J. OPERATOR THEORY
91:1(2024), 239–259
doi: 10.7900/jot.2022apr08.2381

PROPERTIES PRESERVED BY GROUPOID EQUIVALENCE

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Communicated by Stefaan Vaes

ABSTRACT. We show that groupoid equivalence preserves a number of groupoid properties such as properness or the property of being topologically principal.

KEYWORDS: *Groupoids, groupoid equivalence, groupoid $*$-algebras.*

MSC (2020): Primary 46L05, 46L55.

INTRODUCTION

The *raison d'être* for groupoid equivalence is that equivalent second countable groupoids with Haar systems have Morita equivalent groupoid C^*-algebras [12]. Then we can take advantage of the many properties of C^*-algebras preserved by Morita equivalence [2, 32]. This has implications for the groupoids themselves. For example, if G and H are equivalent second countable groupoids with Haar systems, then $C^*(G)$ is GCR if and only if $C^*(H)$ is GCR [32, Proposition 3.2]. However, Clark and the first author have shown that $C^*(G)$ is GCR if and only if each G-orbit in $G^{(0)}$ is locally closed and every isotropy group is GCR — see [17, Theorem 7.1] and [27, Theorem 4.2]. Hence the same must be true for any second countable groupoid H with a Haar system that is equivalent to G.

In this article, we want to examine properties of groupoids that are preserved by groupoid equivalence. Unlike the example above, our methods do not go through C^*-theory. Hence we can separate the property that orbits are locally closed or the type of the isotropy groups and see that these are preserved individually. We can also study properties that have important implications to the structure of the groupoid C^*-algebra, such as the groupoid being a proper groupoid, and show that these properties are preserved as well. Since we do not use C^*-theory, we can avoid requiring that our groupoids are second countable and/or that they have Haar systems. Nevertheless our methods do require that groupoids have open range and source maps. This is clearly the most important class of groupoids as it is implied by the existence of a Haar system.

Our paper is organized as follows. In Section 1 we establish our definitions and recall the notion of the blow-up of a groupoid. Since two groupoids are equivalent if and only if they have isomorphic blow-ups, this allows us to reduce many questions to the case of a blow-up.

In Section 2, we show that groupoid equivalence preserves proper groupoids as well as Cartan groupoids. We also observe that if G and H are equivalent, then every isotropy group of G is isomorphic to an isotropy group of H and vice versa. Hence if G has abelian isotropy, then so does any groupoid H equivalent to G.

In Section 3 we show that equivalence preserves the property that a groupoid is topologically principal in that the set of points $u \in G^{(0)}$ where the isotropy group, $G(u)$, is trivial is dense in the unit space. We show this is also true for the property of being essentially principal meaning points with trivial isotropy are dense in every closed invariant subset of the unit space. Both these assertions also hold when we replace trivial isotropy with the stronger notion of discretely trivial isotropy introduced by Renault in [21].

In Section 4, we see that the orbit spaces of equivalent groupoids are homeomorphic and what equivalence implies about the continuity of the isotropy map $u \mapsto G(u)$. We apply this to show that the property of being proper modulo the isotropy introduced in [28] is preserved by equivalence.

In Section 5 we examine how the concept of integrablity, introduced by Clark and an Huef, behaves under equivalence.

In Section 6, we recall and sharpen some results from [14, Section 3] about equivalence of twists.

ASSUMPTIONS. Here groupoid always means a locally compact Hausdorff groupoid with open range and source maps. An isomorphism of groupoids means an isomorphism of topological groupoids; that is, an algebraic isomorphism that is also a homeomorphism.

1. PRELIMINARIES

Groupoid actions are discussed in detail in [31, Section 2.1]. Here we recall that in order for a groupoid G to act on a space P we require a continuous *moment map* $r_P : P \to G^{(0)}$ so that $\gamma \cdot p$ is defined exactly when $s(\gamma) = r_P(p)$. In most situations, we drop the subscript "P" and simply write $s(\gamma) = r(p)$ and trust the meaning is clear from context. We say that the action is *free* if $\gamma \cdot p = p$ if and only if $\gamma = r(p)$ and that the action is *proper* if the map $\Theta : G * P \to P \times P$ given by $(\gamma, p) \mapsto (\gamma \cdot p, p)$ is a proper map in that $\Theta^{-1}(K)$ is compact whenever $K \subset P \times P$ is compact.

In many places in the literature, moment maps for groupoid actions are also required to be open. For example, this is the case in [12] and [24]. One reason for this is that it is an appropriate assumption in the definition of groupoid equivalence.

DEFINITION 1.1 ([31, Definition 2.29]). Suppose that G and H are locally compact Hausdorff groupoids. Then a locally compact Hausdorff space Z is called a *(G, H)-equivalence* if the following conditions are satisfied:
(E1) Z is a free and proper left G-space;
(E2) Z is a free and proper right H-space;
(E3) the G- and H-actions commute;
(E4) the moment map r_Z is open and induces a homeomorphism of Z/H onto $G^{(0)}$;
(E5) the moment map s_Z is open and induces a homeomorphism of $G\backslash Z$ onto $H^{(0)}$.

REMARK 1.2. Definition 1.1 could be used even when G and H are not required to have open range and source maps. However, when G and H do have open range and source maps, it follows from [31, Remark 2.31] that we can replace (E4) and (E5) with the assertion that the moment maps r_Z and s_Z are open and simply induce bijections. Since moment maps were always assumed to be open in [12] and [24], Definition 1.1 agrees with [12, Definition 2.1] and [24, Definition 1.4].

Since we sometimes construct new groupoids from existing ones, it is necessary to take some care to verify that the new range and source maps are open. As a bit of comfort, we include the following observation which is definitely in the spirit of the central theme of this paper.

LEMMA 1.3. *Suppose that G and H are equivalent locally compact Hausdorff groupoids that do not necessarily have open range and source maps. If H has open range and source maps, then so does G.*

Proof. Let Z be a (G, H) equivalence. It suffices to see that $r : G \to G^{(0)}$ is open. To do this, we use Fell's criterion [31, Proposition 1.1]. Hence if $u_n \to r(\gamma)$ in $G^{(0)}$, it will suffice to pass to a subnet, relabel, and find $\gamma_n \to \gamma$ with $r(\gamma_n) = u_n$. Since r_Z induces a homeomorphism of Z/H onto $G^{(0)}$, there are $z_n \in Z$ with $r_Z(z_n) = u_n$ and $z_n \cdot H \to z \cdot H$ in Z/H with $r_Z(z) = r(\gamma)$. Since H has open range and source maps, the orbit map $q : Z \to Z/H$ is open [31, Proposition 2.12]. Thus we can pass to a subnet, relabel, and assume that there are $\eta_n \in H$ such that $z_n \cdot \eta_n \to z$ in Z. Replacing $z_n \cdot \eta_n$ with z_n, we have $z_n \to z$ with $r_Z(z_n) = u_n$ and $r_Z(z) = r(\gamma)$.

Since s_Z is open by definition, we can pass to another subnet, relabel, and assume that there are $w_n \to \gamma \cdot z$ in Z with $s_Z(w_n) = s_Z(z_n)$. Since $s_Z(w_n) = s_Z(z_n)$ there is a unique $\gamma_n \in G$ such that $w_n = \gamma_n \cdot z_n$. Since Z is a proper G-space, [31, Proposition 2.17] implies that we can pass to a subnet and assume that

$\gamma_n \to \gamma'$ in G. Since $(\gamma_n \cdot z_n)$ converges to both $\gamma' \cdot z$ and $\gamma \cdot z$, we have $\gamma' = \gamma$. But $r(\gamma_n) = r_Z(z_n) = u_n$ and we are done. ∎

A well-known and fundamental example of groupoid equivalence is provided by the *blow-up* construction. If $f : X \to G^{(0)}$ is a continuous *open* surjection, then the blow-up is the groupoid

$$G[X] = \{(x, \gamma, y) \in X \times G \times X : f(x) = r(\gamma) \text{ and } s(\gamma) = f(y)\}$$

equipped with the obvious groupoid structure (see [31, Example 2.37] for details). Then G and $G[X]$ are equivalent groupoids. The proof requires both the openness of f as well as the assumption that G has open range and source maps.

