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HOMOGENEOUS C^*-EXTENSIONS OF $C(X)\otimes K(H)$. PART I.

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The remarkable work of L. G. Brown, R. G. Douglas and P. A. Fillmore ([10], [12]) on extensions of the ideal of compact operators by commutative C^*-algebras has stimulated further research concerning more general extensions ([1], [3], [4], [9], [13], [16], [20], [26], [34—39], [41—47]). This is motivated in part by the desire to extend the Brown-Douglas-Fillmore theory so as to provide a tool for analysing the structure of C^*-algebras.

In particular, such a development might lead to a better understanding of the structure of type I C^*-algebras.

Also we should mention the general program for the study of extensions sketched by L. G. Brown in ref. [9].

A class of extensions to be studied, as suggested in ref. [26], are those of $C(X) \otimes K(H)$. Among these, the homogeneous extensions, considered here, seem to be more tractable. Let us explain what the homogeneity requirement means. Roughly speaking, an extension of $C(X) \otimes K(H)$ by a C^*-algebra A (separable and with unit) gives rise, for each $x \in X$, to an extension of $K(H)$ by some quotient A/J_x of A. The map which associates to $x \in X$ the ideal J_x will be called the ideal symbol of the extension. The extension is called homogeneous if $J_x = 0$ for all $x \in X$. Under a suitable equivalence relation and with some additional conditions on X and A (X finite-dimensional and A nuclear), the homogeneous extensions yield a group $\mathrm{Ext}(X, A)$, which will be the main object of our study. For X reduced to a point, this is just the Brown-Douglas-Fillmore group, but the consideration of the more general $\mathrm{Ext}\,(X, A)$ will be seen (in Part II) to be also of some interest for the study of the usual extensions by $K(H)$.

Passing now to the results of Part I of this paper, we should mention a Weyl-von Neumann type theorem for rather general (not only homogeneous) extensions of $C(X)\otimes K(H)$, a short exact sequence for $\mathrm{Ext}\,(X, A)$ in the A-"variable" for general nuclear C^*-algebras (this is new also for the usual Ext-groups) and the use of this exact sequence in extending the homotopy-invariance result of Salinas ([42]) from nuclear quasi-diagonal C^*-algebras to the class of nuclear C^*-algebras admitting composition series with quasi-diagonal quotients (this includes the type I C^*-algebras).

In more detail, the content of the six sections of Part I is as follows.

§ 1 contains general definitions and some preliminaries.

In § 2, assuming that the ideal symbol of the extension satisfies some lower semicontinuity requirement and X is finite-dimensional, we prove the existence of trivial extensions and a generalization of the Weyl-von Neumann type theorem of [45].

Beginning with § 3 we consider only homogeneous extensions. We use the Choi-Effros theorem [16] to show that Ext (X, A) is a group when X is finite-dimensional and A nuclear. Also in § 3, we prove, under the same requirements, that in each equivalence class of Ext (X, A) there is an extension which can be realized using the norm-continuous $L(H)$-valued functions on X.

In § 4 the short exact sequence in the A-"variable" (A-nuclear) for Ext (X, A) is proved. This generalizes the exact sequence in [10] as well as the subsequent generalization in [9].

In § 5 we deal with homotopy-invariance for Ext (X, A) both in the X-"variable" and in the A-"variable". Both homotopy-invariance properties are proved for nuclear quasi-diagonal C^*-algebras via an adaption of the argument of Salinas [42] and then using § 4 extended to more general C^*-algebras. Let us also mention a brief discussion of quasi-diagonality in C^*-algebras, an adaption of the notion due to P. R. Halmos [27].

In § 6 we prove a short exact sequence for Ext (X, A) in the X-"variable".

Finally we should mention that Part II of this paper is concerned with topological properties of homogeneous extensions of $C(X) \otimes K(H)$.

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§ 1.

Let H be a complex, separable Hilbert space of infinite dimension. Let $L(H)$ denote the bounded operators on H, $K(H)$ the ideal of compact operators and

$$\pi: L(H) \to L/K(H) = L(H)/K(H)$$

the canonical homomorphism of $L(H)$ onto the Calkin algebra.

For X a compact metrizable space, $C_n(X, K(H))$ denotes the C^*-algebra of $K(H)$-valued continuous functions on X, where $K(H)$ is endowed with the norm topology. Similarly, $C_{*s}(X, L(H))$ is the C^*-algebra of $L(H)$-valued continuous functions on X, where the continuity is with respect to the $*$-strong operator-topology on $L(H)$ (of course, the C^*-norm is the sup-norm). Clearly, $C_n(X, K(H))$ is a closed two-sided ideal of $C_{*s}(X, L(H))$. By p we shall denote the canonical homomorphism

$$p: C_{*s}(X, L(H)) \to C_{*s}(X, L(H))/C_n(X, K(H)).$$

For A a separable C^*-algebra with unit, *an extension of* $C_n(X, K(H))$ *by* A, *is a short exact sequence*

$$(*) \qquad 0 \to C_n(X, K(H)) \xrightarrow{\rho} B \xrightarrow{\sigma} A \to 0$$

where B *is a* C^**-algebra with unit,* ρ *and* σ *are* $*$*-homomorphisms,* σ *being unit-preserving.*

For D a C^*-algebra and $M \subset D$ let us denote

$$\operatorname{Ann}(M; D) = \{y \in D;\ My = 0\}.$$

In order not to complicate our study of extensions it is natural to eliminate a certain trivial part of B, by considering only the extensions satisfying the requirements of the following.

1.1. DEFINITION. *An* X*-extension by* A *is an exact sequence* $(*)$ *satisfying the additional requirement:*

$$\operatorname{Ann}(\rho(C_n(X, K(H)));\ B) = 0.$$

The following folklore-type proposition, in fact about multipliers of $C_n(X, K(H))$, gives a more concrete realization of X-extensions by A.

1.2. PROPOSITION. *Let* $(*)$ *be an* X*-extension by* A. *Then there is a unique* $*$*-homomorphism*

$$\varphi\colon B \to C_{*s}(X, L(H))$$

such that $\varphi \circ \rho = i$, *where* i *denotes the inclusion*

$$C_n(X, K(H)) \hookrightarrow C_{*s}(X, L(H)).$$

Moreover φ *is injective and unit-preserving.*

Proof. The closed two-sided ideal $\rho(C_n(X, K(H)))$ of B, being isomorphic with $C_n(X, K(H))$, has a natural faithful non-degenerate $*$-representation on the Hilbert space:

$$\ell^2(X; H) = \{(h_x)_{x\in X};\ h_x \in H,\ \sum \|h_x\|^2 < +\infty\}.$$

Moreover this representation is in the commutant of the natural representation of $\ell^\infty(X)$ on $\ell^2(X; H)$. By [21, Prop. 2.10.4] the representation of $\rho(C_n(X, K(H)))$ has a unique extension to a representation of B (which is unit-preserving), still in the commutant of the representation of $\ell^\infty(X)$. This yields unit-preserving $*$-homomorphisms $\varphi_x\colon B \to L(H)$ such that for $b \in B$ and $f \in C_n(X, K(H))$ we have

$$b\rho(f) = \rho(g)$$

where $g \in C_n(X, K(H))$ is given by $g(x) = \varphi_x(b)f(x)$. Moreover,

$$\varphi_x(\rho(f)) = f(x).$$

Clearly, we may define $\varphi(b)$ by $(\varphi(b))(x) = \varphi_x(b)$ provided we prove that

$$X \ni x \mapsto \varphi_x(b) \in L(H)$$

is strongly continuous (for $*$-strong continuity consider $b^* \in B$). For $\xi \in H$, $\|\xi\| = 1$, let P denote the projection of H onto $\mathbf{C}\xi$ and $f \in C_n(X, K(H))$ the constant function equal to P. Then

$$g(x) = \varphi_x(b)f(x) = \varphi_x(b)P$$

is an element $g \in C_n(X, K(H))$ and this is equivalent to the continuity of the map $X \ni x \mapsto \varphi_x(b)\xi \in H$, i.e. the desired conclusion.

The uniqueness of φ follows from

$$\mathrm{Ann}\,(i(C_n(X, K(H)));\ C_{*s}(X, L(H))) = 0.$$

Indeed, if φ' is another $*$-homomorphism with $\varphi' \circ \rho = i$, then for $b \in B$ we have

$$\varphi(b) - \varphi'(b) \in \mathrm{Ann}\,(i(C_n(X, K(H)));\ C_{*s}(X, L(H))) = 0.$$

Also, if $\varphi(b) = 0$, then $\varphi(b\rho(f)) = 0$ and since $b\rho(f) \in \rho(C_n(X, K(H)))$ we infer $b\rho(f) = 0$. Thus

$$\mathrm{Ker}\,\varphi \subset \mathrm{Ann}\,(\rho(C_n(X, K(H)));\ B) = 0,$$

which gives the desired result about injectivity. Q.E.D.

1.3. REMARK. *In view of the preceding proposition it is clear that, from now on, we may and shall assume, for an X-extension* $(*)$ *by A, that*

$$C_n(X, K(H)) \subset B \subset C_{*s}(X, L(H)).$$

1.4. DEFINITION. *Two X-extensions by A given by exact sequences*

$$0 \to C_n(X, K(H_1)) \hookrightarrow B_1 \xrightarrow{\sigma_1} A \to 0$$

$$0 \to C_n(X, K(H_2)) \hookrightarrow B_2 \xrightarrow{\sigma_2} A \to 0$$

are said to be equivalent, if there is a unitary

$$U \in C_{*s}(X, L(H_1, H_2))$$

such that

$$U^*B_2U = B_1 \text{ and } \sigma_2(b) = \sigma_1(U^*bU) \text{ for } b \in B_2.$$

1.5. PROPOSITION. *There is a one-to-one correspondence between X-extensions by A*

$$0 \to C_n(X, K(H)) \hookrightarrow B \xrightarrow{\sigma} A \to 0$$

and unital $*$-monomorphisms

$$\tau \colon A \to C_{*s}(X, L(H))/C_n(X, K(H)).$$

In this correspondence $B = p^{-1}(\tau(A))$ and σ is obtained from the obvious isomorphisms between $B/C_n(X, K(H))$, $\tau(A)$ and A.

Proof. How τ yields an X-extension by A is quite clear from above, for the converse also, remark that σ gives an isomorphism between

$$B/C_n(X, K(H)) \subset C_{*s}(X, L(H))/C_n(X, K(H))$$

and A, the inverse of which will give the $*$-monomorphism τ.

Since $B \subset C_{*s}(X, L(H))$, it is obvious that

$$\text{Ann}\,(C_n(X, K(H));\ B) = 0. \text{ Q.E.D.}$$

1.6. REMARK. *Proposition 1.5 gives an alternative way of defining X-extensions by A. This will be frequently used in what follows referring to an X-extension as defined by some $*$-monomorphism τ. For a unitary $U \in C_{*s}(X, L(H_1, H_2))$ let $\alpha(U)$ denote the isomorphism*

$$C_{*s}(X, L(H_1)) \ni f \mapsto UfU^* \in C_{*s}(X, L(H_2))$$

and let $\tilde{\alpha}(U)$ denote the isomorphism between $C_{*s}(X, L(H_1))/C_n(X, K(H_1))$ and $C_{*s}(X, L(H_2))/C_n(X, K(H_2))$ induced by $\alpha(U)$. Then the X-extensions defined by

$$\tau_i \colon A \to C_{*s}(X, L(H_i))/C_n(X, K(H_i)), \qquad (i = 1, 2)$$

are equivalent, iff $\tau_2 = \alpha(U) \circ \tau_1$ for some unitary $U \in C_{*s}(X, L(H_1, H_2))$. We shall use the notation $\tau_1 \sim \tau_2$ for the equivalence of the X-extensions defined by τ_1 and τ_2.

Let now for $x \in X$, p_x denote the $*$-homomorphism

$$p_x \colon C_{*s}(X, L(H))/C_n(X, K(H)) \to L/K(H)$$

which associates to $p(f)$ the element $\pi(f(x))$ of $L/K(H)$. We shall also denote by $I(A)$ the set of closed two-sided ideals of $A, \neq A$.

1.7. DEFINITION. *Let $\tau \colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$ be a $*$-monomorphism. Then the map*

$$X \ni x \mapsto \text{Ker}\,(p_x \circ \tau) \in I(A)$$

is called the ideal symbol of the X-extension by A defined by τ. The X-extension defined by τ is called exact if

$$\bigcap_{x \in X} \text{Ker}\,(p_x \circ \tau) = 0.$$

In case $\operatorname{Ker}(p_x \circ \tau) = 0$ *for all* $x \in X$, *the X-extension defined by* τ *is called homogeneous.*

It is easily seen that the equivalence of X-extensions preserves the ideal symbol and hence also exactness and homogeneity.

Given a map

$$X \ni x \mapsto I_x \in I(A)$$

satisfying the exactness condition

$$\bigcap_{x \in X} I_x = 0,$$

we shall denote by

$$\operatorname{Ext}(X; A, (I_x)_{x \in X})$$

the set of equivalence classes of X-extensions by A with ideal symbol $X \ni x \mapsto$ $\mapsto I_x \in I_x(A)$.

Clearly, the X-extensions considered are exact.

In case $I_x = 0$ for all $x \in X$, we shall denote by

$$\operatorname{Ext}(X, A)$$

the set $\operatorname{Ext}(X; A, (I_x)_{x \in X})$.

We do not know what conditions the ideal symbol must satisfy in order that $\operatorname{Ext}(X; A, (I_x)_{x \in X}) \neq 0$, although in § 2 a certain lower semicontinuity for the ideal symbol will be considered which is necessary for the existence of trivial extensions with the given ideal symbol and which will be shown also to be sufficient provided X is finite dimensional.

If τ defines an X-extension by A with ideal symbol $(I_x)_{x \in X}$ then $[\tau] \in \operatorname{Ext}(X; A, (I_x)_{x \in X})$ denotes its equivalence class. Consider also

$$\tau_i : A \to C_{*s}(X, L(H_i))/C_n(X, K(H_i)), \qquad (i = 1, 2)$$

two $*$-monomorphisms with $\operatorname{Ker}(p_x \circ \tau_i) = I_x$, $(i = 1, 2;\ x \in X)$. This yields a natural $*$-monomorphism

$$\tau_1 \oplus \tau_2 : A \to C_{*s}(X, L(H_1 \oplus H_2))/C_n(X, K(H_1 \oplus H_2))$$

with $\operatorname{Ker}(p_x \circ (\tau_1 \oplus \tau_2)) = I_x$ for $x \in X$. Moreover, it is easily seen that $[\tau_1 \oplus \tau_2]$ depends only on $[\tau_1]$, $[\tau_2]$. Thus,

$$[\tau_1] + [\tau_2] = [\tau_1 \oplus \tau_2]$$

is a well-defined operation on $\operatorname{Ext}(X; A, (I_x)_{x \in X})$ and it is easily seen that $\operatorname{Ext}(X; A, (I_x)_{x \in X})$ endowed with this operation is a commutative semigroup.

