{2n} K P_{2n} = [E_{2n,\infty}, F_{2n,\infty}]$.

Since $\sigma(K_{00}) = \{0\}$, by Proposition 4.1, there exist idempotents $E_{n,0}$, $F_{n,0} \in \mathcal{B}(\mathcal{H}_0)$ such that $K_{00} = \lim_n [E_{n,0}, F_{n,0}]$, and each $[E_{n,0}, F_{n,0}]$ is nilpotent.

Then $Q_n := E_{2n,\infty} \oplus E_{n,0}$ and $R_n := F_{2n,\infty} \oplus F_{n,0}$ are idempotents in $\mathcal{B}(\mathcal{H})$ such that

$$\lim_n \| L_{2n} - [Q_n, R_n] \| = 0.$$

Now $[Q_n, R_n] = [E_{2n,\infty}, F_{2n,\infty}] \oplus [E_{n,0}, F_{n,0}] \in \mathfrak{C}_{\mathfrak{E}}$, and therefore any operator similar to $[Q_n, R_n]$ is also in $\mathfrak{C}_{\mathfrak{E}}$. Since $\sigma(P_{2n} K P_{2n}|_{\mathcal{M}_{2n}}) \cap \{0\} = \varnothing$, a simple argument using Rosenblum's operator (see [14, Corollary 3.2]) shows that

$$[Q_n, R_n] \sim X_n := \begin{bmatrix} K_{11} & K_{12} & K_{13} & \cdots & K_{1,2n} & 0 & \cdots & K_{1,0} \\ 0 & K_{22} & K_{23} & \cdots & K_{2,2n} & 0 & \cdots & K_{2,0} \\ 0 & 0 & \ddots & \cdots & & 0 & \cdots & \vdots \\ 0 & 0 & 0 & \ddots & & 0 & \cdots & \vdots \\ 0 & 0 & 0 & 0 & K_{2n,2n} & 0 & \cdots & K_{2n,0} \\ \vdots & \vdots & & & & 0 & \ddots & 0 \\ 0 & \cdots & & & & 0 & \cdots & [E_{n,0}, F_{n,0}] \end{bmatrix}.$$

Hence $X_n \in \mathfrak{C}_{\mathfrak{E}}$ for all $n \geqslant 1$.

Note that

$$\lim_n \|X_n - (P_{2n} + P_0)K(P_{2n} + P_0)\| = \lim_n \|K_{00} - [E_{n,0}, F_{n,0}]\| = 0.$$

Since $K = \lim_n (P_{2n} + P_0)K(P_{2n} + P_0)$, we conclude that

$$K = \lim_n X_n \in \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}}). \quad \blacksquare$$

We shall have reason to appeal to the next result of Herrero's more than once below. We first recall that a *Cauchy domain* is an open set $\Omega \subseteq \mathbb{C}$ such that $\Omega$ has finitely many components, the closures of any two of which are disjoint, and whose boundary $\partial(\Omega)$ consists of a finite number of closed, rectifiable Jordan curves, any two of which are disjoint.

PROPOSITION 4.4 ([14, Corollary 1.6]). *Let $X, Y \in \mathcal{B}(\mathcal{H})$. If $\sigma \neq \varnothing$ is a relatively closed and open subset of $\sigma(X)$, and $\Omega$ (a Cauchy domain) is a neighbourhood of $\sigma$ satisfying $(\sigma(X) \setminus \sigma) \cap \overline{\Omega} = \varnothing$, then*

(i) $\|X - Y\| < \text{MIN}\{\|(\lambda I - X)^{-1}\|^{-1} : \lambda \in \partial(\Omega)\}$ *implies that* $\sigma' := \sigma(Y) \cap \Omega \neq \varnothing$*; and*

(ii) $\dim\, \mathcal{H}(\sigma; X) = \dim\, \mathcal{H}(\sigma'; Y)$.

PROPOSITION 4.5. *Let $K \in \mathcal{K}(\mathcal{H}) \cap \text{CLOS}\,(\mathfrak{C}_{\mathfrak{E}})$. Then $K \in \text{BAL}(\mathcal{H})$.*

*Proof.* Let $0 \neq \lambda \in \sigma(K)$. It suffices to prove that $-\lambda \in \sigma(K)$ and that

$$\dim \mathcal{H}(\{\lambda\}; K) = \dim \mathcal{H}(\{-\lambda\}; K).$$

Let $\Omega := \Omega_1 \cup \Omega_2 \cup \Omega_3$ be the *disjoint* union of three open sets satisfying:

(I) $\Omega_3 = -\Omega_1$;

(II) $\lambda \in \Omega_1$ (and thus $-\lambda \in \Omega_3$); and

(III) $\sigma(K) \setminus \{\lambda, -\lambda\} \subseteq \Omega_2$.

That this is possible is clear, since $\sigma(K)$ is at most an infinite sequence of isolated points converging to zero. In fact, we can do this while choosing $\Omega_1$ to be a disc of arbitrarily small radius centred at $\lambda$.

Let $\varepsilon := \inf\{\|(\alpha I - K)^{-1}\|^{-1} : \alpha \in \partial\Omega\} > 0$. By Proposition 4.4, if $T \in \mathcal{B}(\mathcal{H})$ and $\|T - K\| < \varepsilon$, then

(I) $\sigma(T) \cap \Omega_1 \neq \varnothing$;
(II) $\dim(\mathcal{H}(\{\lambda\}; K)) = \dim(\mathcal{H}(\Omega_1; T))$; and
(III) $\dim(\mathcal{H}(\Omega_3; K)) = \dim(\mathcal{H}(\Omega_3; T))$.