The unit space of $G[X]$ is $\{(x, f(x), x) : x \in X\}$ which we often silently identify with X.

An important feature of blow-ups, especially for our needs here, is the following result. A proof is given in [31, Theorem 2.52].

THEOREM 1.4. *Two groupoids G and H are equivalent if and only if they have isomorphic blow-ups.*

REMARK 1.5 (Standard Technique). In the sequel, we will repeatedly use the following observation. Since topological properties of groupoids are clearly preserved by isomorphism, to show that such a property is preserved by equivalence of groupoids, it follows from Theorem 1.4 that it suffices to show that G has the property if and only if any blow-up $G[X]$ has the same property.

2. PROPER AND CARTAN GROUPOIDS

Recall that we call a groupoid G is *proper* if it acts properly on its unit space. Our next example illustrates that interesting proper groupoids exist in abundance.

EXAMPLE 2.1. Suppose that G acts properly on the left of the space P. Let $G \rtimes P = \{(p, \gamma, q) \in P \times G \times P : p = \gamma \cdot q\}$ be the action groupoid as in [31, Definition 2.5]. Then $G \rtimes P$ is a proper groupoid — see [31, Example 2.1.14]. (Note that this includes the assertion that $G \rtimes P$ has open range and source maps even if the moment map is not open. See [31, Remark 2.14].)

A slightly weaker notion is that of a *Cartan* groupoid. In analogy with [19, Definition 1.1.2], a G-space P is called a *Cartan G-space* if every point in P has a compact neighborhood K such that $\Theta^{-1}(K \times K)$ is compact [31, Definition 2.22]. We call a groupoid G Cartan if $G^{(0)}$ is a Cartan G-space for the usual G-action. Just as in Example 2.1, if P is a Cartan G-space, then the action groupoid $G \rtimes P$ is a Cartan groupoid. Clearly, every proper G-space is Cartan, but the converse can fail even for group actions — see [19, p. 298].

We can now state our first permanence result.

THEOREM 2.2. *Suppose that G and H are equivalent groupoids:*
(i) *G is proper if and only if H is proper;*
(ii) *G is Cartan if and only if H is Cartan.*

We start with some preliminary observations. We let $\pi : G \to G^{(0)} \times G^{(0)}$ be given by $\pi(\gamma) = (r(\gamma), s(\gamma))$.

LEMMA 2.3. *A groupoid G is proper if and only if whenever a net $(\gamma_n) \subset G$ is such that $\pi(\gamma_n) \to (u, v) \in G^{(0)} \times G^{(0)}$, then (γ_n) has a convergent subnet.*

Proof. Since $r(\gamma_n) = \gamma_n \cdot s(\gamma_n)$, the lemma is just a restatement of [31, Proposition 2.17(PA3)]. ∎

LEMMA 2.4. *Suppose that $f : X \to G^{(0)}$ is a continuous, open surjection. Then G is proper if and only if $G[X]$ is proper.*

Proof. We use the criterion given in Lemma 2.3.

Suppose that G is proper and that $((x_n, \gamma_n, y_n)) \subset G[X]$ is such that $x_n \to x$ and $y_n \to y$ in X, respectively. Then $r(\gamma_n) \to f(x)$ and $s(\gamma_n) \to f(y)$. Since G is proper, (γ_n) has a convergent subnet. But then $((x_n, \gamma_n, y_n))$ also has a convergent subnet. Hence $G[X]$ is proper.

Conversely, suppose that $G[X]$ is proper and that (γ_n) is a net in G such that $(\pi(\gamma_n))$ converges in $G^{(0)} \times G^{(0)}$. Since f is open, we can pass to a subnet, relabel, and assume that there are $x_n \in X$ such that $f(x_n) = r(\gamma_n)$ and $x_n \to x$ in X. Similarly, we can pass to another subnet, relabel, and assume that there are $y_n \in X$ such that $f(y_n) = s(\gamma_n)$ and $y_n \to y$. Since $G[X]$ is proper and $(x_n, \gamma_n, y_n) \cdot y_n = x_n$, the net $((x_n, \gamma_n, y_n))$ must have a convergent subnet. Since the topology on $G[X]$ is the relative product topology, (γ_n) must have a convergent subnet. Hence G is proper as claimed. ∎

REMARK 2.5. Although the first part of the proof of Lemma 2.4 did not require the openness of f, it is still required to see that G and $G[X]$ are equivalent.

The following criterion follows from [31, Definition 2.22 and Lemma 2.23].

LEMMA 2.6. *A G-space P is Cartan if and only if every $p \in P$ has a compact neigbhorhood K such that the following is compact in G:*

$$P(K, K) = \{\gamma \in G : \gamma \cdot K \cap K \neq \emptyset\}.$$

LEMMA 2.7. *Suppose that $f : X \to G^{(0)}$ is a continuous, open surjection. Then G is Cartan if and only if $G[X]$ is Cartan.*

Proof. We will use the criterion from Lemma 2.6.

Suppose that G is Cartan and $x \in X$. Let $u = f(x)$. By assumption, there is a compact neighborhood K of u such that

$$P(K, K) = \{\gamma \in G : s(\gamma) \in K \text{ and } r(\gamma) \in K\}$$

is compact in G. Since f is continuous and open, there is a compact neighborhood, K', of x in X such that $f(K') = K$. (Note that every point $y \in f^{-1}(K)$ has a compact neighborhood K_y. Hence there are finitely many y_i such that $K \subset \bigcup_i f(K_{y_i})$. Then $K' = f^{-1}(K) \cap \left(\bigcup K_{y_i} \cup K_x \right)$ is compact.)

Suppose that $((x_i, \gamma_i, y_i))$ is a net in $P(K', K')$. Then (x_i) and (y_i) are nets in K' while (γ_i) is a net in $P(K, K)$. Passing to multiple subnets, and relabeling, we can assume that $(x_i, \gamma_i, y_i) \to (x, \gamma, y)$ in $G[X]$. Since $P(K', K')$ is closed, it must be compact. Since $x \in X$ was arbitrary, $G[X]$ is Cartan.

Conversely, suppose that $G[X]$ is Cartan. Fix $u \in G^{(0)}$. Let $x \in f^{-1}(u)$, and let K' be a compact neighborhood of x such that $P(K', K')$ is compact in $G[X]$. Let $K = f(K')$. Since f is continuous and open, K is a compact neighborhood of u. Let (γ_i) be a net in $P(K, K)$. Therefore, $r(\gamma_i)$ and $s(\gamma_i)$ are both in K. Let $x_i, y_i \in K'$ be such that $f(x_i) = r(\gamma_i)$ and $f(y_i) = s(\gamma_i)$. Then $((x_i, \gamma_i, y_i))$ is a net in the compact set $P(K', K')$. Therefore (γ_i) has a convergent subnet and the closed set $P(K, K)$ must be compact. ∎

Proof of Theorem 2.2. The theorem now follows immediately from Lemmas 2.4 and 2.7 as in Remark 1.5. ∎

Free Cartan spaces play a significant role due to the following observation. If P is a free G-space, then we let

$$R(G, P) = \{(p, q) \in P \times P : G \cdot p = G \cdot q\}.$$

Then since the action is free, there is a well-defined map $\tau : R(G, P) \to G$, called the *translation map*, such that $\tau(p, q) \cdot q = p$.