Let X, Y be compact metrizable spaces and let $g: X \to Y$ be a continuous map. This yields a $*$-homomorphism

$$G: C_{*s}(Y, L(H)) \to C_{*s}(X, L(H))$$

defined by $G(f) = f \circ g$ for $f \in C_{*s}(X, L(H))$. Clearly,

$$G(C_n(Y, K(H))) \subset C_n(X, K(H))$$

and we have and induced $*$-homomorphism

$$\tilde{G}: C_{*s}(Y, L(H))/C_n(Y, K(H)) \to C_{*s}(X, L(H))/C_n(X, K(H)).$$

Let $\tau: A \to C_{*s}(Y, L(H))/C_n(Y, K(H))$ define an Y-extension by A with ideal symbol $(I_y)_{y\in Y}$; then $\tilde{G}\circ\tau = g^*(\tau)$ is a $*$-homomorphism of A into $C_{*s}(X, L(H))/C_n(X, K(H))$ and $\operatorname{Ker}(p_x \circ (g^*(\tau))) = I_{g(x)}$. So, in case $\bigcap_{x\in X} I_{g(x)} = 0$, there is a well-defined map, still denoted by g^*, $[\tau] \mapsto [g^*(\tau)] = g^*[\tau]$,

$$g^*: \operatorname{Ext}(Y; A, (I_y)_{y\in Y}) \to \operatorname{Ext}(X; A, (I_{g(x)})_{x\in X}),$$

which is a homomorphism. In particular, for A fixed, $\operatorname{Ext}(X, A)$ becomes a contravariant functor from nonvoid compact metrizable spaces to commutative semigroups.

§ 2.

This section is devoted to the study of trivial X-extensions with given ideal symbol. In case X is finite-dimensional and the ideal symbol lower semicontinuous in an appropriate sense, we shall prove the existence of trivial extensions and also a generalization of the Weyl-von Neumann type theorem of [45] (see also ref. [4]), which will show that $\operatorname{Ext}(X; A, (I_x)_{x\in X})$ is a semigroup with unit in this case.

We recall that the compact metrizable space X is of dimension $\leqslant n$ if for every covering by open sets of X there is a finite open covering refining it, that has order $\leqslant n$ (Ch. V of [29]).

The appearance of finite-dimensionality requirements in the study of X-extensions should be traced back to a continuous selection theorem of E. Michael [33], which is also used in the related subject of continuous fields of Hilbert spaces (see 10.1.2, 10.8.6, 10.8.7 and 10.10.9 in ref. [21]).

From now on, throughout the rest of this paper it will be assumed that the compact metrizable space X has finite dimension.

If $K \subset \mathbf{R}^m$ is a compact subset, then given a covering by open subsets of $\mathbf{R}^m$ there is $\varepsilon > 0$ and a refinement consisting of cubes with edges, parallel to the coordinate axes, of length 2ε and centers in εZ^m. This, together with the fact that a compact metrizable space of dimension $\leqslant n$ can be imbedded in $\mathbf{R}^{2n+1}$ [Thm. V. 3 in ref. 29], easily yields the following useful fact, we shall record as:

2.1. REMARK. *If X has dimension $\leqslant n$, then every open covering of X has a refinement each open set of which intersects no more than $3^{2n+1}-1$ other open sets of it.*

2.2. DEFINITION. *For X a compact metrizable space, a map*

$$X \ni x \mapsto I_x \in I(A)$$

is called lower semicontinuous (abbreviated l.s.c) if for every convergent sequence $(x_n)_{n=1}^{\infty} \subset X, \lim_{n\to\infty} x_n = x_0$ we have

$$\bigcap_{n=1}^{\infty} I_{x_n} \subset I_{x_0}.$$

Denoting for $a \in A$ and $J \in I(A)$ by a/J the image of a in A/J, it is easily seen that the l.s.c. condition 2.2 is equivalent to the following: whenever $x_n \to x_0$ and $a \in A$, we have

$$\liminf_{n\to\infty} \|a/I_{x_n}\| \geqslant \|a/I_{x_0}\|$$

(use the fact that $\|a/\bigcap_n I_{x_n}\| = \sup_n \|a/I_{x_n}\|$ and consider subsequences).

2.3. DEFINITION. *An X-extension by A defined by the $*$-monomorphism τ, with exact ideal symbol $(I_x)_{x\in X}$ is called trivial if there is a unital $*$-homomorphism*

$$\mu\colon A \to C_{*s}(X, L(H))$$

such that $p \circ \mu = \tau$ and $\operatorname{Ker}(d_x \circ \mu) = I_x$ *for all $x \in X$, where $d_x\colon C_{*s}(X, L(H)) \to$ $\to L(H)$ is the map $d_x(f) = f(x)$.*

It is easily seen that $p \circ \mu = \tau$ implies

$$\operatorname{Ker}(d_x \circ \mu) \subset \operatorname{Ker}(p_x \circ \tau) = I_x$$

so, for homogeneous X-extensions by A, the condition $\operatorname{Ker}(d_x \circ \mu) = 0$ follows from the first condition.

Also, *for the existence of trivial X-extensions it is necessary that the ideal symbol be l.s.c.* Indeed, we have for $x_n \to x_0$, that $(d_{x_n} \circ \mu)(a)$ is strongly convergent to $(d_{x_0} \circ \mu)(a)$, so that

$$\liminf_{n\to\infty} \|(d_{x_n} \circ \mu)(a)\| \geqslant \|(d_{x_0} \circ \mu)(a)\|$$

which is equivalent to

$$\liminf_{n\to\infty} \|a/I_{x_n}\| \geqslant \|a/I_{x_0}\|,$$

establishing our assertion.

To prove that for X of finite dimension and l.s.c. ideal symbol there exist trivial extensions we need some preparations.

For the next Lemma A is unital and separable (as always), $E(A)$ is the state space of A and for $J \in I(A)$, $E(A/J)$ will be considered as a subset of $E(A)$.

2.4. LEMMA. *Suppose X has finite dimension and let $X \ni x \mapsto I_x \in I(A)$ be l.s.c. Then given a state φ of A such that $\varphi \mid I_{x_0} = 0$, there is a map $X \ni x \mapsto$ $\mapsto \omega_x \in E(A)$, continuous for the weak topology on $E(A)$, such that $\omega_x \mid I_x = 0$ for all $x \in X$ and $\omega_{x_0} = \varphi$.*

Proof. The idea is to use the selection theorem of E. Michael [33] for the set-valued map

$$X \ni x \mapsto E(A/I_x) \subset E(A).$$

To this end we give $E(A)$ the metric

$$d(f, g) = \sum_{n=1}^{\infty} 2^{-n} |f(a_n) - g(a_n)|$$

where $\|a_n\| \leqslant 1$ and $(a_n)_{n=1}^{\infty}$ is total in A. Clearly d induces the weak topology on $E(A)$ and $E(A)$ is a complete metric space since $E(A)$ is compact for the weak topology. Moreover the balls with respect to d are convex, so that their intersections with the $E(A/I_x)$ are convex and hence contractible.

Thus, the only thing still to be checked is the lower semicontinuity (in the sense of Michael) for $X \ni x \mapsto E(A/I_x) \subset E(A)$. The lower semicontinuity condition is

given $\varepsilon > 0$, $y \in X$ and $f \in E(A/I_y)$ there is
a neighborhood $V \subset X$ of y such that

$$E(A/I_x) \cap \{g \in E(A);\ d(f, g) < \varepsilon\} \neq \emptyset$$

for all $x \in V$.

This is easily seen to be equivalent for metrizable X with

whenever $x_n \to y$ and $f \in E(A/I_y)$, there are

$$f_n \in E(A/I_{x_n}) \text{ such that } f_n \to f \text{ weakly.}$$

Now, for this reformulation it is easily seen that it will be sufficient to prove it only for f in some subset of $E(A/I_y)$. Thus we may assume $f = k^{-1}(g_1 + \ldots + g_k)$ where $g_j \in E(A/I_y)$, $(j = 1, \ldots, k)$, and pure. But this makes a second reduction possible, namely we may assume f is pure. Then, considering π_n any representations of A with $\operatorname{Ker} \pi_n = I_{x_n}$, we have $f \mid \bigcap_n \operatorname{Ker} \pi_n = 0$, since $f \mid I_y = 0$ and

$$I_y \supset \bigcap_n I_{x_n} = \bigcap_n \operatorname{Ker} \pi_n.$$

Now f being pure, our assertion follows from (3.4.2.(ii) in [21]).

Thus the Lemma follows by applying the theorem of Michael. Q.E.D.

Let $C(X, E(A))$ denote the set of continuous maps from X to $E(A)$, $E(A)$ being endowed with the weak topology. We consider on $C(X, E(A))$ the topology given by the metric

$$\delta(F, G) = \sup_{x \in X} d(F(x), G(x)),$$

where d is the metric on $E(A)$ considered in the proof of Lemma 2.4. Further consider the closed subset $\Omega \subset C(X, E(A))$ defined by

$$\Omega = \{F \in C(X, E(A));\ F(x)|\, I_x = 0, \quad (\forall)\, x \in X\}.$$

2.5. LEMMA. *Suppose X has finite dimension and $X \ni x \mapsto I_x \in I(A)$ is l.s.c. Then there is a sequence $\{\omega_j\}_{j=1}^{\infty}$ of continuous maps $\omega_j \colon X \to E(A)$ such that*

$$\bigcap_{j=1}^{\infty} \operatorname{Ker} \omega_j(x) = I_x \textit{ for every } x \in X.$$

Proof. In view of Lemma 2.4, $\{\omega_j\}_{j=1}^{\infty}$ may be any dense sequence in Ω. Thus all we have to prove is that Ω is separable when the metric δ is given. But since Ω is a closed subset of $C(X, E(A))$ it is clearly sufficient to prove that $C(X, E(A))$ is separable. This can be easily seen as follows. The space X being compact and metrizable, fix a metric on X and consider $\{V_k^{(j)}\}_{k=1}^{n(j)}$ open coverings ($j \in \mathbf{N}$), by open balls of radius $1/j$. Let further $\{\varphi_k^{(j)}\}_{k=1}^{n(j)}$ be a partition of unit subordinated to $\{V_k^{(j)}\}_{k=1}^{n(j)}$ and $\Theta \subset E(A)$ a countable dense subset of $E(A)$. Then it is easily seen that the maps $F \in C(X, E(A))$ of the form

$$F(x) = \sum_{k=1}^{n(j)} \varphi_k^{(j)}(x)\, \theta_k,$$

(with $j \in \mathbf{N}$, $\theta_k \in \Theta$), form a countable dense subset of Ω. Q.E.D.

2.6. THEOREM. *For X of finite dimension and $X \ni x \mapsto I_x \in I(A)$ l.s.c., $\bigcap_{x \in X} I_x = 0$, there exists a trivial X-extension by A, with ideal symbol $X \in x \mapsto I_x \in I(A)$.*

Proof. Consider $\{\omega_j\}_{j=1}^{\infty}$ a sequence of $E(A)$-valued functions satisfying the conditions in Lemma 2.5, and where each term appears an infinite number of times. Let then $\pi_x^{(j)}$ denote the representation of A on $H_x^{(j)}$ with cyclic vector $\xi_x^{(j)}$ associated with $\omega_j(x)$ by the Gelfand-Naimark-Segal construction. Consider further

$$H_x = \bigoplus_{j=1}^{\infty} H_x^{(j)}\ , \quad \pi_x = \bigoplus_{j=1}^{\infty} \pi_x^{(j)}$$

and let

$$\Gamma_0 \subset \prod_{y \in X} H_x$$

be the set of those $(h_x)_{x \in X}$ such that

$$h_x = \bigoplus_{j=1}^{\infty} \left(\sum_{j=1}^{\infty} \varphi_{ij}(x) \pi_x^{(j)}(a_i) \right) \xi_x^{(j)}$$

for some $n \in \mathbf{N}$, $\varphi_{ij} \in C(X)$, $a_i \in A$, $(1 \leqslant i \leqslant n, j \in \mathbf{N})$, and where $\varphi_{ij} \neq 0$ only for a finite number of j. Clearly, Γ_0 is a vector subspace and since

$$\|h_x\|^2 = \sum_{j=1}^{\infty} \sum_{1 \leqslant p,\, q \leqslant n} \overline{\varphi_{pj}(x)} \varphi_{qj}(x)\, (\omega_j(x))(a_p^* a_q)$$

we infer that $X \ni x \mapsto \|h_x\| \in \mathbf{R}$ is continuous for $(h_x)_{x \in X} \in \Gamma_0$. Define now

$$\Gamma \subset \prod_{x \in X} H_x$$

as the set of those $(h_x)_{x \in X}$ such that for every $\varepsilon > 0$ there is $(h'_x)_{x \in X} \in \Gamma_0$ satisfying

$$\sup_{x \in X} \|h_x - h'_x\| < \varepsilon.$$

It is easy to check that $((H_x)_{x \in X}, \Gamma)$ is a continuous field of Hilbert spaces ([21], 10.1.2) which is also separable ([21],10.2.1). Moreover if $a \in A$ and $(h_x)_{x \in X} \in \Gamma$ then also $(\pi_x(a)h_x)_{x \in X} \in \Gamma$.

By (10.8.7 in ref. [21]) the continuous field of Hilbert spaces $((H_x)_{x \in X}, \Gamma)$ is trivial ([21], 10.1.4). Thus there are unitary maps U_x of H_x onto H such that the set of maps $X \ni x \mapsto U_x h_x \in H$, where $(h_x)_{x \in X}$ runs over Γ, coincides with the set of all continuous H-valued functions on X. Moreover, for $a \in A$ the function $\mu(a) \colon X \to L(H)$, $(\mu(a))(x) = U_x \pi_x(a)\, U_x^*$, has the property that $\mu(a) f \in C(X, H)$ for every $f \in C(X, H)$. Taking also $\mu(a^*)$ into account, this gives $\mu(a) \in C_{*s}(X, L(H))$. Then $\tau = p \circ \mu$ is a trivial X-extension by A with ideal symbol $(I_x)_{x \in X}$ as can be easily seen since $\operatorname{Ker} \pi_x = I_x$ and π_x is of infinite multiplicity for every $x \in X$. Q.E.D.

Our next aim is to prove the Weyl-von Neumann type theorem. This will also require several steps.

2.7. PROPOSITION. *Suppose X has finite dimension, let*

$$0 \to C_n(X, K(H)) \hookrightarrow B \xrightarrow{\sigma} A \to 0$$

be an exact X-extension by A with ideal symbol $X \ni x \mapsto I_x \in I(A)$ and let $\omega \in C(X, E(A))$ be such that $\omega(x) | I_x = 0$ for all $x \in X$. Then given $\varepsilon > 0$ and $V \subset H$, $1 \in W \subset B$ finite dimensional subspaces, there is $h \in C(X, H)$ such that

$$\|h(x)\| = 1, \quad h(x) \perp V, \quad (\forall)\, x \in X,$$

$$|(\omega(x))(\tau(b)) - \langle b(x)h(x), h(x)\rangle| \leqslant \varepsilon \|b\|, \quad (\forall)\ x \in X, \quad (\forall) b \in W$$

and the linear span of $\{h(x)\}_{x \in X}$ is finite dimensional.

Proof. By N we shall denote an integer $N \geqslant 3^{2n+1}$ where n is $\geqslant$ than the dimension of X. Let

$$\tau : A \to C_{*s}(X, L(H))/C_n(X, K(H))$$

be the $*$-monomorphism which defines our X-extension. Since $\tau \circ \sigma = p | B$ and $\pi \circ d_x = p_x \circ p$, we have

$$p_x \circ \tau \circ \sigma = p_x \circ (p | B) = \pi \circ (d_x | B).$$

It follows that

$$(\pi \circ d_x)(B) = (p_x \circ \tau)(A).$$

Also since

$$\operatorname{Ker}(p_x \circ \tau) = I_x,$$

there is

$$\omega'(x) \in E((p_x \circ \tau)(A)) = E((\pi \circ d_x)(B))$$

such that

$$\omega'(x) \circ p_x \circ \tau = \omega(x).$$

Considering now the state $\omega'(x) \circ \pi$ on $d_x(B) \subset L(H)$ and using (11.2.1 in [21]), it is easily seen that we can find a subspace $R_x \subset H$, $\dim R_x = N \dim W + 1$ such that $R_x \perp V$ and

$$|\langle d_x(b)\xi, \xi\rangle - (\omega'(x))((\pi \circ d_x)(b))| \leqslant \frac{\varepsilon}{2} \|b\|$$

for every $b \in W$ and $\xi \in R_x$, $\|\xi\| = 1$. This can be also written:

$$|\langle b(x)\xi, \xi\rangle - (\omega(x))(\sigma(b))| \leqslant \frac{\varepsilon}{2} \|b\|$$

for $b \in W$, $\xi \in R_x$, $\|\xi\| = 1$.

