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EMBEDDING FROM BERGMAN SPACES INTO TENT SPACES

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ABSTRACT. Let A^p_ω denote the Bergman space in the unit disc induced by a radial weight ω with the doubling property $\int\limits_r^1 \omega(s)\mathrm{d}s \leqslant C \int\limits_{(1+r)/2}^1 \omega(s)\mathrm{d}s$. The tent space $T^q_s(\nu,\omega)$ consists of functions such that $\int\limits_{\mathbb{D}} \Big(\int\limits_{\Gamma(\zeta)} |f(z)|^s \mathrm{d}\nu(z)\Big)^{q/s} \cdot \omega(\zeta)\mathrm{d}A(\zeta) < \infty$, where $\Gamma(\zeta)$ is a non-tangential approach region with vertex ζ in the punctured unit disc $\mathbb{D}\setminus\{0\}$. We characterize the positive Borel measures ν such that A^p_ω is embedded into the tent space $T^q_s(\nu,\omega)$ for $0<q<p<\infty$, by considering a generalized area operator.

KEYWORDS: *Bergman space, tent space, Carleson measure, area operator.*

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1. INTRODUCTION

Let $\mathbb{D}$ be the unit disc in the complex plane. For a non-negative function $\omega \in L^1[0,1)$, its extension to $\mathbb{D}$, defined by $\omega(z)=\omega(|z|)$ for all $z\in\mathbb{D}$, is called a radial weight. As in [28], a radial weight ω belongs to the class $\widehat{\mathcal{D}}$ if $\widehat{\omega}(r) = \int\limits_r^1 \omega(s)\mathrm{d}s$ satisfies the doubling condition

$$\sup_{0\leqslant r<1} \frac{\widehat{\omega}(r)}{\widehat{\omega}((1+r)/2)} < \infty.$$

The class $\widehat{\mathcal{D}}$ appears naturally when studying the boundedness of the Bergman projection acting on weighted Bergman spaces [26]. Given $0<p<\infty$, the weighted Bergman space A^p_ω is the set of all holomorphic functions f on $\mathbb{D}$ such that

$$\|f\|_{A^p_\omega} = \Big(\int\limits_{\mathbb{D}} |f(z)|^p \omega(z)\mathrm{d}A(z)\Big)^{1/p} < \infty,$$

where $\mathrm{d}A$ is the normalized area measure on $\mathbb{D}$. When $\omega(z) = (1-|z|^2)^\alpha$ with $\alpha > -1$, one obtains the standard Bergman spaces A^p_α, whose theory has been extensively developed, see [9, 12, 31] for example. Recently, Bergman spaces A^p_ω with more general weights ω have been studied in [22, 23, 24, 27, 28].

For $\alpha \in (0,\pi)$, the non-tangential approach region with vertexes in the unit disc is defined to be

$$\Gamma_\alpha(\zeta) = \Big\{ z \in \mathbb{D} : |\theta - \arg(z)| < \alpha\Big(1 - \frac{|z|}{r}\Big)\Big\}, \quad \zeta = r\mathrm{e}^{\mathrm{i}\theta} \in \mathbb{D}\setminus\{0\},$$

and the related tent is defined by

$$T_\alpha(z) = \{\zeta \in \mathbb{D} : z \in \Gamma_\alpha(\zeta)\}, \quad z \in \mathbb{D}\setminus\{0\}.$$

Let ν be a positive Borel measure on $\mathbb{D}$, finite on compact sets, and let $\omega \in \widehat{\mathcal{D}}$. For $0 < q, s < \infty$, $0 < \alpha < \pi$, the tent space $T^q_{s,\alpha}(\nu,\omega)$ consists of ν-equivalence classes of ν-measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|^q_{T^q_{s,\alpha}(\nu,\omega)} = \int_{\mathbb{D}} \Big(\int_{\Gamma_\alpha(\zeta)} |f(z)|^s \mathrm{d}\nu(z) \Big)^{q/s} \omega(\zeta)\mathrm{d}A(\zeta) < \infty.$$

Tent spaces were introduced in the paper of Coifman, Meyer and Stein [6] in order to study problems in harmonic analysis, and further studied by Cohn and Verbitsky [5] among others. These spaces turned to be quite useful in developing further the classical theory of Hardy spaces, closely related to tent spaces due to the importance of the use of maximal and square area functions and other objects from harmonic analysis [10]. The recent studies [14, 15] show that tent spaces have natural analogues for Bergman spaces, and they may play an important role in the theory of weighted Bergman spaces, similar to that of the original tent spaces in the Hardy space case. The results in [24] also show that the tent space $T^q_{s,\alpha}(\nu,\omega)$ is independent of the aperture α of the lens appearing in the definition, and the quasi-norms obtained are equivalent. In view of that, we drop the parameter α from the notations, and simply write $T^q_s(\nu,\omega)$ to indicate the case $\alpha = 1/2$. Also we use the notation $\Gamma(\zeta) := \Gamma_{1/2}(\zeta)$ and $T(z) := T_{1/2}(z)$.

Let $0 < p, q < \infty$. A positive Borel measure μ on $\mathbb{D}$ is a q-Carleson measure for A^p_ω, if there exists a constant $C > 0$ such that $\|f\|_{L^q_\mu} \leqslant C\|f\|_{A^p_\omega}$ for all $f \in A^p_\omega$, where L^q_μ is the family of all μ-measurable functions f on $\mathbb{D}$ such that

$$\|f\|_{L^q_\mu} = \Big(\int_{\mathbb{D}} |f(z)|^q \mathrm{d}\mu(z) \Big)^{1/q} < \infty.$$

In [27], Peláez, Rättyä and Sierra obtained a characterization of the embedding from Bergman spaces A^p_ω into the tent spaces $T^q_s(\nu,\omega)$, for weights ω in the class $\widehat{\mathcal{D}}$, under the assumption that $1 + s/p - s/q > 0$.

THEOREM A. *Let $0 < p, q, s < \infty$ such that $1 + s/p - s/q > 0$, $\omega \in \widehat{\mathcal{D}}$, and let ν be a positive Borel measure on $\mathbb{D}$, finite on compact subsets, such that $\nu(\{0\}) = 0$. Let*

ν^ω be the measure given by $\mathrm{d}\nu^\omega(z) = \omega(T(z))\mathrm{d}\nu(z)$ for all $z \in \mathbb{D}$. Then the following assertions hold:

(i) $\mathrm{Id} : A^p_\omega \to T^q_s(\nu,\omega)$ *is bounded if and only if ν^ω is a $(p+s-(ps/q))$-Carleson measure for A^p_ω. Moreover,*

$$\|\mathrm{Id}\|^s_{A^p_\omega \to T^q_s(\nu,\omega)} \asymp \|\mathrm{Id}\|^{p+s-(ps/q)}_{A^p_\omega \to L^{p+s-(ps/q)}_{\nu^\omega}};$$

(ii) $\mathrm{Id} : A^p_\omega \to T^q_s(\nu,\omega)$ *is compact if and only if* $\mathrm{Id} : A^p_\omega \to L^{p+s-(ps/q)}_{\nu^\omega}$ *is compact.*

Here, $\omega(T(z)) = \int\limits_{T(z)} \omega(\zeta)\mathrm{d}A(\zeta)$ for $z \neq 0$. We also set

$$\omega(T(0)) = \lim_{r\to 0^+} \omega(T(r))$$

to deal with the origin. Also, the notation $A \asymp B$ means that the two quantities are comparable. The requirement $\nu(\{0\}) = 0$ in Theorem A, which does not carry any real restriction, is a technical hypotheses caused by the geometry of the tents $\Gamma(z)$. Theorem A can be interpreted as a characterization of Carleson measures, and it follows from a characterization of the boundedness and compactness of some type of area operators, that we are going to define next.

For $0 < s < \infty$, the generalized area operator induced by positive Borel measures μ and ν on $\mathbb{D}$, is defined as

$$G^\nu_{\mu,s}(f)(\zeta) = \Big(\int\limits_{\Gamma(\zeta)} |f(z)|^s \frac{\mathrm{d}\mu(z)}{\nu(T(z))} \Big)^{1/s}, \quad \zeta \in \mathbb{D}\backslash\{0\}.$$

Area operators acting on Hardy spaces and on standard Bergman spaces have been studied in [4, 11, 14, 16, 17, 29]. Write μ^ω_ν for the positive measure given by

$$\mathrm{d}\mu^\omega_\nu(z) = \frac{\omega(T(z))}{\nu(T(z))}\mathrm{d}\mu(z)$$

for almost every $z \in \mathbb{D}$. In [27], a characterization of the boundedness and compactness on Bergman spaces of the generalized area operator $G^\nu_{\mu,s}$ for $0 < p,q,s < \infty$ in the case $1 + s/p - s/q > 0$ was obtained. It was proved that, for positive Borel measures μ,ν on $\mathbb{D}$, satisfying

$$\mu(\{z \in \mathbb{D} : \nu(T(z)) = 0\}) = 0 = \mu(\{0\}), \tag{1.1}$$

and $\omega \in \widehat{\mathcal{D}}$, one has that $G^\nu_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded if and only if μ^ω_ν is a $(p+s-(ps/q))$-Carleson measure for A^p_ω; and moreover, one has the estimate $\|G^\nu_{\mu,s}\|^s_{A^p_\omega\to L^q_\omega} \asymp \|\mathrm{Id}\|^{p+s-(ps/q)}_{A^p_\omega \to L^q_{\mu^\omega_\nu}}$. Also, $G^\nu_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact if and only if $\mathrm{Id} : A^p_\omega \to L^{p+s-(ps/q)}_{\mu^\omega_\nu}$ is compact.

It is natural to ask for a characterization in the case $1 + s/p - s/q \leqslant 0$. The purpose of this work is to answer this question and obtain the corresponding descriptions for all $0 < p, q, s < \infty$. In order to state our main result, for positive Borel measures μ, ν on $\mathbb{D}$, finite on compact sets, and $\zeta \in \mathbb{D}\backslash\{0\}$ we define

$$B_\mu^\nu(\zeta) = \int_{\Gamma(\zeta)} \frac{\mathrm{d}\mu(z)}{\nu(T(z))} \quad \text{and} \quad \mathrm{d}\mu_\nu(\zeta) = \frac{\mathrm{d}\mu(\zeta)}{\nu(T(\zeta))}.$$

When $0 < p \leqslant q < \infty$ and $0 < s < \infty$, it is clear that $1 + s/p - s/q > 0$. Thus, we only need to deal with the case $0 < q < p < \infty$. We state our main result as follows.