LEMMA 2.8. *Let P be a free G-space. Then P is a Cartan G-space if and only if its translation map is continuous.*

This lemma is [31, Proposition 2.24].

Recall that if $u \in G^{(0)}$, then $G(u) = \{\gamma \in G : r(\gamma) = u = s(\gamma)\}$ is called the *isotropy group* of G at u. We need the following which will be used repeatedly in the sequel.

PROPOSITION 2.9. *Suppose that Z is a (G, H)-equivalence. Then the isotropy groups $G(r(z))$ and $H(s(z))$ are isomorphic.*

Proof. This is [31, Example 2.3.7]. We supply a proof for convenience. Let $u = r(z)$ and $v = s(z)$. Define $\phi : G(u) \to H(v)$ by $\phi(g) = \tau_H(z, g \cdot z)$ where τ_H is the translation map for the free and proper right H-action on Z (see [31, Lemma 2.42]). Then ϕ is continuous and $\phi(g)$ is the unique element of $H(v)$ such that $z \cdot \phi(g) = g \cdot z$. Therefore ϕ is injective and $z \cdot \phi(gg') = (gg') \cdot z = g \cdot (g' \cdot z) = g \cdot (z \cdot \phi(g')) = (g \cdot z) \cdot \phi(g') = z \cdot (\phi(g)\phi(g'))$. Therefore ϕ is a homomorphism. We can also define a continuous map $\psi : H(v) \to G(u)$ by $\psi(h) = \tau_G(z \cdot h, z)$. It is not hard to check that $\psi = \phi^{-1}$. ∎

Recall that a groupoid G is principal if the action of G on $G^{(0)}$ is free. It follows from Proposition 2.9 that equivalence preserves principal groupoids. Moreover, if G is a principal Cartan groupoid, then $G^{(0)}$ is a free Cartan G-space. Hence using Proposition 2.9, we get the following corollary of Theorem 2.2(ii).

COROLLARY 2.10. *Suppose that G and H are equivalent groupoids. Then H is a principal Cartan groupoid if and only if G is a principal Cartan groupoid.*

REMARK 2.11. We can prove Corollary 2.10 without reference to Theorem 2.2 as follows. If $G[X]$ is a blow-up of a principal groupoid G, then $G[X]$ is principal and the two translation maps on the unit spaces are related by

$$\tau_{G[X]}(x,y) = (x, \tau_G(f(x), f(y)), y).$$

Now we can employ Lemma 2.6 and use Remark 1.5 as usual.

3. TOPOLOGICALLY AND ESSENTIALLY PRINCIPAL GROUPOIDS

There are many situations where having a dense set of points where the isotropy is trivial is a crucial hypothesis. Therefore, we make the following definitions.

DEFINITION 3.1. A groupoid G is *topologically principal* if $\{u \in G^{(0)} : G(u) = \{u\}\}$ is dense in $G^{(0)}$. We say that G is *essentially principal* if for every G-invariant closed set $F \subset G^{(0)}$, $\{u \in F : G(u) = \{u\}\}$ is dense in F.

REMARK 3.2. The terminology adopted here is far from being universally accepted. In fact, the literature is rather chaotic in naming these properties. What is called here "topologically principal" has been called "topologically free" in many places and certainly that term has been conflated with "essentially free" as well. The second author used "essentially free" in place of "essentially principal" in [31, Chapter 11]. Renault used "essentially free" in [21, Definition 4.8] for the stronger property that points with discretely trivial isotropy are dense in closed invariant sets (which was called "strongly essentially free" in [31, Proposition 11.35]). Unfortunately, it seems "essentially free" has a different meaning in group dynamics. Renault also uses the term topologically principal in [20, Definition 3.5], but topologically principal was often called essentially principal in articles such as [3] and topologically free in [25].

REMARK 3.3. Note that G is essentially principal if and only if the reduction, $G(F) = \{\gamma \in G : r(\gamma) \in F \text{ and } s(\gamma) \in F\}$, is topologically principal for every G-invariant closed subset $F \subset G^{(0)}$. (Since F is saturated, $G(F)$ has open range and source maps.)

THEOREM 3.4. *Suppose that G and H are equivalent groupoids:*

(i) *G is topologically principal if and only if H is topologically principal;*

(ii) *G is essentially principal if and only if H is essentially principal.*

As in the previous section, we will work first with blow-ups. We will repeatedly use the observation that if $f : X \to G^{(0)}$ is a continuous open surjection, then the isotropy group $G[X](x)$ is trivial if and only if the isotropy group $G(f(x))$ is trivial.

LEMMA 3.5. *Suppose that $f : X \to G^{(0)}$ is a continuous, open surjection. Then G is topologically principal if and only if $G[X]$ is topologically principal.*

Proof. If G is topologically principal and $x \in X$, then there are $u_i \to f(x)$ with $G(u_i)$ trivial. Since f is open, we can pass to subnet, relabel, and assume that there are $x_i \to x$ with $f(x_i) = u_i$. Then $G[X](x_i)$ is trivial, and it follows that $G[X]$ is topologically principal.

Conversely, suppose that $G[X]$ is topologically principal and that $f(x) \in G^{(0)}$. Then there is a sequence $(x_i) \subset X$ with $x_i \to x$ and each $G[X](x_i)$ trivial. But then $f(x_i) \to f(x)$ and each $G(f(x_i))$ is trivial. Therefore G is topologically principal. ∎

To get a similar result for essentially principal groupoids, we can use the observation that if $F \subset G^{(0)}$ is closed and $C = f^{-1}(F)$, then the restriction $f|_C : C \to F$ is open and

$$G[X](C) = G(F)[C]. \tag{3.1}$$

COROLLARY 3.6. *Suppose that $f : X \to G^{(0)}$ is a continuus, open surjection. Then G is essentially principal if and only if $G[X]$ is essentially principal.*

Proof. Suppose that $G[X]$ is essentially principal. Let F be a closed G-invariant subset of $G^{(0)}$. Then $C := f^{-1}(F)$ is closed. Suppose $x \in C$ and $(x, \gamma, y) \in G[X]$. Then $f(x) = r(\gamma)$ and $s(\gamma) = f(y)$. But $r(\gamma) \in F$ implies $s(\gamma) \in F$. Hence $y \in C$. Therefore C is $G[X]$-invariant. Then by assumption $G[X](C)$ is topologically principal. By (3.1), $G(F)[C]$ is topologically principal and $G(F)$ is too by Lemma 3.5. Since F was arbitrary, this shows that G is essentially principal.

Conversely, suppose that G is essentially principal and that C is a closed $G[X]$-invariant subset of X. We claim that $F := f(C)$ is closed in $G^{(0)}$. Suppose $f(x_i) \to u$ with each $x_i \in C$. Let $u = f(x)$. Since f is open, we can pass to subnet, relabel, and assume that there are $y_i \in X$ with $y_i \to x$ and $f(y_i) = f(x_i)$. But then $(x_i, f(x_i), y_i) \in G[X]$. Since C is $G[X]$-invariant and $x_i \in C$, this means each $y_i \in C$. Since C is closed, $x \in C$ and $u \in F$. Thus F is closed.

To see that F is G-invariant, suppose $r(\gamma) \in F$. Therefore there is a $x \in C$ such that $f(x) = r(\gamma)$. Let y be such that $f(y) = s(\gamma)$. Then $(x, \gamma, y) \in G[X]$ and $x \in C$. Since C is $G[X]$-invariant, we must have $y \in C$. This means that $s(\gamma) \in F$, and F is a closed G-invariant subset of $G^{(0)}$.