Consider also an open neighborhood G'_x of x, such that

$$\|b(y)\xi - b(x)\xi\| \leqslant \frac{\varepsilon}{4(N+1)} \|b\|$$

$$\|(\omega(y))(\sigma(b)) - (\omega(x))(\sigma(b))\| \leqslant \frac{\varepsilon}{4} \|b\|$$

whenever $y \in G'_x$, $b \in W$, $\xi \in R_x$, $\|\xi\| = 1$.

Since X has topological dimension $\leqslant n$, there is a refinement $(G_k)_{k=1}^q$, $G_k \subset G'_{x_k}$ of the open covering $(G'_x)_{x \in X}$ such that each G_k meets at most N other G_j's.

We shall now prove the existence of $\xi_k \in R_{x_k}$, $k = 1, \ldots, q$, such that

$$d_{x_j}(W)\xi_j \perp \xi_k \text{ whenever } G_j \cap G_k \neq \emptyset, \text{ and } \|\xi_k\| = 1.$$

Remark that this implies $\xi_j \perp \xi_k$ for $G_j \cap G_k \neq \emptyset$, since $1 \in W$.

The ξ_k's will be chosen by induction. For ξ_1 we may take any vector $\xi_1 \in R_{x_1}$, $\|\xi_1\|=1$. Suppose $\xi_1, \dots, \xi_j$ have been chosen, then consider $1 \leqslant i_1 < i_2 < \dots < i_m \leqslant j$ those indices for which $G_{i_s} \cap G_{j+1} \neq \emptyset$. Clearly $m \leqslant N$ by Remark 2.1. It follows that

$$\sum_{s=1}^{m} \dim (d_{x_{i_s}}(W)\xi_{i_s}) \leqslant N \dim W < \dim R_{x_{j+1}}$$

so we can find $\xi_{j+1} \in R_{x_{j+1}}$, $\|\xi_{j+1}\| = 1$ and such that

$$d_{x_{i_s}}(W)\,\xi_{i_s} \perp \xi_{j+1}, \quad (1 \leqslant s \leqslant m).$$

Consider now $\{\varphi_k\}_{k=1}^{q}$ a partition of unity subordinated to the covering $(G_k)_{k=1}^{q}$. Then we define

$$h(x) = \sum_{k=1}^{q} \varphi_k^{1/2}(x)\xi_k.$$

Since $\xi_j \perp \xi_k$ whenever $\varphi_k^{1/2}(x)\varphi_j^{1/2}(x) \neq 0$, it follows that $\|h(x)\| = 1$ for all $x \in X$. It is also obvious that $h(x) \perp V$ for all $x \in X$ and that the linear span of $\{h(x)\}_{x \in X}$ is finite-dimensional.

We have

$$\begin{aligned}
&|\langle b(x)h(x), h(x)\rangle - (\omega(x))(\sigma(b))| \leqslant \\
&\leqslant \sum_{G_k \cap G_j \neq \emptyset} \varphi_k^{1/2}(x)\varphi_j^{1/2}(x)|\langle (b(x) - b(x_k))\xi_k, \xi_j\rangle| + \\
&+ \sum_{k=1}^{q} \varphi_k(x)|\langle b(x_k)\xi_k, \xi_k\rangle - (\omega(x_k)(\sigma(b))| + \\
&+ \sum_{k=1}^{q} \varphi_k(x)|(\omega(x) - \omega(x_k))(\sigma(b))| \leqslant \\
&\leqslant \sum_{G_k \cap G_j \neq \emptyset} \varphi_k^{1/2}(x)\varphi_j^{1/2}(x)\frac{\varepsilon\|b\|}{4(N+1)} + \\
&+ \sum_{k=1}^{q} \varphi_k(x)\frac{\varepsilon\|b\|}{2} + \sum_{k=1}^{q} \varphi_k(x)\frac{\varepsilon\|b\|}{4} = \\
&= \frac{3\varepsilon}{4}\|b\| + \frac{\varepsilon\|b\|}{4(N+1)}\left(\sum_{k=1}^{q} \varphi_k^{1/2}(x)\right)^2 \leqslant \\
&\leqslant \frac{3\varepsilon}{4}\|b\| + \frac{\varepsilon\|b\|}{4(N+1)}(N+1)\sum_{k=1}^{q} \varphi_k(x) = \varepsilon\|b\|.
\end{aligned}$$

This ends the proof. Q.E.D.

For the next Proposition, let $M_n = L(\mathbf{C}^n)$ be the C^*-algebra of $n \times n$ matrices with the system of matrix units $(e_{ij})_{1 \leqslant i, j \leqslant n}$.

Let also $Cp(A, M_n)$ denote the set of completely positive unital maps from A to M_n endowed with the point-norm topology.

2.8. PROPOSITION. *Suppose X has finite dimension, let*

$$0 \to C_n(X, K(H)) \hookrightarrow B \xrightarrow{\sigma} A \to 0$$

be an exact X-extension by A with ideal symbol $X \ni x \mapsto I_x \in I(A)$ and let $\Psi: X \to \to Cp(X, M_n)$ be a continuous map such that $\Psi(x)| I_x = 0$ for all $x \in X$. Then given $\varepsilon > 0$ and $V \subset H$, $1 \in W \subset B$ finite-dimensional subspaces, there is $U: X \to L(\mathbf{C}^n, H)$ a norm-continuous map such that

$$U^*(x)U(x) = \mathrm{I}_{\mathbf{C}^n},\ U(x)\ (\mathbf{C}^n) \perp V, \qquad (\forall)\ x \in X,$$

$$\|U^*(x)b(x)U(x) - (\Psi(x))(\sigma(b))\| \leqslant \varepsilon \|b\|, \qquad (\forall)\ x \in X, \qquad (\forall)\ b \in W,$$

and the linear span of $\{U(x)(\mathbf{C}^n)\}_{x \in X}$ is finite-dimensional.

Proof. There is a natural isomorphism ([17, 4])

$$\Lambda: Cp(A, M_n) \to Cp(A \otimes M_n, \mathbf{C}) = E(A \otimes M_n)$$

given by

$$\Lambda(\Psi)(\sum_{i,j} a_{ij} \otimes e_{ij}) = \frac{1}{n} \sum_{i,j} \Psi_{ij}(a_{ij})$$

where Ψ_{ij} are the components of Ψ; i.e. $\Psi(a) = \sum_{i,j} \Psi_{ij}(a)e_{ij}$.

Consider the exact sequence

$$0 \to C_n(X, K(H)) \otimes M_n \to B \otimes M_n \xrightarrow{\sigma \otimes id_{M_n}} A \otimes M_n \to 0.$$

Identifying $C_n(X, K(H)) \otimes M_n$ and $C_n(X, K(H^n))$, this sequence can be viewed as an exact X-extension by $A \otimes M_n$ with ideal symbol $X \ni x \mapsto I_x \otimes M_n \in I(A \otimes M_n)$. Consider then $\omega = \Lambda(\Psi) \in E(A \otimes M_n)$ and apply Proposition 2.7. This gives a continuous function $h = (h_1, \ldots, h_n): X \to H^n$ such that

$$\|h(x)\| = 1, \quad h(x) \perp V \otimes M_n, \quad (\forall)\ x \in X,$$

$$|(\omega(x))((\sigma \otimes id_{M_n})(b)) - \langle b(x)h(x), h(x)\rangle| \leqslant \frac{\varepsilon^2}{16n^3}\|b\|, \qquad (\forall)\ x \in X, b \in W \otimes M_n$$

and the linear span of $\{h(x)\}_{x \in X}$ is finite-dimensional.

Let us define

$$S: X \to L(\mathbf{C}^n, H) \text{ by } S(x)e_j = n^{1/2}h_j(x)$$

where $\{e_1, \dots, e_n\}$ is the canonical orthonormal basis of $\mathbf{C}^n$. Then we have

$$S^*(x)S(x) = n \sum_{i,j} \langle h_j(x), h_i(x) \rangle e_{ij} = n \sum_{i,j} \langle (1 \otimes e_{ij})h(x), h(x) \rangle e_{ij}$$

so that

$$\| S^*(x)S(x) - 1_{M_n} \| \leqslant n \sum_{i,j} \left| \langle (1 \otimes e_{ij})h(x), h(x) \rangle - \frac{\delta_{ij}}{n} \right| =$$

$$= n \sum_{i,j} | \langle (1 \otimes e_{ij})h(x), h(x) \rangle - (\omega(x))(1 \otimes e_{ij})| \leqslant$$

$$\leqslant n \cdot n^2 \cdot \frac{\varepsilon^2}{16n^3} = \frac{\varepsilon^2}{16}.$$

Supposing $\varepsilon \leqslant 1$ (which means no loss of generality), we have

$$\| (S^*(x)S(x))s^{-1/2} - 1_{M_n} \| \leqslant$$

$$\leqslant \max \left\{ \left(1 - \left(1 + \frac{\varepsilon^2}{16} \right)^{-1/2}, \left(\left(1 - \frac{\varepsilon^2}{16} \right)^{-1/2} - 1 \right) \right\} \leqslant \frac{\varepsilon}{4},$$

so that

$$\| S(x) - S(x)(S^*(x)S(x))^{-1/2} \| \leqslant \|S(x)\| \, \| 1_{M_n} - (S^*(x)S(x))^{-1/2} \| \leqslant$$

$$\leqslant \frac{\varepsilon}{4} \left(1 + \frac{\varepsilon^2}{16} \right)^{1/2} \leqslant \frac{\varepsilon}{3}.$$

Finally, if

$$U(x) = S(x)(S^*(x)S(x))^{-1/2},$$

then $U(x)$ is an isometry and clearly depends continuously on $x \in X$. We have

$$\| U^*(x)b(x)U(x) - (\Psi(x))(\sigma(b)) \| \leqslant$$

$$\leqslant \|U(x) - S(x)\| \|b\| (1 + \|S(x)\|) + \| S^*(x)b(x)S(x) - (\Psi(x))(\sigma(b)) \| \leqslant$$

$$\leqslant \left(2 + \frac{\varepsilon}{4} \right) \frac{\varepsilon}{3} \|b\| + n \sum_{i,j} \left| \langle b(x)h_j(x), h_i(x) \rangle - \frac{1}{n} (\Psi_{ij}(x))(\sigma(b)) \right| =$$

$$= \left(\frac{2\varepsilon}{3} + \frac{\varepsilon^2}{12} \right) \|b\| + n \sum_{i,j} | \langle (b(x) \otimes e_{ij})h(x), h(x) \rangle - (\omega(x))((\sigma \otimes id_{M_n})(b \otimes e_{ij})) | \leqslant$$

$$\leqslant \left(\frac{2\varepsilon}{3} + \frac{\varepsilon^2}{12} + n \cdot n^2 \cdot \frac{\varepsilon^2}{16n^3} \right) \|b\| \leqslant \varepsilon \|b\|.$$

Also, since $U(x)(\mathbf{C}^n) = S(x)(\mathbf{C}^n)$, it is obvious that $U(x)(\mathbf{C}^n) \perp V$ and the linear span of $\{U(x)(\mathbf{C}^n)\}_{x\in X}$ is finite-dimensional. Q.E.D.

Let $X \ni x \mapsto I_x \subset I(A)$ be an l.s.c. map with $\bigcap_{x\in X} I_x = 0$ and let

$$0 \to C_n(X, K(H_1)) \hookrightarrow B_1 \xrightarrow{\sigma_1} A \to 0$$

be a trivial X-extension by A with ideal symbol $X \ni x \mapsto I_x \in I(A)$. Let also $\mu_1 \colon A \to B_1 \subset C_{*s}(X, L(H))$ be the $*$-monomorphism implementing the triviality of this X-extension by A (i.e. $\sigma_1 \circ \mu_1 = \mathrm{id}_A$ and $\mathrm{Ker}\,(d_x \circ \mu) = I_x$, $(\forall)\ x \in X$). Consider also

$$0 \to C_n(X, K(H)) \hookrightarrow B \xrightarrow{\sigma} A \to 0$$

an arbitrary X-extension by A with ideal symbol $X \ni x \mapsto I_x \in I(A)$.

With these notations, we have

2.9. PROPOSITION. *There is* $S \in C_{*s}(X, L(H_1, H))$ *such that*

$$S^*(x)S(x) = I, \qquad (\forall)\ x \in X,$$
$$S\mu_1(\sigma(b)) - bS \in C_n(X, K(H_1, H)), \qquad (\forall)\ b \in B.$$

Proof. There is an increasing sequence $0 = A_0 \leqslant A_1 \leqslant A_2 \leqslant \ldots$, $\|A_n\| \leqslant 1$, of elements of $C_n(X, K(H_1))$ which are *constant on* X and *of finite rank* such that

$$\lim_{j\to\infty} \|A_j k - k\| = 0, \qquad (\forall)\ k \in C_n(X, K(H_1))$$
$$\lim_{j\to\infty} \|A_j b - bA_j\| = 0, \qquad (\forall)\ b \in B_1.$$

Since $C_n(X, K(H_1))$ has an approximate unit which is an increasing sequence of constant finite rank elements, this follows from [4, remarks after the proof of Thm. 1]. Consider also $\{b_j\}_{j=1}^{\infty}$, $b_j = b_j^*$, a total sequence in B. Then replacing $\{A_n\}_{n=0}^{\infty}$ by some subsequence, we may suppose that

$$\|[\mu_1(\sigma(b_k)), (A_j - A_{j-1})^{1/2}]\| \leqslant 2^{-j} \text{ for } 1 \leqslant k \leqslant j.$$

Consider further $P_j \in C_n(X, K(H_1))$ constant projections such that

$$A_j P_j = P_j A_j = A_j.$$

Using Proposition 2.8 several times one can easily construct norm-continuous maps $X \ni x \mapsto U_j(x) \in L(H_1, H)$ and finite rank projections $R_j \in L(H)$, $R_1 \leqslant R_2 \leqslant \ldots$, $\ldots$, such that

(*i*) $U_j^* U_j = P_j$, $(j \in \mathbf{N})$
(*ii*) $U_j(x)(H_1) \subset (R_{j+1} - R_j)(H)$, $(\forall)\ x \in X$
(*iii*) $\|(I - R_{j+1}) b_k(x) R_j\| \leqslant 2^{-j}$, $(\forall)\ x \in X$ and $1 \leqslant k \leqslant j$
(*iv*) $\|P_j \mu_1(\sigma(b_k)) P_j - U_j^* b_k U_j\| \leqslant 2^{-j}$ for $1 \leqslant k \leqslant j$.

The sum

$$\sum_{j=1}^{\infty} U_j(x)(A_j - A_{j-1})^{1/2} = S(x)$$

is easily seen to be strongly convergent and $S^*(x)S(x) = I_{H_1}$. Also since the A_j's are constant and because of (*ii*) it is easily checked that the sum defining $S(x)$ is uniformly $*$-strongly convergent on X, thus defining an element $S \in C_{*s}(X, L(H_1, H))$.