THEOREM 1.1. *Let $0 < q < p < \infty$, $0 < s < \infty$, $\omega \in \widehat{\mathcal{D}}$, and let μ, ν be positive Borel measures on $\mathbb{D}$, with μ_ν^ω finite on compact sets, satisfying* (1.1). *Then the following conditions are equivalent:*

(i) $G_{\mu,s}^\nu : A_\omega^p \to L_\omega^q$ *is bounded;*
(ii) $G_{\mu,s}^\nu : A_\omega^p \to L_\omega^q$ *is compact;*
(iii) $\mathrm{Id} : A_\omega^p \to T_s^q(\mu_\nu, \omega)$ *is bounded;*
(iv) $\mathrm{Id} : A_\omega^p \to T_s^q(\mu_\nu, \omega)$ *is compact;*
(v) $B_\mu^\nu \in L_\omega^{(pq)/(s(p-q))}$.

Moreover, we have

$$\|G_{\mu,s}^\nu\|_{A_\omega^p \to L_\omega^q}^s \asymp \|\mathrm{Id}\|_{A_\omega^p \to T_s^q(\mu_\nu,\omega)}^s \asymp \|B_\mu^\nu\|_{L_\omega^{(pq)/(s(p-q))}}. \tag{1.2}$$

Note that when $0 < q < p < \infty$ with $1 + s/p - s/q > 0$, it follows from [24, Theorem 8] that μ_ν^ω is a $(p + s - (ps/q))$-Carleson measure for A_ω^p, if and only if, $B_\mu^\nu \in L_\omega^{(pq)/(s(p-q))}$.

Throughout the paper, constants are often given without computing their exact values, and the value of a constant C may change from one occurrence to the next. We use the notation $a \lesssim b$ to indicate that there is a constant $C > 0$ with $a \leqslant Cb$. Also, we recall that the notation $a \asymp b$ means that the two quantities are comparable.

2. SOME BACKGROUND

A sequence of points $\{a_k\}$ in the unit disc $\mathbb{D}$ is said to be separated (in the Bergman metric) if there exists $\delta > 0$ such that $\beta(a_i, a_j) \geqslant \delta$ for all i and j with $i \neq j$, where $\beta(z, w)$ denotes the Bergman metric on $\mathbb{D}$. We use the notation

$$D(a, r) = \{z \in \mathbb{B}_n : \beta(z, a) < r\}$$

for the Bergman metric ball of radius $r > 0$ centered at a point $a \in \mathbb{D}$. For $r > 0$, a sequence $\{a_k\}$ in $\mathbb{D}$ is said to be an *r-lattice* if $\mathbb{D} = \bigcup_k D(a_k, r)$, the sets $D(a_k, r/4)$ are pairwise disjoints, and there exists a positive integer N such that every $z \in \mathbb{D}$ belongs to at most N of the sets $D(a_k, 4r)$. By [30, Theorem 2.23], there are r-lattices on $\mathbb{D}$ for every $r > 0$. It is also clear that any r-lattice is separated.

A particularly important case of tent spaces is given when $\nu = \sum_k \delta_{a_k}$, where $Z = \{a_k\}$ is a separated sequence and δ_{a_k} are the usual Dirac point masses at points a_k. We say that $\lambda = \{\lambda_k\} \in T_q^p(Z, \omega)$ if

$$\|\lambda\|^p_{T_q^p(Z,\omega)} := \int_{\mathbb{D}} \Big(\sum_{a_k \in \Gamma(\zeta)} |\lambda_k|^q \Big)^{p/q} \omega(\zeta) \mathrm{d}A(\zeta) < \infty, \quad 0 < p, q < \infty,$$

and $\lambda = \{\lambda_k\} \in T_\infty^p(Z, \omega)$ if

$$\|\lambda\|^p_{T_\infty^p(Z,\omega)} := \int_{\mathbb{D}} \Big(\sup_{a_k \in \Gamma(\zeta)} |\lambda_k| \Big)^{p} \omega(\zeta) \mathrm{d}A(\zeta) < \infty.$$

In case that $q = p$, by Fubini's theorem, we have

$$\|\lambda\|^p_{T_p^p(Z,\omega)} \asymp \sum_k |\lambda_k|^p \, \omega(T(a_k)). \tag{2.1}$$

We will need the following duality result for the tent spaces of sequences. For the proof, see Theorem 4 and Proposition 1 in [24]. Recall that, if $1 < p < \infty$, its conjugate exponent p' is given by $p' = p/(p-1)$.

THEOREM B. *Let $1 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$, and $Z = \{a_k\}$ be a separated sequence. If $1 < q < \infty$, then the dual of $T_q^p(Z, \omega)$ is isomorphic to $T_{q'}^{p'}(Z, \omega)$ under the pairing*

$$\langle \lambda, \beta \rangle_{T_2^2(Z,\omega)} = \sum_k \lambda_k \, \overline{\beta}_k \; \omega(T(a_k)),$$

for $\lambda = \{\lambda_k\} \in T_q^p(Z, \omega)$, and $\beta = \{\beta_k\} \in T_{q'}^{p'}(Z, \omega)$. If $0 < q \leqslant 1$, then the dual of $T_q^p(Z, \omega)$ is isomorphic to $T_\infty^{p'}(Z, \omega)$ under the same pairing.

We will use the following result concerning factorization of sequence tent spaces. The proof is similar to the one in [18, Proposition 6]. A factorization result for tent spaces of functions over the upper half-space was proven in [5] by Cohn and Verbitsky.

PROPOSITION 2.1. *Let $0 < p, q < \infty$, $\omega \in \widehat{\mathcal{D}}$, and $Z = \{a_k\}$ be an r-lattice. Suppose $p < p_1, p_2 < \infty$, $q < q_1, q_2 < \infty$ and satisfy*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$T_q^p(Z, \omega) = T_{q_1}^{p_1}(Z, \omega) \cdot T_{q_2}^{p_2}(Z, \omega).$$

That is, if $\alpha \in T_{q_1}^{p_1}(Z, \omega)$ and $\beta \in T_{q_2}^{p_2}(Z, \omega)$, then $\alpha \cdot \beta \in T_q^p(Z, \omega)$ with

$$\|\alpha \cdot \beta\|_{T_q^p(Z,\omega)} \lesssim \|\alpha\|_{T_{q_1}^{p_1}(Z,\omega)} \cdot \|\beta\|_{T_{q_2}^{p_2}(Z,\omega)};$$

and conversely, if $\lambda \in T_q^p(Z, \omega)$, then there exist sequences $\alpha \in T_{q_1}^{p_1}(Z, \omega)$ and $\beta \in T_{q_2}^{p_2}(Z, \omega)$ such that $\lambda = \alpha \cdot \beta$, and $\|\alpha\|_{T_{q_1}^{p_1}(Z,\omega)} \cdot \|\beta\|_{T_{q_2}^{p_2}(Z,\omega)} \lesssim \|\lambda\|_{T_q^p(Z,\omega)}$.

We also need two important inequalities. Consider a sequence of Rademacher functions $r_k(s)$. For almost every $s \in [0,1]$ the sequence $\{r_k(s)\}$ consists of signs ± 1. We state first the classical Khinchine's inequality (see [8, Appendix A] for example).

2.1. KHINCHINE'S INEQUALITY. Let $0 < p < \infty$. Then for any sequence $\{c_k\}$ of complex numbers, we have

$$\Big(\sum_k |c_k|^2\Big)^{p/2} \asymp \int_0^1 \Big|\sum_k r_k(s)\, c_k\Big|^p \mathrm{d}s.$$

The next result is known as Kahane's inequality, and it will be usually applied in connection with Khinchine's inequality. For a reference, see Lemma 5 in Luecking's paper [15] or the paper of Kalton [13].

2.2. KAHANE'S INEQUALITY. Let X be a quasi-Banach space, and $0 < p, q < \infty$. For any sequence $\{x_k\} \subset X$, one has

$$\Big(\int_0^1 \Big\|\sum_k r_k(s)\, x_k\Big\|_X^q \mathrm{d}s\Big)^{1/q} \asymp \Big(\int_0^1 \Big\|\sum_k r_k(s)\, x_k\Big\|_X^p \mathrm{d}s\Big)^{1/p}.$$

Moreover, the implicit constants can be chosen to depend only on p and q, and not on the quasi-Banach space X.

3. KEY RESULTS

In this section we are going to obtain the key results for the proof of Theorem 1.1. We start with some lemmas.

LEMMA 3.1. *Let μ be a positive Borel measure on $\mathbb{D}$, finite on compact subsets of $\mathbb{D}$. For $t > 0$, let $\{a_k\}$ be a t-lattice on $\mathbb{D}$. Then*

$$\mu(\mathbb{D}) \lesssim \sum_{a_k \in A} \frac{\mu(D(a_k, t))^2}{\mu(D(a_k, 2t))},$$

where $A = \{z \in \mathbb{D} : \mu(D(z,t)) > 0\}$.

Proof. As μ is finite on compact sets, by a standard approximation argument, it is enough to prove the inequality, assuming that $\mu(\mathbb{D}) < \infty$. Being $\{a_k\}$ a t-lattice, for $t \leqslant r \leqslant 4t$, we have

$$\mu(\mathbb{D}) \asymp \sum_{a_k \in A} \mu(D(a_k, r)).$$

By Cauchy–Schwarz inequality, we have

$$\mu(\mathbb{D}) \asymp \sum_{a_k\in A} \mu(D(a_k,t)) \leqslant \Big(\sum_{a_k\in A} \mu(D(a_k,2t))\Big)^{1/2}\Big(\sum_{a_k\in A} \frac{\mu(D(a_k,t))^2}{\mu(D(a_k,2t))}\Big)^{1/2}$$
$$\lesssim \mu(\mathbb{D})^{1/2}\Big(\sum_{a_k\in A} \frac{\mu(D(a_k,t))^2}{\mu(D(a_k,2t))}\Big)^{1/2},$$

and the inequality follows. ∎

We need the following estimate from [24, Lemma 4].

LEMMA A. *Let $\omega \in \widehat{\mathcal{D}}$ and $0 < p < \infty$. Let ν be a positive Borel measure on $\mathbb{D}$, finite on compact sets. Then there exists $\lambda_0 = \lambda_0(p,\omega) \geqslant 1$ such that*

$$\int_{\mathbb{D}} \Big(\int_{\mathbb{D}} \Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\lambda} \mathrm{d}\nu(z)\Big)^{p} \omega(\zeta)\,\mathrm{d}A(\zeta) \asymp \int_{\mathbb{D}} \nu(\Gamma(\zeta))^{p}\,\omega(\zeta)\,\mathrm{d}A(\zeta) + \nu(\{0\})$$

for each $\lambda > \lambda_0$.

If $\mathrm{spt}(\mu)$ denotes the support of the measure μ, it is then clear that $\mu(D(z,t)) \neq 0$ for $z \in \mathrm{spt}(\mu)$. We need the following inequality, that can be interesting by itself.