Suppose that $y \in f^{-1}(F)$. Then there is a $x \in C$ such that $f(x) = f(y)$. Then $(x, f(x), y) \in G[X]$. Since C is $G[X]$-invariant, $y \in C$. Hence $C = f^{-1}(F)$.

By assumption, $G(F)$ is topologically principal. By Lemma 3.5, $G(F)[C]$ is too. By (3.1), $G[X](C)$ is also topologically principal. Since C was arbitrary, $G[X]$ is essentially principal. ∎

A locally compact Hausdorff *étale* groupoid G is called *effective* if the interior of $\mathrm{Iso}(G) = \{\gamma \in G : r(\gamma) = s(\gamma)\}$ is reduced to $G^{(0)}$. It follows from [20, Proposition 3.6] that topologically principal étale groupoids are always effective, and the converse holds when G is also second countable. However, our next example shows that the blow-up of an étale groupoid need not be étale. Hence the property of being étale is not preserved by equivalence.

EXAMPLE 3.7 (Blow-ups of étale groupoids). Suppose that $f : X \to G^{(0)}$ is a continuous, open surjection such that there is a $u \in G^{(0)}$ such that $f^{-1}(u)$ is not discrete. Then $G[X]$ is not étale. To see this note that there is a sequence $(x_n) \subset f^{-1}(u) \setminus \{x\}$ such that $x_n \to x$. But then $(x_n, u, x) \to (x, u, x)$ in $G[X]$ and $((x_n, u, x)) \subset G[X] \setminus G[X]^{(0)}$. Hence $G[X]^{(0)}$ is not open in $G[X]$ and $G[X]$ is not étale. We make some additional comments on equivalence and étale groupoids in an appendix.

However we can still make the following observation.

REMARK 3.8 (Effective). If G and H are equivalent *second countable* étale groupoids, then one is effective if and only if the other is by Theorem 3.4 and the remarks preceding Example 3.7. We do not know whether this holds if we drop the second countability assumption. Our methods break down since Example 3.7 shows that the blow-up of an étale groupoid need not be étale.

In [21], Renault defines the isotropy of G to be *discretely trivial at $v \in G^{(0)}$* if for every compact set $K \subset G$, there is a neighborhood V of v in $G^{(0)}$ such that $u \in V$ implies that $K \cap G(u) \subset \{u\}$. It is not hard to see that if G is discretely trivial at v, then $G(v) = \{v\}$ — if not, let $K = \{\gamma\}$ for $\gamma \in G(v) \setminus \{v\}$. Since every neighborhood of v contains v, we obtain a contradiction. If G is étale, then the converse holds [31, Lemma 11.19(c)]. However, the converse fails in general [31, Example 11.23]. The concept of discrete triviality plays a key role in extending some simplicity and structure results to non-étale groupoids in [21]. (See also Corollary 4.3.)

PROPOSITION 3.9. *Suppose that Z is a (G, H)-equivalence. Then the isotropy of G is discretely trivial at $r(z)$ if and only if the isotropy of H is discretely trivial at $s(z)$.*

We first prove the result for blow-ups. Define $\mathfrak{c} : G[X] \to G$ by $\mathfrak{c}(x, \gamma, y) = \gamma$. Clearly, $\mathfrak{c}$ is a groupoid homomorphism.

LEMMA 3.10. *Suppose that $f : X \to G^{(0)}$ is a continuous, open surjection. Then the isotropy of $G[X]$ is discretely trivial at $y \in X$ if and only if the isotropy of G is discretely trivial at $f(y)$.*

Proof. Suppose that G is discretely trivial at $v \in G^{(0)}$ and $f(y) = v$. Let C be a compact set in $G[X]$. Then $K := \mathfrak{c}(C)$ is compact and there is a neighborhood V of v in $G^{(0)}$ such that $u \in V$ implies that $K \cap G(u) \subset \{u\}$. Let $V' = f^{-1}(V)$. Then V' is a neighborhood of y. Suppose that $x \in V'$ and $\sigma = (x, \gamma, x) \in C \cap G[X](x)$. Then $\gamma \in G(f(x)) \cap K$. Therefore $\gamma = f(x)$ and $\sigma = (x, f(x), x)$. Since C was arbitrary, $G[X]$ is discretely trivial at y.

Conversely, suppose that $G[X]$ is discretely trivial at $y \in X$. Let $v = f(y)$ and suppose to the contrary of what we want to prove that G is not discretely trivial at v. Then there is a compact set K such that given any neighborhood V of v in $G^{(0)}$, there is $u_V \in V$ and $\gamma_V \neq u_V$ in $K \cap G(u_V)$. After passing to a subnet, and relabeling, we can assume that $u_V \to v$, $\gamma_V \to \gamma \in G(v)$. Furthermore, since f is open, we can also assume there are $x_V \to y$ in X with $f(x_V) = u_V$. Then $(x_V, \gamma_V, x_V) \in G[X](x_V) \subset G[X]$ and $(x_V, \gamma_V, x_V) \to (y, \gamma, y)$. Let C be a compact neighborhood of (y, γ, y) in $G[X]$. Since $G[X]$ is discretely trivial at y there is a neighborhood V' of y such that $x \in V'$ implies that $C \cap G[X](x) \subset \{(x, f(x), x)\}$. This leads to a contradiction as we eventually have (x_V, γ_V, x_V) in C. ∎

Proof of Proposition 3.9. Let $G[Z]$ be the blow-up of G with respect to the moment map $r_Z : Z \to G^{(0)}$ and let $H[Z]$ be the blow-up of H with respect to the moment map $s_Z : Z \to H^{(0)}$. Then as in the proof of [31, Theorem 2.52], we get an isomorphism $\phi : G[Z] \to H[Z]$ given by

$$\phi(z, \gamma, w) = (z, \tau_H(z, \gamma \cdot w), w).$$

Since $\phi(z, r_Z(z), z) = (z, s_Z(z), z)$, the isotropy of $G[Z]$ is discretely trivial at $(z, r_Z(z), z)$ if and only if the isotropy of $H[Z]$ is discretely trivial at $(z, s_Z(z), z)$. Now the proposition follows from Lemma 3.10. ∎

Recall that if G is étale, then the isotropy at $v \in G^{(0)}$ is discretely trivial if and only if $G(v) = \{v\}$ [31, Lemma 11.19(c)]. Then using Proposition 2.9, we obtain the following corollary.

COROLLARY 3.11. *Suppose that G and H are equivalent groupoids such that H is étale. Then the isotropy of G is discretely trivial at $u \in G^{(0)}$ if and only if $G(u) = \{u\}$.*

DEFINITION 3.12. A groupoid G is *strongly topologically principal* if the set of points with discretely trivial isotropy is dense in $G^{(0)}$. We say that G is *strongly essentially principal* if the points with discretely trivial isotropy are dense in every closed G-invariant subset of $G^{(0)}$.

REMARK 3.13. What we are calling strongly essentially principal here is what Renault called essentially free in [21]. Note that if G is equivalent to an étale groupoid, then Definition 3.12 reverts to Definition 3.1 by Corollary 3.11.

With straightforward modifications to Lemma 3.5 and Corollary 3.6 using Lemma 3.10, we obtain the following analogue of Theorem 3.4.

THEOREM 3.14. *Suppose that G and H are equivalent groupoids:*
(i) *G is strongly topologically principal if and only if H is strongly topologically principal;*
(ii) *G is strongly essentially principal if and only if H is strongly essentially principal.*

A concept related to discretely trivial isotropy — brought to our attention by Jean Renault — is that of discrete isotropy. Renault defines G to have *discrete isotropy* if $G^{(0)}$ is open in the isotropy subgroupoid $\mathrm{Iso}(G) = \{\gamma \in G : s(\gamma) = r(\gamma)\}$. Unlike the situation for étale groupoids, it is not hard to check that G has discrete isotropy if and only if any blow-up of G does. Then we obtain the following result.

