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A LIFTING THEOREM FOR OPERATOR MODELS OF FINITE RANK ON MULTIPLY-CONNECTED DOMAINS

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INTRODUCTION

The purpose of this paper is to prove a version of the Sz.-Nagy-Foiaş lifting theorem in the context of an operator model on a multiply connected domain R. When R is assumed to be the unit disk D, various complications arising from the multiple connectedness of the underlying domain disappear; there results a new duality proof of the general Sz.-Nagy-Foiaş lifting theorem [15] which specializes to that of Sarason [14] for the scalar C_{00} case. We also obtain a version for multiply connected domains of the characterization of the compact operators in the commutant of a C_{00} contraction operator due to Muhly [12].

Here R is a bounded domain in the complex plane bounded by $n+1$ analytic nonintersecting Jordan curves, and $\mathrm{Rat}(\bar{R})$ is the uniform closure on $\bar{R}$ of the algebra of rational function with poles off of $\bar{R}$. Let $C(\partial R)$ be the C^*-algebra of functions continuous on ∂R. For $\mathscr{H}$ a separable Hilbert space, $\mathscr{L}(\mathscr{H})$ is the algebra of (bounded, linear) operators on $\mathscr{H}$. Let $\rho\colon C(\partial R)\to\mathscr{L}(\mathscr{K})$ be a $*$-representation of $C(\partial R)$, let $\mathscr{M}$ and $\mathscr{N}$ subspaces of the Hilbert space $\mathscr{K}$ invariant under $\rho(f)$ for all f in $\mathrm{Rat}(\bar{R})$ such that $\mathscr{N}\subset\mathscr{M}$, and let $\mathscr{H}=\mathscr{M}\ominus\mathscr{N}$. Define $\sigma\colon \mathrm{Rat}(\bar{R})\to\mathscr{L}(\mathscr{H})$ by $\sigma(f)=P_{\mathscr{H}}\rho(f)|\mathscr{H}$. Then σ defines a completely contractive unital (c.c.u.) representation of $\mathrm{Rat}(\bar{R})$, and, by a result of Arveson [6], any c.c.u. representation arises in this way. The operator model discussed here, first introduced by Abrahamse and Douglas [4] and studied further by Abrahamse [1] and the author [8], is a canonical model for such a c.c.u. representation. A precise statement and sketch of the proof of this fact, known to experts in the area but only hinted at in the literature (see ref. [4]), is given in Section 2 of this paper. When R is the unit disk D, the model coincides with the canonical model of Sz.-Nagy and Foiaş for a completely nonunitary contraction operator [16].

Let σ be a c.c.u. representation of $\mathrm{Rat}(\bar{R})$ arising as $\sigma(f)=P_{\mathscr{H}}\sigma(f)|\mathscr{H}$ as above. Given a $T\in\mathscr{L}(\mathscr{H})$ commuting with $\sigma(f)$ for all $f\in\mathrm{Rat}(\bar{R})$, the lifting question asks whether there is a $X\in\mathscr{L}(\mathscr{K})$ such that X commutes with all $\rho(f)$, X leaves $\mathscr{M}$ and $\mathscr{N}$ invariant, and T arises as $P_{\mathscr{H}}X|\mathscr{H}$. For the case $R=D$, the scalar C_{00} case is due to Sarason [14], the general case to Sz.-Nagy and Foiaş [15], where in each instance it is shown in addition that one can choose X with $\|X\| = \|T\|$. For the scalar C_{00} case with a general R, the result is due to Abrahamse [1], who also shows that it is not possible in general to choose X with $\|X\| = \|T\|$. In the analogous situation on the polydisk $D^N (N > 1)$, Clark [9] has shown that the answer is no; however, if one permits certain types of unbounded X, the answer is yes [10].

In Section 3 of this paper, we recover the positive part of Clark's result by adapting his argument to our situation. In Section 4 we use duality techniques to show that if the operator model is of finite rank (or, equivalently, the normal operator $\rho(z)$ has a bounded spectral multiplicity function), then the lift X can be arranged to be bounded. The key ingredients of the proof are a generalization of Sarason's version of the Riesz factorization lemma [14, p. 198] and a result of the author [7] on the finiteness of a supremum of a collection of operator valued outer functions. The generalization of the Riesz factorization lemma needed is derived in Section 1; the analysis hinges on results of Moore, Rosenblum and Rovnyak [13] on factorization of Toeplitz operators associated with isometries. Finally, in Section 5, we show that the compact operators in the commutant of a C_{00} c.c.u. representation of finite rank arise from an appropriate analogue of $H^\infty + C$, and thus generalize the result of Muhly [12].

Definitions and notation from [7] will be used here with limited explanation. All Hilbert spaces are assumed to be complex, separable.

1. THE RIESZ FACTORIZATION LEMMA FOR ISOMETRIES

Let V_1 and V_2 be isometries acting on separable Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$ respectively. Following Moore, Rosenblum and Rovnyak [13], let $V_i = S_i \oplus U_i$ $(i = 1, 2)$ be the Wold decomposition of V_i into its shift part S_i and its unitary part U_i. We say that $S_1 < S_2$ if dim Ker $S_1^* \leqslant$ dim Ker S_2^*. For $A \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$, $T \in \mathscr{L}(\mathscr{H}_1)$, A is called

(i) (V_1, V_2)-analytic if $AV_1 = V_2 A$.

(ii) (V_1, V_2)-inner if A is partially isometric and (V_1, V_2)-analytic, and

(iii) (V_1, V_2)-outer if A is (V_1, V_2)-analytic and $(A\mathscr{H}_1)^-$ (the closure of the range of A) reduces V_2.

The operator T is said to be V_1-Toeplitz if $V_1^* T V_1 = T$.

The following generalization of the Riesz factorization lemma will be needed in the sequel.

THEOREM 1.1. *Let $F \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ be (V_1, V_2)-analytic and suppose $S_1 < S_2$. Then $F = GH$ where $G \in \mathscr{L}(\mathscr{H}_2)$ is (V_2, V_2)-analytic, $H \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is (V_1, V_2)-outer $(F^*F)^{1/2} = H^*H$ and $G^*G = HH^*$.*

If $F = G_1H_1$ is another factorization of F having all the properties stated above, then there is a unique $B \in \mathscr{L}(\mathscr{H}_2)$ such that both B and B^* are (V_2, V_2)-inner, B has initial space equal to $(H\mathscr{H}_1)^-$ and final space equal to $[\text{Ker } G]^\perp$, $H_1 = BH$ *and* $G_1 = GB^*$.

Proof. We follow the proof of Sarason [14, Theorem 4], using the results of ref. [13] to make the necessary modifications. Consider $T = F^*F$. By *i)* ⇒ *ii)* of Theorem 4 of [13], there is an operator $C \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ such that $C|(T^{1/2}\mathscr{H}_1)^-$ is one-to-one and $CT^{1/2}$ is (V_1, V_2)-analytic. Set $C' = CT^{1/4}$. Then we also have $C'|(T^{1/2}\mathscr{H}_1)^-$ is one-to-one and $C'T^{1/4}$ is (V_1, V_2)-analytic. Hence, by *ii)* ⇒ *i)* of Theorem 4 of [13], $T^{1/2} = H^*H$ where H is (V_1, V_2)-outer. Now let $F = JT^{1/2}$ be the polar decomposition of F, where J is a partial isometry whose initial space is $(T^{1/2}\mathscr{H}_1)^-$ and whose final space is $(F\mathscr{H}_1)^-$. If we set $G = JH^*$, then $F = GH$. It remains to show that G is (V_2, V_2)-analytic and that $G^*G = HH^*$.

We first prove the latter fact. By definition of G,

$$(*) \qquad G^*G = HJ^*JH^*.$$

Now J^*J is the orthogonal projection of $\mathscr{H}_1$ onto $J^*\mathscr{H}_2$. But $J^*\mathscr{H}_2$ is the initial space of J, or $(T^{1/2}\mathscr{H}_1)^-$. But the equality $T^{1/2} = H^*H$ shows that the ranges of $T^{1/2}$ and of H^* have the same closures, and hence $J^*JH^* = H^*$. From this and $(*)$, it follows that $G^*G = HH^*$.

To prove that G is (V_2, V_2)-analytic, we note that, since H is (V_1, V_2)-outer, $(H\mathscr{H}_1)^-$ is a reducing subspace $\mathscr{L}$ for V_2. The equality $G = JH^*$ shows that the null space of G is equal to $\mathscr{L}^\perp$. Thus it suffices to show that $GV_2x = V_2Gx$ for all x in $\mathscr{L}$. Any such x can be approximated by a y of the form $y = Hz$ with z in $\mathscr{H}_1$. For any such y, we have

$$GV_2y = GV_2Hz = GHV_1z = FV_1z = V_2Fz = V_2GHz = V_2Gy$$

as desired. The existence assertion follows.

If $F = G_1H_1$ is another such factorization, then $T^{1/2} = H^*H$ and $H_1^*H_1$. Then we must have $H_1 = BH$ where both B and B^* are (V_2, V_2)-inner, B has initial space equal to $(H\mathscr{H}_1)^-$ and final space equal to $(\ker H_1)^\perp$ ([13], Theorem 3). From this it follows also that $G_1 = GB^*$. The theorem follows.