LEMMA 3.2. *Let $0 < p < \infty$, $\omega \in \widehat{\mathcal{D}}$ and $f \in A^p_\omega$. Then, for a positive Borel measure μ on $\mathbb{D}$ finite on compact sets, and $t > 0$, we have*

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|f'(z)|^2\,(1-|z|^2)^2}{\mu(D(z,t))}\,\mathrm{d}\mu(z)\Big)^{p/2} \omega(\zeta)\,\mathrm{d}A(\zeta) \lesssim \|f\|^p_{A^p_\omega}.$$

Proof. By subharmonicity, we have

$$\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|f'(z)|^2(1-|z|^2)^2}{\mu(D(z,t))}\mathrm{d}\mu(z) \lesssim \int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \Big(\int_{D(z,t/2)} |f'(\xi)|^2 \mathrm{d}A(\xi)\Big)\frac{\mathrm{d}\mu(z)}{\mu(D(z,t))}.$$

For $\xi \in D(z,t/2)$, we have $D(\xi,t/2) \subset D(z,t)$, and hence

$$\mu(D(z,t)) \geqslant \mu(D(\xi,t/2)) > 0.$$

Also, for $z \in \Gamma(\zeta)$ and $\xi \in D(z,t/2)$, we have (see [30, estimate (2.20)])

$$|1-\overline{\zeta}\xi| \asymp |1-\overline{\zeta}z| \lesssim 1-|z| \asymp 1-|\xi|.$$

Thus

$$\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|f'(z)|^2(1-|z|^2)^2}{\mu(D(z,t))}\mathrm{d}\mu(z) \lesssim \int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \Big(\int_{D(z,t/2)} \Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\lambda} \frac{|f'(\xi)|^2 \mathrm{d}A(\xi)}{\mu(D(\xi,t/2))}\Big)\mathrm{d}\mu(z)$$

for $\lambda > \lambda_0$, where λ_0 is the number given by Lemma A. Set

$$A = \{\xi \in \mathbb{D} : \mu(D(\xi,t/2)) > 0\}.$$

Therefore, by Fubini's theorem, we have

$$\int\limits_{\Gamma(\zeta)\cap\operatorname{spt}(\mu)} \frac{|f'(z)|^2(1-|z|^2)^2}{\mu(D(z,t))}\mathrm{d}\mu(z) \lesssim \int\limits_{\mathbb{D}\cap A}\Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\lambda}\frac{|f'(\xi)|^2}{\mu(D(\xi,t/2))}\Big(\int\limits_{D(\xi,t/2)}\mathrm{d}\mu(z)\Big)\mathrm{d}A(\xi)$$
$$\leqslant \int\limits_{\mathbb{D}}\Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\lambda}|f'(\xi)|^2\,\mathrm{d}A(\xi).$$

Hence, applying Lemma A with the measure $\mathrm{d}\nu = |f'|^2\mathrm{d}A$, we obtain

$$\int\limits_{\mathbb{D}}\Big(\int\limits_{\Gamma(\zeta)\cap\operatorname{spt}(\mu)}\frac{|f'(z)|^2\,(1-|z|^2)^2}{\mu(D(z,t))}\,\mathrm{d}\mu(z)\Big)^{p/2}\omega(\zeta)\,\mathrm{d}A(\zeta)$$
$$\lesssim \int\limits_{\mathbb{D}}\Big(\int\limits_{\Gamma(\zeta)}|f'(\xi)|^2\,\mathrm{d}A(\xi)\Big)^{p/2}\omega(\zeta)\,\mathrm{d}A(\zeta) \lesssim \|f\|^p_{A^p_\omega},$$

after using the area function description of A^p_ω-spaces (see [23, Theorem 4.2] for example). This finishes the proof. ∎

Recall that

$$B^{\nu}_{\mu}(\zeta) = \int\limits_{\Gamma(\zeta)}\frac{\mathrm{d}\mu(z)}{\nu(T(z))} = \int\limits_{\Gamma(\zeta)}\frac{\mathrm{d}\mu^{\omega}_{\nu}(z)}{\omega(T(z))}, \quad \zeta\in\mathbb{D}\backslash\{0\},$$

with

$$\mathrm{d}\mu^{\omega}_{\nu}(z) = \frac{\omega(T(z))}{\nu(T(z))}\mathrm{d}\mu(z).$$

Next result together with Proposition 3.5 will be the key for our characterization of the boundedness of generalized area operators acting on weighted Bergman spaces.

PROPOSITION 3.3. *Let $1 \leqslant \sigma < \infty$, $\omega \in \widehat{\mathcal{D}}$, and let μ be a positive Borel measure on $\mathbb{D}$, finite on compact sets of $\mathbb{D}$, with $\mu(\{0\}) = 0$. The following conditions are equivalent:*

(i) $B^{\omega}_{\mu} \in L^1_\omega$;

(ii) *for any $t > 0$, there exists some $C > 0$ such that*

$$\int\limits_{\mathbb{D}}\Big(\int\limits_{\Gamma(\zeta)\cap\operatorname{spt}(\mu)}\frac{|f'(z)|(1-|z|)\mathrm{d}\mu(z)}{\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}}\Big)^{2\sigma/(2+\sigma)}\omega(\zeta)\mathrm{d}A(\zeta)\leqslant C\|f\|^{2\sigma/(2+\sigma)}_{A^\sigma_\omega} \tag{3.1}$$

for all $f \in A^\sigma_\omega$. Moreover, one has

$$\|B^{\omega}_{\mu}\|^{\sigma/(2+\sigma)}_{L^1_\omega} \asymp C_\mu,$$

where C_μ denotes the infimum over all constants C satisfying (3.1).

Proof. First, we show the statement (i) implies (ii). By Cauchy–Schwarz inequality, the left quantity in (3.1) is no more than constant times

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|f'(z)|^2\,(1-|z|)^2\,\mathrm{d}\mu(z)}{\mu(D(z,t))} \Big)^{\sigma/(2+\sigma)} (B^{\omega}_{\mu}(\zeta))^{\sigma/(2+\sigma)} \omega(\zeta)\,\mathrm{d}A(\zeta)$$
$$\lesssim \|f\|^{2\sigma/(2+\sigma)}_{A^{\sigma}_{\omega}} \cdot \|B^{\omega}_{\mu}\|^{\sigma/(2+\sigma)}_{L^1_{\omega}},$$

after the use of Hölder's inequality (as the conjugate exponent of $(2+\sigma)/2$ is $(2+\sigma)/\sigma$) together with Lemma 3.2.

Conversely, let $Z = \{a_k\}$ be a t-lattice on $\mathbb{D}$, and for $s \in [0,1]$, we take the test function

$$F_s(z) = \sum_k r_k(s)\,\lambda_k \Big(\frac{1-|a_k|}{1-\bar{a}_k z}\Big)^{\gamma}, \tag{3.2}$$

where $r_k(s)$ are Rademacher functions, $\gamma > \lambda_0$ (with λ_0 being given by Lemma A) and $\lambda = \{\lambda_k\} \in T^{\sigma}_2(Z,\omega)$. Then we have $F_s \in A^{\sigma}_{\omega}$ with $\|F_s\|_{A^{\sigma}_{\omega}} \lesssim \|\lambda\|_{T^{\sigma}_2(Z,\omega)}$ which follows from [24, Lemma 6]. For every $s \in [0,1]$, statement (ii) implies

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \Big| \sum_k \frac{r_k(s)\lambda_k(1-|a_k|)^{\gamma}(1-|z|)\bar{a}_k}{(1-\bar{a}_k z)^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}} \Big| \mathrm{d}\mu(z) \Big)^{2\sigma/(2+\sigma)} \omega(\zeta)\mathrm{d}A(\zeta)$$
$$\leqslant C\,\|F_s\|^{2\sigma/(2+\sigma)}_{A^{\sigma}_{\omega}}.$$

By Kahane's inequality, we have that

$$\int_0^1 \Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \Big| \sum_k \frac{r_k(s)\,\lambda_k\,(1-|a_k|)^{\gamma}\,(1-|z|)\,\bar{a}_k}{(1-\bar{a}_k z)^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}} \Big| \mathrm{d}\mu(z) \Big)^{2\sigma/(2+\sigma)} \mathrm{d}s$$

is comparable to

$$\Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \int_0^1 \Big| \sum_k \frac{r_k(s)\,\lambda_k(1-|a_k|)^{\gamma}\,(1-|z|)\,\bar{a}_k}{(1-\bar{a}_k z)^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}} \Big| \mathrm{d}s\,\mathrm{d}\mu(z) \Big)^{2\sigma/(2+\sigma)},$$

and, by Khinchine's inequality, this is comparable to

$$\Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \Big(\sum_k \frac{|\lambda_k|^2(1-|a_k|)^{2\gamma}(1-|z|)^2|a_k|^2}{|1-\bar{a}_k z|^{2(\gamma+1)}\mu(D(z,t))\omega(T(z))} \Big)^{1/2} \mathrm{d}\mu(z) \Big)^{2\sigma/(2+\sigma)}.$$

Hence, integrating with respect to s our inequality, using Fubini's theorem and the previous obtained estimates, and taking into account that $\|F_s\|_{A^{\sigma}_{\omega}} \lesssim \|\lambda\|_{T^{\sigma}_2(Z,\omega)}$, we get

$$\int_{\mathbb{D}}\Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)}\Big(\sum_k \frac{|\lambda_k|^2(1-|a_k|)^{2\gamma}(1-|z|)^2|a_k|^2}{|1-\overline{a}_k z|^{2(\gamma+1)}\mu(D(z,t))\omega(T(z))}\Big)^{1/2}\mathrm{d}\mu(z)\Big)^{2\sigma/(2+\sigma)}\omega(\zeta)\mathrm{d}A(\zeta)$$
$$\lesssim C\,\|\lambda\|^{2\sigma/(2+\sigma)}_{T_2^\sigma(Z,\omega)}.$$

For $\eta > \lambda_0$, using Lemma A we obtain

$$\int_{\mathbb{D}}\Big(\int_{\mathbb{D}\cap \mathrm{spt}(\mu)}\Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\eta}\Big(\sum_k \frac{|\lambda_k|^2(1-|a_k|)^{2\gamma}(1-|z|)^2|a_k|^2}{|1-\overline{a}_k z|^{2(\gamma+1)}\mu(D(z,t))\omega(T(z))}\Big)^{1/2}\mathrm{d}\mu(z)\Big)^{2\sigma/(2+\sigma)}\omega(\zeta)\mathrm{d}A(\zeta)$$
$$\lesssim C\,\|\lambda\|^{2\sigma/(2+\sigma)}_{T_2^\sigma(Z,\omega)}.$$