PROPOSITION 3.15. *If G and H are equivalent groupoids, then G has discrete isotropy if and only if H has discrete isotropy.*

It appears to be unknown whether every groupoid with discrete isotropy (such as a principal groupoid) is equivalent to an étale groupoid.

4. ISOTROPY AND ORBITS

REMARK 4.1. One important consequence of Proposition 2.9 is that generic properties of the isotropy of equivalent groupoids are preserved. For example, if G has abelian isotropy, then the same is true of any groupoid H equivalent to G. Naturally, similar statements can be made if the isotropy groups are all amenable, GCR, or CCR.

If G is a groupoid, the topology of the orbit space $G\backslash G^{(0)} = G^{(0)}/G$ plays a prominent role in deciphering the structure of $C^*(G)$. Hence the following will be useful.

PROPOSITION 4.2. *If G and H are equivalent groupoids then the orbit spaces $G\backslash G^{(0)}$ and $H\backslash H^{(0)}$ are homeomorphic. Moreover if Z is a (G,H)-equivalence, then the map $G \cdot r(z) \mapsto s(z) \cdot H$ is well-defined and gives such a homeomorphism.*

Proof. It suffices to prove the last assertion which is exactly [31, Lemma 2.41]. We provide the proof for convenience. If Z is a (G,H)-equivalence, then we get a G-equivariant homeomorphism $\bar{r} : Z/H \to G^{(0)}$. Since the orbit maps are open by [31, Proposition 2.12], we get a homeomorphism $\underline{r} : G\backslash(Z/H) \to G\backslash G^{(0)}$. Similarly, we get a homeomorphism $\underline{s} : (G\backslash Z)/H \to H^{(0)}/H$ such that $\underline{s}((G \cdot z) \cdot H) = s(z) \cdot H$. Since $(G \cdot z) \cdot H \mapsto G \cdot (z \cdot H)$ gives a homeomorphism of $(G\backslash Z)/H$ onto $G\backslash(Z/H)$, this suffices. ∎

Recall that the action of G on $G^{(0)}$ is *minimal* if every orbit $G \cdot u$ is dense in $G^{(0)}$. Since this is equivalent to saying that every point in $G\backslash G^{(0)}$ is dense, the next corollary is immediate.

COROLLARY 4.3. *Suppose that G and H are equivalent groupoids. Then G acts minimally on $G^{(0)}$ if and only if H acts minimally on $H^{(0)}$.*

In the same spirit, recall that a topological space is T_0 if distinct points have distinct closures, T_1 if points are closed, and T_2 if the space is Hausdorff. Applied to the orbit space $G\backslash G^{(0)}$, the latter is T_0 exactly when distinct orbits in $G^{(0)}$ have distinct closures. It is T_1 when orbits are closed in $G^{(0)}$. Now we can apply Proposition 4.2 to obtain the following corollary.

COROLLARY 4.4. *Suppose that G and H are equivalent groupoids. Then $G\backslash G^{(0)}$ is T_0 (respectively, T_1, respectively, T_2) if and only if $H\backslash H^{(0)}$ is T_0 (respectively, T_1, respectively, T_2).*

Recall from [31, Section 3.4] that the space Σ_0 of closed subgroups of a groupoid G has a locally compact topology which is the relative topology coming from the Fell topology on the compact space $\mathcal{C}(G)$ of closed subsets of G. We say that the isotropy of G is continuous at $u \in G^{(0)}$ if $G(u_n) \to G(u)$ in Σ_0 for any net (u_n) in $G^{(0)}$ converging to u in $G^{(0)}$.

PROPOSITION 4.5. *Suppose that Z is a (G, H)-equivalence. Then the isotropy of G is continuous at $r(z)$ if and only if the isotropy of H is continuous at $s(z)$.*

Before proceeding with the proof, we need a preliminary result involving blow-ups. For a blow-up $G[X]$, the isotropy groups are related as follows:

$$G[X](x) = \{(x, \gamma, x) \in G[X] : \gamma \in G(f(x))\}. \tag{4.1}$$

LEMMA 4.6. *Suppose that $f : X \to G^{(0)}$ is a continuous open surjection. Then the isotropy of G is continuous at $f(x)$ if and only if the isotropy of $G[X]$ is continuous at x.*

Proof. Let $u = f(x)$. Suppose that the isotropy is continuous at u and that (x_i) is a net converging to x in X. We claim that $G[X](x_i) \to G[X](x)$. We apply the criteria in [31, Lemma 3.22]. We can assume that we have already passed to a subnet and relabeled. Suppose that $(x_i, \gamma_i, x_i) \in G[X](x_i)$ and $(x_i, \gamma_i, x_i) \to (y, \gamma, z)$ in $G[X]$. Then we must have $y = x = z$. Furthermore $\gamma_i \in G(f(x_i))$ and $\gamma_i \to \gamma$. By assumption, $G(f(x_i)) \to G(f(x))$. Hence $\gamma \in G(f(x))$. Thus part (a) of [31, Lemma 3.22] is satisfied.

For part (b), suppose $(x, \gamma, x) \in G[X](x)$. Then $\gamma \in G(f(x))$ and we still have $G(f(x_i)) \to G(f(x))$. Hence we can pass to a subnet, relabel, and assume that there are $\gamma_i \in G(f(x_i))$ with $\gamma_i \to \gamma$. Then $(x_i, \gamma_i, x_i) \to (x, \gamma, x)$ and we have shown that the isotropy is continuous at x.

Conversely, suppose that the isotropy of $G[X]$ is continuous at x and that $u_i \to u = f(x)$ in $G^{(0)}$. We want to show that $G(u_i) \to G(u)$. Again, we can assume that we have passed to a subnet and relabeled. Since f is open, we can pass to a further subnet if necessary, and assume that there are $x_i \in X$ such that $x_i \to x$ and $f(x_i) = u_i$. Then by assumption, $G[X](x_i) \to G[X](x)$. If $\gamma_i \in G(u_i)$

and $\gamma_i \to \gamma$ in G, then $(x_i, \gamma_i, x_i) \to (x, \gamma, x)$ in $G[X]$. Hence, $(x, \gamma, x) \in G[X](x)$ and $\gamma \in G(u)$. If $\gamma \in G(u)$, then $(x, \gamma, x) \in G[X](x)$ and we can pass to a subnet, relabel, and find $(x_i, \gamma_i, x_i) \in G[X](x_i)$ such that $(x_i, \gamma_i, x_i) \to (x, \gamma, x)$. But then, $\gamma_i \to \gamma$ and $G(u_i) \to G(u)$ as claimed. ∎

Proof of Proposition 4.5. The result now follows using the proof of [31, Theorem 2.52] exactly as in the proof of Proposition 3.9. ∎

COROLLARY 4.7. *Suppose that G and H are equivalent groupoids. Then $u \mapsto G(u)$ is continuous on $G^{(0)}$ if and only if $v \mapsto H(v)$ is continuous on $H^{(0)}$.*

As an application of these ideas, we recall a condition on a groupoid with abelian isotropy the authors introduced in [28] — called *proper modulo the isotropy* — generalizing a proper groupoid with abelian isotropy. This allowed us to describe the primitive ideal space $\mathrm{Prim}(C^*(G))$ as a topological space, generalizing work of [6] and [16] in the case of abelian isotropy.