To make use of this, we wish to put it in a more functional form. Let $\mathscr{K}$ and $\mathscr{K}_*$ be separable Hilbert spaces such that $\dim \mathscr{K} \leqslant \dim \mathscr{K}_*$. Let $C^p(\mathscr{K}_*, \mathscr{K})$ be the class of operators from $\mathscr{K}_*$ into $\mathscr{K}$ of Schatten p-class $(1 \leqslant p \leqslant \infty)$. The space of operator-valued functions $H^q_{C^p(\mathscr{K}_*, \mathscr{K})}$ $(1 \leqslant p, q \leqslant \infty)$ and of vector-valued functions $H^q_{\mathscr{K}}$ defined on the unit disk is defined as in Section 1 of ref. [7]. Functions in these spaces will also be considered defined on ∂D via nontangential boundary values when convenient. Let Θ be a fixed element of $H^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}$ with

$\|\Theta\| \leqslant 1$ and let $\Delta \in L^\infty_{\mathscr{L}(\mathscr{K})}$ be defined by

$$\Delta(z) = [I - \Theta(z)^* \Theta(z)]^{1/2}$$

for $|z| = 1$. We wish to apply Theorem 1.1 to the case where V_1 is multiplication by z on $\mathscr{H}_1 = H^2_{\mathscr{K}}$ and V_2 is multiplication by z on $\mathscr{H}_2 = H^2_{\mathscr{K}_*} \oplus (\Delta L^2_{\mathscr{K}})^-$. Since we are assuming $\dim \mathscr{K} \leqslant \dim \mathscr{K}_*$, we have that $S_1 < S_2$ as required. If we write elements of $H^2_{\mathscr{K}_*} \oplus (\Delta L^2_{\mathscr{K}})^-$ as column vectors, an operator $F \in \mathscr{L}(\mathscr{H}_1, \mathscr{H}_2)$ is (V_1, V_2)-analytic if and only if F is multiplication by an operator-valued function of the form

$$F(z) = \begin{bmatrix} F_1(z) \\ F_2(z) \end{bmatrix} (|z| = 1),$$

where $F_1 \in H^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}$, $F_2 \in \Delta L^\infty_{\mathscr{L}(\mathscr{K})}$ (by this is meant that $F_2 \in L^\infty_{\mathscr{L}(\mathscr{K})}$ and $(F_2(z)\mathscr{K})^- \subset (\Delta(z)\mathscr{K})^-$ for a.e. z on ∂D). We write

$$F \in \begin{bmatrix} H^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)} \\ \\ \Delta L^\infty_{\mathscr{L}(\mathscr{K})} \end{bmatrix}$$

for such an F. Using similar notation and the analysis in ref. [14], one concludes that $G \in \mathscr{L}(\mathscr{H}_2)$ is (V_2, V_2)-analytic if and only if

$$G \in \begin{bmatrix} H^\infty_{\mathscr{L}(\mathscr{K}_*)} & 0 \\ \\ \Delta L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} & \Delta L^\infty_{\mathscr{L}(\mathscr{K})} \Delta \end{bmatrix}$$

and that $B \in \mathscr{L}(\mathscr{H}_2)$ is (V_2, V_2)-inner with adjoint also (V_2, V_2)-inner if and only if B is multiplication by an operator-valued function of the form

$$B(z) = \begin{bmatrix} U & 0 \\ 0 & V(z) \end{bmatrix}, \quad |z| = 1$$

where U is a partial-isometry on $\mathscr{K}_*$ and V is a measurable function on ∂D whose values are partial isometries on $\mathscr{K}$. Finally, in Theorem 1.1, if in addition F is trace-class, then the factors G and H are Hilbert-Schmidt class. Thus we obtain the following

THEOREM 1.2. *Let $\mathscr{K}$ and $\mathscr{K}_*$ be separable Hilbert spaces with $\dim \mathscr{K} \leqslant \dim \mathscr{K}_*$. Then any operator-valued function F in*

$$\begin{bmatrix} H^\infty_{C^1(\mathscr{K}, \mathscr{K}_*)} \\ \\ \Delta L^\infty_{C^1(\mathscr{K})} \end{bmatrix}$$

factors as $F = GH$ *with*

$$G \in \begin{bmatrix} H^{\infty}_{C^2(\mathscr{K}_*)} & 0 \\ \Delta L^{\infty}_{C^2(\mathscr{K}_*, \mathscr{K})} & \Delta L^{\infty}_{C^2(\mathscr{K})} \Delta \end{bmatrix}, \ H \in \begin{bmatrix} H^{\infty}_{C^2(\mathscr{K}, \mathscr{K}_*)} \\ \Delta L^{\infty}_{C^2(\mathscr{K})} \end{bmatrix},$$

H is (V_1, V_2)-outer, and also $(F^*F)^{1/2} = H^*H, G^*G = HH^*$.

If $F = G_1H_1$ is another such factorization, then there is a parital-isometry-valued function B of the form

$$B(z) = \begin{bmatrix} U & 0 \\ 0 & V(z) \end{bmatrix} (|z| = 1)$$

such that $H_1 = BH$ and $G_1 = GB^*$.

2. THE FUNCTIONAL MODEL

Let R be a bounded domain in the complex plane whose boundary consists of $n + 1$ analytic nonintersecting Jordan curves, let $C_1, \ldots, C_n$ be n cuts in R such that $R \setminus \bigcup_{i=1}^{n} C_i$ is simply-connected, and let t be an arbitrary but fixed point in R. We borrow definitions and notation from ref. [7] without explanation. For $\mathscr{K}$ a Hilbert space, and $\alpha \in \mathrm{Hom}\,(\pi_0(R), \mathscr{U}(\mathscr{K}))$, harmonic measure m on ∂R for the point t is used to define the norm of the spaces $H^2_{\mathscr{K}}(\alpha)$ and $L^2_{\mathscr{K}}(\partial R)$.

It is well-known that there is function $v(z)$ meromorphic on $\bar{R}$ with only pole a simple pole at t and n zeroes in R and no zeroes on ∂R such that

$$\frac{1}{2\pi i} v(z)\, dz = dm(z) \qquad \text{for } z \text{ on } \partial R.$$

Let $\mathscr{K}_*$ be another Hilbert space, and let $\Theta \in H^{\infty}_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta)$ for some $\alpha \in \mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$ and $\beta \in \mathrm{Hom}\,(\pi_0(R), \mathscr{U}(\mathscr{K}_*))$ have $\|\Theta\|_{\infty} \leqslant 1$. We show how such a Θ gives rise to a c.c.u. representation of $\mathrm{Rat}(\bar{R})$.

Let $\Delta(z) = [I - \Theta(z)^*\Theta(z)]^{1/2}$ for $z \in \partial R$ and let $\mathscr{M} = H^2_{\mathscr{K}_*}(\alpha) \oplus (\Delta L^2_{\mathscr{K}}(\partial R))^-$, $\mathscr{N} = \{\Theta f \oplus \Delta f : f \in H^2_{\mathscr{K}}(\beta)\} \subset \mathscr{M}$, and $\mathscr{D} = L^2_{\mathscr{K}_*}(\partial R) \oplus (\Delta L^2_{\mathscr{K}}(\partial R))^-$. Then both $\mathscr{M}$ and $\mathscr{N}$ are closed subspaces of D invariant under the operator M_f of multiplication by $f(z)$ for any f in $\mathrm{Rat}(\bar{R})$. If we set $\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$ and define $\sigma_{\Theta} : \mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{H})$ by

$$\sigma_{\Theta}(f) = P_{\mathscr{H}} M_f \mid \mathscr{H},$$

then σ_{Θ} is a c.c.u. representation of $\mathrm{Rat}\,(\bar{R})$. Furthermore, σ_{Θ} is ∂R-pure in the sense that there is no non-zero subspace $\mathscr{H}_0 \subset \mathscr{H}$ such that σ^0_{Θ} defined by $\sigma^0_{\Theta}(f) =$

$= \sigma_\Theta(f) | \mathscr{H}_0$ is ∂R-normal (i.e. the restriction to Rat $(\bar{R})$ of a $*$-representation of $C(\partial R)$). That these properties characterize representations of the form σ_Θ is the content of the following

THEOREM 2.1. *Let σ be any ∂R-pure c.c.u. representation of* $\mathrm{Rat}(\bar{R})$. *Then there are Hilbert spaces $\mathscr{K}$ and $\mathscr{K}_*$, group representations α and β in* $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}_*))$ *and* $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$ *respectively, and an element Θ in the unit ball of $H^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta)$, such that σ is unitarily equivalent to σ_Θ.*

Sketch of the Proof. Let σ be a c.c.u. $\mathscr{L}(\mathscr{H})$-valued representation of $\mathrm{Rat}(\bar{R})$. By the result of Arveson [6], there is a ∂R-normal representation $\tau\colon \mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{D})$ and $\tau(\mathrm{Rat}(\bar{R})$-invariant subspaces $\mathscr{N} \subset \mathscr{M} \subset \mathscr{D}$ such that $\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$ and $\sigma(f) = P_{\mathscr{H}} \tau(f) | \mathscr{H}$ for all f in Rat $(\bar{R})$. Since σ is ∂R-pure, if τ is chosen to be minimal, then $\sigma_{\mathscr{N}}$ defined by $\sigma_{\mathscr{N}}(f) = \tau(f)|\mathscr{N}$ is necessarily a pure subnormal representation of $\mathrm{Rat}(\bar{R})$. By results of Abrahamse and Douglas (see refs. [3, 4]), up to unitary equivalence $\sigma_{\mathscr{N}}$ has the form $\sigma_{\mathscr{N}}(f) = f_+ = M_f | H^2_\beta(\mathscr{K})$ for a Hilbert space $\mathscr{K}$ and some $\beta \in \mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$. The representation $\sigma_{\mathscr{M}}$ (defined by $\sigma_{\mathscr{M}}(f) = \tau(f) | \mathscr{M}$ need not be pure in general.