Set $D_k = D(a_k,t)$. Since $\{a_k\}$ is a t-lattice, for any $z\in\mathbb{D}$ we have

$$\sum_k \chi_{D_k}(z)\leqslant N,$$

as any $z\in\mathbb{D}$ belongs to at most N of the sets D_k. Then, by Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
&\sum_k \int_{D_k\cap \mathrm{spt}(\mu)}\Big(\frac{1-|a_k|^2}{|1-\overline{\zeta}a_k|}\Big)^{\eta}\frac{|\lambda_k|(1-|a_k|)^{\gamma}(1-|z|)|a_k|}{|1-\overline{a}_k z|^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}}\mathrm{d}\mu(z)\\
&\quad\asymp \sum_k \int_{D_k\cap \mathrm{spt}(\mu)}\Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\eta}\frac{|\lambda_k|(1-|a_k|)^{\gamma}(1-|z|)|a_k|}{|1-\overline{a}_k z|^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}}\mathrm{d}\mu(z)\\
&\quad= \int_{\mathbb{D}\cap \mathrm{spt}(\mu)}\Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\eta}\Big(\sum_k \frac{|\lambda_k|(1-|a_k|)^{\gamma}(1-|z|)|a_k|}{|1-\overline{a}_k z|^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}}\chi_{D_k}(z)\Big)\mathrm{d}\mu(z)\\
&\quad\leqslant N^{1/2}\int_{\mathbb{D}\cap \mathrm{spt}(\mu)}\Big(\frac{1-|z|^2}{|1-\overline{\zeta}z|}\Big)^{\eta}\Big(\sum_k \frac{|\lambda_k|^2(1-|a_k|)^{2\gamma}(1-|z|)^2|a_k|^2}{|1-\overline{a}_k z|^{2(\gamma+1)}\mu(D(z,t))\omega(T(z))}\Big)^{1/2}\mathrm{d}\mu(z).
\end{aligned}$$

Putting this in our previous estimate, and taking into account that $|1-\overline{\zeta}a_k|\lesssim 1-|a_k|$ for $a_k\in\Gamma(\zeta)$, we arrive at the inequality

$$\int_{\mathbb{D}}\Big(\sum_{a_k\in\Gamma(\zeta)\cap A}\int_{D_k}\frac{|\lambda_k|(1-|a_k|)^{\gamma}(1-|z|)|a_k|}{|1-\overline{a}_k z|^{\gamma+1}\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}}\mathrm{d}\mu(z)\Big)^{2\sigma/(2+\sigma)}\omega(\zeta)\mathrm{d}A(\zeta)$$
$$\lesssim C\,\|\lambda\|^{2\sigma/(2+\sigma)}_{T_2^\sigma(Z,\omega)},$$

where $A=\{z\in\mathbb{D}:\mu(D(z,t))>0\}$. Now, notice that, for $z\in D_k$, we have $|1-\overline{a}_k z|\asymp 1-|a_k|$, and $\mu(D(z,t))\leqslant \mu(D(a_k,2t))$. Also, since $\omega\in\widehat{\mathcal{D}}$, we have $\omega(T(z))\asymp\omega(T(a_k))$ for $z\in D_k$ (see [27] for example). Therefore, we obtain the estimate

$$\int_{\mathbb{D}} \Big(\sum_{a_k \in \Gamma(\zeta)\cap A} |\lambda_k| \frac{\mu(D_k)}{\mu(D(a_k, 2t))^{1/2}\omega(T(a_k))^{1/2}} \Big)^{2\sigma/(2+\sigma)} \omega(\zeta) \mathrm{d}A(\zeta)$$

$$\lesssim C \|\lambda\|^{2\sigma/(2+\sigma)}_{T_2^\sigma(Z,\omega)}. \tag{3.3}$$

It is clear that $B_\mu^\omega \in L^1_\omega$ if and only if μ is a finite measure with $\|B_\mu^\omega\|_{L^1_\omega} \asymp \mu(\mathbb{D})$. By Lemma 3.1, we have

$$\|B_\mu^\omega\|_{L^1_\omega} \asymp \mu(\mathbb{D}) \lesssim \sum_{a_k \in A} \frac{\mu(D_k)^2}{\mu(D(a_k, 2t))}.$$

Thus, bearing in mind (2.1), in order to prove that $B_\mu^\omega \in L^1_\omega$ with $\|B_\mu^\omega\|_{L^1_\omega} \lesssim C^{(2+\sigma)/\sigma}$, it suffices to show that

$$\Big\{\gamma_k\Big\}_{k: a_k\in A} := \Big\{ \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,\, 2t))^{1/4}\omega(T(a_k))^{1/4}} \Big\}_{k: a_k \in A} \in T_4^4(Z,\, \omega)$$

with $\|\gamma\|^4_{T_4^4(Z,\omega)} \lesssim C^{(2+\sigma)/\sigma}$. By the duality of tent sequence spaces given in Theorem B, we only need to prove that

$$\sum_{k:\, a_k \in A} |\alpha_k| \, \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,\, 2t))^{1/4}\omega(T(a_k))^{1/4}} \, \omega(T(a_k)) \lesssim C^{(2+\sigma)/4\sigma} \, \|\alpha\|_{T_{4/3}^{4/3}(Z,\omega)} \tag{3.4}$$

for each $\alpha = \{\alpha_k\} \in T_{4/3}^{4/3}(Z,\, \omega)$. By Fubini's theorem, we have

$$\sum_{k:\, a_k \in A} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k, 2t))^{1/4}\omega(T(a_k))^{1/4}} \omega(T(a_k))$$

$$\asymp \int_{\mathbb{D}} \Big(\sum_{k:\, a_k \in A\cap\Gamma(\zeta)} |\alpha_k| \, \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,\, 2t))^{1/4}\omega(T(a_k))^{1/4}} \Big) \omega(\zeta) \, \mathrm{d}A(\zeta). \tag{3.5}$$

Using the factorization of tent sequence spaces given in Proposition 2.1, we can factorize $\alpha_k = \beta_k \cdot \lambda_k^{1/2}$, with $\lambda \in T_2^\sigma(Z,\, \omega)$, and $\beta \in T_2^{4\sigma/(3\sigma-2)}(Z,\, \omega)$. Moreover, we have

$$\|\beta\|_{T_2^{4\sigma/(3\sigma-2)}(Z,\omega)} \cdot \|\lambda\|^{1/2}_{T_2^\sigma(Z,\omega)} \lesssim \|\alpha\|_{T_{4/3}^{4/3}(Z,\omega)}. \tag{3.6}$$

Then

$$\sum_{k:\, a_k \in A\cap\Gamma(\zeta)} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,\, 2t))^{1/4}\omega(T(a_k))^{1/4}}$$

$$\leqslant \Big(\sum_{k:\, a_k \in A\cap\Gamma(\zeta)} |\beta_k|^2 \Big)^{1/2} \Big(\sum_{k:\, a_k \in A\cap\Gamma(\zeta)} \frac{|\lambda_k| \mu(D_k)}{\mu(D(a_k,\, 2t))^{1/2}\omega(T(a_k))^{1/2}} \Big)^{1/2}.$$

Putting this estimate into (3.5), and using Hölder's inequality with exponent $4\sigma/(2+\sigma)$ (that has conjugate exponent $4\sigma/(3\sigma-2)$), we get

$$\begin{aligned}
&\sum_{k:\, a_k\in A} |\alpha_k| \, \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,\, 2t))^{1/4}\omega(T(a_k))^{1/4}}\, \omega(T(a_k)) \\
&\quad\lesssim \|\beta\|_{T_2^{4\sigma/(3\sigma-2)}(Z,\omega)} \Big(\int_{\mathbb{D}} \Big(\sum_{k:\, a_k\in A\cap\Gamma(\zeta)} \frac{|\lambda_k|\ \mu(D_k))}{\mu(D(a_k,\, 2t))^{1/2}\omega(T(a_k))^{1/2}} \Big)^{2\sigma/(2+\sigma)} \\
&\qquad\qquad \cdot \omega(\zeta) \mathrm{d}A(\zeta) \Big)^{(2+\sigma)/4\sigma} \\
&\quad\lesssim C^{(2+\sigma)/4\sigma}\ \|\beta\|_{T_2^{4\sigma/(3\sigma-2)}(Z,\omega)} \cdot \|\lambda\|^{1/2}_{T_2^{\sigma}(Z,\omega)},
\end{aligned}$$

where the last estimate follows from (3.3). Finally, by (3.6), we obtain the desired result (3.4), finishing the proof. ∎

As a consequence, we obtain the following result, that we will use in order to obtain the case $pq = s(p-q)$ in Theorem 1.1.

COROLLARY 3.4. *Let $\omega \in \widehat{\mathcal{D}}$, and let μ be a positive Borel measure on $\mathbb{D}$, finite on compact sets of $\mathbb{D}$, with $\mu(\{0\}) = 0$. Then $B^{\omega}_{\mu} \in L^1_{\omega}$ if and only if, for $p \in (0,\infty)$ and $t > 0$, we have*

$$\int_{\mathbb{D}} \int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|g(z)|^{p/4}|f'(z)|(1-|z|)\mathrm{d}\mu(z)}{\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}} \omega(\zeta)\mathrm{d}A(\zeta) \leqslant K\, \|g\|^{p/4}_{A^p_\omega}\ \|f\|_{A^4_\omega}. \tag{3.7}$$

Moreover, one has

$$\|B^{\omega}_{\mu}\|^{1/2}_{L^1_\omega} \asymp K_\mu,$$

where K_μ is the infimum over all constants K satisfying (3.7).