We outline the details from [28, Section 9]. We let $R \subset G^{(0)} \times G^{(0)}$ be the image of $\pi : G \to G^{(0)} \times G^{(0)}$ where $\pi(\gamma) = (r(\gamma), s(\gamma))$. Then with the relative product topology, R is a topological groupoid which can violate our standing assumptions as R need not be locally compact, nor need it have open range and source maps. However, if $G \backslash G^{(0)}$ is Hausdorff, then R is at least closed in $G^{(0)} \times G^{(0)}$. If G has abelian isotropy, then there is a well-defined action of R on Σ_0 given by $\pi(\gamma) \cdot H = \gamma \cdot H = \gamma H \gamma^{-1}$. We say that G *is proper modulo its isotropy* if G has abelian isotropy, $G \backslash G^{(0)}$ is Hausdorff, and R acts continuously on Σ_0. As observed in [28, Example 9.5], there is a large class of groupoids, such as those studied in [7, 8, 9], that are proper modulo their isotropy.

For the remainder of this section we assume that G has abelian isotropy. Then if H is equivalent to G, it has abelian isotropy as well by Proposition 2.9.

THEOREM 4.8. *Suppose that G and H are equivalent groupoids. Then G is proper modulo its isotropy if and only if H is proper modulo its isotropy.*

Since Corollary 4.4 implies that $G \backslash G^{(0)}$ is Hausdorff if and only if $H \backslash H^{(0)}$ is, to prove the theorem it will suffice to prove the following lemma and appeal to Remark 1.5 as usual.

PROPOSITION 4.9. *Suppose that $f : X \to G^{(0)}$ is a continuous, open surjection. Let $R = \pi(G)$ as above, and let $R' = \pi'(G[X])$ where $\pi' : G[X] \to X \times X$ is the corresponding map for $G[X]$. Then R acts continuously on Σ_0 if and only if R' acts continuously on Σ_0' where Σ_0' is the space of closed subgroups of $G[X]$.*

Before proceeding with the proof of the lemma, we introduce some notation and make a few observations. Let $p_0 : \Sigma_0 \to G^{(0)}$ be given by $p_0(H) = u$ when $H \subset G(u)$. Let $p_0' : \Sigma_0' \to X$ be the corresponding map for $G[X]$. Using (4.1), there is a map $\rho : \Sigma_0' \to \Sigma_0$ such that if $p_0'(H') = x$, then

$$H' = \{(x, \gamma, x) : \gamma \in \rho(H')\}.$$

Note that $p_0(\rho(H')) = f(p_0'(H'))$. Using [31, Lemma 3.22] it is not hard to establish the following lemma.

LEMMA 4.10. *A net* $H_n' \to H'$ *in* Σ_0' *if and only if* $\rho(H_n') \to \rho(H')$ *in* Σ_0 *and* $p_0'(H_n') \to p_0'(H')$.

LEMMA 4.11. *The maps* $\rho : \Sigma_0' \to \Sigma_0$ *and* $\mathfrak{c} : G[X] \to G$ *are continuous open surjections.*

Proof. To see that ρ is open, we use Fell's criterion. Suppose that $H_n \to \rho(H')$. Let $u_n = p_0(H_n) = f(p_0'(H'))$. Since f is open, we can pass to a subnet, relabel, and assume that there are $x_n \to p_0'(H')$ such that $f(x_n) = u_n$. Let $H_n' = \{(x_n, \gamma, x_n) : \gamma \in H_n\}$. Now we can use Lemma 4.10 to show that $H_n' \to H'$. Since $\rho(H_n') = H_n$ by construction, this proves openness.

The proof of continuity for ρ is similar, but easier, as are the assertions about $\mathfrak{c}$. ∎

Observe that if $H' \in \Sigma_0'$ and $\alpha \in G[X]$ is such that $s(\alpha) = p_0'(H')$, then

$$\alpha \cdot H' = H''$$

where $\rho(H'') = \mathfrak{c}(\alpha) \cdot \rho(H')$.

Proof of Proposition 4.9. Suppose that R acts continuously on Σ_0. We want to show that R' acts continuously on Σ_0'. To this end, suppose that $H_n' \to H_0'$ in Σ_0' with $x_n = p_0'(H_n')$. Suppose we also have $\pi'(\alpha_n) = (y_n, x_n) \to \pi'(\alpha_0) = (y_0, x_0)$. We want to see that $\pi'(\alpha_n) \cdot H_n' \to \pi'(\alpha_0) \cdot H_0'$. Let $H_n = \rho(H_n')$. Then the continuity of ρ implies that $H_n \to H_0$. Moreover, $u_n = f(x_n) = p_0(H_n) \to u_0 = f(x_0) = p_0(H_0)$. Hence $\pi(\mathfrak{c}(\alpha_n)) \cdot H_n \to \pi(\mathfrak{c}(\alpha_0)) \cdot H_0$. Since $\rho(\pi'(\alpha_n) \cdot H_n') = \pi(\mathfrak{c}(\alpha_n)) \cdot \rho(H_n')$, Lemma 4.10 implies that $\pi'(\alpha_n) \cdot H_n' \to \pi'(\alpha_0) \cdot H_0'$ as required.

Conversely, assume that R' acts continuously on Σ_0'. To show that R then acts continuously on Σ_0, we assume $H_n \to H_0$ with $p_0(H_n) = u_n$. We also suppose $\pi(\gamma_n) = (v_n, u_n) \to (v_0, u_0)$. We need to establish that $\pi(\gamma_n) \cdot H_n \to \pi(\gamma_0) \cdot H_0$. For this, it suffices to see that any subnet has a subnet converging to $\pi(\gamma_0) \cdot H_0$. Since f is open, we can pass to a subnet, relabel, and assume that there are $x_n \to x_0$ in X such that $f(x_n) = u_n$. Passing to still another subnet, we can also assume that there are $y_n \to y_0$ in X such that $f(y_n) = v_n$. Then $\alpha_n = (y_n, \gamma_n, x_n) \in G[X]$ and $\alpha_n \to \alpha_0$. Let $H_n' = \{(x_n, \gamma, x_n) : \gamma \in H_n\}$. Then $H_n' \to H_0'$ with $p_0'(H_n') = x_n$. By assumption, $\pi'(\alpha_n) \cdot H_n' \to \pi'(\alpha_0) \cdot H_0'$. Since $\mathfrak{c}(\alpha_n) = \gamma_n$ by construction, the continuity of ρ implies that $\pi(\gamma_n) \cdot H_n \to \pi(\gamma_0) \cdot H_0$. This completes the proof. ∎

5. HAAR SYSTEMS AND INTEGRABLE GROUPOIDS

In this section, we will be primarily working with second countable groupoids with Haar systems. The following is the main result in [30].

THEOREM 5.1 ([30, Theorem 2.1]). *Suppose that G and H are equivalent second countable groupoids. Then G has a Haar system if and only if H has a Haar system.*

If (G, λ) is a groupoid with a Haar system $\lambda = \{\lambda_u\}_{u \in G^{(0)}}$, then building on [23] and [1], in [18, Definition 3.1], Clark and an Huef define a groupoid G to be *integrable* if for every compact set $N \subset G^{(0)}$,

$$\sup_{u \in N} \lambda^u(s^{-1}(N)) < \infty. \tag{5.1}$$

As stated, integrability does not appear to be a purely topological property since *a priori*, it depends on a choice of Haar system. However, if G is a second countable principal groupoid, then Clark and an Huef prove that G is integrable if and only if $C^*(G, \lambda)$ has bounded trace [18, Theorem 4.4]. It follows from the Equivalence Theorem that the Morita equivalence class of $C^*(G)$ is invariant under any choice of a Haar system [31, Proposition 2.74]. Since the property of having bounded trace is preserved by Morita equivalence [2, Proposition 7], it follows that a principal groupoid G is integrable if and only if (5.1) holds for some, and hence any, Haar system on G. Furthermore, if G is equivalent to H and if H admits a Haar system, then so does G by Theorem 5.1. In particular, we have the following proposition.