Since $K \in \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})$, there exists $X \in \mathfrak{C}_{\mathfrak{E}}$ such that $\|X - K\| < \varepsilon$. But $X \in \mathfrak{C}_{\mathfrak{E}}$ implies that $X \sim -X$, and thus $X \in \mathrm{BAL}(\mathcal{H})$. Hence

$$\begin{aligned}\dim(\mathcal{H}(\Omega_3; K)) = \dim(\mathcal{H}(\Omega_3; X)) &= \dim(\mathcal{H}(-\Omega_3; X)) = \dim(\mathcal{H}(\Omega_1; X)) \\ &= \dim(\mathcal{H}(\{\lambda\}; K)) > 0.\end{aligned}$$

It follows that $-\lambda \in \sigma(K)$ and that

$$\dim(\mathcal{H}(\{-\lambda\}; K)) = \dim(\mathcal{H}(\{\lambda\}; K)).$$

In other words, $K \in \mathrm{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H})$. ∎

COROLLARY 4.6.

$$\mathcal{K}(\mathcal{H}) \cap \mathrm{BAL}(\mathcal{H}) = \mathcal{K}(\mathcal{H}) \cap \operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}}).$$

*Proof.* This is an immediate consequence of Proposition 4.3 and Proposition 4.5. ∎

COROLLARY 4.7. *We have:*

$$\begin{aligned}[\operatorname{CLOS}(\mathfrak{C}_{\mathfrak{E}})] \cap \mathcal{K}(\mathcal{H}) &= [\operatorname{CLOS}(\mathrm{NEG}_{\mathrm{INVS}}(\mathcal{H}))] \cap \mathcal{K}(\mathcal{H}) \\ &= [\operatorname{CLOS}(\mathrm{NEG}_{\mathrm{S}}(\mathcal{H}))] \cap \mathcal{K}(\mathcal{H}) = \mathrm{BAL}(\mathcal{H}) \cap \mathcal{K}(\mathcal{H}).\end{aligned}$$

EXAMPLE 4.8. Let $K = \bigoplus_n \begin{bmatrix} 0 & \frac{1}{n} \\ 0 & 0 \end{bmatrix}$, so that $K$ is compact and of infinite rank. Then $K = [E, F]$, where $E := \bigoplus_n \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $F := \bigoplus_n \begin{bmatrix} 1 & \frac{1}{n} \\ 0 & 0 \end{bmatrix}$ are idempotents.

Note, however, that if $E_1$ and $F_1$ are *compact* idempotents, then they are necessarily of finite rank, and thus $[E_1, F_1]$ is also a finite-rank operator. Thus $\mathfrak{C}_{\mathfrak{E}} \cap \mathcal{K}(\mathcal{H}) \neq \{[E, F] : E, F \text{ are compact idempotents}\}$.

The following simple lemma will be of use to us in the next example.

LEMMA 4.9. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and $T_1 = R_1 \oplus 0, T_2 = R_2 \oplus 0 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$, where $R_1, R_2 \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ are invertible. If $T_1$ and $T_2$ are similar, then so are $R_1$ and $R_2$.*

*Proof.* Since $T_1$ and $T_2$ are similar, there exists an invertible operator $S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ such that $ST_1 = T_2 S$. Thus

$$ST_1 = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} R_1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} AR_1 & 0 \\ CR_1 & 0 \end{bmatrix}, \quad T_2 S = \begin{bmatrix} R_2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} R_2 A & R_2 B \\ 0 & 0 \end{bmatrix}.$$

From this we see that $AR_1 = R_2 A, CR_1 = 0, R_2 B = 0$.

As $R_1$ and $R_2$ are invertible, we conclude that $B = 0$ and $C = 0$. This in turn implies that $A$ is invertible in $\mathcal{B}(\mathcal{H}_1)$, whence $R_1$ is similar to $R_2$ via $A$. ∎

EXAMPLE 4.10. We now produce an example of a compact operator $T$ such that $T$ is unitarily equivalent to $-T$ but it is not a commutator of idempotents in $\mathcal{B}(\mathcal{H})$. While it is based upon an example from [8], the extension of their result to infinite dimensions requires a surprisingly long argument.

Let $A_0 = \begin{bmatrix} \frac{i}{2} & 1 \\ 0 & \frac{i}{2} \end{bmatrix} \in \mathbb{M}_2(\mathbb{C})$. If $T_0 = A_0 \oplus -A_0 \in \mathbb{M}_4(\mathbb{C})$, then by Example 10 of [8], $T_0$ is similar to $-T_0$ via an involution (this is trivial), but $T_0$ is not a commutator of idempotents.

Let $A = A_0 \oplus 0^{(\infty)}$, and let $T = A \oplus -A$. Observe that $\operatorname{RANK} T = 4$. Again, it is trivial to see that $T$ is similar to $-T$ via an involution, namely $J = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$, but we claim that $T$ is not a commutator of idempotents. (Note that $J$ is in fact a unitary involution.)

The proof below is an adaptation of the proof of Proposition 5 of [8].

Suppose to the contrary that $T = [E, F]$ where $E$ and $F$ are idempotents. After conjugating by an appropriate similarity $S$, we may write

$$S^{-1}TS = [P, Q],$$

where $P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{2}I + B & X \\ -Y & \frac{1}{2}I - C \end{bmatrix}$ are *idempotents* (but $Q$ is not necessarily a projection). Let us assume that this decomposition of $P$ and $Q$ is relative to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ of the Hilbert space.

A calculation (which is used in [8] and which is not hard to verify) yields that $BX = XC$ and $YB = CY$. It follows that $C(\ker X) \subseteq \ker X$ and that $B(\ker Y) \subseteq \ker Y$.

Now

$$\operatorname{RANK}[P, Q] = 4 = \operatorname{RANK} \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix} = \operatorname{RANK} X + \operatorname{RANK} Y.$$

Furthermore,

$$\operatorname{RANK}(XY \oplus YX) = \operatorname{RANK}([P, Q]^2) = \operatorname{RANK}(S^{-1}T^2S) = \operatorname{RANK}(S^{-1}(A^2 \oplus A^2)S),$$

and therefore $\operatorname{RANK} XY + \operatorname{RANK} YX = 4$ and $\sigma(XY \oplus YX) = \sigma(T^2) = \{-\frac{1}{4}, 0\}$. But $\sigma(XY) \cup \{0\} = \sigma(YX) \cup \{0\}$, and thus $-\frac{1}{4} \in \sigma(XY) \cap \sigma(YX)$, implying that $XY \neq 0 \neq YX$.

Now $\operatorname{RANK} X + \operatorname{RANK} Y = 4$ from above, and neither operator is zero. Suppose $\operatorname{RANK} X = 1$. Then $\operatorname{RANK} XY \leqslant 1$, $\operatorname{RANK} YX \leqslant 1$ and so $\operatorname{RANK}[P, Q]^2 \leqslant 2 \neq \operatorname{RANK} T^2$, a contradiction. Hence $\operatorname{RANK} X = \operatorname{RANK} Y = 2$.