Nevertheless, by the generalized Wold decomposition of [3, Theorem 11], up to unitary equivalence $\sigma_{\mathscr{M}}$ has the form

$$\sigma_{\mathscr{M}}(f) = M_f | H^2_{\mathscr{K}_*}(\alpha) \oplus \mathscr{R},$$

where $\mathscr{K}_*$ is some other Hilbert space, α is some group representation in $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}_*))$ and $f \to M_f | \mathscr{R}$ is a ∂R-normal representation. That it may now be assumed that $\mathscr{M}$ has the form $H^2_{\mathscr{K}_*}(\alpha) \oplus (\Delta L^2_{\mathscr{K}}(\partial R))^-$, and that $\mathscr{N} = H^2_{\mathscr{K}}(\beta)$ sits inside $\mathscr{M}$ as $\{\Theta f \oplus \Delta f \colon f \in H^2_{\mathscr{K}}(\beta)\}$ for some Θ in the unit ball of $H^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta)$ follows by making the appropriate modifications to the argument of Douglas [11, pp. 191— 193], where the case $R = D$ is worked out with the generalization to a general R in mind.

3. UNBOUNDED LIFTINGS

The purpose of this Section is to show that the lifting problem discussed in the introduction always has a solution, if we allow certain types of unbounded operators. The basic idea again comes from Clark's discussion of the lifting theorem for the polydisk [10].

Now consider Hilbert spaces $\mathscr{K}$ and $\mathscr{K}_*$, group representations $\alpha \in \mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}_*))$ and $\beta \in \mathrm{Hom}\,(\pi_0(R), \mathscr{U}(\mathscr{K}))$ and a fixed element Θ of the unit ball of $H^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta)$. It will be convenient to represent the elements of the Hilbert spaces mentioned in the previous section as column vectors rather than in the notation of direct sums. Thus we have

$$\mathscr{D} = \begin{bmatrix} L^2_{\mathscr{K}_*} \\ \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}$$

$$\mathscr{M} = \begin{bmatrix} H^2_{\mathscr{K}_*}(\alpha) \\ \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}$$

$$\mathscr{N} = \begin{bmatrix} \Theta \\ \Delta \end{bmatrix} H^2_{\mathscr{K}}(\beta)$$

and $\mathscr{H} = \mathscr{M} \ominus \mathscr{N}$.

In addition set

$$\mathscr{R} = \begin{bmatrix} 0 \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}.$$

Then operators on any of these spaces can be represented by 2×2 operator matrices, and this convention we follow in this Section and the next.

Next consider the class

$$\mathscr{I}^2 = \begin{bmatrix} H^2_{\mathscr{L}(\mathscr{K}_*)}(\alpha, \alpha) & 0 \\ \\ \Delta L^2_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} & \Delta L^\infty_{\mathscr{L}(\mathscr{K})}\Delta \end{bmatrix}$$

of operator matrices Ψ of the form

$$\Psi = \begin{bmatrix} \psi_{21} & 0 \\ \psi_{21} & \psi_{22} \end{bmatrix}$$

where $\psi_{11} \in H^2_{\mathscr{L}(\mathscr{K}_*)}(\alpha, \alpha)$, $\psi_{21} \in L^2_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})}$ with $(\psi_{21}(z)\mathscr{K}_*)^- \subset (\Delta(z)\mathscr{K})^-$ for a.e z on ∂R, and $\psi_{22} \in L^\infty_{\mathscr{L}(\mathscr{K})}$ with $\operatorname{Ker} \Delta(z) \subset \operatorname{Ker} \psi_{22}(z)$ and $(\psi_{22}(z)\mathscr{K})^- \subset (\Delta(z)\mathscr{K})^-$. for a.e. z in ∂R. A $\Psi \in \mathscr{I}^2$ is said to be *Θ-compatible* if $\Psi\mathscr{N} \subset \mathscr{N}$; by this we mean that, if f is any element of $\mathscr{N}$ such that Ψf is in $\mathscr{M}$, then in fact Ψf is in $\mathscr{N}$. If D is any subspace of $\mathscr{M}$, let D_B be the manifold of all functions in D of the form $P_D g$ where g ranges over

$$\begin{bmatrix} H^\infty_{\mathscr{K}_*}(\alpha) \\ \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}$$

and P_D is the orthogonal projection of $\mathcal{M}$ onto D. We note that by the definition of $\mathcal{I}^2$, and Ψ in $\mathcal{I}^2$ maps

$$\begin{bmatrix} H^\infty_{\mathcal{K}_*}(\alpha) \\ (\Delta L^2_{\mathcal{K}})^- \end{bmatrix}$$

into $\mathcal{M}$. In particular, for $D=\mathcal{H} \subset \mathcal{M}$, $\mathcal{H}_B$ is dense in $\mathcal{H}$. Finally, if Ψ is a Θ-compatible element of $\mathcal{I}^2$, define T_Ψ on $\mathcal{H}_B$ by $T_\Psi f = P_{\mathcal{H}} \Psi g$ where $f = P_{\mathcal{H}} g$ and

$$g \in \begin{bmatrix} H^\infty_{\mathcal{K}_*}(\alpha) \\ (\Delta L^2_{\mathcal{K}})^- \end{bmatrix}.$$

It is easy to check that the Θ-compatibility of Ψ implies that T_Ψ is well-defined on $\mathcal{H}_B$, and thus T_Ψ is defined from a dense subset of $\mathcal{H}$ into $\mathcal{H}$. In the case where T_Ψ has a (necessarily unique) bounded extension to all of $\mathcal{H}$, we will say that $\Psi \in \mathcal{B}(\Theta)$. This however does not preclude the possibility that Ψ itself is unbounded as an operator on $\mathcal{M}$.

THEOREM 3.1. *Any operator T on $\mathcal{H}$ commuting with $\sigma_\Theta(\mathrm{Rat}(\bar{R}))$ has the form $T = T_\Psi$ for some Ψ in $\mathcal{B}(\Theta)$. Conversely, every such T_Ψ commutes with $\sigma_\Theta(\mathrm{Rat}\ (\bar{R}))$.*

Proof. If Ψ is in $\mathcal{B}(\Theta)$, the check that T_Ψ commutes with $\sigma_\Theta(\mathrm{Rat}(\bar{R}))$ proceeds as in the polydisc setting [9].

Conversely, suppose that $T \in \mathcal{L}(\mathcal{H})$ commutes with $\sigma_\Theta(\mathrm{Rat}(\bar{R}))$. We wish to define

$$\Psi(z) = \begin{bmatrix} \psi_{11}(z) & 0 \\ \psi_{21}(z) & \psi_{22}(z) \end{bmatrix}, \quad (z \in \partial R)$$

so that $\Psi \in \mathcal{B}(\Theta)$ and $T = T_\Psi$. We first define the components ψ_{11} and ψ_{21} as follows.

Let E_α be a fixed element of $H^\infty_{\mathcal{L}(\mathcal{K}_*)}(\alpha, e) \cap (H^\infty_{\mathcal{L}(\mathcal{K}_*)}(e, \alpha))^{-1}$ (e the identity element of Hom $(\pi_0(R), \mathcal{U}(\mathcal{K}_*))$. For $k_* \in \mathcal{K}_*$ and $z \in \partial R$, set

$$\begin{bmatrix} \psi_{11}(z) k_* \\ \psi_{21}(z) k_* \end{bmatrix} = \left(TP_{\mathcal{H}} \begin{bmatrix} E_\alpha E_\alpha(z)^{-1} k_* \\ 0 \end{bmatrix} \right)(z).$$

Letting z vary now, we verify that

$$\begin{bmatrix} \psi_{11} \\ \psi_{21} \end{bmatrix} E_\alpha k_* = TP_{\mathcal{H}} \begin{bmatrix} E_\alpha k_* \\ 0 \end{bmatrix} \in \mathcal{M}.$$

Hence $\psi_{11}E_\alpha$ is an operator-valued function mapping $\mathscr{K}_*$ into $H^2_{\mathscr{K}_*}(\alpha)$ and $\psi_{21}E_\alpha$ is an operator-valued function mapping $\mathscr{K}_*$ into $(\Delta L^2_{\mathscr{K}})^-$. It follows that $\psi_{11}E_\alpha$ and $\psi_{21}E_\alpha$ are square-integrable in operator norm and that $(\psi_{21}(z)\mathscr{K}_*)^- \subset (\Delta(z)\mathscr{K})^-$ for a.e. z Then also $\psi_{11} = (\psi_{11}E_\alpha)E_\alpha^{-1}$ and $\psi_{21} = (\psi_{21}E_\alpha)E_\alpha^{-1}$ are square-integrable in operator norm.

To show that $\Psi \in \mathscr{I}^2$, it remains only to define ψ_{22} appropriately. By identifying $(\Delta L^2_{\mathscr{K}})^-$ with $\mathscr{R}$, we see that it suffices to define a $\psi_{22} \in \mathscr{L}(\mathscr{R})$ which commutes with $M_f|\mathscr{R}$ for all f in $\mathrm{Rat}(\bar{R})$. To do this we first need some information regarding the geometry of our model space. Analogous facts for the Sz.-Nagy-Foiaş model (i.e. the case $R = D$) are discussed in ref. [16, Chapter II].