PROPOSITION 5.2. *Suppose that G is a second countable principal groupoid admitting a Haar system, and that H is a second countable groupoid equivalent to G. If G is integrable, then H admits a Haar system and H is integrable.*

REMARK 5.3. It would be interesting to determine if the property of being integrable is independent of the choice of a Haar system in the general case. Even more tempting, it would be nice to find a purely topological criterion that does not depend on the existence of a Haar system. One possibility is suggested by [18, Definition 3.6] and [18, Proposition 3.11]. Unfortunately, the converse of [18, Proposition 3.11] is only known in the principal case.

6. TWISTS

The notion of a *twist* or a $\mathbf{T}$-*groupoid* originated in [10] and twists now play a key role in many studies — for example, see [4, 5, 11, 13, 15, 20, 26]. Recall that a twist E over G is given by a central groupoid extension

$$G^{(0)} \times \mathbf{T} \xrightarrow{\ \iota\ } E \xrightarrow{\ j\ } G \tag{6.1}$$

where ι and j are continuous groupoid homomorphisms such that ι is a homeomorphism onto its range, j is an open surjection inducing a homeomorphism of the unit space of E with $G^{(0)}$, and with kernel equal to the range of ι. We identify the unit space of E with $G^{(0)}$. Furthermore

$$\iota(r(e), t)e = e\iota(s(e), t) \quad \text{for all } e \in E \text{ and } t \in \mathbf{T}.$$

Note that E becomes a principal $\mathbf{T}$-bundle with respect to the action given by $t \cdot e = \iota(r(e), t)e$.

Conversely, we can think of a twist as a groupoid E admitting a free left $\mathbf{T}$-action that is compatible with the groupoid structure as in [14, Section 3]. Specifically, we have the following observation.

LEMMA 6.1. *Suppose* $\mathbf{T}$ *acts freely on a groupoid* E *such that* $r(t \cdot e) = r(e)$, $s(t \cdot e) = s(e)$, *and* $(t \cdot e)(t' \cdot f) = (tt') \cdot ef$. *Let* G *be the orbit space* $\mathbf{T}\backslash E$, *and let* $j : E \to G$ *the orbit map. Then* G *is a locally compact groupoid with respect to the operations* $j(e)j(f) = j(ef)$ *with* $(e, f) \in E^{(2)}$, *and* $j(e)^{-1} = j(e^{-1})$. *Furthermore* E *is twist over* G *with* $\iota(u, t) = t \cdot u$.

Sketch of the Proof. Since $\mathbf{T}$ is compact, G is locally compact Hausdorff. It is routine to verify that G is a groupoid with $G^{(0)}$ identified with $E^{(0)}$ via $u \mapsto j(u)$, and that (6.1) is exact. Since $\mathbf{T}$ is compact, it follows that $r_G(j(e)) = r_E(e)$ is open as is s_G. ∎

If E is a twist acting on the left of a space Z, then we get a $\mathbf{T}$-action on the left of Z by $t \cdot z = \iota(r(z), t) \cdot z$. Similarly, if Z is a right E-space, we get a right $\mathbf{T}$-action on Z by $z \cdot t = z \cdot \iota(s(z), t)$. In particular, if Z is an equivalence between two twists, then it is both a left $\mathbf{T}$-space and a right $\mathbf{T}$-space.

DEFINITION 6.2 ([14, Definition 3.1]). Suppose that E and E' are twists over G and G', respectively. We say that (E, E')-equivalence Z is an (E, E')-*twist equivalence over* (G, G') if $t \cdot z = z \cdot t$ for all $t \in \mathbf{T}$ and $z \in Z$.

REMARK 6.3. In [14] (E, E')-twist equivalence was called (E, E')-$\mathbf{T}$-equivalence. In [14, Theorem 3.2], it is shown that if E and E' are second countable twists such that G and G' have Haar systems, then the restricted groupoid C^*-algebras $C^*(G; E)$ and $C^*(G'; E')$ are Morita equivalent.

Let Z be (E, E')-twist equivalence over (G, G'). Let $Z_{\mathbf{T}}$ be the locally compact Hausdorff quotient $\mathbf{T}\backslash Z = Z/\mathbf{T}$. If $z \in Z$, we let $[z] = z \cdot \mathbf{T} = \mathbf{T} \cdot z$.

Suppose $e, f \in E$ and $z, w \in Z$ are such that $j(e) = j(f)$ and $[z] = [w]$. Then there are $t, t' \in \mathbf{T}$ such that $f = t \cdot e$ and $w = t' \cdot z$. Then

$$[f \cdot w] = [(t \cdot e) \cdot (t' \cdot z)] = [t \cdot (e \cdot z) \cdot t'] = [e \cdot z].$$

With a similar observation when $j'(e') = j'(f')$ in G', we have a left G-action on $Z_{\mathbf{T}}$ and a right action of G' on $Z_{\mathbf{T}}$ given by

$$j(e) \cdot [z] = [e \cdot z] \quad \text{and} \quad [z] \cdot j'(e') = [z \cdot e'],$$

respectively. Note that if r_Z is the moment map for the left E-action on Z, then the moment map for the left G-action on $Z_{\mathbf{T}}$ is given by $r_{Z_{\mathbf{T}}}([z]) = r_Z(z)$. Usually, we will abuse notation and simply write r for these maps and s for the corresponding moment maps for the right actions.

PROPOSITION 6.4. *Let Z be (E, E')-twist equivalence over (G, G'), and let $Z_{\mathbf{T}}$ be the quotient (G, G')-space as above. Then $Z_{\mathbf{T}}$ is a (G, G')-equivalence. In particular, G and G' are equivalent groupoids.*

Proof. To see that the left G-action is continuous, suppose that $j(e_i) \to j(e)$ in G and that $[z_i] \to [z]$ in $Z_{\mathbf{T}}$ with $s(e_i) = r(z_i)$. It will suffice to see that every subnet of $([e_i \cdot z_i])$ has a subnet converging to $[e \cdot z]$. Since the orbit maps are open, after passing to a subnet, we can pass to a further subnet, relabel, and assume that $e_i \to e$ while $z_i \to z$. But then $e_i \cdot z_i \to e \cdot z$. Therefore the left G-action is continuous. The proof for the right G'-action is similar.

To see that $Z_{\mathbf{T}}$ is a (G, G')-equivalence, we verify (E1)–(E5) Definition 1.1. To see that the left G-action is proper, suppose that $[z_i] \to [z]$ and that $j(e_i) \cdot [z_i] \to [w]$. After passing to a subnet and relabeling, we can assume that $z_i \to z$ while $e_i \cdot z_i \to w$. Then (e_i) must have a convergent subnet which implies that $(j(e_i))$ does as well. This suffices by [31, Proposition 2.17]. Since the G-actions is clearly free, $Z_{\mathbf{T}}$ is a free and proper left G-space. A similar argument shows that it is a free and proper right G'-space. This establishes (E1) and (E2).

Furthermore, these actions commute: $(j(e) \cdot [z]) \cdot j'(e') = [e \cdot z] \cdot j'(e') = [(e \cdot z) \cdot e'] = [e \cdot (z \cdot e')] = j(e) \cdot ([z] \cdot j'(e'))$. So (E3) holds.

To see that the moment map for the G-action is open, suppose that $u_i \to r(e) = r(j(e))$. Since the moment map for the E-action is open, we can pass to a subnet, relabel, and assume that $e_i \to e$ with $r(e_i) = u_i$. But then $j(e_i) \to j(e)$. Similarly, the moment map for the G' action is open.