Let $\mathcal{M}_1 := (\ker X)^\perp$ and $\mathcal{M}_2 := \mathcal{H}_2 \ominus \mathcal{M}_1$. Let $\mathcal{N}_1 := \ker Y$ and $\mathcal{N}_2 := \mathcal{H}_1 \ominus \mathcal{N}_1$. Relative to the decomposition $\mathcal{H} = \mathcal{N}_1 \oplus \mathcal{N}_2 \oplus \mathcal{M}_1 \oplus \mathcal{M}_2$, we may write

$$P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

and recalling that $B\mathcal{N}_1 \subseteq \mathcal{N}_1$ and $C\mathcal{M}_2 \subseteq \mathcal{M}_2$ from above,

$$Q = \begin{bmatrix} \frac{1}{2}I + B_1 & B_2 & X_1 & 0 \\ 0 & \frac{1}{2}I + B_4 & X_3 & 0 \\ 0 & -Y_2 & \frac{1}{2}I - C_1 & 0 \\ 0 & -Y_4 & -C_2 & \frac{1}{2}I - C_4 \end{bmatrix}.$$

Thus, with respect to the decomposition $\mathcal{H} = \mathcal{N}_1 \oplus \mathcal{M}_2 \oplus \mathcal{M}_1 \oplus \mathcal{N}_2$, we have

$$P = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix},$$

and recalling that $B\mathcal{N}_1 \subseteq \mathcal{N}_1$ and $C\mathcal{M}_2 \subseteq \mathcal{M}_2$ from above,

$$Q = \begin{bmatrix} \frac{1}{2}I + B_1 & 0 & X_1 & B_2 \\ 0 & \frac{1}{2}I - C_4 & -C_2 & -Y_4 \\ 0 & 0 & \frac{1}{2}I - C_1 & -Y_2 \\ 0 & 0 & X_3 & \frac{1}{2} + B_4 \end{bmatrix}.$$

This shows that

$$[P, Q] = \begin{bmatrix} 0 & 0 & X_1 & 0 \\ 0 & 0 & 0 & Y_4 \\ 0 & 0 & 0 & Y_2 \\ 0 & 0 & X_3 & 0 \end{bmatrix}.$$

Now $\dim(\mathcal{M}_1 \oplus \mathcal{N}_2) = 4$ (since each of these spaces has dimension 2), and

$$[P, Q]^2 = \begin{bmatrix} 0 & 0 & 0 & X_1 Y_2 \\ 0 & 0 & Y_4 X_3 & 0 \\ 0 & 0 & Y_2 X_3 & 0 \\ 0 & 0 & 0 & X_3 Y_2 \end{bmatrix}.$$

This is similar to $T^2$ which has four eigenvalues all equal to $-\frac{1}{4}$, and so $\sigma(Y_2 X_3) = \{-\frac{1}{4}\} = \sigma(X_3 Y_2)$. In particular, $X_3$ and $Y_2$ are invertible, and thus so is $\begin{bmatrix} 0 & Y_2 \\ X_3 & 0 \end{bmatrix}$. It follows (using Rosenblum's Theorem [14, Corollary 3.2]) that $[P, Q]$ is similar to

$$R := \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Y_2 \\ 0 & 0 & X_3 & 0 \end{bmatrix}.$$

But then by Lemma 4.9, $\begin{bmatrix} 0 & Y_2 \\ X_3 & 0 \end{bmatrix}$ is similar to $A_0 \oplus -A_0$, proving that $A_0 \oplus -A_0$ is (similar to) the commutator of the idempotents

$$P_0 := \begin{bmatrix} 0 & 0 \\ 0 & I \end{bmatrix} \quad \text{and} \quad Q_0 := \begin{bmatrix} \frac{1}{2}I - C_1 & -Y_2 \\ X_3 & \frac{1}{2}I + B_4 \end{bmatrix}.$$

NOTATION 4.11. Given $\varnothing \neq L \subseteq \mathbb{C}$ and $\varepsilon > 0$, we define $L_\varepsilon := \{z \in \mathbb{C} : \operatorname{DIST}(z, L) < \varepsilon\}$.

Recall that if $\Delta := \{A \subseteq \mathbb{C} : A \text{ is compact}\}$, then the *Hausdorff metric* on $\Delta$ is the metric defined by

$$d_H(A, B) := \operatorname{MAX}\left(\operatorname{MAX}_{a \in A} \operatorname{DIST}(a, B), \operatorname{MAX}_{b \in B} \operatorname{DIST}(b, A)\right).$$

THEOREM 4.12. *Let $K \in \operatorname{CLOS}(\mathfrak{E} - \mathfrak{E}) \cap \mathcal{K}(\mathcal{H})$, and write*

$$K = \begin{bmatrix} K_1 & K_2 \\ 0 & K_4 \end{bmatrix}$$

*relative to the decomposition $\mathcal{H} = \mathcal{H}(\{-1, 1\}; K) \oplus (\mathcal{H}(\{-1, 1\}; K))^\perp$. By considering $-K$ instead of $K$ if necessary, we may assume without loss of generality that*

$$\operatorname{TR}(K_1) \geqslant 0.$$

*Then*

(i) $\operatorname{NUL}(K - I) \geqslant \operatorname{TR} K_1$*; and*
(ii) $K_4$ *is balanced.*

REMARKS 4.13. An equivalent formulation of (ii) is that if $0 \neq \alpha \in \sigma(K) \setminus \{-1, 1\}$, then $-\alpha \in \sigma(K)$ and $\mu(\alpha) = \mu(-\alpha)$. Also, if $\sigma(K) \cap \{-1, 1\} = \varnothing$, then $K_1$, $K_2$ above are absent and $K = K_4$ in the argument below, meaning that one is only required to prove that $K_4$ is balanced. As such, the first half of the proof (regarding $K_1$) only applies if $\sigma(K) \cap \{-1, 1\} \neq \varnothing$.

