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A COMPACTNESS CONDITION FOR LINEAR OPERATORS ON FUNCTION SPACES

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Let $(\underline{\overline{X}}, F, m)$ and $(\underline{\overline{Y}}, G, n)$ be separable measure spaces. For $1 < p < \infty$ and $1 \leqslant q < \infty$ we consider a pair of bounded linear operators T and P from $L^p(m)$ into $L^q(n)$. We assume P is positivity preserving and that:

For each $f \in L^p(m)$,

$$0 \leqslant |Tf(y)| \leqslant P|f|(y), \quad \text{a.e. } [n]. \tag{1}$$

We will prove

THEOREM 1. *If* (1) *is satisfied and if P is compact then T is compact.*

Comments. When P is an integral operator Theorem 1 is known. It is stated in ref.[6, p. 94] and, in the context of integral operators on Banach function spaces, it is implicit in ref. [4, p. 175]. This note presents a new argument of some general interest which does not require that P be an integral operator.

The referee has drawn to my attention that Dodds and Fremlin [2] have also proved a version of Theorem 1 in the broader context of Banach lattices. Professor Dodds was kind enough to send me copies of this work. It is worth mentioning here that a primary step in proving their lattice theoretic results is a reduction to the case where T and P map an abstract M-space E into an abstract L-space F. This corresponds to our Theorem 1 with $p = \infty$ and $q = 1$. Unfortunately, the proof given here breaks down precisely when $p = \infty$. It can be modified but the necessary changes seem to obfuscate the basic simplicity of the present arguments.

It is my feeling that this paper and that of Dodds and Fremlin are largely complementary and that both have something to offer. Our work here is a simple proof of an important special case of the Dodds-Fremlin results. Rather than push for generality here I have tried for simplicity.

To indicate the scope of the arguments, I have included the statement (Theorem 2) of an extension of Theorem 1 to the Banach function space context of [4]. This extension requires no essential changes and seems natural.

For an application of Theorem 1 in a physical context see ref. [1]. Thanks are due to my colleague Ira Herbst for bringing this problem to my attention and for simplifications in the proof.

Notations. The norm of an operator $A: L^p(m) \to L^q(n)$ is $\|A\|$.

The characteristic function of a set E is written as χ_E, as is the operator of multiplication by χ_E. The complement of a set E is E^c.

DEFINITION 1. An operator $A: L^p(m) \to L^q(n)$ is said to have *absolutely continuous norm* if for each $\varepsilon > 0$ there exist measurable sets $\bar{F} \subset \underline{X}$ and $\bar{G} \subset \underline{Y}$ with $m(\bar{F}) + m(\bar{G}) < \infty$ and there is a $\delta > 0$ such that whenever $F \in \mathscr{F}$ and $G \in \mathscr{G}$,

$$m(\bar{F} \cap F) < \delta \text{ implies } \|A\chi_F\| < \varepsilon \tag{2}$$

and

$$n(\bar{G} \cap G) < \delta \text{ implies } \|\chi_G A\| < \varepsilon. \tag{3}$$

This terminology is not standard. The next lemma is basic.

LEMMA 1. *If $A: L^p(m) \to L^q(n)$ is compact it has absolutely continuous norm.*

Proof. This is a consequence of Lemma 7.1 in ref. [4] which is attributed to I. Gelfand. For $f \in L^p(m)$ and $\varepsilon > 0$ we can find a set $\bar{G}_f$ and a $\delta_f > 0$ so that $n(\bar{G}_f \cap G) < \delta_f$ implies $\|\chi_G A f\|_q < \varepsilon/2$. If $\|g - Af\|_q < \varepsilon/2$ then $\|\chi_G g\| < \varepsilon$. Since $\{Af: \|f\|_p \leqslant 1\}$ is precompact in $L^q(n)$, (3) follows readily.

The result (2) follows from (3) by duality.

We now apply Lemma 1 to the operator P. Since P is compact we can find sets $\bar{F}$ and $\bar{G}$ with $\|P\chi_{\bar{F}^c}\| < \varepsilon$ and $\|\chi_{\bar{G}^c} P\| < \varepsilon$. Setting $T_0 = \chi_{\bar{G}} T \chi_{\bar{F}}$ and $P_0 = \chi_{\bar{G}} P \chi_{\bar{F}}$ it follows that P_0 is compact and from (1) we have

$$|T_0 f(y)| \leqslant P_0 |f|(y) \quad \text{a.e.} \quad [n] \tag{4}$$

and that $\|T_0 - T\| \leqslant 3\varepsilon$. Since the norm limit of compact operators is compact we only need show that T_0 is compact. Since T_0 and P_0 are naturally identifiable as operators from $L^p(\bar{F}, m)$ to $L^q(\bar{G}, n)$ it follows from (4) that we may assume that m and n are finite measures.