Using (*ii*), (*iii*) and $b_k = b_k^*$ we have

$$\sum_{i \neq j} \| U_i^* b_k U_j \| = 2 \sum_{i > j} \| U_i^* b_k U_j \| \leqslant$$
$$\leqslant 2 \sum_{1 \leqslant i,\, j \leqslant k} \| U_i^* b_k U_j \| + 2 \sum_{i > j \geqslant 1} 2^{-(i-1)}$$
$$\leqslant 2 \sum_{1 \leqslant i,\, j \leqslant k} \| U_i^* b_k U_j \| + 2 \sum_{i=1}^{\infty} i \cdot 2^{-(i-1)} < +\infty.$$

Also using (*iv*) we have

$$\sum_{j=1}^{\infty} \| (A_j - A_{j-1})^{1/2} (U_j^* b_k U_j - \mu_1(\sigma(b_k)))(A_j - A_{j-1})^{1/2} \| < +\infty$$

and using the inequalities for $\| [(A_j - A_{j-1})^{1/2}, \mu_1(\sigma(b_k))] \|$ we have

$$\sum_{j=1}^{\infty} \| \mu_1(\sigma(b_k))(A_j - A_{j-1}) - (A_j - A_{j-1})^{1/2} \mu_1(\sigma(b_k))(A_j - A_{j-1})^{1/2} \| < +\infty.$$

Thus we have

$$\sum_{j=1}^{\infty} \| (A_j - A_{j-1})^{1/2} U_j^* b_k U_j (A_j - A_{j-1})^{1/2} - \mu_1(\sigma(b_k))(A_j - A_{j-1}) \| +$$
$$+ \sum_{i \neq j} \| (A_i - A_{i-1})^{1/2} U_i^* b_k U_j (A_j - A_{j-1})^{1/2} \| < +\infty.$$

This proves that

$$S^* b_k S - \mu_1(\sigma(b_k)) \in C_n(X, K(H_1)) \text{ for all } k \in \mathbf{N}.$$

Since $\{b_k\}_{k=1}^{\infty}$ is total in B we infer

$$S^* b S - \mu_1(\sigma(b)) \in C_n(X, K(H_1)) \text{ for all } b \in B.$$

It follows that:

$$(bS - S\mu_1(\sigma(b)))^*(bS - S\mu_1(\sigma(b))) =$$
$$= (S^* b^* b S - \mu_1(\sigma(b^* b))) +$$
$$+ \mu_1(\sigma(b^*))(\mu_1(\sigma(b)) - S^* b S) +$$
$$+ (\mu_1(\sigma(b^*)) - S^* b S)\mu_1(\sigma(b)) \in C_n(X, K(H_1)).$$

But this is equivalent with

$$bS - S\mu_1(\sigma(b)) \in C_n(X, K(H_1, H)). \qquad \text{Q.E.D.}$$

2.10. THEOREM. *Suppose X has finite dimension and let $X \ni x \mapsto I_x \in I(A)$ be an exact l.s.c. ideal symbol. Then the trivial X-extension by A with ideal symbol $X \ni x \leftarrow I_x \in I(A)$ are all equivalent and their class is a neutral element in the semigroup* $\mathrm{Ext}(X; A, (I_x)_{x \in X})$.

Proof. Let $\tau, \tau_1 : A \to C_{*s}(X, L(H))/C_n(X, K(H))$ be $*$-monomorphisms defining X-extensions by A with ideal symbol $X \ni x \to I_x \in I(A)$. Then assuming that τ_1 defines a trivial X-extension by A with the given ideal symbol, we shall prove that $[\tau \oplus \tau_1] = [\tau]$. This will show that $[\tau_1]$ is a neutral element for $\mathrm{Ext}(X; A, (I_x)_{x \in X})$ and since two neutral elements must coincide, also the other assertion of the theorem will follow.

Consider the exact sequences

$$0 \to C_n(X, K(H) \hookrightarrow B \to A \to 0$$

$$0 \to C_n(X, K(H)) \hookrightarrow B_1 \xrightarrow{\sigma_1} A \to 0$$

corresponding to the X-extensions by A, defined *via* τ and τ_1. Denoting by H_1 the Hilbert space $H \oplus H \oplus \ldots$, by $\mu_2 : A \to C_{*s}(X, L(H_1))$ the $*$-monomorphism

$$(\mu_2(a))(x) = (\mu_1(a)(x)) \oplus (\mu_1(a)(x)) \oplus \ldots,$$

and by B_2 the C^*-algebra

$$B_2 = \mu_2(A) + C_n(X, L(H_1)) \subset C_{*s}(X, L(H_1)),$$

we obtain an exact sequence

$$0 \to C_n(X, K(H_1)) \to B_2 \to A \to 0$$

defining a trivial X-extension by A with ideal symbol $X \ni x \mapsto I_x \in I(A)$.

By Proposition 2.9 there is $S \in C_{*s}(X, L(H_1, H))$ such that

$$S\mu_2(\sigma(b)) - bS \in C_n(X, K(H_1, H)) \quad \text{for all} \quad b \in B.$$

Denote by $V \in C_{*s}(X, L(H_1))$ the constant isometry

$$V(x)(h_1 \oplus h_2 \oplus \ldots) = 0 \oplus h_1 \oplus h_2 \oplus \ldots$$

and by $P \in C_{*s}(X, L(H_1, H))$ the constant co-isometry

$$P(x)(h_1 \oplus h_2 \oplus \ldots) = h_1.$$

Clearly V commutes with $\mu_2(A)$ and hence with B_2 modulo $C_n(X, K(H_1))$. Similarly, P intertwines μ_2 and μ_1. Consider then $U(x)\colon H \to H \oplus H$ defined by

$$U(x)(h) = ((I - S(x)S^*(x))h + S(x)V^*(x)S^*(x)h) \oplus P(x)S^*(x)h.$$

Then U is unitary, $U \in C_{*s}(X, L(H, H \oplus H))$ and $\tilde{\alpha}(U)\circ\tau = \tau \oplus \tau_1$. Q.E.D.

§ 3.

Beginning with this section we shall consider only homogeneous X-extensions by A. Assuming that A is nuclear, we shall apply the Choi-Effros completely positive lifting theorem ([16], see also [4, 46]) to prove that $\mathrm{Ext}(X, A)$ for finite-dimensional X is a group. Using this fact we shall also prove that every homogeneous X-extension by A is equivalent to the one for which

$$C_n(X, K(H)) \subset B \subset C_n(X, L(H)).$$

3.1. LEMMA. *Let $\Psi\colon A \to C_{*s}(X, L(H))$ be a completely positive map. Then there exists a separable Hilbert space $H_1 \supset H$ and a unital $*$-monomorphism μ: $A \to C_{*s}(X, L(H_1))$ such that*

$$(\Psi(a))(x) = P(\mu(a))(x)\,|\,H \text{ for every } a \in A,\ x \in X$$

(P denotes the orthogonal projection of H_1 onto H).

Proof. For each $x \in X$ let $\Psi_x\colon A \to L(H)$ denote the completely positive map

$$\Psi_x(a) = (\Psi(a))(x).$$

Let $\mu_x'\colon A \to L(H_x)$, $H_x \supset H$, be the Stinespring minimal dilation of Ψ_x. Let further $\mu''\colon A \to L(H_2)$ be a unital $*$-monomorphism, where H_2 is separable and infinite-dimensional. Consider $H_x' = H_x \oplus H_2$, P_x the orthogonal projection of H_x' onto $H = H \oplus 0 \subset H_x \oplus H_2$ and let $\tilde{\mu}_x = \mu_x' \oplus \mu''$. Obviously

$$\Psi_x(a) = P_x\tilde{\mu}_x(a)|H.$$

Let $\Gamma \subset \prod_{x\in X}(H_x' \ominus H)$ be the uniform closure of the linear span of the elements of the form $((I - P_x)\tilde{\mu}_x(a)h(x))_{x\in X}$ where $a \in A$, $h \in C(X, H \oplus H_2)$. Then if

$$h_i = h_i' \oplus h_i'' \in C_n(X, H \oplus H_2),$$

we have

$$\left\| \sum_{i=1}^{n} (I - P_x)\tilde{\mu}_x(a_i)h_i(x) \right\|^2 = \left\| \sum_{i=1}^{n} \mu''(a_i)h_i'' \right\|^2 + \sum_{1\leqslant i,j\leqslant n} \langle \Psi_x(a_j^*a_i)h_i'(x), h_j'(x)\rangle -$$

$$- \sum_{1\leqslant i,j\leqslant n} \langle \Psi_x(a_i)h_i'(x), \Psi_x(a_j)h_j'(x)\rangle,$$

which is clearly a continuous function of $x \in X$. It is easy to check now that $((H'_x \ominus H)_{x \in X}, \Gamma)$ is a continuous field of Hilbert spaces (10.1.1 in [21]). Since X is finite-dimensional and this field is separable and each $H'_x \ominus H$ is separable infinite-dimensional, it follows by ([21], 10.8.7) that we have a trivial field. Hence there are unitary operators $U_x \colon H'_x \ominus H \to H_3$ such that the set of functions $X \ni x \mapsto U_x h_x \in H_3$ where $(h_x)_{x \in X}$ runs over $C(X, H'_x \ominus H)$, is just the set of all continuous H_3-valued functions $C(X, H_3)$. Defining $H_1 = H \oplus H_3$, $V_x \colon H \oplus (H'_x \ominus H) \to H \oplus H_3$, $V_x = I_H \oplus U_x$, and $(\mu(a))(x) = V_x \tilde{\mu}_x(a) V_x^*$, we shall see that

$$X \ni x \mapsto (\mu(a))(x) \in L(H_1)$$

has the desired properties. Indeed, since $\tilde{\mu}(a)$ maps $C(X, H) \oplus \Gamma$ into itself, it follows that $\mu(a)$ maps $C(X, H_1)$ into itself which entails the strong continuity of $X \in x \mapsto (\mu(a))(x)$. Also the dilation property of μ is quite obvious. Q.E.D.

3.2. THEOREM. *Suppose A is nuclear and X finite-dimensional. Then* $\mathrm{Ext}(X, A)$ *is a group.*

Proof. The proof is the same as that outlined in ([3]), only one must use Lemma 3.1 instead of the Stinespring dilation theorem.

Indeed, let

$$\tau \colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$$

define a homogenous X-extension by A. By the Choi-Effros theorem there is a completely positive map

$$\Psi \colon A \to C_{*s}(X, L(H))$$

such that $p \circ \Psi = \tau$. Using Lemma 3.1 for Ψ we get

$$\mu \colon A \to C_{*s}(X, L(H_1)),\ H_1 \supset H,$$

dilating Ψ. Let Φ denote the completely positive map

$$\Phi \colon A \to C_{*s}(X, L(H_1 \ominus H))$$

which is the compression of μ to $H_1 \ominus H$. Then $[(p \circ \Phi) \oplus \tau_0]$, where τ_0 is any trivial homogeneous X-extension by A, will be an inverse for $[\tau]$. Q.E.D.

Since $\mathrm{Ext}(X, A)$ is a group, it is time to mention that keeping X fixed we get a contravariant functor from the category of separable nuclear C^*-algebras with unit, the morphisms being the unit-preserving $*$-homomorphisms to the category of abelian groups. This depends in fact on Thm. 2.10. For $\zeta \colon A \to B$ a unit-preserving $*$-homomorphism, $\zeta_* \colon \mathrm{Ext}(X, B) \to \mathrm{Ext}(X, A)$ is defined by

$$\zeta_*[\tau] = [(\tau \circ \zeta) \oplus \tau_0]$$

where τ_0 is any trivial homogeneous X-extension by A.

3.3. THEOREM. *Suppose X is finite-dimensional and A nuclear, and let*

$$\tau \colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$$

define a homogeneous X-extension by A. Then there is

$$\tau_1 \colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$$

such that $[\tau] = [\tau_1]$ *and*

$$\tau_1(A) \subset C_n(X, L(H))/C_n(X, K(H)) \subset C_{*s}(X, L(H))/C_n(X, K(H)).$$

Proof. Consider a completely positive lifting

$$\Phi \colon A \to C_{*s}(X, L(H))$$

for τ and consider also

$$\Psi \colon A \to C_{*s}(X, L(H'))$$

a completely positive lifting for some inverse of $[\tau]$ so that there is a unital$*$-homomorphism $\rho \colon A \to C_{*s}(X, L(H \oplus H'))$ such that

$$\rho(a) - \Phi(a) \oplus \Psi(a) \in C_n(X, K(H \oplus H'))$$

for every $a \in A$. Let P and P' be the projections of $H \oplus H'$ onto H and respectively H'. Consider

$$\tilde{\Phi} \colon A \to C_{*s}(X, L(H \oplus (H \oplus H') \oplus (H \oplus H') \oplus \ldots))$$

defined by

$$\tilde{\Phi}(a) = \Phi(a) \oplus \rho(a) \oplus \rho(a) \oplus \ldots .$$

By Thm. 2.10. we have $[\tau] = [p \circ \tilde{\Phi}]$.
Consider also

$$\tilde{\rho} \colon A \to C_{*s}(X, L((H \oplus H') \oplus (H \oplus H') \oplus \ldots))$$

defined by $\tilde{\rho} = \rho \oplus \rho \oplus \ldots$, and let

$$G \in C_{*s}(X, L(H \oplus H') \oplus (H \oplus H') \oplus \ldots, H \oplus (H \oplus H') \oplus (H \oplus H') \oplus \ldots)$$

be the constant unitary operator such that

$$(G(x))(h_1 \oplus h_1') \oplus (h_2 \oplus h_2') \oplus \ldots) = h_1 \oplus (h_2 \oplus h_1') \oplus (h_3 \oplus h_2') \oplus \ldots$$

The map

$$\eta \colon A \to C_{*s}(X, L((H \oplus H') \oplus (H \oplus H') \oplus \ldots)$$

defined by

$$(\eta(a))(x) = (\tilde{\rho}(a))(x) - (P(\rho(a))(x)\,P' + P'(\rho(a))(x)\,P) \oplus$$
$$\oplus\, (P(\rho(a))\,(x)P' + P'(\rho(a))(x)P) \oplus \ldots +$$
$$+\, G^*(x)[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P) \oplus$$
$$\oplus\, (P(\rho(a))(x)P' + P'(\rho(a))(x)P \oplus \ldots]\, G(x)$$

is such that $[p \circ \eta] = [\tau]$. Indeed,

$$(\eta(a))(x) - G^*(x)\,(\tilde{\Phi}(a))(x)G(x) =$$
$$= (\tilde{\rho}(a))(x) - [(P(\rho(a))(x)\,P' + P'(\rho(a))(x)P) \oplus \ldots] +$$
$$+\, G^*(x)[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P) \oplus \ldots]\, G(x) -$$
$$-\, G^*(x)\,[(\Phi(a))(x) \oplus (P(\rho(a))(x)P + P'(\rho(a))(x)P') \oplus \ldots]\, G(x) -$$
$$-\, G^*(x)\,[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P) \oplus \ldots]\, G(x) =$$
$$= (P(\rho(a))(x)P + P'(\rho(a))(x)P') \oplus (P(\rho(a))(x)P + P'(\rho(a))(x)P') \oplus \ldots -$$
$$-\, \Phi(x) + P'(\rho(a))(x)P') \oplus (P(\rho(a))(x)P + P'(\rho(a))(x)P' \oplus \ldots) =$$
$$= (P(\rho(a))(x)P - \Phi(x)) \oplus 0 \oplus 0 \ldots,$$

so that clearly

$$\eta(a) - G^*\tilde{\Phi}(a)G \in C_n(X, K(H \oplus H') \oplus (H \oplus H') \oplus \ldots)).$$