*Proof.* Since $K$ is compact, we know that $0 \neq \alpha \in \sigma(K)$ implies that $\alpha$ is isolated, and thus there exists $\delta > 0$ such that if $G_{-1} := \{-1\}_\delta$, $G_1 := \{1\}_\delta$ and $G_0 := (\sigma(K) \setminus \{-1, 1\})_\delta$, then (using $\sqcup$ to denote the *disjoint* union of sets)

$$\sigma(K) \subseteq G_{-1} \sqcup G_1 \sqcup G_0.$$

Let $(T_m)_m$ be a sequence in $\mathfrak{E} - \mathfrak{E}$ such that $K = \lim_m T_m$. By Proposition 4.4, there exists $m_0 \geqslant 1$ such that $m \geqslant m_0$ implies that:

(a) $\sigma(T_m) \subseteq G_{-1} \sqcup G_1 \sqcup G_0$;
(b) $\sigma(T_m) \cap G_{-1} \neq \varnothing$, $\sigma(T_m) \cap G_1 \neq \varnothing$, $\sigma(T_m) \cap G_0 \neq \varnothing$; and
(c) $\dim \mathcal{H}(G_1; T) = \dim \mathcal{H}(G_1; K) < \infty$, $\dim \mathcal{H}(G_{-1}; T) = \dim \mathcal{H}(G_{-1}; K) < \infty$.

Clearly $\operatorname{TR} K_1 = \dim\, \mathcal{H}(G_1; K) - \dim\, \mathcal{H}(G_{-1}; K)$.

Meanwhile, if we write

$$T_m = \begin{bmatrix} T_1^{(m)} & T_2^{(m)} \\ 0 & T_4^{(m)} \end{bmatrix}$$

relative to the decomposition $\mathcal{H} = \mathcal{H}(G_1 \sqcup G_{-1}; T_m) \oplus (\mathcal{H}(G_1 \sqcup G_{-1}; T_m))^{\perp}$, then $T_1^{(m)}$ acts on a finite-dimensional space and $\operatorname{TR}(T_1^{(m)}) = \sum\{\mu(\beta)\beta : \beta \in \sigma(T_m) \cap (G_1 \sqcup G_{-1})\}$.

Since $\sigma(-T_1^{(m)}) \cap \sigma(T_4^{(m)}) = \varnothing$, by Lemma 2 of Hartwig and Putcha [13] (the reader should be aware that there is a typographical error in the statement of their lemma — the correct hypothesis there should be that $\sigma(-P) \cap \sigma(Q) = \varnothing$, as they require in their proof), $T_1^{(m)} \in \mathfrak{E} - \mathfrak{E}$. From this and their characterisation of $\mathfrak{E} - \mathfrak{E}$ (i.e. the eigenvalues of $T_1^{(m)}$ which are different from $0, -1$ and $1$ come in pairs when counted with algebraic multiplicity) and Lemma 3.10, it follows that:

$$\begin{aligned}
\operatorname{TR} K_1 &= \dim \mathcal{H}(G_1; K) - \dim \mathcal{H}(G_{-1}; K) = \dim \mathcal{H}(G_1; T_m) - \dim \mathcal{H}(G_{-1}; T_m) \\
&= \dim \mathcal{H}(G_1; T_1^{(m)}) - \dim \mathcal{H}(G_{-1}; T_1^{(m)}) \\
&= \dim \mathcal{H}(\{1\}; T_1^{(m)}) - \dim \mathcal{H}(\{-1\}; T_1^{(m)}) \\
&= \operatorname{TR}(T_1^{(m)}) \leqslant \operatorname{NUL}(T_1^{(m)} - I) = \operatorname{NUL}(T_m - I).
\end{aligned}$$

But $K = \lim_m T_m$, and thus $\operatorname{NUL}(K - I) \geqslant \operatorname{NUL}(T_m - I)$, whence $\operatorname{NUL}(K - I) \geqslant \operatorname{TR} K_1$, proving (i) above.

There remains to show that $K_4$ is balanced. Obviously it suffices to consider the case where $K \neq 0$. Fix a strictly decreasing sequence $(\delta_n)_n$ of strictly positive real numbers satisfying:

(i) $\delta_2 < 1 < \delta_1 < \delta_0 := \|K\| + 1$;
(ii) $\alpha \in \sigma(K)$ and $\delta_2 < |\alpha| < \delta_1$ implies that $|\alpha| = 1$;
(iii) $\alpha \in \sigma(K)$ implies that $|\alpha| \notin \{\delta_n\}_n$; and
(iv) $\lim_n \delta_n = 0$.

Since $\sigma(K)$ is a sequence converging to 0, this is easy to do.

Given $0 < r < s$, let $A_{r,s} := \{z \in \mathbb{C} : r < |z| < s\}$ be the open annulus centred at 0 of inner radius $r$ and outer radius $s$. Abbreviate this to $\Omega_n := A_{\delta_{n+1},\delta_n}$, $n \geqslant 0$.

For any $n \geqslant 0$, it is clear that $\dim \mathcal{H}(\Omega_n; K) < \infty$. Also, if $\{-1, 1\} \cap \sigma(K) \neq \varnothing$, then $\{-1, 1\} \cap \sigma(K) \subseteq \Omega_1$.

Let $\varepsilon > 0$ and choose $N \geqslant 1$ such that $\delta_N < \varepsilon$. Let $\Omega := \bigcup_{n=0}^{N} \Omega_n$, and let $\Gamma := \{z \in \mathbb{C} : |z| < \delta_{N+1}\}$. Then $\Omega, \Gamma$ are open and disjoint, and $\sigma(K) \subseteq \Omega \cup \Gamma$. In fact, for $0 \leqslant j \leqslant N$, $\sigma(K) \cap \Omega_n$ is a finite set (including multiplicity). Define

$$\begin{aligned}
\eta_1 &:= \frac{1}{2} \min\{|\alpha - \beta| : \alpha, \beta \in \Omega \cap \sigma(K), \alpha \neq \beta\}, \\
\eta_2 &:= \min\{\operatorname{DIST}(\alpha, \partial\Omega_n) : \alpha \in \Omega_n \cap \sigma(K), 0 \leqslant n \leqslant N\},
\end{aligned}$$

and choose $0 < \eta < \min(\eta_1, \eta_2)$.