LEMMA 3.2. *Let*

$$\mathscr{L} = \{k \in \mathscr{K} : \Theta(z)k = 0 \text{ for a.e. } z \in \partial R\}$$

and let $P_{\mathscr{L}}$ be the orthogonal projection of $\mathscr{K}$ onto $\mathscr{L}$. Then

(*i*) $P_{\mathscr{L}}\Delta(z) = \Delta(z)P_{\mathscr{L}} = P_{\mathscr{L}}$ *for* a.e. $z \in \partial R$

(*ii*) $P_{\mathscr{L}}\beta(A) = \beta(A)P_{\mathscr{L}}$ *for all* $A \in \pi_0(R)$

and

(*iii*) $\mathscr{R} \ominus (P_{\mathscr{R}}\mathscr{H})^- = \left\{\begin{bmatrix} 0 \\ h \end{bmatrix},\ h \in H^2_{\mathscr{L}}(\beta)\right\}.$

Proof. Since $\Delta(z) = (I - \Theta(z)^*\Theta(z))^{1/2}$ by definition, (*i*) is clear. If $k \in \mathscr{L}$, then for any A in $\pi_0(R)$, $\Theta \circ A^{-1}(z)k = 0$ for a.e. z, or $\alpha(A^{-1})\ \Theta(z)\beta(A^{-1})^*k = \alpha(A)^*\ \Theta(z)\beta(A)k = 0$, which shows that $\beta(A)k \in \mathscr{L}$. This proves (*ii*). If $k \in \mathscr{R}\ominus(P_{\mathscr{R}}\mathscr{H})^-$, then k is orthogonal to $\mathscr{H}$ and hence is in $\mathscr{N}$. Since $k \in \mathscr{R}$, k has the form $\begin{bmatrix} 0 \\ \ell \end{bmatrix}$, and since $k \in \mathscr{N}$, k has the form $\begin{bmatrix} \Theta h \\ \Delta h \end{bmatrix}$ for some $h \in H^2_{\mathscr{K}}(\beta)$. Hence we must have $\Theta h = 0$ which implies $h(z) \in \mathscr{L}$ a.e. Then by (*i*), $\Delta(z)h(z) = h(z)$, and hence $k = \begin{bmatrix} 0 \\ h \end{bmatrix}$. The argument is reversible and hence any such k is in $\mathscr{R} \ominus (P_{\mathscr{R}}\mathscr{H})^-$, and (*iii*) follows.

LEMMA 3.3. *For any f in $\mathscr{H}$,*

$$\|P_{\mathscr{R}}f\| = \lim_{n\to\infty} \|\sigma_\Theta(\varphi^n)^* f\|,$$

where φ is any inner function in $\mathrm{Rat}(\bar{R})$ (*for example, the Ahlfors function associated with a point t in R* [5]).

Proof. The proof is a straightforward computation. The corresponding fact for the disk is Proposition II.3.2. of ref. [16].

For $k \in \mathscr{R}$ of the form $k = P_{\mathscr{R}}h$ for some $h \in \mathscr{H}$, we define $\psi_{22}^{*}k = P_{\mathscr{R}}T^{*}h$. The computation, using Lemma 3.3,

$$\|\psi_{22}^{*}k\| = \lim_{n\to\infty}\|\sigma_{\Theta}(\varphi^{n})^{*}\, T^{*}h\| = \lim_{n\to\infty}\|T^{*}\sigma_{\Theta}(\varphi^{n})^{*}\, h\| \leqslant \|T^{*}\|\lim_{n\to\infty}\|\sigma_{\Theta}(\varphi^{n})^{*}h\| =$$

$$= \|T^{*}\|\,\|k\|$$

shows that ψ_{22}^{*} is well-defined and has a unique bounded extension to $(P_{\mathscr{R}}\mathscr{H})^{-}$ with norm bounded by $\|T^{*}\|$. Now by Lemma 3.2,

$$\mathscr{R} \ominus (P_{\mathscr{R}}\mathscr{H})^{-} = \left\{\begin{bmatrix} 0 \\ h \end{bmatrix} : h \in H_{\mathscr{L}}^{2}(\beta)\right\}.$$

Hence the representation $\tau_{0}^{*}: f \to M_{\bar{f}} \mid (P_{\mathscr{R}}\mathscr{H})^{-}$ is ∂R-subnormal with ∂R-normal extension $\tau^{*}: f \to M_{\bar{f}}|\mathscr{R}, f \in \mathrm{Rat}(\bar{R})$. Furthermore the computation

$$M_{f}\,\psi_{22}^{*}P_{\mathscr{R}}h = M_{\bar{f}}P_{\mathscr{R}}T^{*}h = P_{\mathscr{R}}\sigma_{\Theta}(f)^{*}\, T^{*}h = P_{\mathscr{R}}T^{*}\,\sigma_{\Theta}(f)^{*}\, h =$$

$$= \psi_{22}^{*}P_{\mathscr{R}}\sigma_{\Theta}(f)^{*}\, h = \psi_{22}^{*}M_{\bar{f}}P_{\mathscr{R}}h$$

shows that ψ_{22}^{*} commutes with $\tau_{0}^{*}(\mathrm{Rat}(\bar{R}))$ on $(P_{\mathscr{R}}\,\mathscr{H})^{-}$. To extend ψ_{22}^{*} to all of $\mathscr{R}$ in such a way that

(*i*) $\psi_{22}^{*}M_{\bar{f}} = M_{\bar{f}}\psi_{22}^{*}$ on $\mathscr{R}$ for $f \in \mathrm{Rat}\ (\bar{R})$,

(*ii*) $\psi_{22}^{*}P_{\mathscr{R}}P_{\mathscr{H}} = P_{\mathscr{R}}T^{*}P_{\mathscr{H}}$

and

(*iii*) $\|\psi_{22}^{*}\| \leqslant \|T^{*}\|$, it suffices to invoke the following general fact.

LEMMA 3.4. *If* $\pi_{0}: \mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{H})$ *is a* ∂R*-subnormal representation with* ∂R*-normal extension* τ: $\mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{K})$ *and* $T \in \mathscr{L}(\mathscr{H})$ *commutes with* $\tau_{0}(\mathrm{Rat}(\bar{R}))$, *then there is an* $X \in \mathscr{L}(\mathscr{K})$ *such that* $X\mathscr{H} \subset \mathscr{H}$, $X|\mathscr{H} = T$, *and* $\|X\| = \|T\|$.

Proof. The class of ∂R-subnormal representations is characterized by Theorem 11 of ref. [3]. For the case where τ_{0} is in addition ∂R-pure, the above Lemma follows from the results of Section 1.6 of ref. [3]. The general case follows by making appropriate modifications.

We now have ψ_{22}^{*} defined on R and satisfying (*i*), (*ii*) and (*iii*). We let ψ_{22} be the adjoint of ψ_{22}^{*}, and now consider ψ_{22} as defined on $(\Delta L_{\mathscr{K}}^{2})^{-}$. By (*i*), ψ_{22} commutes with all scalar multiplication operators on $(\Delta L_{\mathscr{K}}^{2})^{-}$, and hence must itself be multiplication by an operator-valued function $\psi_{22}(z)$ with $\mathrm{Ker}\ \psi_{22}(z) \subset \mathrm{Ker}\ \Delta(z)$ and $(\psi_{22}(z)\mathscr{K})^{-} \subset (\Delta(z)\mathscr{K})^{-}$ for a.e.z Set

$$\Psi(z) = \begin{bmatrix} \psi_{11}(z) & 0 \\ \psi_{21}(z) & \psi_{22}(z) \end{bmatrix}$$

Then $\Psi \in \mathscr{I}^2$. It remains to show that $\Psi \in \mathscr{B}(\Theta)$ and that $T = T_\Psi$.

We note that the operator Ψ of multiplication by $\Psi(z)$ is defined on the dense subset

$$\mathscr{D}_\Psi = \begin{bmatrix} H^\infty_{\mathscr{K}_*}(\alpha) \\ \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}$$

of $\mathscr{M}$. We next show that for $f \in \mathscr{D}_\Psi$, (*iv*) $P_{\mathscr{K}} \Psi f = TP_{\mathscr{K}} f$. This equation readily implies that Ψ is Θ-compatible and that $T = T_\Psi$, and hence the theorem.

First, if

$$f = \begin{bmatrix} \sum_{i=1}^N r_i E_\alpha k_i \\ \\ 0 \end{bmatrix}$$

for $r_i \in \operatorname{Rat}(\overline{R})$ and $k_i \in \mathscr{K}_*$, then

$$\begin{aligned} TP_{\mathscr{K}} f &= T \sum_{i=1}^N \sigma_\Theta(r_i)\, P_{\mathscr{K}} \begin{bmatrix} E_\alpha k_i \\ 0 \end{bmatrix} \\ &= \sum_{i=1}^N \sigma_\Theta(r_i)\, TP_{\mathscr{K}} \begin{bmatrix} E_\alpha k_i \\ 0 \end{bmatrix} \\ &= P_{\mathscr{K}} \sum_{i=1}^N r_i\, \Psi \begin{bmatrix} E_\alpha k_i \\ 0 \end{bmatrix} \\ &= P_{\mathscr{K}} \sum_{i=1}^N \Psi\, r_i \begin{bmatrix} E_\alpha k_i \\ 0 \end{bmatrix} \\ &= P_{\mathscr{K}} \Psi f. \end{aligned}$$

Next we note from (*ii*) that $P_{\mathscr{K}} \psi_{22} P_{\mathscr{R}} = TP_{\mathscr{K}} P_{\mathscr{R}}$, and, since Ψ is lower-triangular, $\psi_{22} P_{\mathscr{R}} = \Psi P_{\mathscr{R}}$. Hence $P_{\mathscr{K}} \Psi P_{\mathscr{R}} = TP_{\mathscr{K}} P_{\mathscr{R}}$, and (*iv*) follows for $f \in \mathscr{R}$ as well.