Consider a $*$-monomorphism

$$\rho_0 \colon A \to C_{*s}(X, L(H))$$

which is *constant* ($\rho_0(a)$ is constant for each $a \in A$) and such that

$$\rho_0(A) \cap C_n(X, K(H)) = 0.$$

Clearly, $[p \circ \rho_0] = 0$ and $\rho_0(A) \subset C_n(X, L(H))$. By Thm. 2.10, there is a unitary $U \in C_{*s}(X, L(H \oplus H', H))$ such that

$$U\rho(a)U^* - \rho_0(a) \in C_n(X, K(H)) \text{ for every } a \in A.$$

Consider also

$$\tilde{U} \in C_{*s}(X, L((H \oplus H') \oplus (H \oplus H') \oplus \ldots, H \oplus H \oplus \ldots))$$

defined by

$$\tilde{U}(x) = U(x) \oplus U(x) \oplus \ldots .$$

To prove the theorem it will be sufficient to show that

$$\tilde{U}\eta(a)\tilde{U}^* \in C_n(X, L(H \oplus H \oplus \ldots)).$$

We have

$$\tilde{U}(x)(\eta(a))(x)\tilde{U}^*(x) = \tilde{U}(x)(\tilde{\rho}(a))(x)\,\tilde{U}^*(x) -$$

$$- \tilde{U}(x)\,[(P(\rho(a))(x)\,(P') + P'(\rho(a))(x)P) \oplus \ldots]\,\tilde{U}^*(x) +$$

$$+ \tilde{U}(x)G^*[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P) \oplus \ldots]\,G\,\tilde{U}^*(x) =$$

$$= U(x)(\rho(a))(x)U^*(x) \oplus U(x)(\rho(a))(x)U^*(x) \oplus \ldots -$$

$$- [U(x)(P(\rho(a))(x)\,P' + P'(\rho(a))(x)\,P)U^*(x) \oplus \ldots] +$$

$$+ \tilde{U}(x)G^*[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P \oplus \ldots]\,G\,\tilde{U}^*(x).$$

Since

$$U\rho(a)U^* \in C_n(X, L(H)),$$

it is clear that the first term is a norm-continuous function of x. Also,

$$\rho(a) - \Phi(a) \oplus \Psi(a) \in C_n(X, K(H \oplus H'))$$

implies that

$$P\rho(a)P' + P'\rho(a)P \in C_n(X, K(H \oplus H')).$$

Since U is $*$-strongly continuous it follows also

$$U(P\rho(a)P' + P'\rho(a)P)U^* \in C_n(X, K(H))$$

so that also the second term in the expression of $\tilde{U}\eta(a)\tilde{U}^*$ is norm continuous.

For the third term, let us first make some computations:

$$\tilde{U}(x)G^*[0 \oplus (P(\rho(a))\,(x)P' + P'(\rho(a))(x)P) \oplus \ldots]\,G\tilde{U}^*(x)(h_1 \oplus h_2 \oplus \ldots) =$$

$$= \tilde{U}(x)G^*[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P) \oplus \ldots] \times$$

$$\times\,(PU^*(x)h_1 \oplus (PU^*(x)h_2 + P'U^*(x)h_1) \oplus (PU^*(x)h_3 + P'U^*(x)h_2) \oplus \ldots) =$$

$$= \tilde{U}(x)G^*[0 \oplus (P(\rho(a))(x)P'U^*(x)h_1 + P'(\rho(a))(x)PU^*(x)h_2) \oplus$$

$$\oplus\,(P(\rho(a))(x)P'U^*(x)h_2 + P'(\rho(a))(x)PU^*(x)h_3) \oplus \ldots] =$$

$$= U(x)P'(\rho(a))(x)PU^*(x)h_2 \oplus (U(x)P(\rho(a))(x)P'U^*(x)h_1 +$$

$$+ U(x)P'(\rho(a))(x)PU^*(x)h_3) \oplus \ldots,$$

hence

$$\tilde{U}(x)G^*[0 \oplus (P(\rho(a))(x)P' + P'(\rho(a))(x)P)\oplus \ldots]\, G\tilde{U}^*(x) =$$

$$= (U(x)P'(\rho(a))(x)PU^*(x) \ \oplus U(x)P'(\rho(a))(x)PU^*(x) \oplus \ldots) \circ S^* +$$

$$+ (U(x)P(\rho(a))(x)P'U^*(x) \oplus U(x)P(\rho(a))(x)P'U^*(x) \oplus \ldots) \circ S$$

where $S \in L(H \oplus H \oplus \ldots)$ is the shift $S(h_1 \oplus h_2 \oplus \ldots) = 0 \oplus h_1 \oplus h_2 \oplus \ldots$.

Since we have seen that $U(x)P'(\rho(a))(x)PU^*(x)$ and $U(x)P(\rho(a))(x)P'U^*(x)$ are norm-continuous functions of $x \in X$, this ends the proof. Q.E.D.

Note that for τ defining a homogeneous X-extension by A each $p_x \circ \tau$ defines an extension of $K(H)$ by A and denoting by $\mathrm{Ext}(A)$ the Brown-Douglas-Fillmore Ext for A, the preceding theorem implies the following corollary:

3.4. COROLLARY. *Suppose X is finite-dimensional, A nuclear and $\tau: A \to C_{*s}(X, L(H)) / C_n(X, K(H))$ defines a homogeneous X-extension by A. Then the map $X \ni x \mapsto [p_x \circ \tau] \in \mathrm{Ext}(A)$ is continuous.*

For what follows we shall also *define* $\mathrm{Ext}(X, x_0; A)$ *where* (X, x_0) *is a pointed compact metrizable space, as the set of those* $[\tau] \in \mathrm{Ext}(X, A)$ *for which* $[p_{x_0} \circ \tau] = 0$. Clearly this is a semigroup and, if X is finite-dimensional and A nuclear, it is a group.

§ 4.

This section is devoted to the proof of the following theorem.

4.1. THEOREM. *Let A be a nuclear C^*-algebra with unit, $J \subset A$ a proper ($1 \notin J$) closed two-sided ideal and (X, x_0) a pointed finite-dimensional metrizable compact space. Consider $\tilde{J} = J + \mathbf{C} \cdot 1_A$ and $i: \tilde{J} \to A$, $q: A \to A/J$ the canonical $*$-homomorphisms. Then the sequence*

$\mathrm{Ext}\ (X, x_0; A/J) \xrightarrow{q_*} \mathrm{Ext}\ (X, x_0; A) \xrightarrow{i_*} \mathrm{Ext}\ (X, x_0; \tilde{J})$ *is exact.*

The proof is quite long and will be carried out through a sequence of lemmas.

First some remarks.

Since A is nuclear, A/J and $\tilde{J}$ are nuclear [48] so that the considered Ext's are groups.

Remark also that the non-pointed version of Thm. 4.1, trivially implied by Thm. 4.1, implies in fact Thm. 4.1. Than can be seen as follows. Since $i_* \circ q_* = 0$ is quite obvious in both cases, we have only to prove that $\mathrm{Ker}\ i_* \subset \mathrm{Im}\ q_*$ in the pointed case follows from the non-pointed case. Let $\alpha: \{x_0\} \to X$ be inclusion, $\beta: X \to \{x_0\}$ the constant map and $[\tau] \in \mathrm{Ext}\ (X, A)$ such that $\alpha^*[\tau] = 0$ and $i_*[\tau] = 0$.

Assuming the non-pointed version of Thm. 4.1 holds, there is $[\sigma] \in \text{Ext}\,(X, A/J)$ such that $q_*[\sigma] = [\tau]$. But then

$$[\sigma] - (\beta^* \circ \alpha^*)\,[\delta] \in \text{Ext}\,(X, x_0; A/J)$$

and

$$\begin{aligned} q_*([\sigma] - (\beta^* \circ \alpha^*)[\sigma]) &= [\tau] - q_*((\beta^* \circ \alpha^*)\,[\sigma]) = \\ &= [\tau] - (\beta^* \circ \alpha^*)\, q_*[\sigma] = \\ &= [\tau] - (\beta^* \circ \alpha^*)\,[\tau] = [\tau] \end{aligned}$$

which is the desired result.

Thus let

$$\tau\colon A \to C_{*s}(X, L(H))/C_n(X, K(H))$$

be a $*$-monomorphism defining a homogeneous X-extension by A, such that $i_*[\tau] = 0$. All we must prove is the existence of $[\sigma] \in \text{Ext}\,(X, A/J)$ such that $q_*[\sigma] = [\tau]$, and this will be achieved in the remaining part of this section.

Since A is nuclear there is a unital completely positive map $\Psi\colon A \to C_{*s}(X, L(H))$ such that $p \circ \Psi = \tau$ ([16]). Moreover, since i is injective, $i_*[\tau] = [\tau \circ i]$ so that using Thm. 2.10 and replacing $[\tau]$ by some equivalent homogeneous X-extension we may assume there is a *constant* $*$-homomorphism implementing the triviality of $[\tau \circ i]$, i.e. there is a constant $*$-monomorphism

$$\rho_0\colon \tilde{J} \to C_{*s}(X,\ L(H))$$

such that

$$\rho_0(a) - \Psi(a) \in C_n(X, K(H)) \text{ for all } a \in \tilde{J} \subset A.$$

Consider also $\rho\colon A \to C_{*s}(X, L(H))$ the constant, possibly non-unital, $*$-homomorphism generated by $(\rho_0 \mid J)$ with the same null-space as $(\rho_0 \mid J)$ ([21], 2.10.3).

Let

$$0 \leqslant u_1 \leqslant u_2 \leqslant \dots,\ \|u_j\| \leqslant 1,$$

be an approximate unit of J such that

$$u_{j+1}u_j = u_j, \qquad (j \in \mathbf{N}).$$

Consider $E_j \in C_{*s}(X, L(H))$ the constant element which is the spectral projection of $\rho(u_j)$ for the set $\{1\}$. Since $\rho(u_{j+1})\,\rho(u_j) = \rho(u_j)$ we infer $E_{j+1}\rho(u_j) = \rho(u_j)$. Also clearly $\rho(u_j)E_j = E_j$ and $E_j \leqslant \rho(u_j) \leqslant E_{j+1}$.

Let now $\{a_j\}_{j\in\mathbf{N}} \subset J$, $\{b_j\}_{j\in\mathbf{N}} \subset A$ be total sequences of hermitian elements of J and respectively A.

Since

$$\|\rho(a_k)\,(I - E_j)\| \leqslant \|\rho(a_k(1 - u_{j-1})\| \leqslant \|\,a_k(1 - u_{j-1})\|$$

and

$$\|(I - E_j)\rho(b_k)E_i\| \leqslant \|\,I - E_j)\,\rho(b_k u_i)\| \leqslant \|(1 - u_{j-1})\,(b_k u_i)\|$$

we may replace $\{u_j\}_{j\in \mathbf{N}}$ by some subsequence so that

(1) $$\|\rho(a_k) - \rho(a_k) E_j\| \leqslant 2^{-j} \quad \text{for} \quad 1 \leqslant k \leqslant j,$$

(2) $$\|(I - E_{j+1}) \rho(b_k) E_j\| \leqslant 2^{-j} \quad \text{for} \quad 1 \leqslant k \leqslant j.$$

Also it is clear that if E is the strong limit of the constant projections E_j, then $(I - E)$ is the orthogonal projection onto the null-space of ρ, in particular $(I-E)\rho(A) = 0$.

Let also
$$P_j = E_j - E_{j-1}, (E_0 = 0),$$
and consider
$$Y_k = \rho(b_k) - \sum_{j\geqslant 1} (I - E_{j+1})\rho(b_k) P_j - \sum_{j\geqslant 1} P_j\rho(b_k)(I - E_{j+1}).$$

4.2. Lemma. *Let* $0 \leqslant Q_j \leqslant P_j$, $Q_j \in C_n(X, K(H))$. *Then for*
$$Q = \sum_{j\geqslant 1} Q_j \in C_{*s}(X(L(H))$$
we have $(k \in \mathbf{N})$:
$$\rho(a_k)Q \in C_n(X, K(H)) \text{ and } (\rho(b_k) - Y_k)Q \in C_n(X, K(H))$$

Proof. Since
$$\sum_{j\geqslant 1} \|\rho(a_k)Q_j\| \leqslant \sum_{j=1}^{k} \|\rho(a_k)Q_j\| + \sum_{j>k} \|\rho(a_k)(I - E_{j-1})\| \leqslant$$
$$\leqslant \sum_{j=1}^{k} \|\rho(a_k)Q_j\| + \sum_{j>k} 2^{-j} < +\infty,$$
it follows that $\sum_{j\geqslant 1} \rho(a_k)Q_j$ is norm convergent and hence
$$\rho(a_k)Q = \sum_{j\geqslant 1} \rho(a_k)Q_j \in C_n(X, K(H)).$$
A similar argument gives also
$$(\rho(b_k) - Y_k)\, Q \in C_n(X, K(H))$$
since:
$$\sum_{j\geqslant 1} \|(\rho(b_k) - Y_k)Q_j\| \leqslant$$
$$\leqslant \sum_{j\geqslant 1} \|\sum_{i\geqslant 1} (I - E_{i+1}) \rho(b_k)P_iQ_j\| + \sum_{j\geqslant 1} \|\sum_{i\geqslant 1} P_i\rho(b_k)(I - E_{i+1})\, Q_j\| =$$
$$= \sum_{j\geqslant 1} \|(I - E_{j+1})\, \rho(b_k)Q_j\| + \sum_{j\geqslant 1} \|\sum_{i=1}^{j-2} P_i\rho(b_k)Q_j\| \leqslant$$
$$\leqslant \sum_{j\geqslant 1} \|(I - E_{j+1})\, \rho(b_k)E_j\| + \sum_{j\geqslant 1} \|E_{j-2}\, \rho(b_k)\, (I - E_{j-1})\| < +\infty.$$
Q.E.D.

4.3. LEMMA. *There are constant finite rank projections* $Q_j \leqslant P_j$ *such that*

$$(\Psi(b_k) - \rho(b_k))(E - \sum_{j \geqslant 1} Q_j) \in C_n(X, K(H))$$

for all $k \in \mathbf{N}$.

Proof. Since

$$(\Psi(b_k) - \rho(b_k))P_j = (\Psi(b_k) - \rho(b_k))\rho(u_j)P_j =$$

$$= [\Psi(b_k)(\rho(u_j) - \Psi(u_j)) + (\Psi(b_k)\Psi(u_j) - \Psi(b_k u_j)) +$$

$$+ (\Psi(b_k u_j) - \rho(b_k u_j))]P_j \in C_n(X, K(H)),$$

there are finite rank constant projections $Q_j \leqslant P_j$ such that

$$\|(\Psi(b_k) - \rho(b_k))(P_j - Q_j)\| \leqslant 2^{-j} \text{ for } 1 \leqslant k \leqslant j.$$

It follows that the series

$$\sum_{j \geqslant 1} (\Psi(b_k) - \rho(b_k))(P_j - Q_j)$$

is norm-convergent to

$$(\Psi(b_k) - \rho(b_k))(E - \sum_{j \geqslant 1} Q_j)$$

and so

$$(\Psi(b_k) - \rho(b_k))(E - \sum_{j \geqslant 1} Q_j) \in C_n(X, K(H)). \qquad \text{Q.E.D.}$$

We now construct recurrently a set of constant finite-rank self-adjoint projections $\{R_{i,j}\}_{i,j \geqslant 1}$. Let $R_{1,j}$ be the projection Q_j provided by Lemma 4.3. Since Y_k and P_j are constant, once $R_{i,j}$ are constructed for fixed i and all $j \in \mathbf{N}$, we can find $R_{i+1,j}$ a constant finite-rank selfadjoint projection such that $R_{i+1,j}$ is the constant finite-rank selfadjoint projection onto the linear span of the ranges of the $P_j Y_k R_{i,s}$, $(1 \leqslant k \leqslant i + j + 4, /s - j/ \leqslant 1)$, and of $R_{i,j}$ (with convention $R_{i,0} = 0$). Note that $R_{i,j} \leqslant P_j$, $R_{i,j} \leqslant R_{i+1,j}$ and, since $Y_k P_j = (P_{j-1} + P_j + P_{j+1}) Y_k P$ we also have

$$Y_k R_{i,j} = (R_{i+1,j-1} + R_{i+1,j} + R_{i+1,j+1}) Y_k R_{i,j} \text{ for } 1 \leqslant k \leqslant i + j + 2.$$

Consider also:

$$B = \sum_{j \geqslant 1} \left(\frac{1}{j} \sum_{i=1}^{j} R_{i,j} \right),$$

$$Q = \sum_{j \geqslant 1} R_{1,j}, \quad Q' = \sum_{j \geqslant 1} R_{j,j}, \quad Q'' = \sum_{j \geqslant 1} R_{j+2,j}$$

which are constant elements of $C_{*s}(X, L(H))$. Then $Q \leqslant B \leqslant Q' \leqslant Q'' \leqslant E$, and $(I - Q'')Y_k Q' \in C_n(X, K(H))$ for all $k \in \mathbf{N}$. Also clearly Q, Q', Q'' are projections.