Then we may find open sets $\Omega_j^{(n)}$, $1 \leqslant j \leqslant r_n$, $0 \leqslant n \leqslant N$ such that:

(I) $\sigma(K) \cap \Omega_j^{(n)}$ contains exactly one element for all $1 \leqslant j \leqslant r_n$, $0 \leqslant n \leqslant N$ (though possibly with algebraic multiplicity greater than one);

(II) $\sigma(K) \cap \Omega = \bigcup_{n=0}^{N} \bigcup_{j=1}^{r_n} (\sigma(K) \cap \Omega_j^{(n)})$;

(III) $\text{DIAM}\, \Omega_j^{(n)} < \eta$ for all $1 \leqslant j \leqslant r_n$, $0 \leqslant n \leqslant N$; and

(IV) $\text{DIAM}\, \Omega_j^{(n)} = \text{DIAM}\, \Omega_k^{(n)}$ for all $1 \leqslant j, k \leqslant r_n$, $0 \leqslant n \leqslant N$.

(In essence, we take a ball of radius $\eta$ around each $\alpha \in \sigma(K) \cap \Omega$, and observe that $\eta < \eta_1$ ensures that each such ball only contains one element of $\sigma(K)$, and that no two such balls intersect. Furthermore, if $\alpha \in \Omega_n$ for some $0 \leqslant n \leqslant N$, then $\eta < \eta_2$ implies that the entire ball of radius $\eta$ centred at $\alpha$ is contained in that $\Omega_n$.)

By the upper-semicontinuity of the spectrum and (an induction argument using Proposition 4.4), there exists $\zeta > 0$ such that if $T \in \mathcal{B}(\mathcal{H})$ and $\|T - K\| < \zeta$, then

(a) $\sigma(T) \subseteq \left( \bigcup_{0 \leqslant n \leqslant N} \left( \bigcup_{1 \leqslant j \leqslant r_n} \Omega_j^{(n)} \right) \right) \cup \Gamma$; and

(b) $\dim \mathcal{H}(\Omega_j^{(n)}; T) = \dim \mathcal{H}(\Omega_j^{(n)}; K) < \infty$, $1 \leqslant j \leqslant r_n$, $0 \leqslant n \leqslant N$.

In particular, since $K \in \text{CLOS}\,(\mathfrak{E} - \mathfrak{E})$, we may assume that $T \in \mathfrak{E} - \mathfrak{E}$ and that $\|T - K\| < \zeta$.

The fact that $\sigma(T) \subseteq \bigcup_{n=0}^{N} \Omega_n \cup \Gamma$ and that these sets are open and disjoint ensures that relative to the decomposition $\mathcal{H} = \bigoplus_{n=0}^{N} \mathcal{H}(\Omega_n; T) \oplus \mathcal{H}(\Gamma; T)$, we may write $T$ as an upper-triangular operator matrix

$$T = [T_{i,j}].$$

Note that for all $0 \leqslant n \leqslant N$, $\sigma(T_{n,n}) \subseteq \Omega_n$, and thus if $0 \leqslant i \neq j \leqslant N$, then $\sigma(-T_{i,i}) \cap \sigma(T_{j,j}) = \varnothing$. Furthermore, $\Omega_n \cap \Gamma = \varnothing$ for all $0 \leqslant n \leqslant N$, and $\sigma(T_{N+1,N+1}) \subseteq \Gamma$. From this we conclude that

$$T \sim \text{DIAG}(T_{0,0}, T_{1,1}, T_{2,2}, \ldots, T_{N,N}, T_{N+1,N+1}).$$

By the Hartwig–Putcha Theorem [13], since $T \in \mathfrak{E} - \mathfrak{E}$, it follows that each $T_{n,n} \in \mathfrak{E} - \mathfrak{E}$, $0 \leqslant n \leqslant N + 1$. But $T_{n,n} \in \mathcal{B}(\mathcal{H}(\Omega_n; T))$, and $\dim \mathcal{H}(\Omega_n; T) < \infty$, $0 \leqslant n \leqslant N$. By the Hartwig–Putcha characterisation of DOIs in the finite-dimensional setting, each $\sigma(T_{n,n})$ is balanced, $0 \leqslant n \leqslant N$, $n \neq 1$, and $\sigma(T_{1,1}) \setminus \{-1, 1\}$ is balanced.

Let $\alpha \in \sigma(K) \cap \Omega$, $\alpha \notin \{-1, 1\}$, and choose $0 \leqslant n \leqslant N$, $1 \leqslant j \leqslant r_n$ such that $\alpha \in \Omega_j^{(n)}$. From (b) above,

$$\dim \mathcal{H}(\Omega_j^{(n)}; T) = \dim \mathcal{H}(\Omega_j^{(n)}; K) = \mu(\alpha),$$

the algebraic multiplicity of $\alpha$ in $\sigma(K)$. Since $T$ is balanced,

$$\dim \mathcal{H}(\Omega_j^{(n)};T) = \dim \mathcal{H}(-\Omega_j^{(n)};T).$$

Now from condition (a) above, the Hausdorff metric

$$\text{DIST}_H(\sigma(T)\cap\Omega, \sigma(K)\cap\Omega) < \eta,$$

and thus there exists $\beta \in \sigma(K)$ such that $|\beta - (-\alpha)| < 2\eta$. But $\eta > 0$ can be chosen arbitrarily small, implying that $-\alpha \in \sigma(K)\cap\Omega_n$. Since $\alpha \neq 0$, $-\alpha \in \Omega_i^{(n)}$ for some $1 \leqslant i \neq j \leqslant r_n$ and hence $\Omega_i^{(n)} = -\Omega_j^{(n)}$.

But then

$$\begin{aligned}\dim \mathcal{H}(\Omega_j^{(n)};K) &= \dim \mathcal{H}(\Omega_j^{(n)};T) = \dim \mathcal{H}(-\Omega_j^{(n)};T) = \dim \mathcal{H}(\Omega_i^{(n)};T)\\ &= \dim \mathcal{H}(\Omega_i^{(n)};K) = \dim \mathcal{H}(-\Omega_j^{(n)};K),\end{aligned}$$

which implies that $(\sigma(K)\setminus\{-1,1\})\cap\Omega$ is balanced.