4. BOUNDED LIFTINGS

The purpose of this Section is to show that, if in addition it is assumed that both $\dim \mathscr{K} < \infty$ and $\dim \mathscr{K}_* < \infty$, then the $\Psi \in \mathscr{B}(\Theta)$ of the previous Section may actually be chosen to lie in

$$\mathscr{I}^\infty = \begin{bmatrix} H^\infty_{\mathscr{L}(\mathscr{K}_*)}(\alpha, \alpha) & 0 \\ \\ \Delta L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} & \Delta L^\infty_{\mathscr{L}(\mathscr{K})} \Delta \end{bmatrix}$$

and thus to define a bounded multiplication operator on

$$\mathscr{M} = \begin{bmatrix} H^2_{\mathscr{K}_*}(\alpha) \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}$$

We first need to establish some duality relations among $\mathscr{I}^\infty$ and other related spaces. For the C_{00} case ($\Delta \equiv 0$), these relations are indicated by Muhly [12] for the case $R = D$, and by Abrahamse [1] for a general R but with scalar valued functions.

The space $\mathscr{I}^\infty$ defined above is a Banach space when given the operator norm inherited from $\mathscr{L}(\mathscr{M})$ (or equivalently, the essential supremum of the pointwise operator norm). One can also view $\mathscr{I}^\infty$ isometrically as a space of multiplication operators on the space

$$\mathscr{D} = \begin{bmatrix} L^2_{\mathscr{K}_*} \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}.$$

Then $\mathscr{I}^\infty$ is a subalgebra of the commutant of $\{M_f | \mathscr{D} : f \in \mathrm{Rat}(\bar{R})\}$. This latter commutant in turn consists of multiplication operators whose multipliers belong to

$$\tilde{\mathscr{I}}^\infty = \begin{bmatrix} L^\infty_{\mathscr{L}(\mathscr{K}_*)} & L^\infty_{\mathscr{L}(\mathscr{K}, \mathscr{K}_*)}\Delta \\ \Delta L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} & \Delta L^\infty_{\mathscr{L}(\mathscr{K}}\cdot\Delta \end{bmatrix}$$

$\tilde{\mathscr{I}}^\infty$ is a Banach space when given the operator norm inherited from $\mathscr{L}(\mathscr{D})$, or equivalently, the essential supremum of the pointwise operator norms. It also happens that $\tilde{\mathscr{I}}^\infty$ can be viewed as the dual of a Banach space of trace-class multiplication operators

$$\tilde{\mathscr{T}} = \begin{bmatrix} L^1_{C^1(\mathscr{K}_*)} & L^1_{C^1(\mathscr{K}, \mathscr{H}_*)}\,\Delta \\ \Delta L^1_{C^1(\mathscr{K}_*, \mathscr{K})} & \Delta L^1_{C^1(\mathscr{H})}\,\Delta \end{bmatrix}.$$

For $F \in \tilde{\mathscr{I}}^\infty$ and $f \in \tilde{\mathscr{T}}$, we define the duality by the bilinear form

$$[F,f] = \mathrm{tr}(Ff) = \frac{1}{2\pi i}\int_{\partial R} \mathrm{tr}(F(z)f(z))\, v(z)\, dz$$

where the appropriate definition of $\|f\|$ is

$$\|f\| = \mathrm{Tr} f = \frac{1}{2\pi i}\int_{\partial R} \mathrm{Tr}(f(z))\, v(z)\, dz.$$

For group representations $\alpha \in \mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}_*))$ and $\beta \in \mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$, and $1 \leqslant p, q \leqslant \infty$, define the space

$$\mathrm{K}^p_{OCq(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta) = v^{-1} H^p_{Cq(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta).$$

Then one can check that $\mathscr{I}^\infty$ is the annihilator of

$$\begin{bmatrix} K^1_{OC^1(\mathscr{K}, \mathscr{K}_*)}(\alpha, \alpha) & 0 \\ \Delta L^1_{C^1(\mathscr{K}_*, \mathscr{K})} & 0 \end{bmatrix}$$

in the above pairing between $\tilde{\mathscr{I}}^\infty$ and $\tilde{\mathscr{T}}$.

Next, the set of operator matrices $[L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} L^\infty_{\mathscr{L}(\mathscr{K})} \; \Delta]$ maps

$$\begin{bmatrix} L^2_{\mathscr{K}_*} \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}$$

into $L^2_{\mathscr{K}}$ via multiplication, and is the dual, via the bilinear form above, of

$$\begin{bmatrix} L^1_{C^1(\mathscr{K}, \mathscr{K}_*)} \\ \Delta L^1_{C^1(\mathscr{K})} \end{bmatrix},$$

the space of trace-class multiplication operators mapping $L^2_{\mathscr{K}}$ into

$$\begin{bmatrix} L^2_{\mathscr{K}_*} \\ (\Delta L^2_{\mathscr{K}})^- \end{bmatrix}.$$

The subspace $[H^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})}(\beta, \alpha) \; 0]$ of $[L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} L^\infty_{\mathscr{L}(\mathscr{K})} \; \Delta]$ is precisely the annihilator of

$$\begin{bmatrix} K^1_{OC^1(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta) \\ \Delta L^1_{C^1(\mathscr{K})} \end{bmatrix}.$$

Furthermore the operator $\begin{bmatrix} \Theta \\ \Delta \end{bmatrix}$ gives an isometric injection of $[H^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})}(\beta, \alpha) \quad 0]$ into

$$\begin{bmatrix} H^\infty_{\mathscr{L}(\mathscr{K}_*)}(\alpha, \alpha) & 0 \\ \Delta L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} & \Delta L_{\mathscr{L}(\mathscr{K})} \Delta \end{bmatrix}.$$

The following lemma follows from the discussion above and general Banach space facts.

LEMMA 4.1. *The quotient Banach space*

$$\begin{bmatrix} H^{\infty}_{\mathcal{L}(\mathcal{K}_*)}(\alpha, \alpha) & 0 \\ \Delta L^{\infty}_{\mathcal{L}(\mathcal{K}_*, \mathcal{K})} & \Delta L^{\infty}_{\mathcal{L}(\mathcal{K})} \Delta \end{bmatrix} \Big/\!\!\Big/ \begin{bmatrix} \Theta \\ \Delta \end{bmatrix} [H^{\infty}_{\mathcal{L}(\mathcal{K}_*, \mathcal{K})}(\beta, \alpha) \quad 0]$$

is the dual of the quotient Banach space

$$\begin{bmatrix} K^{1}_{OC^1(\mathcal{K}, \mathcal{K}_*)}(\alpha, \beta) \\ \Delta L^{1}_{C^1(\mathcal{K})} \end{bmatrix} [\Theta^* \Delta] \Big/\!\!\Big/ \begin{bmatrix} K^{1}_{OC^1(\mathcal{K}_*)}(\alpha, \alpha) & 0 \\ \Delta L^{1}_{C^1(\mathcal{K}_*, \mathcal{K})} & 0 \end{bmatrix}.$$

The following generalization of the Riesz factorization lemma is a key ingredient for the proof of the lifting theorem to follow.

LEMMA 4.2. *Assume that* $\dim \mathcal{K} \leqslant \dim \mathcal{K}_* < \infty$. *Then any element* F *of*

$$\begin{bmatrix} K^{\infty}_{OC^1(\mathcal{K}, \mathcal{K}_*)}(\alpha, \beta) \\ \Delta L^{\infty}_{C^1(\mathcal{K})} \end{bmatrix}$$

has a factorization $F = F_1 F_2$ *where*

$$F_1 \in \begin{bmatrix} H^{\infty}_{C^2(\mathcal{K}_*)}(\alpha, e) & 0 \\ \Delta L^{\infty}_{C^2(\mathcal{K}_*, \mathcal{K})} & \Delta L^{\infty}_{C^2(\mathcal{K})} \end{bmatrix},$$

$$F_1 \in \begin{bmatrix} K^{\infty}_{OC^2(\mathcal{K}, \mathcal{K}_*)}(e, \beta) \\ L^{\infty}_{C^2(\mathcal{K})} \end{bmatrix}$$

and $\max \{\|F_1\|_2^2, \|F_2\|_2^2\} \leqslant M^2 \|F\|_1$ *where* M *is a constant independent of* F *(2-norms are Hilbert-Schmidt norms, 1-norms are trace norms).*

Proof. We consider R as embedded as a fundamental polygon for a group G of linear fractional transformation on the universal covering surface D for R (see Section 3.1 of ref. [7]). The group G is isomorphic in a canonical way to the fundamental group $\pi_0(R)$, and this isomorphism induces a correspondence $\alpha \to \hat{\alpha}$ of $\mathrm{Hom}(\pi_0(R), \mathcal{U}(\mathcal{K}))$ onto $\mathrm{Hom}\,(G, \mathcal{U}(\mathcal{K}))$ ($\mathcal{K}$ any Hilbert space). For group homomorphisms $\hat{\alpha} \in \mathrm{Hom}\,(G, \mathcal{U}(\mathcal{K}_*))$ and $\hat{\beta} \in \mathrm{Hom}(G, \mathcal{U}(\mathcal{K}))$, spaces of operator-valued functions $H^{p}_{C^q(\mathcal{K}, \mathcal{K}_*)}(\hat{\alpha}, \hat{\beta})$ and $H^{p}_{\mathcal{L}(\mathcal{K}, \mathcal{K}_*)}(\hat{\alpha}, \hat{\beta})$ and of vector-valued function