4.4. LEMMA. *$[Y_k, B] \in C_n(X, K(H))$ for all $k \in \mathbf{N}$.*

Proof. Consider

$$S_{i,j} = R_{i,j} - R_{i-1,j}, (R_{0,j} = 0).$$

Then the $S_{i,j}$ form a family of pairwise orthogonal selfadjoint constant finite-rank projections. Also B can be written as:

$$B = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} S_{i,j} \right).$$

Note that

$$S_{s,t} Y_k S_{i,j} = S_{s,t} P_t Y_k P_j S_{i,j},$$

so that

$$S_{s,t} Y_k S_{i,j} = 0 \text{ whenever } |t-j| \geqslant 2.$$

Also, if

$$\max\,(i+j, s+t) \geqslant k+2 \text{ and } |i-s| \geqslant 2,$$

then

$$S_{i,j}\, Y_k S_{s,t} = 0.$$

Indeed, since

$$(S_{s,t} Y_k S_{i,j})^* = S_{i,j} Y_k S_{s,t},$$

it will be sufficient to prove this only in case $i - s \geqslant 2$. Now, if $i + j \leqslant s + t$, then $t - j \geqslant 2$ and the assertion follows from the preceding discussion. Thus we are left with the case when $i - s \geqslant 2$, $i + j > s + t$ and $|t - j| \leqslant 1$. But then

$$(i-2) + t + 2 \geqslant i + j - 1 > k$$

and hence

$$S_{i,j} Y_k S_{s,t} = S_{i,j} Y_k R_{i-2,t} S_{s,t} =$$

$$= S_{i,j}(R_{i-1,t+1} + R_{i-1,t} + R_{i-1,t-1}) Y_k R_{i-2,t} S_{s,t} = 0.$$

Moreover,

$$Y_k S_{i,j} = Y_k R_{k+i,j} S_{i,j} =$$

$$= (R_{k+i+1,j+1} + R_{k+i+1,j} + R_{k+i+1,j-1}) Y_k R_{k+i,j} S_{i,j}$$

which can be expressed as a finite sum of $S_{s,t} Y_k S_{i,j}$.

Thus it follows that for $i+j \geqslant k+2$ we have:

$$Y_k S_{i,j} = \sum_{|\alpha| \leqslant 1,\, |\beta| \leqslant 1} S_{i+\alpha, j+\beta} Y_k S_{i,j},$$

$$S_{i,j}\, Y_k = \sum_{|\alpha| \leqslant 1,\, |\beta| \leqslant 1} S_{i,j} Y_k S_{i+\alpha, j+\beta},$$

(with the convention $S_{i,j} = 0$ whenever $i \leqslant 0$ or $j \leqslant 0$).

We have

$$[Y_k, B] = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} [Y_k, S_{i,j}] \right) =$$

$$= \sum_{j=1}^{k+2} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} [Y_k, S_{i,j}] \right) +$$

$$+ \sum_{j=k+3}^{\infty} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} (\sum_{|\alpha| \leqslant 1,\, |\beta| \leqslant 1} (S_{i+\alpha, j+\beta} Y_k S_{i,j} - S_{i,j} Y_k S_{i+\alpha, j+\beta})) \right) =$$

$$= \sum_{j=1}^{k+2} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} [Y_k, S_{i,j}] \right) +$$

$$+ \sum_{j=k+3}^{\infty} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} (\sum_{|\alpha| \leqslant 1,\, |\beta| \leqslant 1} (S_{i+\alpha, j+\beta} Y_k S_{i,j} - S_{i,j} Y_k S_{i-\alpha, j-\beta})) \right).$$

Thus, to prove that $[Y_k, B] \in C_n(X, K(H))$, it will be sufficient to prove that for $|\alpha| \leqslant 1$, $|\beta| \leqslant 1$, we have

$$C_{\alpha,\beta} = \sum_{j=k+3}^{\infty} \left(\sum_{i=1}^{j} \frac{j-i+1}{j} (S_{i+\alpha, j+\beta} Y_k S_{i,j} - S_{i,j} Y_k S_{i-\alpha, j-\beta}) \right) \in C_n(X, K(H)).$$

But we can write

$$C_{\alpha,\beta} = - \sum_{j=k+3-\beta}^{\infty} \left(\sum_{i=1-\alpha}^{j+\beta} \frac{j+\beta-i-\alpha+1}{j+\beta} S_{i+\alpha, j+\beta} Y_k S_{i,j} \right) +$$

$$+ \sum_{j=k+3}^{\infty} \left(\sum_{i=1-\alpha}^{j} \frac{j-i+1}{j} S_{i+\alpha, j+\beta} Y_k S_{i,j} \right).$$

Using the notations

$$T_{i,j} = S_{i+\alpha, j+\beta} Y_k S_{i,j},\ r_{i,j} = \frac{j+\beta-i-\alpha+1}{j+\beta} \text{ and } s_{i,j} = \frac{j-i+1}{j},$$

we have

$$C_{\alpha,\beta} = \sum_{j=k+5}^{\infty} \left(\sum_{i=1}^{j} s_{i,j}T_{i,j} - \sum_{i=1-\alpha}^{j+\beta} r_{i,j}T_{i,j} \right) + D_{\alpha,\beta}$$

where $D_{\alpha,\beta}$ is a finite sum of constant finite-rank elements and hence clearly $D_{\alpha,\beta} \in C_n(XK(H))$.

Now remarking that $T_{i,j} = 0$ unless $i \geqslant 1 - \alpha$ and $i \geqslant 1$, it follows that

$$C_{\alpha,\beta} - D_{\alpha,\beta} = \sum_{j=k+5}^{\infty} \left(\sum_{i=\max(1,1-\alpha)}^{j-2} (s_{i,j} - r_{i,j})T_{i,j} \right) +$$

$$+ \sum_{j=k+5}^{\infty} \left(\sum_{i=j-1}^{j} s_{i,j}T_{i,j} - \sum_{i=j-1}^{j+\beta} r_{i,j}T_{i,j} \right).$$

The first sum defines an element of $C_n(XK(H))$ since $(s_{i,j} - r_{i,j})T_{i,j}$ are constant finite-rank, $(i,j) \neq (m,n) \Rightarrow T_{i,j}^{*}T_{m,n} = T_{i,j}T_{m,n}^{*} = 0$ and $\|(s_{i,j} - r_{i,j})\, T_{i,j}\| \to 0$ whenever $1 \leqslant i \leqslant j$ and $i + j \to +\infty$. The same kind of argument shows also that the second sum is in $C_n(X(K(H))$. Q.E.D.

We introduce now the following notations:

$$Q = (I - E) + Q, \qquad \tilde{Q}' = (I - E) + Q', \qquad \tilde{Q}'' = (I - E) + Q'',$$

$$\tilde{B} = (I - E) + B.$$

The properties of these elements are summarized in the following lemma.

4.5. Lemma. *We have* :

(*i*) $\rho(J)\tilde{Q}'' \in C_n(X, K(H))$;

(*ii*) $I - E \leqslant \tilde{Q} \leqslant \tilde{B} \leqslant \tilde{Q}' \leqslant \tilde{Q}''$ *and* $\tilde{Q}, \tilde{Q}', \tilde{Q}''$ *are selfadjoint projections*;

(*iii*) $(I - \tilde{Q}'')\Psi(A)\tilde{Q}' \in C_n(X(K(H))$;

(*iv*) $(\Psi(a) - \rho(a))(I - \tilde{Q}) \in C_n(X, K(H))$ *for every* $a \in A$;

(*v*) $[\rho(A), \tilde{B}] \subset C_n(X, K(H))$;

(*vi*) $[\Psi(A), \tilde{B}] \subset C_n(X, K(H))$.

Proof. (*i*) By the first part of Lemma 4.2, we have $\rho(J)Q'' \in C_n(X, K(H))$. Moreover, $\rho(J)(I - E) = 0$, which makes our assertion obvious.

(*ii*) follows immediately from the fact that Q, Q', Q'' are selfadjoint projections and

$$0 \leqslant Q \leqslant B \leqslant Q' \leqslant Q'' \leqslant E.$$

(*iv*) is a transcription of Lemma 4.3, since

$$(I-\tilde{Q}) = I-(I-E)-Q = E-Q.$$

(*iii*) By the second part of Lemma 4.2 we have $(\rho(b_k)-Y_k)Q' \in C_n(XK(H))$. Also we know that $(I-Q'')Y_kQ' \in C_n(X, K(H))$ so that $(I-Q'')\rho(b_k)Q' \in C_n(X,K(H))$. Since $(I-E)\rho(A) = \rho(A)(I-E) = 0$, we infer $(I-\tilde{Q}'')\rho(b_k)\tilde{Q}' \in C_n(X, K(H))$ and since $\{b_k\}_{k\in\mathbf{N}}$ is total in A it follows that $(I-\tilde{Q}'')\rho(A)\tilde{Q}' \subset C_n(X, K(H))$. Since $0 \leqslant I-\tilde{Q}'' \leqslant I-\tilde{Q}$, it follows by (iv) that $(I-\tilde{Q}'')(\rho(a)-\Psi(a)) \in \in C_n(X, K(H))$ for all $a \in A$. Hence we have

$$(I-\tilde{Q}'')\,\Psi(A)\tilde{Q}' \subset C_n(X, K(H)).$$

(*v*) We have

$$[\rho(b_k), \tilde{B}] = [\rho(b_k), B] = [\rho(b_k)-Y_k, B] + [Y_k, B] =$$

$$= [\rho(b_k)-Y_k, Q'BQ'] + [Y_k, B]$$

so $[\rho(b_k), \tilde{B}] \in C_n(X, K(H))$ by the second part of Lemma 4.2 and by Lemma 4.4.

(*vi*) We have

$$[\Psi(a), \tilde{B}] = [\Psi(a)-\rho(a), \tilde{B}] + [\rho(a), \tilde{B}] = [\rho(a)-\Psi(a), I-\tilde{B}] + [\rho(a), \tilde{B}],$$

where $a \in A$. Since $I-\tilde{B} = (I-\tilde{Q})\,(I-\tilde{B})\,(I-\tilde{Q})$, assertion (*vi*) follows from (*iv*) and (*v*). Q. E. D.

Using now the Choi-Effros theorem, there is a unital completely positive map $\varphi\colon A/J \to A$ such that $q \circ \varphi = \mathrm{id}_{A/J}$. Consider also the completely positive map

$$\Phi = \Psi \circ \varphi : A/J \to C_{*s}(X, L(H)).$$

Since $\varphi(q(a))-a \in J$ for every $a \in A$, using Lemma 4.5. (i) we have

$$(*) \qquad \Phi(q(a))\tilde{Q}''-\Psi(a)\tilde{Q}'' \in C_n(X, K(H))$$

(recall also that $\rho(a)-\Psi(a) \in C_n(X, K(H))$ when $a \in J$).

Since $\tilde{Q}''$ is a constant projection we may use Lemma 3.1. for the compression of Φ to the range of $\tilde{Q}''$. This yields a Hilbert space H_1, a unital *-homomorphism

$$\mu\colon A/J \to C_{*s}(X, L(H_1))$$

and a constant partial isometry

$$W \in C_{*s}(X, L(H, H_1))$$

such that

$$W^*W = \tilde{Q}'' \text{ and } W^*\mu(q(a))W = \tilde{Q}''\Phi(q(a))\tilde{Q}''.$$

Consider $\tilde{B}_1, \tilde{Q}_1, \tilde{Q}'_1, \tilde{Q}''_1 \in C_{*s}(X, L(H_1))$ the constant elements defined as follows:

$$\tilde{B}_1 = W\tilde{B}W^*, \quad \tilde{Q}_1 = W\tilde{Q}W^*, \quad \tilde{Q}'_1 = W\tilde{Q}'W^*, \quad \tilde{Q}''_1 = W\tilde{Q}''W^* = WW^*.$$

Note that $\tilde{Q}_1$, $\tilde{Q}'_1$, $\tilde{Q}''_1$ are projections, $\tilde{Q}_1 \leqslant \tilde{B}_1 \leqslant \tilde{Q}'_1 \leqslant \tilde{Q}''_1$ and that:

$$(**) \qquad \tilde{B}_1 W = W\tilde{B}, \quad W^*\tilde{B}_1 = \tilde{B}W^*.$$

4.6. LEMMA. *We have*

$$(I - \tilde{Q}'')\mu(A/J)\tilde{Q}'_1 \subset C_n(X, K(H_1)),$$

$$[\mu(A/J), \tilde{B}_1] \subset C_n(X, K(H_1)).$$

Proof. We have:

$$((I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1)^*\, ((I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1) =$$

$$= \tilde{Q}'_1 \mu(q(a^*a))\, \tilde{Q}'_1 - \tilde{Q}'_1 \mu(q(a^*))\, \tilde{Q}''_1 \mu(q(a))\, \tilde{Q}'_1 =$$

$$= W\tilde{Q}'\Phi(q(a^*a))\, \tilde{Q}'W^* - W\tilde{Q}'\Phi(q(a^*))\, \tilde{Q}''\Phi(q(a))\, \tilde{Q}'W^*.$$

Since $\Phi(q(a))\, \tilde{Q}'' - \Psi(a)\, \tilde{Q}'' \in C_n(X, K(H))$ and $\tilde{Q}' \leqslant \tilde{Q}''$, we infer that

$$((I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1)^*\, ((I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1) - W\tilde{Q}'\Psi(a^*a)\, \tilde{Q}'W^* +$$

$$+ W\tilde{Q}'\Psi(a^*)\, \tilde{Q}''\Psi(a)\, \tilde{Q}'W^* \in C_n(X, K(H_1)).$$

But $\Psi(a^*a) - \Psi(a^*)\Psi(a) \in C_n(X, K(H))$, and so

$$W\tilde{Q}'\Psi(a^*a)\, \tilde{Q}'W^* - W\tilde{Q}'\Psi(a^*)\, \tilde{Q}''\Psi(a)\, \tilde{Q}'W^* =$$

$$= W\tilde{Q}'(\Psi(a^*a) - \Psi(a^*)\, \Psi(a))\, \tilde{Q}'W^* +$$

$$W((I - \tilde{Q}'')\, \Psi(a)\, \tilde{Q}')^*\, ((I - \tilde{Q}'')\, \Psi(a)\, \tilde{Q}') \in C_n(X, K(H_1))$$

by Lemma 4.5. (iii).