Further, since a finite separable measure algebra is isomorphic ([3], pp. 171—174) to a measure algebra of the form $([0, 1], \mathscr{B}, m)$ where m is a finite Borel measure on the interval $[0, 1]$, it is no loss in generality to take $\underline{X} = \underline{Y} = [0, 1]$ and m and n to be finite Borel measures. In the remainder of the paper we will make these simplifying assumptions.

For $0 < a < b \leqslant 1$ we set $\chi_{a,b}(x) = \chi_{(a,b]}(x)$. When $a = 0 \leqslant b$ we set $\chi_{0,b}(x) = \chi_{[0,b]}(x)$. For $0 \leqslant x,\ y \leqslant 1$ define

$$F(x, y) = \int_{[0,1]} \chi_{0,y}(t)\, P\chi_{0,x}(t)\, \mathrm{d}n. \tag{5}$$

Then $0 \leqslant F(y, x)$ and F satisfies for $x \leqslant x_1$ and $y \leqslant y_1$

$$F(y_1, x_1) - F(y_1, x) - F(y, x_1) + F(y, x) = \int_{[0,1]} \chi_{y, y_1}(t)\, P\chi_{x, x_1}(t) \mathrm{d}n \geqslant 0. \tag{6}$$

The function $F(y, x)$ is obviously right continuous and from (6) it follows that $F(y, x)$ is the joint distribution function of a unique finite measure μ on $[0, 1]^2 = [0, 1] \times [0, 1]$, with

$$F(y, x) = \mu\{[0, y] \times [0, x]\}. \tag{7}$$

From (5) we have

$$\int_{[0,1]} g(y)\, Pf(y)\, \mathrm{d}n = \int_{[0,1]^2} g(y) f(x)\, \mathrm{d}\mu, \tag{8}$$

for all $f \in L^p(m)$ and $g \in L^{q'}(n)$, where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$.

Using the operator T we define another set function

$$\nu\{(a, b] \times (c, d]\} = \int \chi_{a, b}(t)\, T\chi_{c, d}(t)\, \mathrm{d}n \tag{9}$$

on half open rectangles $(a, b] \times (c, d]$. Extending ν additively to the field $\mathscr{R}_0$ of finite unions of disjoint half open rectangles we see from (1) and (9) that

$$|\nu(R)| \leqslant \mu(R) \quad \text{for each } R \in \mathscr{R}_0.$$

It follows that ν is countably additive on $\mathscr{R}_0$ and extends to a complex valued signed Borel measure which we also denote as ν. The signed measure ν is absolutely continuous with respect to μ, and the Radon-Nikodym derivative $b(y, x) = \dfrac{\mathrm{d}\nu}{\mathrm{d}\mu}$ satisfies

(10) $\|b\|_\infty \leqslant 1$, here $\|b\|_\infty$ is the $L^\infty(\mu)$ norm, and

$$\int f(y, x)\, \mathrm{d}\nu = \int f(y, x)\, b(y, x)\, \mathrm{d}\mu$$

if f is ν integrable.

In particular, if $f \in L^p(m)$ and $g \in L^{q'}(n)$,

$$\int_{[0,1]} g(y)\, Tf(y)\, \mathrm{d}n = \int_{[0,1]^2} g(y)\, b(y, x) f(x)\, \mathrm{d}\mu. \tag{11}$$

Note that T determines b and is determined by b. Also, if T satisfies (1) the correspondence $T \leftrightarrow b$ is linear. We let $\mathscr{B}$ denote the class of all measurable b satisfying (10). It is elementary to deduce from (11) and (8) that each $b \in \mathscr{B}$ corresponds to a unique operator T satisfying (1). We denote this operator as T_b. We now define a norm for T_b when $b \in \mathscr{B}$, by

$$\|T_b\|_1 = \int_{[0,1]^2} |b(y, x)| \, \mathrm{d}\mu. \tag{12}$$

The next lemma is our basic estimate.

LEMMA 2. *If P has absolutely continuous norm there will exist a function $\psi(u) > 0$ defined for $u > 0$ and satisfying $\psi(0+) = 0$ such that for $b \in \mathscr{B}$,*

$$\|T_b\| \leqslant \psi(\|T_b\|_1). \tag{13}$$

Proof. For fixed $f \in L^p(m)$ and $g \in L^{q'}(n)$ with $\|f\|_p \leqslant 1$ and $\|g\|_{q'} \leqslant 1$ we introduce the sets

$$F = F_\lambda = \{x : |f(x)| > \lambda\} \text{ and } G = G_\lambda = \{y : |g(y)| > \lambda\}.$$

By Tchebychev's inequality we have

$$m(F) \leqslant \lambda^{-p} \text{ and } n(G) \leqslant \lambda^{-q'}, \text{ (if } q = 1,\ n(G) = 0 \text{ for } \lambda \geqslant 1). \tag{14}$$

Then

$$\int g T_b f \, \mathrm{d}n = \int \chi_G g T_b f \, \mathrm{d}n + \int \chi_{G^c} g T_b \chi_F f \, \mathrm{d}n + \int \chi_{G^c} g T_b \chi_{F^c} f \, \mathrm{d}n.$$