Proof. Consider the measure

$$\mathrm{d}\lambda_f(z) = |f'(z)|\, \mu(D(z,t))^{-1/2}\, \omega(T(z))^{1/2}(1-|z|)\chi_{\mathrm{spt}(\mu)}(z)\, \mathrm{d}\mu(z).$$

Then the inequality is

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)} |g(z)|^{p/4} \frac{\mathrm{d}\lambda_f(z)}{\omega(T(z))} \Big) \omega(\zeta)\mathrm{d}A(\zeta) \leqslant K\, \|f\|_{A^4_\omega}\, \|g\|^{p/4}_{A^p_\omega}.$$

By Fubini's theorem,

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)} |g(z)|^{p/4} \frac{\mathrm{d}\lambda_f(z)}{\omega(T(z))} \Big) \omega(\zeta)\mathrm{d}A(\zeta) \asymp \int_{\mathbb{D}} |g(z)|^{p/4}\, \mathrm{d}\lambda_f(z).$$

Hence the inequality is equivalent to

$$\int_{\mathbb{D}} |g(z)|^{p/4}\, \mathrm{d}\lambda_f(z) \lesssim K\, C_f\, \|g\|^{p/4}_{A^p_\omega}$$

with $C_f \asymp \|f\|_{A^4_\omega}$. By [27, Theorem 3] (see also [24, Theorem 8]), this is equivalent to $B^\omega_{\lambda_f}$ being in $L^{4/3}_\omega$ with $\|B^\omega_{\lambda_f}\|_{L^{4/3}_\omega} \lesssim K\|f\|_{A^4_\omega}$. Then, applying Proposition 3.3 with $\sigma = 4$, we obtain the desired result. ∎

The non-tangential maximal operator (in the punctured unit disc) is defined by

$$N(f)(\zeta) = \sup_{z\in\Gamma(\zeta)} |f(z)|, \quad \zeta \in \mathbb{D}\setminus\{0\}.$$

It is known [23, Lemma 4.4] that, for $0 < p < \infty$, the operator $N : A^p_\omega \to L^p_\omega$ is bounded for each radial weight ω and $\|N(f)\|_{L^p_\omega} \asymp \|f\|_{A^p_\omega}$. We have the following result.

PROPOSITION 3.5. *Let $0 < q < p < \infty$, $\omega \in \widehat{\mathcal{D}}$, and let μ be a positive Borel measure on $\mathbb{D}$, finite on compact sets of $\mathbb{D}$ with $\mu(\{0\}) = 0$. Suppose $s > 0$, $r/s < 1$ with $r = pq/(p-q)$. For any positive integer m with*

$$mq > 2 \quad \text{and} \quad m(p-q)s > 2q, \tag{3.8}$$

set

$$\sigma = \frac{2m(sp - sq - pq)}{ms(p-q) - 2q}. \tag{3.9}$$

We have $B^\omega_\mu \in L^{r/s}_\omega$ if and only if, there is a constant $D > 0$ such that

$$\int_{\mathbb{D}} \int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|g(z)|^\sigma |f'(z)|^\sigma (1-|z|)^\sigma \mathrm{d}\mu(z)}{\mu(D(z,t))^{\sigma/2}\omega(T(z))^{(2-\sigma)/2}} \omega(\zeta)\mathrm{d}A(\zeta) \leqslant D\|g\|^\sigma_{A^{2mp}_\omega} \cdot \|f\|^\sigma_{A^{2mp}_\omega}. \tag{3.10}$$

Moreover, one has

$$\|B^\omega_\mu\|^{1-\sigma/2}_{L^{r/s}_\omega} \asymp D_\mu,$$

where D_μ is the infimum over all constants D satisfying (3.10).

Proof. Observe that, $s(p-q) - pq > 0$ as $r/s < 1$. This, together with (3.8) tells us that $\sigma > 0$. As $mp > 2$, we have $\sigma < 2$.

First, suppose $B^\omega_\mu \in L^{r/s}_\omega$. By Hölder's inequality with exponent $2/\sigma > 1$, we obtain

$$\begin{aligned}
&\int_{\mathbb{D}} \int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|g(z)|^\sigma\, |f'(z)|^\sigma (1-|z|)^\sigma \,\mathrm{d}\mu(z)}{\mu(D(z,t))^{\sigma/2}\,\omega(T(z))^{(2-\sigma)/2}}\ \omega(\zeta)\mathrm{d}A(\zeta) \\
&\lesssim \int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|g(z)|^2 |f'(z)|^2 (1-|z|)^2 \mathrm{d}\mu(z)}{\mu(D(z,t))} \Big)^{\sigma/2} B^\omega_\mu(\zeta)^{(2-\sigma)/2}\omega(\zeta)\mathrm{d}A(\zeta).
\end{aligned}$$

Since $r < s$, it follows that $2r/s(2-\sigma) > 1$. Also, notice that

$$\frac{2r\sigma}{2r - s(2-\sigma)} = mp.$$

Hence, an application of Hölder's inequality with exponent $2r/s(2-\sigma)$ gives

$$\begin{aligned}
&\int_{\mathbb{D}} \int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|g(z)|^{\sigma}\,|f'(z)|^{\sigma}(1-|z|)^{\sigma}\,\mathrm{d}\mu(z)}{\mu(D(z,t))^{\sigma/2}\,\omega(T(z))^{(2-\sigma)/2}}\;\omega(\zeta)\mathrm{d}A(\zeta)\\
&\lesssim \Big(\int_{\mathbb{D}}\Big(\int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|g(z)|^2|f'(z)|^2(1-|z|)^2\mathrm{d}\mu(z)}{\mu(D(z,t))}\Big)^{mp/2}\omega(\zeta)\mathrm{d}A(\zeta)\Big)^{\sigma/mp}\\
&\qquad\cdot \|B_{\mu}^{\omega}\|_{L_{\omega}^{r/s}}^{(2-\sigma)/2}. \qquad (3.11)
\end{aligned}$$

Using Cauchy–Schwarz inequality, the boundedness of the operator N and Lemma 3.2, we have

$$\begin{aligned}
&\int_{\mathbb{D}}\Big(\int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|g(z)|^2|f'(z)|^2(1-|z|)^2\mathrm{d}\mu(z)}{\mu(D(z,t))}\Big)^{mp/2}\omega(\zeta)\mathrm{d}A(\zeta)\\
&\leqslant \int_{\mathbb{D}} N(g)(\zeta)^{mp}\Big(\int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|f'(z)|^2(1-|z|)^2\mathrm{d}\mu(z)}{\mu(D(z,t))}\Big)^{mp/2}\omega(\zeta)\mathrm{d}A(\zeta)\\
&\leqslant \|Ng\|_{L_{\omega}^{2mp}}^{mp}\Big(\int_{\mathbb{D}}\Big(\int_{\Gamma(\zeta)\cap \operatorname{spt}(\mu)} \frac{|f'(z)|^2(1-|z|)^2\mathrm{d}\mu(z)}{\mu(D(z,t))}\Big)^{mp}\omega(\zeta)\mathrm{d}A(\zeta)\Big)^{1/2}\\
&\lesssim \|g\|_{A_{\omega}^{2mp}}^{mp}\;\|f\|_{A_{\omega}^{2mp}}^{mp}.
\end{aligned}$$

Thus, putting this inequality into (3.11), we see that (3.10) holds, and moreover

$$D_{\mu} \lesssim \|B_{\mu}^{\omega}\|_{L_{\omega}^{r/s}}^{(2-\sigma)/2}.$$

Conversely, suppose that (3.10) holds for all functions $f, g \in A_{\omega}^{2mp}$. Consider the measure λ_f given by

$$\mathrm{d}\lambda_f(z) = |f'(z)|^{\sigma}\,\mu(D(z,t))^{-\sigma/2}\,\omega(T(z))^{\sigma/2}(1-|z|)^{\sigma}\,\chi_{\operatorname{spt}(\mu)}(z)\,\mathrm{d}\mu(z).$$

Then, the inequality (3.10) is

$$\int_{\mathbb{D}}\Big(\int_{\Gamma(\zeta)} |g(z)|^{\sigma}\,\frac{\mathrm{d}\lambda_f(z)}{\omega(T(z))}\Big)\omega(\zeta)\mathrm{d}A(\zeta) \leqslant D\,\|f\|_{A_{\omega}^{2mp}}^{\sigma}\,\|g\|_{A_{\omega}^{2mp}}^{\sigma}.$$

By Fubini's theorem, this inequality is equivalent to

$$\int_{\mathbb{D}} |g(z)|^{\sigma}\,\mathrm{d}\lambda_f(z) \leqslant C_f\,\|g\|_{A_{\omega}^{2mp}}^{\sigma}$$

with $C_f \asymp D\,\|f\|_{A_{\omega}^{2mp}}^{\sigma}$. Since $2mp > \sigma$, it follows from [27, Theorem 3] that $B_{\lambda_f}^{\omega}$ is in $L_{\omega}^{2mp/(2mp-\sigma)}$ with

$$\|B_{\lambda_f}^{\omega}\|_{L_{\omega}^{2mp/(2mp-\sigma)}} \lesssim C_f \asymp D\,\|f\|_{A_{\omega}^{2mp}}^{\sigma}.$$

That is

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|f'(z)|^\sigma (1-|z|)^\sigma \mathrm{d}\mu(z)}{\mu(D(z,t))^{\sigma/2}\omega(T(z))^{(2-\sigma)/2}} \Big)^{2mp/(2mp-\sigma)} \omega(\zeta)\mathrm{d}A(\zeta) \lesssim D^{2mp/(2mp-\sigma)} \|f\|_{A^{2mp}_\omega}^{2mp\sigma/(2mp-\sigma)}.$$

From here, we follow the same argument as in the proof of Proposition 3.3. Let $Z = \{a_k\}$ be a t-lattice on $\mathbb{D}$, and set $D_k = D(a_k, t)$. For $s \in [0,1]$, we take the test function F_s as in (3.2), where $\lambda = \{\lambda_k\} \in T_2^{2mp}(Z, \omega)$. By [24, Lemma 6], we know that $F_s \in A^{2mp}_\omega$ with $\|F_s\|_{A^{2mp}_\omega} \lesssim \|\lambda\|_{T_2^{2mp}(Z,\omega)}$. Using the argument with Kahane and Khintchine's inequalities, arguing in a similar way as in the proof of Proposition 3.3, we obtain

$$\int_{\mathbb{D}} \Big(\sum_{a_k\in\Gamma(\zeta)\cap A} \frac{|\lambda_k|^\sigma \mu(D_k)}{\mu(D(a_k,2t))^{\sigma/2}\omega(T(a_k))^{(2-\sigma)/2}} \Big)^{2mp/(2mp-\sigma)} \omega(\zeta)\mathrm{d}A(\zeta) \lesssim D^{2mp/(2mp-\sigma)} \|\lambda\|_{T_2^{2mp}(Z,\omega)}^{2mp\sigma/(2mp-\sigma)}, \tag{3.12}$$

where $A = \{z \in \mathbb{D} : \mu(D(z,t)) > 0\}$. Now, we have

$$\|B^\omega_\mu\|^{r/s}_{L^{r/s}_\omega} \asymp \int_{\mathbb{D}} \Big(\sum_{k: a_k\in\Gamma(\zeta)\cap A} \frac{\mu(D_k)}{\omega(T(a_k))} \Big)^{r/s} \omega(\zeta)\,\mathrm{d}A(\zeta).$$

As μ is finite on compact sets of $\mathbb{D}$ and $2/(2-\sigma) > 1$, arguing as in the proof of Lemma 3.1, we get

$$\|B^\omega_\mu\|^{r/s}_{L^{r/s}_\omega} \lesssim \int_{\mathbb{D}} \Big(\sum_{k: a_k\in\Gamma(\zeta)\cap A} \frac{\mu(D_k)^{2/(2-\sigma)}}{\mu(D(a_k,2t))^{\sigma/(2-\sigma)}\,\omega(T(a_k))} \Big)^{r/s} \omega(\zeta)\,\mathrm{d}A(\zeta).$$