Thus

$$((I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1)^*\, ((I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1) \in C_n(X, K(H_1))$$

and hence also

$$(I - \tilde{Q}''_1)\, \mu(q(a))\, \tilde{Q}'_1 \in C_n(X, K(H_1)),$$

thus proving the first assertion of the lemma.

Since $(I-\tilde{Q}_1'')\,\mu(q(a))\,\tilde{Q}_1'\in C_n(X,K(H_1))$, we get

$$\mu(q(a))\,\tilde{Q}_1'-W\Psi(a)\,W^*\tilde{Q}_1'=$$

$$=(I-\tilde{Q}_1'')\,\mu(q(a))\,\tilde{Q}_1'+\tilde{Q}_1''\mu(q(a))\,\tilde{Q}_1'-W\Psi(a)\,W^*\tilde{Q}_1'=$$

$$=(I-\tilde{Q}_1'')\,\mu(q(a))\,\tilde{Q}_1'+W\tilde{Q}''\Phi(q(a))\,\tilde{Q}'W^*-W\tilde{Q}''\Psi(a)\,\tilde{Q}'W^*-$$

$$-W(I-\tilde{Q}'')\,\Psi(a)\,\tilde{Q}'W^*\in C_n(X,K(H_1))$$

by (*) and Lemma 4.5. *(iii)*.

Since $\tilde{B}_1=\tilde{Q}_1\tilde{B}_1\tilde{Q}_1$, it follows that

$$[\mu(q(a)),\tilde{B}_1]-[W\Psi(a)\,W^*,\tilde{B}_1]\in C_n(X,K(H_1)),$$

and using (**) the second assertion follows from Lemma 4.5.*(vi)*. Q.E.D.

Let $G\in C_{*s}(X,L(H,H_1))$ be the constant element

$$G=\tilde{B}_1^{1/2}W(I-\tilde{B})^{1/2}.$$

In view of (**) we get

$$G=\tilde{B}_1^{1/2}(I-\tilde{B}_1)^{1/2}W=W\tilde{B}(I-\tilde{B})^{1/2}=(I-B_1)^{1/2}W\tilde{B}^{1/2},$$

so that

$$\Omega=\begin{pmatrix}\tilde{B} & G^*\\ G & I-\tilde{B}_1\end{pmatrix}\in C_{*s}(X,L(H\oplus H_1))$$

is a constant selfadjoint projection. Note also that

$$Q\oplus 0_{H_1}\leqslant\Omega\leqslant Q''\oplus I_{H_1}.$$

4.7. Lemma. *We have*

(i) $[(\Psi\oplus\mu\circ q)\,(A),\Omega]\subset C_n(X,K(H\oplus H_1))$;

(ii) $((\Psi\oplus\mu\circ q)\,(a)-(\rho\oplus\mu\circ q)\,(a))\,(I-\Omega)\in C_n(X,K(H\oplus H_1))$ *for all* $a\in A$;

(iii) $(\Psi\oplus\mu\circ q)\,(J)\,\Omega\subset C_n(X,K(H\oplus H_1))$.

Proof. Since $I-\Omega\leqslant(I-\tilde{Q})\oplus I_{H_1}$, assertion *(ii)* follows from Lemma 4.5.*(iv)*.

Also since $\Omega\leqslant\tilde{Q}''\oplus I_{H_1}$, assertion *(iii)* follows from Lemma 4.5.*(i)* and the fact that

$$(\rho-\Psi)\,(J)\subset C_n(X,K(H)).$$

In view of Lemma 4.6 and of Lemma 4.5.(*vi*), in order to prove assertion (*i*) it will be sufficient to show that

$$G\Psi(a)-(\mu\circ q)(a)\,G\in C_n(X,K(H,H_1)) \text{ for all } a\in A.$$

But

$$G\Psi(a)-(\mu\circ q)(a)\,G=W\tilde{B}^{1/2}(I-\tilde{B})^{1/2}\Psi(a)-(\mu\circ q)(a)\,W\tilde{B}^{1/2}(I-\tilde{B})^{1/2}=$$

$$=W[\tilde{B}^{1/2}(I-\tilde{B})^{1/2},\Psi(a)]+W\Psi(a)\tilde{B}^{1/2}(I-\tilde{B})^{1/2}-(\mu\circ q)(a)\,W\tilde{B}\,(I-\tilde{B})^{1/2},$$

so that in view of Lemma 4.5.(*vi*) and $\tilde{Q}'\tilde{B}^{1/2}=\tilde{B}^{1/2}$ it will be sufficient to prove that

$$W\Psi(a)\,\tilde{Q}'-(\mu\circ q)(a)\,W\tilde{Q}'\in C_n(X,K(H,H_1)).$$

But this can be seen as follows:

$$W\Psi(a)\,Q'-(\mu\circ q)(a)\,\tilde{Q}'=W(I-\tilde{Q}'')\,\Psi(a)\,\tilde{Q}'-(I-\tilde{Q}_1'')\,(\mu\circ q)(a)\,W\tilde{Q}'+$$

$$+W\tilde{Q}''\Psi(a)\,\tilde{Q}'-\tilde{Q}_1''(\mu\circ q)(a)\,W\tilde{Q}'=$$

$$=W(I-\tilde{Q}'')\,\Psi(a)\,\tilde{Q}'-(I-\tilde{Q}_1'')\,(\mu\circ q)(a)\,\tilde{Q}_1'W+$$

$$+W\tilde{Q}''(\Psi(a)-\Phi(q(a)))\,\tilde{Q}'\in C_n(X,K(H,H_1))$$

by Lemma 4.5.(iii), Lemma 4.6, and (*). Q.E.D.

The next lemma will be the final point in the proof of Thm. 4.1.

4.8. LEMMA. *There is* $[\sigma]\in \mathrm{Ext}\,(X,A/J)$ *such that*

$$q_*[\sigma]=[\tau].$$

Proof. Denote by H_2 the Hilbert space $H_2=H\oplus H_1$ and, since E, Ω, Q, Q', Q'', $\tilde{Q}$, $\tilde{Q}'$, $\tilde{Q}''$ are constant operator-valued functions on X, let E_0, Ω_0, Q_0, Q_0', Q_0'', $\tilde{Q}_0$, $\tilde{Q}_0'$, $\tilde{Q}_0''$ denote the corresponding operators. Consider also the projection $D_0=E_0\oplus I_{H_1}$. Note that

$$I-\Omega_0\leqslant(I-\tilde{Q}_0)\oplus I_{H_1}\leqslant E_0\oplus I_{H_1}=D_0,$$

and that the compression of $\rho\oplus(\mu\circ q)$ to $D_0(H_2)$ is a unital $*$-homomorphism

$$\tilde{\rho}\colon A\to C_{*s}(X,L(D_0,(H_2))).$$

Denote by

$$\chi_1\colon A\to C_{*s}(X,L(\Omega_0(H_2)))$$
$$\chi_2\colon A\to C_{*s}(X,L((I-\Omega_0)(H_2)))$$
$$\theta_1\colon A\to C_{*s}(X,L((D_0-(I-\Omega_0))(H_2)))$$
$$\theta_2\colon A\to C_{*s}(X,L((I-\Omega_0)\,(H_2))\,)$$

the unital completely positive maps defined by:

$$(\chi_1(a))(x) = \Omega_0(\Psi(a) \oplus (\mu \circ q)(a))(x) \mid \Omega_0(H_2)$$

$$(\chi_2(a))(x) = (I - \Omega_0)(\Psi(a) \oplus (\mu \circ q)(a))(x) \mid (I - \Omega_0)(H_2)$$

$$(\theta_1(a))(x) = (D_0 - (I - \Omega_0))(\tilde{\rho}(a))(x) \mid (D_0 - (I - \Omega_0))(H_2)$$

$$(\theta_2(a))(x) = (I - \Omega_0)(\tilde{\rho}(a))(x) \mid (I - \Omega_0)(H_2).$$

By Lemma 4.7.(i) it follows that $p \circ \chi_1$ and $p \circ \chi_2$ are $*$-homomorphisms and by Lemma 4.7. (ii) $p \circ \theta_2$ and hence also $p \circ \theta_1$ are also $*$-homomorphisms.

Moreover by Lemma 4.7. (iii), $(p \circ \chi_1)(J) = 0$. Since

$$D_0 - (I - \Omega_0) \leqslant D_0 - I + Q_0'' \oplus I_{H_1} = Q_0'' \oplus 0,$$

it follows by Lemma 4.2 that

$$\tilde{\rho}(J)(D_0 - (I - \Omega_0)) \in C_n(X, K(D_0(H_2)))$$

and hence

$$(p \circ \theta_1)(J) = 0.$$

Let γ, δ_1, δ_2, γ_1, γ_2 be the homogeneous X-extensions by A determined by $p \circ \tilde{\rho}$, $p \circ \chi_1$, $p \circ \chi_2$, $p \circ \theta_1$, $p \circ \theta_2$, i.e. the homogeneous X-extensions by A obtained by adding to each of the above $*$-homomorphisms a trivial homogeneous X-extension by A.

We have then in Ext (X, A):

$$[\gamma] = [\gamma_1] + [\gamma_2],$$
$$[\tau] = [\delta_1] + [\delta_2],$$
$$[\gamma] = 0.$$

Moreover, there are $[\sigma_1]$, $[\sigma_2] \in \operatorname{Ext}(X, A/J)$ such that

$$q_*[\sigma_1] = [\gamma_1],$$
$$q_*[\sigma_2] = [\delta_1].$$

Also by Lemma 4.7. (ii) we have

$$[\gamma_2] = [\delta_2].$$

It follows that

$$\begin{aligned} [\tau] &= [\delta_1] + [\delta_2] = \\ &= [\delta_1] + [\gamma_2] = \\ &= [\delta_1] - [\gamma_1] = \\ &= q_*[\sigma_2] - q_*[\sigma_1] = \\ &= q_*([\sigma_2] - [\sigma_1]). \end{aligned}$$

Q.E.D.

§ 5

This section deals with the homotopy invariance properties of Ext (X, A) both in the X and in the A-"variabile". In fact these two homotopy-invariance properties are related and their proof reduces in the case of quasidiagonal C^*-algebras to an adaption of the argument of N. Salinas ([42]) for the usual Ext-groups.

The short exact sequence for Ext (X, A) in § 4 enables us to improve the result of Salinas: A may be any C^*-algebra having a composition series with quasidiagonal quotients. In particular, A may be any GCR-C^*-algebra.

First we need a few facts about quasidiagonality in $C_{*s}(X, L(H))$, but since this seems a rather awkward intermediate degree of generality, we prefer to digress a bit, considering a more general situation.

Let L be a unital C^*-algebra (not necessarily separable), $K \subset L$ a closed two-sided ideal and $p: L \to L/K$ the canonical homomorphism (These notations will not cause any confusion since in our applications $L = C_{*s}(X, L(H))$ and $K = C_n(X, K(H))$). We will make the following *assumption about* K:

there is an increasing sequence $P_1 \leqslant P_2 \leqslant \ldots$
of selfadjoint projections in K,
which is an approximate unit of K.

The set $P(K)$ of selfadjoint projections of K is not filtering in general, but has a weaker property. For $\varepsilon > 0$ and $P, Q \in P(K)$ we shall write

$$P \underset{\varepsilon}{\prec} Q \quad \text{iff} \quad \|P - QP\| \leqslant \varepsilon.$$

Then our special assumption on K implies that
for any $Q_1, Q_2 \in P(K)$ and $\varepsilon > 0$
we can find $Q_3 \in P(K)$ such that

$$Q_1 \underset{\varepsilon}{\prec} Q_3, \qquad Q_2 \underset{\varepsilon}{\prec} Q_3.$$

For a bounded function $f: P(K) \to \mathbf{R}$ we define

$$\liminf_{P \in P(K)} f(P)$$

as the greatest lower bound of those $r \in \mathbf{R}$, such that for every $P \in P(K)$ and $\varepsilon > 0$ there is $Q \in P(K)$ such that $f(Q) \leqslant r$ and $P \underset{\varepsilon}{\prec} Q$. Also we define

$$\limsup_{P \in P(K)} f(P) = - \liminf_{P \in P(K)} (-f(P)).$$

For a finite set $\Sigma \subset L$ the modulus of quasitriangularity $q(\Sigma)$ is defined as

$$q(\Sigma) = \liminf_{P \in P(K)} (\max_{a \in \Sigma} \|(I - P)\, aP\|)$$

and the modulus of quasidiagonality $qd(\Sigma)$ is defined as

$$qd(\Sigma) = q(\Sigma \cup \Sigma^*).$$

Remark that

$$qd(\Sigma) = \liminf_{P \in P(K)} (\max_{a \in \Sigma} \|[P, a]\|).$$

also since for $k \in K$ we have

$$\limsup_{P \in P(K)} \|(I - P) kP\| = 0$$

we easily infer that

$$|q((a_i)_{i=1}^n) - q((a_i')_{i=1}^n)| \leqslant \max_{1 \leqslant i \leqslant n} \|p(a_i - a_i')\|,$$

$$|qd((a_i)_{i=1}^n) - qd((a_i')_{i=1}^n)| \leqslant \max_{1 \leqslant i \leqslant n} \|p(a_i - a_i')\|.$$

5.1. LEMMA. *Let* $\{Q_j\}_{j \in \mathbf{N}} \subset P(K)$ *be such that* $Q_j \underset{10^{-j}}{\prec} Q_{j+1}$, *then there are* $\{Q_j'\}_{j \in \mathbf{N}} \subset (P(K)$ *such that*

$$Q_j' \leqslant Q_{j+1}', \ (j \in \mathbf{N}), \textit{ and } \lim_{j \to \infty} \|Q_j - Q_j'\| = 0.$$

Proof. Consider first two projections $P, Q \in P(K)$, $P \underset{\varepsilon}{\prec} Q$, $\varepsilon < 1/2$. Then $(1 - \varepsilon) P \leqslant PQP \leqslant P$ so that we have the polar decomposition $QP = wa$ where $a = (PQP)^{1/2}$ and $w = QP((I - P) + PQP)^{-1/2}$. Then $w^*w = P$ and $ww^* \in P(K)$, $ww^* \leqslant Q$. Also $\|w - P\| \leqslant 3\varepsilon$ and hence $\|ww^* - P\| \leqslant 6\varepsilon$. Denoting by $E(P, Q)$ the projection ww^*, we thus have

$$E(P, Q) \leqslant Q \text{ and } \|E(P, Q) - P\| \leqslant 6\varepsilon.$$

Using this we define recurrently $\{Q_{i,j}\}_{1 \leqslant i \leqslant j} \subset P(K)$ so that

$$Q_{j,j} = Q_j \text{ and } Q_{i,j} = E(Q_{i,j-1}, Q_{i+1,j}), \qquad (1 \leqslant i \leqslant j - 1).$$

Clearly then $Q_{i,j} \leqslant Q_{i+1,j}$ and it is easily seen that

$$\|Q_{i,j} - Q_{i,j+1}\| \leqslant 6^{j+1-i} \cdot 10^{-j}.$$

It follows that

$$\sum_{j=1}^{\infty} \|Q_{i,j} - Q_{i,j+1}\| \leqslant \sum_{j=1}^{\infty} (6/10)^{-j} = (5/2)(3/5)^{-i}.$$

Hence for $j \to +\infty$, $Q_{i,j}$ converges to some projection Q_i'. Clearly $Q_i' \leqslant Q_{i+1}'$ and $\lim_{i \to \infty} \|Q_i - Q_i'\| = 0$. Q.E.D.