$H^p_{\mathscr{K}}(\alpha)$, all functions now defined on D (or ∂D), 'are defined as in [7, Section 3]. Now the function

$$F \in \begin{bmatrix} K^{\infty}_{OC^1(\mathscr{K},\, \mathscr{K}_*)}(\alpha, \beta) \\ \\ \Delta L^{\infty}_{C^1(\mathscr{K})} \end{bmatrix}$$

defined on $\partial R \cap \partial D$ extends to a function

$$\hat{F} \in \begin{bmatrix} K^{\infty}_{OC^1(\mathscr{K},\, \mathscr{K}_*)}(\hat{\alpha}, \hat{\beta}) \\ \\ \hat{\Delta} L^{\infty}_{C^1(\mathscr{K})} \end{bmatrix}$$

defined on ∂D by demanding that

$$\hat{F} \circ A = \begin{bmatrix} \alpha(A) & 0 \\ 0 & \beta(A) \end{bmatrix} \hat{F} \beta(A)^*$$

for all $A \in G$. (Here $K^{\infty}_{OC^1(\mathscr{K},\, \mathscr{K}_*)}(\hat{\alpha}, \hat{\beta}) = \hat{v}^{-1} H^{\infty}_{C^1(\mathscr{K},\, \mathscr{K}_*)}(\hat{\alpha}, \hat{\beta})$ where $\hat{v}$ is the analytic continuation of v to all of D, and $\hat{\Delta}(z) = (I - \hat{\Theta}(z)^* \hat{\Theta}(z))^{1/2}$ where $\hat{\Theta}$ is the analytic continuation of Θ to D.) If we can produce an

$$\hat{F}_1 \in \begin{bmatrix} H^{\infty}_{C^2(\mathscr{K}_*)}(\hat{\alpha}, \hat{e}) & 0 \\ \\ \hat{\Delta} L^{\infty}_{C^2(\mathscr{K}_*,\, \mathscr{K})} & \hat{\Delta} L^{\infty}_{C^2(\mathscr{K})} \end{bmatrix}$$

and an

$$\hat{F}_2 \in \begin{bmatrix} K^{\infty}_{OC^2(\mathscr{K},\, \mathscr{K}_*)}(\hat{e}, \hat{\beta}) \\ \\ L^{\infty}_{C^2(\mathscr{K})} \end{bmatrix}$$

such that $\hat{F} = \hat{F}_1 \hat{F}_2$ and $\max \{\|\hat{F}_1\|^2, \|\hat{F}_2\|^2\} \leqslant M^2 \|\hat{F}\|_1$, the Lemma will follow by restricting $\hat{F}$, $\hat{F}_1$ and $\hat{F}_2$ to $\partial R \cap \partial D$.

Since for the rest of the proof all the action is on ∂D, we now drop the ^ notation. Since $\dim \mathscr{K} \leqslant \dim \mathscr{K}_*$, Theorem 1.2, applied to

$$\begin{bmatrix} (b_2/b_1)\, I & 0 \\ 0 & I \end{bmatrix} F,$$

where b_1 is the Blaschke factor with zeroe at the pole of v and b_2 is the Blaschke product with zeroes at the zeroes of v, implies that F can be factored as $F = GH$

with

$$G = \begin{bmatrix} g_{11} & 0 \\ g_{21} & g_{22} \end{bmatrix} \in \begin{bmatrix} H^{\infty}_{C^2(\mathscr{K}_*)} & 0 \\ \Delta L^{\infty}_{C^2(\mathscr{K}_*, \mathscr{K})} & \Delta L^{\infty}_{C^2(\mathscr{K})} \end{bmatrix},$$

$$H = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \in \begin{bmatrix} v^{-1} H^{\infty}_{C^2(\mathscr{K}, \mathscr{K}_*)} \\ L^{\infty}_{C^2(\mathscr{K})} \end{bmatrix},$$

where $v\,H$ is (V_1, V_2)-outer, $(F^*F)^{1/2} = H^*H$ and $G^*G = HH^*$. We next claim that there is a $\gamma \in \mathrm{Hom}\,(G, \mathscr{U}(\mathscr{K}^*))$ such that $g_{11} \in H^{\infty}_{C^2(\mathscr{K}_*)}(\alpha, \gamma)$ and $h_1 \in v^{-1} H^{\infty}_{C^2(\mathscr{K}, \mathscr{K}_*)}(\gamma, \beta) = K^{\infty}_{OC^2(\mathscr{K}, \mathscr{K}_*)}(\gamma, \beta)$. To see this, note that for A in G, on the one hand

$$(F \circ A(z)^* F \circ A(z))^{1/2} = H \circ A(z)^* H \circ A(z),$$

while on the other,

$$(F \circ A(z)^* F \circ A(z))^{1/2} = \beta(A)(F(z)^* \ F(z))^{1/2} \ \beta(A)^* = \beta(A) \ H(z)^* H(z) \ \beta(A)^*.$$

Since $(b_2/b_1)H\beta(A)^*$ and $(b_2/b_1)H \circ A$ are both (V_1, V_2)-outer, the uniqueness assertion of Theorem 1.2 implies that there is a partial isometry function

$$\gamma(A) = \gamma(A, z) = \begin{bmatrix} \gamma_1(A) & 0 \\ 0 & \gamma_2(A, z) \end{bmatrix}$$

with initial space equal to $[\mathrm{Ran}\, H(z)]^-$ and final space equal to $[\mathrm{Ran}\, H \circ A(z)]^-$ such that $H \circ A(z) = \gamma(A, z) H(z) \, \beta(A)^*$. Then necessarily $\gamma_1(A)$ has initial and final space equal to $[\mathrm{Ran}\, h_1(z)]^- = [\mathrm{Ran}\, h_1 \circ A(z)]^-$. We can redefine $\gamma_1(A)$ to be unitary by setting it equal to the identity on $[\mathrm{Ran}\, h_1(z)]^{\perp}$ without destroying any of the above properties. That γ_1 is a group representation, that is, satisfies the dentity $\gamma_1(AB) = \gamma_1(A)\gamma_1(B)$ follows by computing $h_1 \circ (AB) = (h_1 \circ A) \circ B$ in two different ways, much like a computation in the proof of Theorem 11 of ref. [3]. From $h_1 \circ A = \gamma_1(A) h \beta(A)^*$, $g_{11} h_1 = f_1$ (the first component of F) and $f_1 \circ A = \alpha(A) f_1 \beta(A)^*$, it follows that $g_{11} \circ A = \alpha(A) \, g_{11} \, \gamma_1(A)^*$ as well.

Now by Theorem 3.2 of ref. [7], since $\dim \mathscr{K}^* < \infty$, for each $\gamma \in \mathrm{Hom}\,(G, \mathscr{U}(\mathscr{K}^*))$ there is an

$$E_\gamma \in H^{\infty}_{\mathscr{L}(\mathscr{K}_*)}(\gamma, e) \cap (H^{\infty}_{\mathscr{L}(\mathscr{K}_*)}(e, \gamma))^{-1}$$

such that

$$\sup_{\gamma} \{\|E_\gamma\|, \ \|E_\gamma^{-1}\|\} \leqslant M < \infty.$$

We now set

$$F_1(z) = G(z) \begin{bmatrix} E_\gamma(z) & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$F_2(z) = \begin{bmatrix} E_\gamma(z)^{-1} & 0 \\ 0 & 1 \end{bmatrix} H(z).$$

Then $F = F_1F_2$ (restricted to R) satisfies all the requirements of Lemma 4.1.

The statement and proof of the next theorem is suggested by the lifting theorem proved by Abrahamse [1] for the scalar C_{00} case. The duality techniques go back to the original paper of Sarason [14].

THEOREM 4.3. *Let $\sigma\colon \mathrm{Rat}(R) \to \mathscr{L}(\mathscr{H})$ be a ∂R-pure c.c.u. representation. Suppose that, if $\tau = \mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{D})$ is a minimal ∂R-normal dilation of σ, the normal operator $\tau(z)$ has bounded spectral multiplicity function. Then, for any operator $T \in \mathscr{L}(\mathscr{H})$ such that T commutes with $\sigma(\mathrm{Rat}(\bar{R}))$ there is a $Y \in \mathscr{L}(\mathscr{D})$ commuting with $\tau(\mathrm{Rat}(\bar{R}))$, and a constant M depending only possibly on the bound for the spectral multiplicity function for $\tau(z)$, such that*

(*i*) $\|Y\| \leqslant M\|T\|$

and

(*ii*) $T = P_{\mathscr{H}}Y|\mathscr{H}$.

Proof. By using Theorem 2.1, we can assume that $\sigma = \sigma_\Theta$ for some Θ in the unit ball of $H^\infty_{\mathscr{L}(\mathscr{K},\mathscr{K}_*)}(\alpha, \beta)$. By possibly considering σ^* instead of σ (where $\sigma^*(f) = \sigma(f)^*$), we can assume that $\dim\mathscr{K} \leqslant \dim\mathscr{K}_*$. Finally the boundedness assumption on the spectral multiplicity function of $\tau(z)$ implies that we can assume that $\dim\mathscr{K}^* < \infty$.