Recall that $\Omega = \bigcup_{0\leqslant n\leqslant N} \Omega_n$, and that the only condition on $N \geqslant 1$ was that we must have $\delta_N < \varepsilon$. In particular, we can choose $N$ arbitrarily large, and from this we conclude that if $\alpha \in \sigma(K)\setminus\{-1,1\}$, then $-\alpha \in \sigma(K)$ and $\mu(\alpha) = \mu(-\alpha)$. This completes the proof. ∎

THEOREM 4.14. *Let $K \in \mathcal{K}(\mathcal{H})$ and write*

$$K = \begin{bmatrix} K_1 & K_2 \\ 0 & K_4 \end{bmatrix}$$

*relative to the decomposition $\mathcal{H} = \mathcal{H}(\{-1,1\};K)\oplus(\mathcal{H}(\{-1,1\};K))^\perp$. Without loss of generality (by considering $-K$ instead of $K$ if necessary), we may suppose that* $\text{TR}\,(K_1) \geqslant 0$. *The following are equivalent:*

(i) $K \in \text{CLOS}(\mathfrak{E}-\mathfrak{E})$;

(ii) $\text{NUL}\,(K_1 - I) \geqslant \text{TR}\,(K_1)$ *and $\sigma(K_4)$ is balanced.*

*Proof.* (i) $\Longrightarrow$ (ii) This is Theorem 4.12.

(ii) $\Longrightarrow$ (i) If $\sigma(K)\cap\{-1,1\} = \varnothing$, then $K = K_4$ and there remains only to show that $K_4$ is balanced. Otherwise, observe that since $\sigma(K_1) \subseteq \{-1,1\}$ and $\sigma(K_4) \subseteq \sigma(K)\setminus\{-1,1\}$, we have that $\sigma(-K_1)\cap\sigma(K_4) = \varnothing$, whence $K \sim \begin{bmatrix} K_1 & 0 \\ 0 & K_4 \end{bmatrix}$. Since $\dim \mathcal{H}(\{-1,1\};K) < \infty$, $K_1$ acts on a finite-dimensional space, and so we may apply Lemma 3.9 to conclude that $K_1 \in \text{CLOS}\,(\mathfrak{E}-\mathfrak{E})$.

Since $K_4$ is balanced, $K_4 \in \text{CLOS}\,(\mathfrak{E}-\mathfrak{E})$ by Proposition 4.3.

It is now clear that $K_1 \oplus K_4 \in \text{CLOS}\,(\mathfrak{E}-\mathfrak{E})$. Since $K$ is similar to $K_1\oplus K_4$ and since $\mathfrak{E}-\mathfrak{E}$ is invariant under conjugation by invertible elements, $K \in \text{CLOS}\,(\mathfrak{E}-\mathfrak{E})$. ∎

## 5. COMMUTATORS AND DIFFERENCES OF ORTHOGONAL PROJECTIONS

5.1. Our goal in this section is to describe the sets $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{P}})$ and $\text{CLOS}\,(\mathfrak{P} - \mathfrak{P})$. We are aided by the fact that the sets $\mathfrak{C}_{\mathfrak{P}}$ and $\mathfrak{P} - \mathfrak{P}$ have been completely characterised by Li [20] and Davis [7], respectively. We begin with $\text{CLOS}\,(\mathfrak{C}_{\mathfrak{P}})$.

THEOREM 5.1 (Li). *Let $\mathcal{H}$ be a complex separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is a commutator of two orthogonal projections if and only if*

(i) $T^* = -T$;

(ii) $\|T\| \leqslant \frac{1}{2}$; *and*

(iii) $T \simeq T^*$.

5.2. It is worth observing that when $n := \dim \mathcal{H} < \infty$, both $\mathfrak{C}_{\mathfrak{P}}$ and $\mathfrak{P} - \mathfrak{P}$ are norm-closed. Indeed, if $(P_n)_n$, $(Q_n)_n$ are two sequences of orthogonal projections in $\mathcal{B}(\mathbb{C}^n)$, then the fact that the closed unit ball of $\mathcal{B}(\mathbb{C}^n)$ is compact can be used to prove that there exists a strictly increasing sequence $(n_k)_k$ of positive integers such that $P := \lim_k P_{n_k}$ and $Q := \lim_k Q_{n_k}$ both exist. Clearly both $P$ and $Q$ are orthogonal projections, and thus if $T = \lim_n [P_n, Q_n]$, we conclude that $T = [P, Q]$, while if $R = \lim_n (P_n - Q_n)$, then $R = P - Q$.

5.3. We remark that unlike the situation with idempotents, $\mathfrak{C}_{\mathfrak{P}} \not\subseteq \mathfrak{P} - \mathfrak{P}$. For example, if $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, then

$$[P, Q] = \begin{bmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{bmatrix} \neq [P, Q]^*,$$

while any difference of orthogonal projections is clearly self-adjoint. Indeed, condition (a) of Li's Theorem above implies that $\mathfrak{C}_{\mathfrak{P}} \cap (\mathfrak{P} - \mathfrak{P}) = \{0\}$.

LEMMA 5.2. *Let $\mathcal{H}$ be a complex separable Hilbert space and $K = K^* \in \mathcal{B}(\mathcal{H})$. Suppose furthermore that $K \simeq_{\mathrm{a}} -K$. Given $\varepsilon > 0$, there exists an operator $L_\varepsilon = L_\varepsilon^* \in \mathcal{B}(\mathcal{H})$ satisfying:*

(i) $\|L_\varepsilon\| \leqslant \|K\|$;

(ii) $\|L_\varepsilon - K\| < 2\varepsilon$; *and*

(iii) $L_\varepsilon \simeq -L_\varepsilon$.

*Proof.* Clearly it suffices to consider the case where $\|K\| = 1$.

By hypothesis, $K$ is self-adjoint (hence normal) and $K$ is approximately unitarily equivalent to $-K$. It follows from the Weyl–von Neumann–Berg Theorem (see, e.g. [6, Theorem II.4.4]) that $\sigma(K) = -\sigma(K)$, and if $\alpha \in \sigma(K)$ is isolated, then

$$\text{NUL}\,(K - \alpha I) = \text{NUL}\,(K + \alpha I).$$

Thus, if $0 < \varepsilon < 1$, then $\sigma(K) \cap [\varepsilon, 1] = -(\sigma(K) \cap [-1, -\varepsilon])$, including the multiplicity (finite or infinite) of isolated eigenvalues. Let $\mathcal{M}_\varepsilon^+ := \mathcal{H}(\sigma(K) \cap [\varepsilon, 1]; K)$ and $\mathcal{M}_\varepsilon^- := \mathcal{H}(\sigma(K) \cap [-1, -\varepsilon]; K)$. Set $\mathcal{N}_\varepsilon := \mathcal{H} \ominus (\mathcal{M}_\varepsilon^+ \oplus \mathcal{M}_\varepsilon^-)$.