Thus, in order to prove that $B^\omega_\mu \in L^{r/s}_\omega$ with $\|B^\omega_\mu\|_{L^{r/s}_\omega} \lesssim D^{2/(2-\sigma)}$, it suffices to show that

$$\{\gamma_k\} := \Big\{ \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,2t))^{\sigma/4}\,\omega(T(a_k))^{(2-\sigma)/4}} \Big\}_{k: a_k\in A} \in T^{4r/s(2-\sigma)}_{4/(2-\sigma)}(Z,\, \omega)$$

with

$$\|\gamma\|_{T^{4r/s(2-\sigma)}_{4/(2-\sigma)}(Z,\omega)} \lesssim D^{1/2}.$$

By the duality of tent sequence spaces, we need to prove that

$$\sum_{k:\, a_k\in A} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k,2t))^{\sigma/4}\omega(T(a_k))^{(2-\sigma)/4}} \omega(T(a_k)) \lesssim D^{1/2} \|\alpha\|_{T^{4r/(4r-2s+\sigma s)}_{4/(2+\sigma)}(Z,\omega)} \tag{3.13}$$

for each $\alpha = \{\alpha_k\} \in T_{4/(2+\sigma)}^{4r/(4r-2s+\sigma s)}(Z, \omega)$. By Fubini's theorem, we have

$$\sum_{k:\, a_k \in A} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k, 2t))^{\sigma/4} \omega(T(a_k))^{(2-\sigma)/4}} \, \omega(T(a_k))$$
$$\asymp \int_{\mathbb{D}} \Big(\sum_{k:\, a_k \in A \cap \Gamma(\zeta)} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k, 2t))^{\sigma/4} \omega(T(a_k))^{(2-\sigma)/4}} \Big) \omega(\zeta) \, \mathrm{d}A(\zeta). \tag{3.14}$$

By Proposition 2.1, we have

$$T_{4/(2+\sigma)}^{4r/(4r-2s+\sigma s)}(Z, \omega) = T_2^{4mp/(2mp+\sigma)}(Z, \omega) \cdot T_{4/\sigma}^{4mp/\sigma}(Z, \omega).$$

Notice that $(2mp+\sigma)/4mp + \sigma/4mp = (4r - 2s + \sigma s)/4r$ because of (3.9). Hence, we can factorize

$$\alpha_k = \beta_k \cdot \lambda_k^{\sigma/2}, \quad \lambda \in T_2^{2mp}(Z, \omega), \quad \beta \in T_2^{4mp/(2mp+\sigma)}(Z, \omega),$$

with

$$\|\beta\|_{T_2^{4mp/(2mp+\sigma)}(Z,\omega)} \cdot \|\lambda\|_{T_2^{2mp}(Z,\omega)}^{\sigma/2} \lesssim \|\alpha\|_{T_{4/(2+\sigma)}^{4r/(4r-2s+\sigma s)}(Z,\omega)}. \tag{3.15}$$

This gives

$$\sum_{k:\, a_k \in A \cap \Gamma(\zeta)} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k, 2t))^{\sigma/4} \omega(T(a_k))^{(2-\sigma)/4}}$$
$$\leqslant \Big(\sum_{k:\, a_k \in A \cap \Gamma(\zeta)} |\beta_k|^2 \Big)^{1/2} \Big(\sum_{k:\, a_k \in A \cap \Gamma(\zeta)} \frac{|\lambda_k|^\sigma \mu(D_k)}{\mu(D(a_k, 2t))^{\sigma/2} \omega(T(a_k))^{(2-\sigma)/2}} \Big)^{1/2}.$$

Putting this into (3.14), using Hölder's inequality with exponent $4mp/(2mp+\sigma)$ (that has conjugate exponent $4mp/(2mp-\sigma)$) together with (3.12) and (3.15), we obtain

$$\sum_{k:\, a_k \in A} |\alpha_k| \frac{\mu(D_k)^{1/2}}{\mu(D(a_k, 2t))^{\sigma/4} \omega(T(a_k))^{(2-\sigma)/4}} \, \omega(T(a_k))$$
$$\lesssim \|\beta\|_{T_2^{4mp/(2mp+\sigma)}(Z,\omega)} \Big(\int_{\mathbb{D}} \Big(\sum_{k:\, a_k \in A \cap \Gamma(\zeta)} \frac{|\lambda_k|^\sigma \, \mu(D_k)}{\mu(D(a_k, 2t))^{\sigma/2}}$$
$$\cdot \, \omega(T(a_k))^{(\sigma-2)/2} \Big)^{2mp/(2mp-\sigma)} \omega(\zeta) \mathrm{d}A(\zeta) \Big)^{(2mp-\sigma)/4mp}$$
$$\lesssim \|\beta\|_{T_2^{4mp/(2mp+\sigma)}(Z,\omega)} \, D^{1/2} \, \|\lambda\|_{T_2^{2mp}(Z,\omega)}^{\sigma/2} \lesssim D^{1/2} \, \|\alpha\|_{T_{4/(2+\sigma)}^{4r/(4r-2s+\sigma s)}(Z,\omega)}.$$

This proves (3.13) , finishing the proof of the proposition. ∎

4. AREA OPERATORS ON BERGMAN SPACES

Recall that the generalized area operator induced by positive measures μ and ν on $\mathbb{D}$ is defined by

$$G^{\nu}_{\mu,s}(f)(\zeta) = \Big(\int\limits_{\Gamma(\zeta)} |f(z)|^s \frac{\mathrm{d}\mu(z)}{\nu(T(z))} \Big)^{1/s}, \quad \zeta \in \mathbb{D}\backslash\{0\}.$$

First, we have the following lemma.

LEMMA 4.1. *Let $0 < p, q, s < \infty$, $\omega \in \widehat{\mathcal{D}}$, and let μ be a positive Borel measure on $\mathbb{D}$. If $G^{\omega}_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded, then $G^{\omega}_{\mu,ms} : A^{mp}_\omega \to L^{mq}_\omega$ is bounded for each positive integer m. Moreover, one has*

$$\|G^{\omega}_{\mu,ms}\|_{A^{mp}_\omega \to L^{mq}_\omega} \leqslant \|G^{\omega}_{\mu,s}\|^{1/m}_{A^p_\omega \to L^q_\omega}. \tag{4.1}$$

Proof. If $f \in A^{mp}_\omega$, then $f^m \in A^p_\omega$ with $\|f^m\|_{A^p_\omega} = \|f\|^m_{A^{mp}_\omega}$ and

$$G^{\omega}_{\mu,ms}(f)(\zeta) = [G^{\omega}_{\mu,s}(f^m)(\zeta)]^{1/m}.$$

Therefore,

$$\|G^{\omega}_{\mu,ms}(f)\|^{mq}_{L^{mq}_\omega} = \|G^{\omega}_{\mu,s}(f^m)\|^q_{L^q_\omega} \leqslant \|G^{\omega}_{\mu,s}\|^q_{A^p_\omega \to L^q_\omega} \|f^m\|^q_{A^p_\omega} = \|G^{\omega}_{\mu,s}\|^q_{A^p_\omega \to L^q_\omega} \|f\|^{mq}_{A^{mp}_\omega},$$

which proves the estimate (4.1). ∎

PROPOSITION 4.2. *Let $\omega \in \widehat{\mathcal{D}}$, $0 < q < p < \infty$ and $0 < s < \infty$. Let μ be a positive Borel measure on $\mathbb{D}$. If $B^\omega_\mu \in L^{(pq)/(s(p-q))}_\omega$, then the operator $G^{\omega}_{\mu,s} : A^p_\omega \to L^q_\omega$ is compact. Moreover,*

$$\|G^{\omega}_{\mu,s}\|_{A^p_\omega \to L^q_\omega} \lesssim \|B^\omega_\mu\|^{1/s}_{L^{(pq)/(s(p-q))}_\omega}.$$

Proof. Assuming that $B^\omega_\mu \in L^{r/s}_\omega$, we first proceed to show that $G^{\omega}_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded. For every $f \in A^p_\omega$, we have

$$\begin{aligned} \|G^{\omega}_{\mu,s}(f)\|^q_{L^q_\omega} &= \int\limits_{\mathbb{D}} \Big(\int\limits_{\Gamma(\zeta)} |f(z)|^s \frac{\mathrm{d}\mu(z)}{\omega(T(z))} \Big)^{q/s} \omega(\zeta)\mathrm{d}A(\zeta) \\ &\leqslant \int\limits_{\mathbb{D}} |N(f)(\zeta)|^q B^\omega_\mu(\zeta)^{q/s} \omega(\zeta)\mathrm{d}A(\zeta). \end{aligned}$$

By Hölder's inequality with exponent p/q and the boundedness of the operator N, we obtain

$$\|G^{\omega}_{\mu,s}(f)\|^q_{L^q_\omega} \leqslant \|Nf\|^q_{L^p_\omega} \|B^\omega_\mu\|^{q/s}_{L^{(pq)/(s(p-q))}_\omega} \lesssim \|f\|^q_{A^p_\omega} \|B^\omega_\mu\|^{q/s}_{L^{(pq)/(s(p-q))}_\omega},$$

which shows that $G^{\nu}_{\mu,s}: A^p_\omega \to L^q_\omega$ is bounded with $\|G^\omega_{\mu,s}\|_{A^p_\omega\to L^q_\omega} \lesssim \|B^\omega_\mu\|^{1/s}_{L^{(pq)/(s(p-q))}_\omega}$. Arguing in the same way as in [28, Theorem 8], it follows that $G^\omega_{\mu,s}$ is compact from A^p_ω to L^q_ω. ∎

PROPOSITION 4.3. *Let $0 < q < p < \infty$, $s > 0$, $r/s < 1$ with $r = pq/(p-q)$, and let $\omega \in \widehat{\mathcal{D}}$. Let μ be a positive Borel measure on $\mathbb{D}$, finite on compact sets of $\mathbb{D}$ with $\mu(\{0\}) = 0$. If $G^\omega_{\mu,s} : A^p_\omega \to L^q_\omega$ is bounded, then $B^\omega_\mu \in L^{r/s}_\omega$. Moreover,*

$$\|B^\omega_\mu\|^{1/s}_{L^{r/s}_\omega} \lesssim \|G^\omega_{\mu,s}\|_{A^p_\omega\to L^q_\omega}. \tag{4.2}$$