Using (1), (8) and (11) we get

$$\left| \int g T_b f \, \mathrm{d}n \right| \leqslant \|\chi_G P\| + \|P \chi_F\| + \int_{G^c} \int_{F^c} |g(y)\, b(y, x) f(x)| \, \mathrm{d}\mu.$$

Observing that $|g(y)| \leqslant \lambda$ on G^c and $|f(x)| \leqslant \lambda$ of F^c gives

$$\left| \int g T_b f \, \mathrm{d}n \right| \leqslant \|\chi_G P\| + \|P \chi_F\| + \lambda^2 \|T_b\|_1.$$

Applying Hölder's inequality and choosing $\lambda = \|T\|_1^{-1/3}$ gives

$$\|T\| \leqslant \|\chi_G P\| + \|P \chi_F\| + \|T_b\|_1^{1/3}.$$

Definition 1 and the inequalities (14) complete the proof.

COROLLARY. *If P has absolutely continuous norm, then the set of $b \in B$ for which T_b is compact is closed in the $L^1(\mu)$ topology.*

Proof. If $\{b_n\} \in \mathscr{B}$ are such that $b_n \to b$ in the $L^1(\mu)$ norm $\| \ \|_1$, then

$$\|T_b - T_{b_n}\| = 2 \, \|T_{(b-b_n)/2}\| \leqslant 2\psi(1/2\| b - b_n\|_1) \to 0.$$

Thus if the T_{b_n} are compact so is T_b.

Note that after Lemma 1 our development has only been based on the positivity of P and not its compactness. The compactness enters again now. By the Corollary, in order to complete the proof of Theorem 1 we only need find an $L^1(\mu)$ dense subset $\tilde{\mathscr{B}}$ of $\mathscr{B}$ such that $\tilde{b} \in \tilde{\mathscr{B}}$ will imply $T_{\tilde{b}}$ is compact. A suitable choice for $\tilde{\mathscr{B}}$ is the space of all polynomials $\tilde{b}(y, x) = \sum a_{nm} y^n x^m$ which are bounded on $[0, 1]^2$ by 1. By the Weierstrass approximation theorem $\tilde{\mathscr{B}}$ is uniformly dense in the space $\tilde{C}$ of continuous functions $h(y, x)$ on $[0, 1]^2$ satisfying $\sup |h| \leqslant 1$. $\tilde{C}$ is $L^1(\mu)$ dense in $\mathscr{B}$ and hence $\tilde{\mathscr{B}}$ is $L^1(\mu)$ dense in $\mathscr{B}$. To see that $T_{\tilde{b}}$ is compact for $\tilde{b} \in \tilde{\mathscr{B}}$ it suffices to take $\tilde{b} = y^n x^m$. But for $\tilde{b} = y^n x^m$, we see from (11) that $T_{\tilde{b}} = Y^n PX^n$ where X is the operator of multiplication by x and Y denotes multiplication by y. Since X and Y are bounded operators and P is compact, $Y^n PX^m$ is compact and the proof is complete.

REMARKS AND EXTENSIONS

I. Theorem 1 is true without the condition that $(\underline{\bar{X}}, \mathscr{F}, m)$ and $(\underline{\bar{Y}}, G, n)$ are separable. In fact, if $\{f_n\}$ is a bounded sequence in $L^p(m)$ it is easy to find separable sub σ-fields $\mathscr{F}_0 \subseteq \mathscr{F}$ and $\mathscr{G}_0 \subseteq \mathscr{G}$ so that $\{f_n\} \subset L^p(\underline{\bar{X}}, \mathscr{F}_0, m)$ and both T and P map $L^p(\underline{\bar{X}}, \mathscr{F}_0, m)$ into $L^q(\underline{\bar{Y}}, \mathscr{G}_0, n)$. Theorem 1 implies that the restriction of T to $L^p(\underline{\bar{X}}, \mathscr{F}_0, m)$ is compact. Thus $\{Tf_n\}$ has a convergent subsequence in $L^q(\underline{\bar{Y}}, \mathscr{G}_0, n)$, and T is compact.

II. Theorem 1 is easily modified to hold in the Banach function space context of [4]. If U and V are Banach function spaces in the sense of [4] and $A: U \to V$, then Definition 1 makes sense but Lemma 1 breaks down even for $A: L^1 \to L^1$, but is true in many examples, (see ref. [5], pp. 41—47). In the terminology of [3], we denote the associate spaces of U and V as U' and V'. V^a and $(U')^a$ denote the subspaces of V and U' consisting of functions of absolutely continuous norm. In this setting, Theorem 1 requires that

$$P: U \to V^a \text{ and } P^*: V' \to (U')^a \tag{15}$$

where P^* is the adjoint of P.

We state

THEOREM 2. *Let U and V be Banach function spaces and let $P: U \to V$ be compact, positivity preserving, and satisfy* (15). *If $T: U \to V$ satisfies* (1) *then T is compact.*

We leave the proof to the interested reader.

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