Next note that we may identify the space

$$\bigoplus_1^{\dim\mathscr{K}_*}\mathscr{H} \text{ with } \hat{\mathscr{H}}_1 = \begin{bmatrix} H^2_{C^2(\mathscr{K}_*)}(\alpha, e) \\ \Delta L^2_{C^2(\mathscr{K}_*,\mathscr{K})} \end{bmatrix} \ominus \begin{bmatrix} \Theta \\ \Delta \end{bmatrix} H^2_{C^2(\mathscr{K}_*,\mathscr{K})}(\beta, e)$$

via the correspondence

$$\bigoplus_1^{\dim\mathscr{K}_*} f_i \to \sum_1^{\dim\mathscr{K}_*}\langle\cdot, e_i\rangle f_i$$

where $\{e_i\}$ is a fixed orthonormal basis for $\mathscr{K}_*$. The space $\hat{\mathscr{H}}_1 \oplus \Delta L^2_{C^2(\mathscr{K})}\Delta$ in turn is isometrically isomorphic in a canonical way to

$$\hat{\mathscr{H}} = \begin{bmatrix} H^2_{C^2(\mathscr{K}_*)}(\alpha, e) & 0 \\ \Delta L^2_{C^2(\mathscr{K}_*,\mathscr{K})} & \Delta L^2_{C^2(\mathscr{K})}\Delta \end{bmatrix} \ominus \begin{bmatrix} \Theta & 0 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} H^2_{C^2(\mathscr{K}_*,\mathscr{K})}(\beta, e) & 0 \\ 0 & 0 \end{bmatrix}.$$

By Theorem 3.1 there is a $\Psi \in \mathscr{B}(\Theta)$ such that $T = T_\Psi$. Hence

$$\bigoplus_1^{\dim\mathscr{K}_*} T = \bigoplus_1^{\dim\mathscr{K}_*} T_\Psi$$

defines a bounded operator on $\hat{\mathscr{H}}_1$. By the proof of Theorem 3.1, ψ_{22} (the lower right entry of Ψ) is a bounded operator on $(\Delta L^2_{\mathscr{K}})^-$ with $\|\psi_{22}\| \leqslant \|T\|$. Hence the operator $L_{\psi_{22}}$ on $\Delta L^2_{C^2(\mathscr{K})}\Delta$ of multiplication by ψ_{22} on the left is bounded by $\|T\|$. Thus the operator $\left(\bigoplus_1^{\dim \mathscr{K}_*} T\right) \oplus L_{\psi_{22}}$, which can be considered to be acting on $\hat{\mathscr{H}}$, has norm equal to $\|T\|$.

Now define the operator $\hat{T}_\Psi$ from a dense subset of $\hat{\mathscr{H}}$ into $\hat{\mathscr{H}}$ by

$$\hat{T}_\Psi h = P_{\hat{\mathscr{H}}} \Psi f \text{ if } h = P_{\hat{\mathscr{H}}} f$$

and

$$f \in \begin{bmatrix} H^\infty_{C^2(\mathscr{K}_*)}(\alpha, e) & 0 \\ \Delta L^2_{C^2(\mathscr{K}_*, \mathscr{K})} & \Delta L^2_{C^2(\mathscr{K})}\Delta \end{bmatrix}.$$

Then it is a simple check that $\hat{T}_\Psi$ on its domain agrees with the bounded operator $\left(\bigoplus_1^{\dim \mathscr{K}_*} T\right) \oplus L_{\psi_{22}}$ and thus has a unique bounded extension to all of $\hat{\mathscr{H}}$ with $\|\hat{T}_\Psi\| = \|T\|$. Conversely, if Y is any element of $\mathscr{B}(\Theta)$ such that $\hat{T}_Y = \hat{T}_\Psi$, then $T_Y = T_\Psi = T$. Thus, to prove the Theorem, it suffices to produce a

$$Y \in \begin{bmatrix} H^\infty_{\mathscr{L}(\mathscr{K}_*)}(\alpha, \alpha) & 0 \\ \Delta L^\infty_{\mathscr{L}(\mathscr{K}_*, \mathscr{K})} & \Delta L^\infty_{\mathscr{L}(\mathscr{K})}\Delta \end{bmatrix}$$

such that $\hat{T}_Y = \hat{T}_\Psi$ on $\hat{\mathscr{H}}$.

This reduction enables us to use the duality results established above as follows. Define a linear functional L on

$$\begin{bmatrix} K^\infty_{OC^1(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta) \\ \Delta L^\infty_{C^1(\mathscr{K})}\Delta \end{bmatrix} [\Theta^*\Delta] \Bigg/ \begin{bmatrix} K^\infty_{OC^1(\mathscr{K}_*)}(\alpha, \alpha) & 0 \\ \Delta L^\infty_{C^1(\mathscr{K}_*, \mathscr{K})} & 0 \end{bmatrix}$$

by

$$L(\{f[\Theta^*\Delta]\}) = (2\pi i)^{-1} \int_{\partial R} \operatorname{tr}(\Psi(z) f(z) [\Theta(z)^* \Delta(z)])\, v(z)\, dz.$$

It is a routine check to show that L is well-defined. We wish to show that L has a bounded extension to all of

$$\begin{bmatrix} K^1_{OC^1(\mathscr{K}, \mathscr{K}_*)}(\alpha, \beta) \\ \Delta L^1_{C^1(\mathscr{K})}\Delta \end{bmatrix} [\Theta^*\Delta] \Bigg/ \begin{bmatrix} K^1_{OC^1(\mathscr{K})} & 0 \\ \Delta L^1_{C^1(\mathscr{K}_*, \mathscr{K})} & 0 \end{bmatrix}$$

By Lemma 4.2, any f in

$$\begin{bmatrix} K^{\infty}_{OC^1(\mathcal{K},\mathcal{K}_*)}(\alpha,\beta) \\ \\ \Delta L^{\infty}_{C^1(\mathcal{K})}\Delta \end{bmatrix}$$

has a factorization $f = gh$ where

$$g \in \begin{bmatrix} H^{\infty}_{C^2(\mathcal{K}_*)}(\alpha, e) & 0 \\ \\ \Delta L^{\infty}_{C^2(\mathcal{K}_*,\mathcal{K})} & \Delta L^{\infty}_{C^2(\mathcal{K})} \end{bmatrix}$$

and

$$h \in \begin{bmatrix} K^{\infty}_{OC^2(\mathcal{K},\mathcal{K}_*)}(e,\beta) \\ \\ L^{\infty}_{C^2(\mathcal{K})} \end{bmatrix}$$

and

$$\max\{\|g\|_2^2, \|h\|_2^2\} \leqslant M^2\|f\|_1.$$

Hence

$$L(\{f[\Theta^*\Delta]\}) = \int_{\partial R} \operatorname{tr}(\Psi\, gh[\Theta^*\Delta])\, dm = \int_{\partial R} \operatorname{tr}(h[\Theta^*\Delta]\,\Psi g)\, dm =$$

$$= \left\langle \Psi g, \begin{bmatrix} \Theta \\ \Delta \end{bmatrix} h^* \right\rangle_{L^2_{C^2(\mathcal{K}_*\oplus\mathcal{K})}}.$$

One can check that $\begin{bmatrix} \Theta \\ \Delta \end{bmatrix} h^*$ is orthogonal to $\begin{bmatrix} \Theta \\ \Delta \end{bmatrix} [H^2_{C^2(\mathcal{K}_*,\mathcal{K})}\ 0]$ in $L^2_{C^2\mathcal{K}_*\oplus\mathcal{K})}$, and that Ψg is in

$$\begin{bmatrix} H^2_{C^2(\mathcal{K})}(\alpha, e) & 0 \\ \\ \Delta L^2_{C^2(\mathcal{K}_*,\mathcal{K})} & \Delta L^2_{C^2(\mathcal{K})}\Delta \end{bmatrix}.$$

Hence the above inner product is equal to

$$\left\langle \hat{T}_\Psi P_{\hat{\mathcal{H}}} g, P_{\hat{\mathcal{H}}} \begin{bmatrix} \Theta \\ \Delta \end{bmatrix} h^* \right\rangle_{\hat{\mathcal{H}}} \leqslant \|\hat{T}_\Psi\|\ \|g\|_2\ \|h\|_2 \leqslant \|T\|\ M^2\|f\|_1.$$

It follows that L has a unique bounded extension as claimed. Now by Lemma 4.1, there is a

$$Y \in \begin{bmatrix} H^{\infty}_{\mathcal{L}(\mathcal{K}_*)}(\alpha, \alpha) & 0 \\ \\ \Delta L^{\infty}_{\mathcal{L}(\mathcal{K}_*,\mathcal{K})} & \Delta L^{\infty}_{\mathcal{L}(\mathcal{K})}\Delta \end{bmatrix}$$

such that

$$\|Y\| \leqslant \|T\|\, M^2 \text{ and } L([\{f[\Theta^*\Delta]\}) = \int \operatorname{tr}(Yf[\Theta^*\Delta])\, \mathrm{d}m.$$

By a computation similar to that above, we conclude that $\hat{T}_\Psi = \hat{T}_Y$. Thus $T_Y = T_\Psi = T$ and the Theorem follows.