Proof. Take a positive integer m big enough so that (3.8) holds. By Lemma 4.1, it follows that the operator $G^\omega_{\mu,2ms} : A^{2mp}_\omega \to L^{2mq}_\omega$ is bounded with

$$\|G^\omega_{\mu,2ms}\|_{A^{2mp}_\omega\to L^{2mq}_\omega} \leqslant \|G^\omega_{\mu,s}\|^{1/(2m)}_{A^p_\omega\to L^q_\omega}. \tag{4.3}$$

Set σ as in (3.9). We have

$$\frac{mp(2-\sigma)}{2(mp-\sigma)} = \frac{r}{s} = \frac{pq}{s(p-q)}.$$

Since $r/s < 1$, $mp > 2$ and (3.8), we have $0 < \sigma < 2$. Observe that

$$q = \frac{(2-\sigma)mps}{2mp + ms(2-\sigma) - 2\sigma}. \tag{4.4}$$

For $f, g \in A^{2mp}_\omega$, let

$$D_\mu(f,g) := \int_{\mathbb{D}} \int_{\Gamma(\zeta)\cap\operatorname{spt}(\mu)} \frac{|g(z)|^\sigma\, |f'(z)|^\sigma (1-|z|)^\sigma \mathrm{d}\mu(z)}{\mu(D(z,t))^{\sigma/2}\, \omega(T(z))^{(2-\sigma)/2}}\, \omega(\zeta)\mathrm{d}A(\zeta).$$

By Hölder's inequality with exponent $2/\sigma > 1$, we have

$$\begin{aligned} &D_\mu(f,g) \\ &\quad\leqslant \int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)\cap\operatorname{spt}(\mu)} \frac{|f'(z)|^2(1-|z|)^2 \mathrm{d}\mu(z)}{\mu(D(z,t))} \Big)^{\sigma/2} (G^\omega_{\mu,2\sigma/(2-\sigma)}(g)(\zeta))^\sigma \omega(\zeta)\mathrm{d}A(\zeta). \end{aligned}$$

Applying Hölder's inequality again, now with exponent $2mp/\sigma$, and then using Lemma 3.2 we get

$$D_\mu(f,g) \lesssim \|f\|^\sigma_{A^{2mp}_\omega}\ \|G^\omega_{\mu,2\sigma/(2-\sigma)}(g)\|^\sigma_{L^{2mp\sigma/(2mp-\sigma)}_\omega}. \tag{4.5}$$

If $\sigma = (2-\sigma)ms$, then (4.4) yields that $2mp\sigma/(2mp-\sigma) = 2mq$, and we obtain

$$\begin{aligned} D_\mu(f,g) &\lesssim \|f\|^\sigma_{A^{2mp}_\omega} \cdot \|G^\omega_{\mu,2ms}(g)\|^\sigma_{L^{2mq}_\omega} \lesssim \|f\|^\sigma_{A^{2mp}_\omega} \cdot \|G^\omega_{\mu,2ms}\|^\sigma_{A^{2mp}_\omega\to L^{2mq}_\omega}\ \|g\|^\sigma_{A^{2mp}_\omega} \\ &\lesssim \|G^\omega_{\mu,s}\|^{\sigma/(2m)}_{A^p_\omega\to L^q_\omega}\ \|f\|^\sigma_{A^{2mp}_\omega}\, \|g\|^\sigma_{A^{2mp}_\omega}. \end{aligned}$$

The last inequality is due to (4.3). By Proposition 3.5, we conclude that $B_\mu^\omega \in L_\omega^{r/s}$, with

$$\|B_\mu^\omega\|_{L_\omega^{r/s}}^{1/s} = \|B_\mu^\omega\|_{L_\omega^{r/s}}^{m(2-\sigma)/\sigma} \lesssim \|G_{\mu,s}^\omega\|_{A_\omega^p \to L_\omega^q},$$

finishing the proof of this case.

If $\sigma > (2-\sigma)ms$, then

$$G_{\mu,2\sigma/(2-\sigma)}^\nu(g)(\zeta) \leqslant N(g)(\zeta)^{(\sigma-(2-\sigma)ms)/\sigma} \cdot G_{\mu,2ms}^\nu(g)(\zeta)^{ms(2-\sigma)/\sigma}, \quad \zeta \in \mathbb{D} \setminus \{0\}.$$

Since $mp > 2$ and $0 < \sigma < 2$, it is easy to check that

$$\frac{2mp-\sigma}{\sigma-(2-\sigma)ms} > 1,$$

that has conjugate exponent

$$\Big(\frac{2mp-\sigma}{\sigma-(2-\sigma)ms}\Big)' = \frac{2mp-\sigma}{2mp+(2-\sigma)ms-2\sigma} = \frac{(2mp-\sigma)q}{(2-\sigma)mps}.$$

Thus, we can use Hölder's inequality with those exponents in order to obtain

$$\|G_{\mu,2\sigma/(2-\sigma)}^\omega(g)\|_{L_\omega^{2mp\sigma/(2mp-\sigma)}}^\sigma \leqslant \|Ng\|_{L_\omega^{2mp}}^{\sigma-ms(2-\sigma)} \cdot \|G_{\mu,2ms}^\omega(g)\|_{L_\omega^{2mq}}^{m(2-\sigma)s}.$$

Putting this in our previous estimate (4.5), using the boundedness of the operator N, we have

$$\begin{aligned} D_\mu(f,g) &\lesssim \|f\|_{A_\omega^{2mp}}^\sigma \cdot \|N(g)\|_{L_\omega^{2mp}}^{\sigma-ms(2-\sigma)} \cdot \|G_{\mu,2ms}^\omega(g)\|_{L_\omega^{2mq}}^{(2-\sigma)ms} \\ &\lesssim \|f\|_{A_\omega^{2mp}}^\sigma \cdot \|g\|_{A_\omega^{2mp}}^{\sigma-ms(2-\sigma)} \cdot \|G_{\mu,2ms}^\omega\|_{A_\omega^{2mp}\to L_\omega^{2mq}}^{(2-\sigma)ms} \cdot \|g\|_{A_\omega^{2mp}}^{(2-\sigma)ms}. \end{aligned}$$

From (4.3) and Proposition 3.5, it follows that B_μ^ω is in $L_\omega^{r/s}$, and the estimate (4.2) follows again.

Finally, it remains to deal with the case $\sigma < (2-\sigma)ms$. In this case, we will start by proving the inequality

$$\|B_\mu^\omega\|_{L_\omega^{r/s}} \lesssim \|G_{\mu,s}\|_{A_\omega^p \to L_\omega^q}^s$$

assuming that B_μ^ω is already in $L_\omega^{r/s}$. An application of Hölder's inequality with exponent $ms(2-\sigma)/\sigma$ shows

$$G_{\mu,(2\sigma/(2-\sigma))}^\omega(g)(\zeta) \leqslant G_{\mu,2ms}^\omega(g)(\zeta)\ B_\mu^\omega(\zeta)^{(ms(2-\sigma)-\sigma)/(2\sigma ms)}.$$

Also, as $\sigma < 2$ and $mq > 2$, we can see that

$$\frac{q(2mp-\sigma)}{p\sigma} > 1,$$

and its conjugate exponent is

$$\Big(\frac{q(2mp-\sigma)}{p\sigma}\Big)' = \frac{q(2mp-\sigma)}{q(2mp-\sigma)-p\sigma}.$$

It is possible to check that

$$\frac{pq[ms(2-\sigma)-\sigma]}{q(2mp-\sigma)-p\sigma}=r.$$

Hence, using Hölder's inequality with the previous exponents, we have

$$\begin{aligned}&\|G^{\omega}_{\mu,2\sigma/(2-\sigma)}(g)\|^{2mp\sigma/(2mp-\sigma)}_{L^{2mp\sigma/(2mp-\sigma)}_{\omega}}\\&\leqslant\int_{\mathbb{D}}G^{\omega}_{\mu,2ms}(g)(\zeta)^{2mp\sigma/(2mp-\sigma)}\;B^{\omega}_{\mu}(\zeta)^{p[ms(2-\sigma)-\sigma]/[s(2mp-\sigma)]}\,\omega(\zeta)\mathrm{d}A(\zeta)\\&\leqslant\|G^{\omega}_{\mu,2ms}(g)\|^{2mp\sigma/(2mp-\sigma)}_{L^{2mq}_{\omega}}\cdot\|B^{\omega}_{\mu}\|^{p[ms(2-\sigma)-\sigma]/[s(2mp-\sigma)]}_{L^{r/s}_{\omega}}.\end{aligned}$$

Putting this estimate into (4.5), we get

$$\begin{aligned}D_{\mu}(f,g)&\lesssim\|f\|^{\sigma}_{A^{2mp}_{\omega}}\cdot\|G^{\omega}_{\mu,2ms}(g)\|^{\sigma}_{L^{2mq}_{\omega}}\cdot\|B^{\omega}_{\mu}\|^{(ms(2-\sigma)-\sigma)/(2ms)}_{L^{r/s}_{\omega}}\\&\lesssim\|f\|^{\sigma}_{A^{2mp}_{\omega}}\cdot\|G^{\omega}_{\mu,2ms}\|^{\sigma}_{A^{2mp}_{\omega}\to L^{2mq}_{\omega}}\cdot\|g\|^{\sigma}_{A^{2mp}_{\omega}}\cdot\|B^{\omega}_{\mu}\|^{(ms(2-\sigma)-\sigma)/(2ms)}_{L^{r/s}_{\omega}}.\end{aligned}$$

Hence, applying Proposition 3.5 and (4.3), we obtain

$$\|B^{\omega}_{\mu}\|^{1-\sigma/2}_{L^{r/s}_{\omega}}\asymp\sup_{\|f\|_{A^{2mp}_{\omega}}=\|g\|_{A^{2mp}_{\omega}}=1}D_{\mu}(f,g)\lesssim\|G^{\omega}_{\mu,s}\|^{\sigma/(2m)}_{A^{p}_{\omega}\to L^{q}_{\omega}}\cdot\|B^{\omega}_{\mu}\|^{(ms(2-\sigma)-\sigma)/(2ms)}_{L^{r/s}_{\omega}}.$$

As we are assuming that B^{ω}_{μ} is already in $L^{r/s}_{\omega}$, this gives

$$\|B^{\omega}_{\mu}\|_{L^{r/s}_{\omega}}\lesssim\|G^{\omega}_{\mu,s}\|^{s}_{A^{p}_{\omega}\to L^{q}_{\omega}}.\tag{4.6}$$