Again, by the Weyl–von Neumann–Berg Theorem,

$$K|_{\mathcal{M}_\varepsilon^+} \simeq_{\mathrm{a}} -K|_{\mathcal{M}_\varepsilon^-}.$$

Relative to $\mathcal{H} = \mathcal{M}_\varepsilon^- \oplus \mathcal{N}_\varepsilon \oplus \mathcal{M}_\varepsilon^+$, we may write

$$K = \begin{bmatrix} K_\varepsilon^- & & \\ & K_\varepsilon^\circ & \\ & & K_\varepsilon^+ \end{bmatrix}.$$

From above, $K_\varepsilon^- \simeq_{\mathrm{a}} -K_\varepsilon^+$, and so we can find a unitary operator $V$ such that

$$\|V^*(-K_\varepsilon^+)V - K_\varepsilon^-\| < \varepsilon.$$

Also, $\|K_\varepsilon^\circ\| \leqslant \varepsilon$. Define (with respect to the same decomposition of $\mathcal{H}$) the operator

$$L_\varepsilon := \begin{bmatrix} V^*(-K_\varepsilon^+)V & & \\ & 0 & \\ & & K_\varepsilon^+ \end{bmatrix}.$$

Clearly $L_\varepsilon \simeq -L_\varepsilon$, and $\|L_\varepsilon - K\| \leqslant \varepsilon < 2\varepsilon$. Note also that $\|L_\varepsilon\| = \|K_\varepsilon^+\| \leqslant \|K\|$. ∎

REMARK 5.3. We note that from the construction of $L_\varepsilon$ above, if $\|K\|$ is not an eigenvalue of $K$, then neither $-\|K\|$ nor $\|K\|$ are eigenvalues of $L_\varepsilon$.

We are now in a position to characterise $\mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{P}})$.

THEOREM 5.4. *Let $\mathcal{H}$ be a complex separable Hilbert space. An operator $T \in \mathcal{B}(\mathcal{H})$ is a limit of commutators of projections if and only if it satisfies the following three conditions:*

(i) $T^* = -T$;

(ii) $\|T\| \leqslant \frac{1}{2}$; *and*

(iii) *$T$ is approximately unitarily equivalent to $T^*$.*

*Proof.* Suppose that $T$ satisfies the above three conditions. Let $K := \mathrm{i}T$. Then $K^* = (\mathrm{i}T)^* = -\mathrm{i}(-T) = \mathrm{i}T = K$, $\|K\| = \|T\| \leqslant \frac{1}{2}$, and if $T^* = \lim_n U_n^* T U_n$, where each $U_n$ is unitary, $n \geqslant 1$, then

$$K = K^* = -\mathrm{i}T^* = -\mathrm{i}\lim_n U_n^* T U_n = \lim_n U_n^*(-\mathrm{i}T)U_n = \lim_n U_n^*(-K)U_n.$$

That is, $K$ is self-adjoint and approximately unitarily equivalent to $-K$.

Let $\varepsilon > 0$, and using Lemma 5.2, we may choose $L_\varepsilon = L_\varepsilon^*$ such that:

(a) $\|L_\varepsilon\| \leqslant \|K\| \leqslant \frac{1}{2}$;

(b) $\|L_\varepsilon - K\| < 2\varepsilon$; and

(c) $L_\varepsilon \simeq -L_\varepsilon$.

By Theorem 5.1, $T_\varepsilon := -\mathrm{i}L_\varepsilon \in \mathfrak{C}_{\mathfrak{P}}$ and clearly $\|T - T_\varepsilon\| < 2\varepsilon$. It follows that $T \in \mathrm{CLOS}\,(\mathfrak{C}_{\mathfrak{P}})$.

The reverse containment is straightforward and is left to the reader. ∎

Our next goal is to classify the closure $\operatorname{CLOS}(\mathfrak{P}-\mathfrak{P})$ of the set $\mathfrak{P}-\mathfrak{P}$ of differences of projections in $\mathcal{B}(\mathcal{H})$, where $\mathcal{H}$ is a complex separable Hilbert space. We first recall the Theorem of Davis [7, Theorem 6.1].

THEOREM 5.5 (Davis). *Let $H \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator of norm at most one, and define $\mathcal{H}_0 := (\ker(H+I) \oplus \ker H \oplus \ker(H-I))^{\perp}$. The following are equivalent:*
(i) $H \in \mathfrak{P}-\mathfrak{P}$;
(ii) $H_0 \simeq -H_0$, *where* $H_0 := H|_{\mathcal{H}_0}$.

We require a couple of standard results; the first is due to Newburgh [21] (alternatively, see [12, Problem 105]).

THEOREM 5.6 (Newburgh). *Let $(M_k)_k$ be a sequence of normal operators on $\mathcal{H}$ which converge in norm to $M \in \mathcal{B}(\mathcal{H})$. Then $(\sigma(M_k))_k$ converges to $\sigma(M)$ in the Hausdorff metric.*

The *essential spectrum* of an operator $T \in \mathcal{B}(\mathcal{H})$ is the spectrum $\sigma_{\mathrm{e}}(T) := \sigma(\pi(T))$, where $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the canonical quotient map. When $T$ is a normal operator, the relationship between $\sigma(T)$ and $\sigma_{\mathrm{e}}(T)$ is particularly simple, and is given by the next result [5, Proposition 4.6].

PROPOSITION 5.7. *If $N \in \mathcal{B}(\mathcal{H})$ is a normal operator, then*

$$\sigma(N) \setminus \sigma_{\mathrm{e}}(N) = \{\lambda \in \sigma(N) : \lambda \text{ is an isolated eigenvalue of finite multiplicity of } N\}.$$

We now have all the tools we need to characterise the set $\operatorname{CLOS}(\mathfrak{P}-\mathfrak{P})$.

THEOREM 5.8. *An operator $H \in \mathcal{B}(\mathcal{H})$ is a limit of differences of projections if and only if it satisfies the following two conditions:*
(i) $-I \leqslant H \leqslant I$; *and*
(ii) *if* $\mathcal{N} := (\ker(H^2 - I))^{\perp}$, *and* $H_1 := H|_{\mathcal{N}}$, *then* $H_1 \simeq_{\mathrm{a}} -H_1$.