REMARK. *For $\Psi \in \mathscr{B}(\Theta)$ and any γ in* Hom $(\pi_0(R), \mathscr{U}(\mathscr{K}_*))$, *define $\hat{T}_\Psi^\gamma$ on a dense subset of*

$$\hat{\mathscr{H}}_\gamma = \begin{bmatrix} H^2_{C^2(\mathscr{K}_*)}(\alpha,\gamma) & 0 \\ \Delta L^2_{C^2(\mathscr{K}_*,\mathscr{K})} & \Delta L^2_{C^2(\mathscr{K})}\Delta \end{bmatrix} \ominus \begin{bmatrix} \Theta & 0 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} H^2_{C^2(\mathscr{K}_*,\mathscr{K})}(\beta,\gamma) & 0 \\ 0 & 0 \end{bmatrix}$$

by $\hat{T}_\Psi^\gamma h = P_{\hat{\mathscr{H}}_\gamma} \Psi g$ where g is an element of

$$\begin{bmatrix} H^\infty_{C^2(\mathscr{K}_*)}(\alpha,\gamma) & 0 \\ \Delta L^2_{C^2(\mathscr{K}_*,\mathscr{K})} & \Delta L^2_{C^2(\mathscr{K})}\Delta \end{bmatrix}$$

such that $P_{\hat{\mathscr{H}}_\gamma} g = h$. The proof of Theorem 4.3 shows that

$$\|L\| = \sup\{\|\hat{T}_\Psi^\gamma\| : \gamma \in \operatorname{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))\}.$$

This is the operator generalization of a result of Abrahamse [1] for the scalar C_{00} case. Thus the open question of (bounded) lifting of the commutant for a model of infinite rank is equivalent to the question of whether this supremum is finite in general. It is easily shown (using the Grauert-Bungart theorem, (see ref. [3]) that each $\hat{T}_\Psi^\gamma$ is similar to $\hat{T}_\Psi^e = T_\Psi$, and thus each $\|\hat{T}_\Psi^\gamma\|$ (for a fixed γ) is finite. The proof of Theorem 4.3 also shows that this question has an affirmative answer if the assumptions of finite dimensionality can be removed from Theorem 3.2 of ref. [7].

5. COMPACT OPERATORS IN THE COMMUTANT

A c.c.u. representation σ of Rat$(\bar{R})$ is said to be of class C_{00} (see refs. [4] and [8]) if both $\sigma(f_n)$ and $\sigma(f_n)^*$ tend to zero in the strong operator topology whenever the sequence $\{f_n\} \subset$ Rat$(\bar{R})$ tends to zero pointwise boundedly on R. If we set $\sigma = \sigma_\Theta$ as in Theorem 2.1, then σ is of class C_{00} if and only if the characteristic function $\Theta \in H^\infty_{\mathscr{L}(\mathscr{K},\mathscr{K}_*)}(\alpha,\beta)$ is inner, i.e. is unitary a.e. on ∂R. Since, in particular, this implies $\dim \mathscr{K} = \dim \mathscr{K}_*$, we may also assume that $\mathscr{K} = \mathscr{K}_*$. Since also in this case $\Delta = 0$, the second component of the model space vanishes.

The purpose of this section is to extend the characterization of the compact operators in the commutant of a C_{00} contraction due to Muhly [12] to operators of class C_{00} of finite rank on a multiply connected domain. The result will depend on the lifting theorem (Theorem 4.3) (specialized to C_{00} representations) and the approximation result Theorem 3.6 of ref. [7].

For elements α and β of $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$, let $C_{C^\infty(\mathscr{K})}(\alpha, \beta)$ consist of those functions f in $L^\infty(m)$ which are continuous on $\partial R \setminus (\partial R \cap \bar{C})$, and satisfy $f \circ A = \alpha(A) f \beta(A)^*$ for all A in $\pi_0(R)$. By an argument analogous to that given by Muhly [12] for the disk case, it can be shown that $K^1_{OC^1(\mathscr{K})}(\beta, \alpha)$ is the dual space of $C_{C^\infty(\mathscr{K})}(\alpha, \beta)/A_{C^\infty(\mathscr{K})}(\alpha, \beta)$, and hence $L^\infty_{\mathscr{L}(\mathscr{K})}(m)/H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$ is the second dual of $C_{C^\infty(\mathscr{K})}(\alpha, \beta)/A_{C^\infty(\mathscr{K})}(\alpha, \beta)$. Muhly's arguments also show that the linear manifold $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta) + C_{C^\infty(\mathscr{K})}(\alpha, \beta)$ is closed in $L^\infty_{\mathscr{L}(\mathscr{K})}(m)$. Finally for f a scalar or operator valued function, let M_f be the operator of multiplication by f and f_+ the restricted operator $M_f | H^2_{\mathscr{K}}(\alpha)$.

THEOREM 5.1. *Suppose $\mathscr{K}$ is a finite dimensional Hilbert space, α and β are elements of* $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$, *$\Theta$ is an inner function in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \beta)$, $\mathscr{H}$ is the Hilbert space $H^2_{\mathscr{K}}(\alpha) \ominus \Theta H^2_{\mathscr{K}}(\beta)$, and σ:* $\mathrm{Rat}(\bar{R}) \to \mathscr{L}(\mathscr{H})$ *is the representation defined by*

$$\sigma(f) = P_{\mathscr{H}} M_f | \mathscr{H}, \; f \in \mathrm{Rat}(\bar{R}).$$

Then a compact operator T on $\mathscr{H}$ is in the commutant of $\{\sigma(f) : f \in \mathrm{Rat}(\bar{R})\}$ *if and only if there is a function Ψ in $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \alpha)$ such that $\Psi \Theta H^2_{\mathscr{K}}(\beta) \subset \Theta H^2_{\mathscr{K}}(\beta)$, $\Theta^* \Psi$ is in $H^\infty_{\mathscr{L}(\mathscr{K})}(\beta, \alpha) + C_{C^\infty(\mathscr{K})}(\beta, \alpha)$, and $T = P_{\mathscr{H}} \Psi_+ | \mathscr{H}$.*

Proof of Sufficiency. Suppose $\Theta^* \psi = \Omega + K$ where Ω is in $H^\infty_{\mathscr{L}(\mathscr{K})}(\beta, \alpha)$ and K is in $C_{C^\infty(\mathscr{K})}(\beta, \alpha)$; then $\psi = \Theta(\Omega + K)$. We wish to show that $T = P_{\mathscr{H}} \psi_+ | \mathscr{H}$ is compact. Since Ω is in $H^\infty_{\mathscr{L}(\mathscr{K})}(\beta, \alpha)$, $P_{\mathscr{H}} \Theta_+ \Omega_+ | \mathscr{H} = 0$, and it suffices to show that $P_{\mathscr{H}} \Theta_+ K_+ | \mathscr{H}$ is compact.

Now we note that $C_{C^\infty(\mathscr{K})}(\beta, \alpha)$ is the uniform closure of $[v^{-1} A_{C^\infty(\mathscr{K})}(\alpha, \beta)]^* + A_{C^\infty(\mathscr{K})}(\beta, \alpha)$. (This is an easy operator generalization of Theorem 1.1 in [2].) Since the map $\Psi \to P_{\mathscr{H}} \Psi_+ | \mathscr{H}$ is continuous from $H^\infty_{\mathscr{L}(\mathscr{K})}(\alpha, \alpha)$ into $\mathscr{L}(\mathscr{H})$, we can assume K is in $[v^{-1} A_{C^\infty(\mathscr{K})}(\beta, \alpha)]^* + A_{C^\infty(\mathscr{K})}(\alpha, \beta)$. As above, the $A_{C^\infty(\mathscr{K})}(\alpha, \beta)$ piece induces the zero operator on $\mathscr{H}$; so it suffices to assume that K is in $[v^{-1} A_{C^\infty(\mathscr{K})}(\beta, \alpha)]^*$. But

$$v^{-1} A_{C^\infty(\mathscr{K})}(\alpha, \beta) = \bar{b}_2 b_1 A_{C^\infty(\mathscr{K})}(\gamma, \beta)$$

for an appropriate γ in $\mathrm{Hom}(\pi_0(R), \mathscr{U}(\mathscr{K}))$, where b_1 and b_2 are Blaschke products on R chosen as in the proof of Lemma 4.2. Now by Theorem 3.6 of Ref. [7], any element of $A_{C^\infty(\mathscr{K})}(\gamma, \beta)$ can be approximated uniformly by a finite linear combination of inner functions in $A_{C^\infty(\mathscr{X})}(\gamma, \beta)$. Hence we will be done if we show that

$$P_{\mathscr{H}}\Theta_+\overline{b}_{1+}b_{2+}(B^*)_+ \mid \mathscr{H}$$

is finite rank if B is an inner function in $A_{C^\infty(\mathscr{X})}(\gamma, \beta)$. Since B, b_1 and b_2 are all continuous on $\overline{R}\setminus(\partial R \cap \overline{C})$, one can show that

$$\overline{b}_1 b_2 B^* H^2_{\mathscr{X}}(\alpha) \subset \mathscr{T} \oplus H^2_{\mathscr{X}}(\beta)$$

where $\mathscr{T}$ is a finite-dimensonal subspace of $L^2_{\mathscr{X}}(m)$. It now follows just as in Muhly's argument [12, p. 209] that $P_{\mathscr{H}}\Theta_+\overline{b}_{1+}b_{2+}B^*_+ \mid \mathscr{H}$ has finite rank as asserted.

Proof of Necessity. If T is in the commutant of $\{\sigma(f) : f \in \mathrm{Rat}(\overline{R})\}$, by Theorem 4.3 there is a Ψ in $H^\infty_{\mathscr{L}(\mathscr{X})}(\alpha, \alpha)$ such that $T = P_{\mathscr{H}}\Psi_+ \mid \mathscr{H}$. The problem is to show that if T is compact, then any such Ψ has the property that $\Theta^*\Psi$ is in $H^\infty_{\mathscr{L}(\mathscr{X})}(\beta, \alpha) + C_{C^\infty(\mathscr{X})}(\beta, \alpha)$. We leave it to the reader to check that the duality properties of $H^\infty_{\mathscr{L}(\mathscr{X})}(\beta, \alpha) + C_{C^\infty(\mathscr{X})}(\beta, \alpha)$ mentioned above, together with Lemma 4.2, specialized to the case where Θ is inner, as a substitute for the Riesz factorization lemma, enables one to push Muhly's arguments for the disk case, simplified to the case where $\mathscr{X}$ is finite dimensional, through.

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