For a general μ, since it is finite on compact sets of $\mathbb{D}$, if we consider the measure $\mu_n:=\mu_{|_{\{|z|\leqslant r_n\}}}$, with $r_n=1-(1/n)$, then $B^{\omega}_{\mu_n}$ is in $L^{r/s}_{\omega}$. Hence, (4.6) yields

$$\|B^{\omega}_{\mu_n}\|_{L^{r/s}_{\omega}}\lesssim\|G^{\omega}_{\mu_n,s}\|^{s}_{A^{p}_{\omega}\to L^{q}_{\omega}}\leqslant\|G^{\omega}_{\mu,s}\|^{s}_{A^{p}_{\omega}\to L^{q}_{\omega}}.$$

Finally, this together with Fatou's lemma gives

$$\|B^{\omega}_{\mu}\|_{L^{r/s}_{\omega}}\leqslant\liminf_{n}\|B^{\omega}_{\mu_n}\|_{L^{r/s}_{\omega}}\lesssim\|G^{\omega}_{\mu,s}\|^{s}_{A^{p}_{\omega}\to L^{q}_{\omega}}.$$

This completes the proof of the proposition. ∎

PROPOSITION 4.4. *Let $\omega\in\widehat{\mathcal{D}}$, and $0<q<p<\infty$, $s>0$, such that $r/s=1$ with $r=pq/(p-q)$. Let μ be a positive Borel measure on $\mathbb{D}$, finite on compact sets of $\mathbb{D}$ with $\mu(\{0\})=0$. If $G^{\omega}_{\mu,s}:A^{p}_{\omega}\to L^{q}_{\omega}$ is bounded, then $B^{\omega}_{\mu}\in L^{1}_{\omega}$. Moreover, we have the estimate*

$$\|B^{\omega}_{\mu}\|_{L^{1}_{\omega}}\lesssim\|G^{\omega}_{\mu,s}\|^{s}_{A^{p}_{\omega}\to L^{q}_{\omega}}.$$

Proof. For any $g\in A^{p}_{\omega}$ and $f\in A^{4}_{\omega}$, set

$$K_{\mu}(f,g)=\int_{\Gamma(\zeta)\cap\mathrm{spt}(\mu)}\frac{|g(z)|^{p/4}|f'(z)|(1-|z|)\mathrm{d}\mu(z)}{\mu(D(z,t))^{1/2}\omega(T(z))^{1/2}}\omega(\zeta)\,\mathrm{d}A(\zeta).$$

Applying first the Cauchy–Schwarz inequality, then Hölder's inequality and Lemma 3.2, we have

$$K_\mu(f,g) \lesssim \int\limits_{\mathbb{D}} \Big(\int\limits_{\Gamma(\zeta)} |g(z)|^{p/2} \frac{\mathrm{d}\mu(z)}{\omega(T(z))} \Big)^{1/2} \Big(\int\limits_{\Gamma(\zeta)\cap \mathrm{spt}(\mu)} \frac{|f'(z)|^2(1-|z|)^2 \mathrm{d}\mu(z)}{\mu(D(z,t))} \Big)^{1/2} \cdot \omega(\zeta)\mathrm{d}A(\zeta)$$

$$\lesssim \|f\|_{A^4_\omega} \Big\{ \int\limits_{\mathbb{D}} \Big(\int\limits_{\Gamma(\zeta)} |g(z)|^{p/2} \frac{\mathrm{d}\mu(z)}{\omega(T(z))} \Big)^{2/3} \omega(\zeta)\mathrm{d}A(\zeta) \Big\}^{3/4}.$$

Now we need to estimate

$$K_{\mu,\omega}(g) := \int\limits_{\mathbb{D}} \Big(\int\limits_{\Gamma(\zeta)} |g(z)|^{p/2} \frac{\mathrm{d}\mu(z)}{\omega(T(z))} \Big)^{2/3} \omega(\zeta)\mathrm{d}A(\zeta).$$

If $p = 2s$, as $r = s$, we see that $q = 2s/3$, so that, in this case, we have

$$K_{\mu,\omega}(g) = \|G^\omega_{\mu,s}(g)\|^q_{L^q_\omega}.$$

Thus,

$$K_\mu(f,g) \lesssim \|f\|_{A^4_\omega} \|G^\omega_{\mu,s}(g)\|^{3q/4}_{L^q_\omega} \lesssim \|f\|_{A^4_\omega} \|G^\omega_{\mu,s}\|^{p/4}_{A^p_\omega \to L^q_\omega} \|g\|^{p/4}_{A^p_\omega}.$$

From Corollary 3.4, the desired result follows.

If $p > 2s$, as $pq/(p-q) = s$, we see that $ps/(p+s) = q$. Then, by Hölder's inequality with exponent $3p/(p-2s)$, we have

$$K_{\mu,\omega}(g) \leqslant \int\limits_{\mathbb{D}} |N(g)(\zeta)|^{(p-2s)/3}\, G^\omega_{\mu,s}(g)(\zeta)^{2s/3}\, \omega(\zeta)\mathrm{d}A(\zeta)$$

$$\leqslant \|N(g)\|^{(p-2s)/3}_{L^p_\omega} \cdot \|G^\omega_{\mu,s}(g)\|^{2s/3}_{L^{ps/(p+s)}_\omega} \lesssim \|G^\omega_{\mu,s}\|^{2s/3}_{A^p_\omega \to L^q_\omega} \|g\|^{p/3}_{A^p_\omega}.$$

Hence, we have

$$K_\mu(f,g) \lesssim \|f\|_{A^4_\omega} \|G^\nu_{\mu,s}\|^{s/2}_{A^p_\omega \to L^q_\omega} \|g\|^{p/4}_{A^p_\omega},$$

which gives the desired result from Corollary 3.4.

Finally, consider the case $p < 2s$. Arguing as in the last case of Proposition 4.3, it is enough to prove the estimate $\|B^\omega_\mu\|_{L^1_\omega} \lesssim \|G^\omega_{\mu,s}\|^s_{A^p_\omega \to L^q_\omega}$, assuming that B^ω_μ is already in L^1_ω. By Hölder's inequality with exponent $2s/p > 1$, we have

$$K_{\mu,\omega}(g) \leqslant \int\limits_{\mathbb{D}} G^\omega_{\mu,s}(g)(\zeta)^{p/3}\, B^\omega_\mu(\zeta)^{(2s-p)/(3s)} \omega(\zeta)\mathrm{d}A(\zeta)$$

$$\leqslant \|B^\omega_\mu\|^{(2s-p)/(3s)}_{L^1_\omega} \cdot \|G^\omega_{\mu,s}(g)\|^{p/3}_{L^{ps/(p+s)}_\omega}.$$

As $ps/(p+s)=q$, this together with the boundedness of $G^{\omega}_{\mu,s}$, gives

$$K_{\mu}(f,g) \lesssim \|B^{\omega}_{\mu}\|^{(2s-p)/4s}_{L^1_{\omega}} \cdot \|G^{\omega}_{\mu,s}\|^{p/4}_{A^p_{\omega}\to L^q_{\omega}} \, \|f\|_{A^4_{\omega}} \, \|g\|^{p/4}_{A^p_{\omega}}.$$

Now, applying Corollary 3.4 and the previous estimate, we obtain

$$\|B^{\omega}_{\mu}\|^{1/2}_{L^1_{\omega}} \asymp \sup_{\|f\|_{A^4_{\omega}}=\|g\|_{A^p_{\omega}}=1} K_{\mu}(f,g) \lesssim \|G^{\omega}_{\mu,s}\|^{p/4}_{A^p_{\omega}\to L^q_{\omega}} \cdot \|B^{\omega}_{\mu}\|^{(2s-p)/4s}_{L^1_{\omega}}$$

and, as we are assuming that B^{ω}_{μ} is already in L^1_{ω}, this shows that

$$\|B^{\omega}_{\mu}\|_{L^1_{\omega}} \lesssim \|G^{\omega}_{\mu,s}\|^{s}_{A^p_{\omega}\to L^q_{\omega}} < \infty,$$

finishing the proof of the proposition. ∎

Proof of Theorem 1.1. By the definition of the area operator and tent spaces, one has the equivalences (i) ⇔ (iii) and (ii) ⇔ (iv). It is also clear that (ii) implies (i). As $G^{\nu}_{\mu,s} = G^{\omega}_{\mu^{\omega}_{\nu},s}$ and $B^{\nu}_{\mu} = B^{\omega}_{\mu^{\omega}_{\nu}}$, then it follows from Proposition 4.2 that (v) implies (ii). Also, that (i) implies (v) follows from Propositions 4.3 and 4.4 when $pq/(p-q) \leqslant s$, and by [27, Theorem 6] when $pq/(p-q) > s$. Finally, the estimate (1.2) is a consequence of the corresponding estimates in the mentioned propositions. This finishes the proof. ∎

5. APPLICATIONS TO INTEGRATION OPERATORS

For an analytic function g in $\mathbb{D}$, define the integration operator

$$J_g f(z) = \int_0^z f(\zeta) g'(\zeta)\, \mathrm{d}\zeta, \quad z \in \mathbb{D},$$

After the pioneering works of Aleman, Siskakis and Cima [1, 2, 3] describing the boundedness and compactness of the operator J_g in Hardy and Bergman spaces, the mentioned operator became extremely popular (see [7, 19, 20] and the references therein, for example). As $(J_g f)' = fg'$, from the area function description of weighted Bergman spaces A^p_{ω} with ω in the class $\widehat{\mathcal{D}}$, it follows that $J_g : A^p_{\omega} \to A^q_{\omega}$ is bounded, if and only if

$$\int_{\mathbb{D}} \Big(\int_{\Gamma(\zeta)} |f(z)|^2 \, |g'(z)|^2 \, \mathrm{d}A(z) \Big)^{q/2} \omega(\zeta)\, \mathrm{d}A(\zeta) \lesssim \|f\|^q_{A^p_{\omega}}.$$

That is, $J_g : A^p_{\omega} \to A^q_{\omega}$ is bounded, if and only if the operator $G_{\mu_g,2} : A^p_{\omega} \to L^q_{\omega}$ is bounded, where the measure μ_g is defined as $\mathrm{d}\mu_g(z) = |g'(z)|^2\, \omega(T(z))\, \mathrm{d}A(z)$. Hence, applying our main result together with the area function description of A^p_{ω}, the following description follows directly.

THEOREM 5.1. *Let g be analytic on $\mathbb{D}$, $\omega \in \widehat{\mathcal{D}}$, and $0 < q < p < \infty$. Then $J_g : A^p_\omega \to A^q_\omega$ is bounded, if and only if $g \in A^r_\omega$ with $r = pq/(p-q)$. Moreover,*

$$\|J_g\|_{A^p_\omega \to A^q_\omega} \asymp \|g\|_{A^r_\omega}.$$

This result was previously known for a more restricted class of weights [23, Theorem 4.9], but for the class $\widehat{\mathcal{D}}$ was only known [27] when $r > 2$.

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