*Proof.* Suppose that $K_n \in \mathfrak{P}-\mathfrak{P}$, $n \geqslant 1$ and that $H = \lim_n K_n$. Since $K_n = K_n^*$ for all $n \geqslant 1$, we have that $H = H^*$. Also, since $-I \leqslant K_n \leqslant I$ for all $n \geqslant 1$, we have that $-I \leqslant H \leqslant I$.

It is now a consequence of the Weyl–von Neumann–Berg Theorem [6, Theorem II.4.4] and Proposition 5.7 that to prove that $H_1 \simeq_{\mathrm{a}} -H_1$, it suffices to show that $\sigma(H_1) = -\sigma(H_1)$, and if $\alpha \in \sigma(H_1)$ satisfying $|\alpha| < 1$ is an isolated eigenvalue of finite multiplicity $\mu(\alpha)$, then $-\alpha \in \sigma(H_1)$ is an isolated eigenvalue of the same multiplicity. (Note that $\sigma(H_1) = -\sigma(H_1)$ implies that $\sigma(H_1) \subseteq [-1,1]$ is symmetric about the origin. Furthermore, by definition of $\mathcal{N}$, $1 \in \sigma(H_1)$ if and only if 1 is a limit of a sequence $(\alpha_n)_n$ in $\sigma(H_1) \cap (-1,1)$, in which case $-\alpha_n \in \sigma(H_1)$ for all $n \geqslant 1$, and thus $-1 = \lim_n -\alpha_n \in \sigma(H_1)$ as well.)

Recall that $H = \lim_n K_n$. By Newburgh's Theorem 5.6,

$$\lim_n d_H(\sigma(K_n), \sigma(H)) = 0.$$

Hence, if $\alpha \in \sigma(H)$ with $|\alpha| < 1$, there exists $\alpha_n \in \sigma(K_n)$ with $\lim_n \alpha_n = \alpha$. Of course, this in turn implies that there exists $N_1 \geqslant 1$ such that $n \geqslant N_1$ implies that $|\alpha_n| < 1$.

But $K_n \in \mathfrak{P} - \mathfrak{P}$, and thus by Davis' Theorem 5.5, $-\alpha_n \in \sigma(K_n)$. By Newburgh's Theorem 5.6 (once again), $-\alpha = \lim_n -\alpha_n \in \sigma(H)$. Thus $\sigma(H_1) \cap (-1,1) = -\sigma(H_1) \cap (-1,1)$, and from this we see that $\sigma(H_1) = -\sigma(H_1)$.

Suppose next that $\alpha \in \sigma(H_1) \cap (-1,1)$ is an isolated eigenvalue of finite multiplicity $\mu(\alpha)$. Since $\sigma(H_1) = -\sigma(H_1)$, $-\alpha \in \sigma(H_1)$ is an isolated eigenvalue of multiplicity $\mu(-\alpha)$. Fix $0 < \varepsilon < \min(|\alpha|, 1 - |\alpha|)$ such that

$$(\alpha - \varepsilon, \alpha + \varepsilon) \cap \sigma(H_1) = \{\alpha\}.$$

By the symmetry of $\sigma(H_1) \subseteq \mathbb{R}$ about the origin, we have that

$$(-\alpha - \varepsilon, -\alpha + \varepsilon) \cap \sigma(H_1) = \{-\alpha\}.$$

Furthermore,

$$\mu(\alpha) = \dim \mathcal{H}((\alpha - \varepsilon, \alpha + \varepsilon) \cap \sigma(H_1); H_1).$$

By Proposition 4.4, there exists $N_2 \in \mathbb{N}$ such that $n \geqslant N_2$ implies that

$$\begin{aligned} &\dim \mathcal{H}((\alpha - \varepsilon, \alpha + \varepsilon) \cap \sigma(K_n); K_n) = \mu(\alpha), \quad \text{and} \\ &\dim \mathcal{H}((-\alpha - \varepsilon, -\alpha + \varepsilon) \cap \sigma(K_n); K_n) = \mu(-\alpha). \end{aligned}$$

The fact that each $K_n \in \mathfrak{P} - \mathfrak{P}$ implies (by Davis' Theorem) that

$$\dim \mathcal{H}((\alpha - \varepsilon, \alpha + \varepsilon) \cap \sigma(K_n); K_n) = \dim \mathcal{H}((-\alpha - \varepsilon, -\alpha + \varepsilon) \cap \sigma(K_n); K_n),$$

whence

$$\mu(\alpha) = \mu(-\alpha),$$

completing the proof of the fact that $H \in \mathfrak{P} - \mathfrak{P}$ implies both conditions (i) and (ii).

Conversely, suppose that $H$ satisfies (i) and (ii) above. Of course, $-I < H_1 < I$ is an hermitian operator. Relative to $\mathcal{H} = \mathcal{N}^\perp \oplus \mathcal{N}$, we may write

$$H = \begin{bmatrix} H^\circ & \\ & H_1 \end{bmatrix}.$$

(Note that $H^\circ$ is hermitian with $\sigma(H^\circ) \subseteq \{-1, 1\}$.) Let $\varepsilon > 0$. Using Lemma 5.2, we can find an hermitian operator $L_\varepsilon$ such that:

(a) $\|L_\varepsilon\| \leqslant \|H_1\| \leqslant 1$;
(b) $\|L_\varepsilon - H_1\| < 2\varepsilon$; and
(c) $L_\varepsilon \simeq -L_\varepsilon$.

Let $H_\varepsilon := \begin{bmatrix} H^\circ & \\ & L_\varepsilon \end{bmatrix}$. As noted in Remark 5.3 (and keeping in mind that $\|L_\varepsilon\| \leqslant \|H_1\|$), since 1 is not an eigenvalue of $H_1$, neither $-1$ nor 1 are eigenvalues of $L_\varepsilon$. It is clear that $\|H_\varepsilon - H\| < 2\varepsilon$ and $H_\varepsilon \in \mathfrak{P} - \mathfrak{P}$ by Theorem 5.5.

Thus $H \in \text{CLOS}\,(\mathfrak{P} - \mathfrak{P})$. ∎

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