Moreover if $r([z]) = r([z'])$, then $r(z) = r(z')$ and there is an e' such that $z = z' \cdot e'$. But then $[z] = [z'] \cdot j'(e')$ and r factors through a homeomorphism of $Z_{\mathbf{T}}/G'$ onto $G^{(0)}$. This establishes (E4), and (E5) is proved similarly. ∎

Our next result is essentially a reworking of [14, Theorem 3.5]. We sketch the details here.

THEOREM 6.5. *Suppose that*

$$(G')^{(0)} \times \mathbf{T} \xrightarrow{i'} E' \xrightarrow{j'} G'$$

is a twist over G'. If E is equivalent to E' and if Z is an (E, E')-equivalence, then E admits a principal left $\mathbf{T}$-action so that E is a twist over a groupoid G with unit space Z/E'. Furthermore, with respect to this twist structure on E, Z is (E, E')-twist equivalence over (G, G').

Proof. Using [31, Lemma 2.42 and Remark 2.43] we can assume that $E = (Z *_s Z)/E'$. Recall that the groupoid structure on E is given as follows. We have $[z, w]^{-1} = [w, z]$. Also $[z, w]$ and $[x, y]$ are composible if $w \cdot E' = x \cdot E'$. Then there is a unique $e' \in E'$ such that $x = w \cdot e'$ and $[z, w][x, y] = [z \cdot e', y]$. Then we can identify the unit space of E with Z/E' and then $r([z, w]) = z \cdot E'$ and $s([z, w]) = w \cdot E'$. It is common practice to summarize the composition law as simply $[z,w][w,p] = [z,p]$ with the gyrations

$$[z,w][x,y] = [z,w][w\cdot e',y][z,w][w,y\cdot(e')^{-1}] = [z,y\cdot(e')^{-1}] = [z\cdot e',y]$$

left implicit.

With this identification of E, the moment map r on Z is given by $r(z) = x\cdot E'$. The left action of E on Z is given by $[z,w]\cdot w = z$ (with a similar understanding as with the simplified composition law).

Since $\mathbf{T}$ is abelian, we can define a left $\mathbf{T}$-action on $Z *_s Z$ using the right $\mathbf{T}$-action on Z induced by E': $t\cdot(z,w) = (z\cdot t,w)$. If $e'\in E'$ with $r(e') = s(z)$, then

$$\begin{aligned}((z\cdot t)\cdot e',w\cdot e') &= (z\cdot \iota'(s(z),t)e',w\cdot e') = (z\cdot(e'\iota'(s(e'),t),w\cdot e')\\ &= ((z\cdot e')\cdot t,w\cdot e').\end{aligned}$$

It follows that we get a left action of $\mathbf{T}$ on E given by

$$t\cdot[z,w] = [z\cdot t,w].$$

If $[z\cdot t,w] = [z,w]$, then $t=1$ since the E'-action on Z is free. Hence E becomes a principal $\mathbf{T}$-space. Furthermore,

$$[z\cdot t,w] = [z\cdot\iota'(s(z),t),w] = [z,w\cdot\iota'(s(w),\bar{t})] = [z,w\cdot\bar{t}].$$

Then

$$\begin{aligned}(t\cdot[z,w])(s\cdot[w,p]) &= [z\cdot t,w][w,p\cdot\bar{s}] = [z\cdot t,p\cdot\bar{s}] = [z\cdot(ts),p]\\ &= (ts)\cdot([z,w][w,p]).\end{aligned}$$

Thus the $\mathbf{T}$-action on E is compatible with the groupoid structure and E is a twist over the quotient $G = \mathbf{T}\backslash E$ as above. It particular, $\iota : Z/E'\times\mathbf{T}\to E$ is given by $\iota(z\cdot E',t) = t\cdot[z,z] = [z\cdot t,z]$ Therefore if $t\in\mathbf{T}$ and $z\in Z$, we have

$$t\cdot z = \iota(z\cdot E',t)\cdot z = [z\cdot t,z]\cdot z = z\cdot t.$$

That is Z is (E,E')-twist equivalence. ∎

Note that in Theorem 6.5, the groupoids G and G' are equivalent by Proposition 6.4. Hence we can apply our previous results in examples such as the following.

EXAMPLE 6.6. Suppose that E is a twist over an essentially principal groupoid G. If E' is equivalent to E, then E' is also a twist over an essentially principal groupoid G'.

Of course, we could replace "essentially principal" above with any property preserved by equivalence.

APPENDIX A. ÉTALE GROUPOIDS AND EQUIVALENCE

We have already observed in Example 3.7 that the property of a groupoid being étale need not extend to arbitrary push-outs, let alone equivalent groupoids. To obtain some interesting specific examples, we can appeal to [22, Situation 4].

EXAMPLE A.1. Let H be a closed subgroup of G and let P be any right H-space. Let $Z = G \times P$. Then $(g, p) \cdot h = (gh, P \cdot h)$ is a free and proper H-space (because H acts freely and properly on the right of G). Similarly, $g' \cdot (g, p) = (g'g, p)$ is a free and proper left G-action. Then Green's Symmetric Imprimitivity Theorem [29, Corollary 4.11] implies that

$$C_0(G\backslash Z) \rtimes_{\mathrm{rt}} H \quad \text{and} \quad C_0(Z/H) \rtimes_{\mathrm{lt}} G$$

are Morita equivalent. In fact, in the second countable case, we can also prove this using the observation that the action groupoids $G \rtimes Z/H$ for the left G-action and $G\backslash Z \rtimes H$ for the right H-action are equivalent [31, Example 2.34], and then applying the Equivalence Theorem [31, Theorem 2.70].

In particular, if H is a discrete subgroup of G, then $G\backslash Z \rtimes H$ is étale. But if G is not discrete, then $G \rtimes Z/H$ is not. (This is because $(G \rtimes Z/H)^{(0)} = \{e\} \times Z/H$ is not open in $G \times Z/H$.)

For a specific example, we can let G be the reals and let H be the integers. Then we let H act on $P = \mathbf{T}$ by an irrational rotation. Since $G\backslash \mathbf{Z}$ can be identified with P, $C_0(G\backslash Z) \rtimes_{\mathrm{rt}} H$ is an irrational rotation algebra. But Z/H can be identified with the torus $\mathbf{T}^2$ and the $\mathbf{R}$-action is the flow along an irrational angle.

Example 3.7 shows that the push-out of an étale groupoid can fail to be étale. It is also the case that the push-out of a non-étale groupoid is *never* étale.

LEMMA A.2. *Suppose that $f : X \to G^{(0)}$ is a continuous open surjection and that G is not étale. Then $G[X]$ is not étale.*

Proof. Since we are assuming that $r : G \to G^{(0)}$ is open, G not étale implies that $G^{(0)}$ is not open in G [31, Lemma 1.26]. Thus there is a sequence $(\gamma_n) \subset G \setminus G^{(0)}$ such that $\gamma_n \to u \in G^{(0)}$. Let $f(x) = u$. Since $r(\gamma_n) \to u$, we can pass to a subnet, relabel, and assume that there are $x_n \in X$ with $x_n \to x$ and $f(x_n) = r(\gamma_n)$. Similarly, after passing to a subnet and relabeling, there are $y_n \in G$ such that $y_n \to x$ and $f(y_n) = s(\gamma_n)$. Then $(x_n, \gamma_n, y_n) \to (x, u, x)$ in $G[X]$. Since $(x_n, \gamma_n, y_n) \notin G[X]^{(0)}$, it follows that $G[X]^{(0)}$ is not open in $G[X]$. Hence $G[X]$ is not étale. ∎

Acknowledgements. This research was partially supported by the Edward Shapiro fund at Dartmouth College.

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Received April 8, 2022; revised July 11, 2022.