5.2. LEMMA. *Consider a subset* $\Omega \subset L$, *such that* $p(\Omega)$ *is separable. Then the following assertions are equivalent*:

(*i*) *For every finite subset* $\Sigma \subset \Omega$ *we have* $qd(\Sigma) = 0$;

(*ii*) *There is an approximate unit* $\{Q_j\}_{j \in \mathbf{N}} \subset P(K)$ *such that* $Q_j \leqslant Q_{j+1}$, $(j \in \mathbf{N})$, *and* $\lim_{j \to \infty} \|[Q_j, a]\| = 0$ *for all* $a \in \overline{\Omega + K}$.

Proof. That $(ii) \Rightarrow (i)$ is immediate.

For the converse it is clear, assuming (i), that for $\{a_k\}_{k\in\mathbf{N}} \subset \Omega$ a sequence with $\{p(a_k)\}_{k\in\mathbf{N}}$ dense in $p(\Omega)$, there is a sequence $\{Q'_j\}_{j\in\mathbf{N}} \subset P(K)$ which is an approximate unit of K, such that $\lim_{j\to\infty} \| [Q'_j, a_k] \| = 0$, for all $k \in \mathbf{N}$, and $Q'_k \underset{10^{-h}}{\prec} Q'_{k+1}$. Then (ii) follows using Lemma 5.1 and the fact that $\lim_{j\to\infty} \| [Q'_j, b] \| = 0$ for every $b \in K$. Q.E.D.

A subset $\Omega \subset L$, with $p(\Omega)$ separable will be called almost diagonal if it satisfies the equivalent conditions of Lemma 5.2.

5.3. Definition. *A homogeneous X-extension by A, defined by $\tau: A \to C_{*s}(X, L(H))/C_n(X, K(H))$ is called quasidiagonal, if $p^{-1}(\tau(A))$ is almost diagonal with respect to the ideal $C_n(X, K(H))$.*

It is easily seen that if τ_1, τ_2 define equivalent homogeneous X-extensions by A, then τ_1 is quasidiagonal if and only if τ_2 is. Thus we can speak about quasidiagonal elements of Ext (X, A).

Also, as for the usual extensions by $K(H)$, it is obvious that the quasidiagonal elements of Ext (X, A) form a semigroup.

Recall from ([42]) that a unital separable C^*-algebra A is called quasidiagonal if there is a *-monomorphism $\rho: A \to L(H)$, $\rho(A) \cap K(H) = 0$ such that $\rho(A)$ is almost diagonal with respect to the ideal $K(H)$ (i.e., in the usual sense).

In view of Thm. 2.10, if A is quasidiagonal and X finite-dimensional, then any trivial homogeneous X-extension by A is quasidiagonal. Moreover it is also clear that the existence of a trivial homogeneous X-extension by A which is quasi-diagonal, insures the quasidiagonality of A.

5.4. Proposition. *Let A be a nuclear quasidiagonal C^*-algebra and X a finite-dimensional metrizable compact space. Let $[\tau] \in$ Ext $(X \times [0, 1], A)$ be such that $i_0^*([\tau]) = 0$, where*

$$i_0: X \times \{0\} \to X \times [0, 1]$$

is the natural inclusion. Then it follows that $[\tau]$ is quasidiagonal.

Proof. In view of Thm. 3.3, we may assume

$$\tau(A) \subset C_n(X \times [0, 1], L(H))/C_n(X \times [0, 1], K(H)).$$

Let also

$$\varphi: A \to C_n(X \times [0, 1], L(H))$$

be a completely positive lifting for τ. Denote further by

$$i_t: X \times \{t\} \to X \times [0, 1]$$

the natural inclusion and by

$$j_t: X \times [0, 1] \to X \times \{t\}$$

the natural projection.

Fix $a_1, \dots, a_m \in A$ and $\varepsilon > 0$. Since $\varphi(a_i) \in C_n(X \times [0, 1], L(H))$, there is a natural number n such that

$$\|(\varphi(a_i))(x, t) - (\varphi(a_i))(x, t')\| < \varepsilon, \quad (1 \leqslant i \leqslant m),$$

whenever $|t - t'| \leqslant 2/n$.

Using Thm. 3.2, Thm. 3.3 and the Choi-Effros theorem, there is a completely positive map

$$\eta: A \to C_n(X \times [0, 1], L(H'))$$

such that $p \circ \eta$ defines a homogeneous $(X \times [0, 1])$-extension by A and

$$[p \circ (\varphi \oplus \eta)] = 0.$$

The completely positive map

$$\theta: A \to C_n \left(X \times [0, 1], L \left(H \oplus \frac{(H \oplus H') \oplus \dots \oplus (H \oplus H')}{n\text{-times}} \right) \right)$$

pefined by

$$(\theta(a))(x, t) = (\varphi(a))(x, t) \oplus \bigoplus_{k=1}^{n} \left((\varphi(a)) \left(x, \frac{k}{n} \right) \oplus (\eta(a)) \left(x, \frac{k}{n} \right) \right)$$

determines an extension, and

$$[p \circ \theta] = [\tau] + \sum_{k=1}^{n} (j_{k/n}^* \circ i_{k/n}^*)([\tau] + [p \circ \eta]) = [\tau].$$

Consider also the completely positive map

$$\psi: A \to C_n \left(X \times [0, 1], L \left(H \oplus \frac{(H \oplus H') \oplus \dots \oplus (H \oplus H')}{n\text{-times}} \right) \right)$$

defined by

$$(\psi(a))(x, t) = (\varphi(a))(x, 0) \oplus \bigoplus_{k=1}^{n} \left((\varphi(a)) \left(x, \frac{k}{n} \right) \oplus (\eta(a)) \left(x, \frac{k}{n} \right) \right).$$

Clearly $p \circ \psi$ defines a homogeneous $(X \times [0, 1])$-extension by A and $[p \circ \psi] = 0$.

For $(k - 1)/n \leqslant t \leqslant k/n$, $(1 \leqslant k \leqslant n)$, define the unitary operator

$$U_t \in L \left(H \oplus \frac{(H \oplus H') \oplus \dots \oplus (H \oplus H')}{n\text{-times}} \right)$$

by

$$U_t(f_0 \oplus (f_1 \oplus f_1') \oplus \dots \oplus (f_n \oplus f_n')) = g_0 \oplus (g_1 \oplus g_1') \oplus \dots \oplus (g_n \oplus g_n')$$

where

$$g'_j = f'_j, \quad (1 \leqslant j \leqslant n);$$

$$g_j = f_{j+1}, \quad (0 \leqslant j \leqslant k-2);$$

$$g_{k-1} = (k - nt)^{1/2} f_k + (nt - k + 1)^{1/2} f_0;$$

$$g_k = -(nt - k + 1)^{1/2} f_k + (k - nt)^{1/2} f_0;$$

$$g_j = -f_j, \quad (k + 1 \leqslant j \leqslant n).$$

It is easy to see that U_t depends continuously on $t \in [0, 1]$.

Consider also the unitary

$$V \in C_n\left(X \times [0, 1], L\left(H \oplus \underbrace{(H \oplus H') \oplus \ldots \oplus (H \oplus H')}_{n\text{-times}}\right)\right)$$

defined by

$$V(x, t) = U_t.$$

With these definitions it is now easy to see that

$$\| V\theta(a_i)\, V^* - \psi(a_i)\| \leqslant 5\varepsilon \quad \text{for} \quad 1 \leqslant i \leqslant m.$$

Since $p \circ \psi$ defines a homogeneous $(X \times [0, 1])$-extension by A and $[p \circ \psi] = 0$, it follows because A is quasidiagonal that

$$qd(\psi(a_1), \ldots, \psi(a_m)) = 0$$

and hence

$$qd(\theta(a_1), \ldots, \theta(a_m)) \leqslant \max_{1 \leqslant j \leqslant m} \| V\theta(a_j)\, V^* - \psi(a_j)\| \leqslant 5\varepsilon.$$

But since

$$[p \circ \theta] = [\tau] = [p \circ \varphi],$$

we infer

$$qd(\varphi(a_1), \ldots, \varphi(a_m)) = qd(\theta(a_1), \ldots, \theta(a_m)) \leqslant 5\varepsilon.$$

Hence since $\varepsilon > 0$ was arbitrary we must have

$$qd(\varphi(a_1), \ldots, \varphi(a_m)) = 0,$$

which is the desired result. Q.E.D.

5.5. PROPOSITION. *Let A be a nuclear quasidiagonal C^*-algebra and assume X is finite-dimensional. Consider also the map*

$$i_t : X \to X \times [0, 1], \qquad i_t(x) = (x, t).$$

Then we have :

$$i_0^*([\tau]) = 0 \Rightarrow i_1^*([\tau]) = 0 \quad \textit{for} \quad [\tau] \in \mathrm{Ext}\,(X \times [0,1],\ A).$$

Proof. Let $[\tau] \in \mathrm{Ext}\ (X \times [0, 1], A)$ be such that $i_0^*([\tau]) = 0$. In view of Thm. 3.3. we mai assume

$$\tau(A) \subset C_n(X \times [0,1],\ L(H))/C_n(X \times [0, 1], K(H)).$$

Let further

$$\Phi : A \to C_n(X \times [0, 1], L(H))$$

be a completely positive lifting for τ. Let also $\{a_j\}_{j \in \mathrm{N}} \subset A$ be a total sequence in A. Denote further by

$$\Phi^t : A \to C_n(X, L(H))$$

the completely positive map

$$(\Phi^t(a))\,(x) = (\Phi(a))\,(x, t).$$

By Proposition 5.4, $[\tau]$ is quasidiagonal. Thus we can find an increasing sequence $0 = P_0 \leqslant P_1 \leqslant P_2 \leqslant \ldots$ of selfadjoint projections in $C_n(X, K(H))$, which is an approximate unit for $C_n(X, K(H))$ and satisfies the following conditions:

(1) $$\|\,[P_j, \Phi(a_k)]\,\| \leqslant 2^{-j}, \qquad 1 \leqslant k \leqslant j;$$

(2) $$\|(I - P_j)\,(\Phi(a_i)\,\Phi(a_k) - \Phi(a_i a_k))\,\| < 2^{-j}, \qquad 1 \leqslant i,\ k \leqslant j.$$

For $j \geqslant 0$ there is an integer $N_j \geqslant 3$ such that $|t - t'| \leqslant 2/N_j$ implies

(3) $$\begin{aligned} &\|P_k(x, t) - P_k(x, t')\| < (10(j + 1))^{-2}, && 1 \leqslant k \leqslant j + 1; \\ &\|(\Phi(a_k))\,(x, t) - (\Phi(a_k))\,(x, t')\| < (j + 1)^{-2}, && 1 \leqslant k \leqslant j + 1. \end{aligned}$$

Defining

$$\Phi_j(a) = (I - P_{j+1})\,\Phi(a)\,(I - P_{j+1}) \ + (P_{j+1} - P_j)\ \Phi(a)\,(P_{j+1} - P_j),$$

for $j \geqslant 0$, we have $p \circ \Phi_j = p \circ \Phi = \tau$ and Φ_j is completely positive.

Also we can find $V_j \in C_{*s}(X \times [0, 1], L(H))$ such that $V_j^* V_j = I$ and $V_j V_j^* = (I - P_j)$, $j \geqslant 0$. Indeed it suffices to use (10.8.7 in [21]) for the continuous field of Hilbert spaces

$$(((I - P_j(y))\, H)_{y \in X \times [0,1]},\ (I - P_j)\, C(X \times [0, 1], H)).$$

Consider then the completely positive maps

$$\Psi_{j,k}\colon A \to C_{*s}(X, L(H))$$

defined by

$$(\Psi_{j,k}(a))\,(x) = (V_j^* \Phi_j(a)\, V_j)\,(x, k/N_j).$$

Also consider

$$H' = \bigoplus_{j \geqslant 0} \left(\bigoplus_{k=1}^{N_j - 1} H^{j,k} \right)$$

where the $H^{j,k}$ are copies of H, and let

$$\Psi\colon A \to C_{*s}(X, L(H))$$

be the unital completely positive map

$$\Psi = \bigoplus_{j \geqslant 0} \left(\bigoplus_{k=1}^{N_j - 1} \Psi_{j,k} \right).$$

Remark that $p \circ \Psi$ defines a homogeneous X-extension by A. This follows from (1), (2), since

$$\Psi_{j,k}(a_i a_s) - \Psi_{j,k}(a_i)\, \Psi_{j,k}(a_s) \in C_n(X, K(H))$$

and

$$\begin{aligned}
&\|\Psi_{j,k}(a_i a_s) - \Psi_{j,k}(a_i)\, \Psi_{j,k}(a_s)\| \leqslant \\
&\leqslant \|(I - P_{j+1})\, (\Phi(a_i a_s) - \Phi(a_i)\, \Phi(a_s))\| + \\
&+ \|(P_{j+1} - P_j)\, (\Phi(a_i a_s) - \Phi(a_i)\, \Phi(a_s))\| + \\
&+ \|(I - P_{j+1})\, \Phi(a_i)\, P_{j+1}\, \Phi(a_s)\| + \\
&+ \|(P_{j+1} - P_j)\, \Phi(a_i)\, (I - P_{j+1} + P_j)\, \Phi(a_s)\|
\end{aligned}$$

which, by (1), (2), is $\leqslant 5 \cdot 2^{-j}$ if $1 \leqslant i,\ s \leqslant j$.

Let also $P_{j,k} \in C_n(X, K(H))$ and $V_{j,k} \in C_{*s}(X, L(H))$ be defined by

$$P_{j,k}(x) = P_j(x, k/N_j) \quad \text{for} \quad j \geqslant 0,\ 0 \leqslant k \leqslant N_j,$$

$$V_{j,k}(x) = V(x, k/N_j) \quad \text{for} \quad j \geqslant 0,\ 1 \leqslant k \leqslant N_j - 1.$$

Because of (3) there are unitaries $U_{j,k} \in C_n(X, K(H)) + I$, $j \geqslant 0$, $1 \leqslant k \leqslant N_j$, such that

$$(4) \qquad \begin{aligned} &U_{j,k}(P_{j+1,k-1} - P_{j,k-1})\, U_{j,k}^{*} = P_{j+1,k} - P_{j,k}, \\ &\|U_{j,k} - I\| < (j+1)^{-2}, \quad j \geqslant 0, \quad 1 \leqslant k \leqslant N_j. \end{aligned}$$

This can be done by standard arguments (compare with the first part of the proof of Lemma 5.1) taking for $U_{j,k}$ the sum of the partial isometries in the polar decompositions of $(P_{j+1,k}-P_{j,k})\,(P_{j+1,k-1}-P_{j,k-1})$ and $(I-P_{j+1,k}+P_{j,k})\,(I-P_{j+1,k-1}+P_{j,k-1})$.

We shall now construct

$$R \in C_{*s}(X, L(H, H'))$$

$$S \in C_{*s}(X, L(H', H'))$$

$$T \in C_{*s}(X, L(H', H))$$

which will then be used to construct a certain unitary

$$U \in C_{*s}(X, L(H \oplus H')).$$

Since

$$H' = \bigoplus_{j \geqslant 0} \left(\bigoplus_{k=1}^{N_j - 1} H^{j,k} \right),$$

it will be sufficient to describe the components

$$R_{j,k} \in C_{*s}(X, L(H, H^{j,k}))$$

of R. These are:

$$R_{j,k} = 0 \text{ if } k \geqslant 2 \text{ and } R_{j,1} = V_{j,1}^{*}\, U_{j;1}(P_{j+1,0} - P_{j,0}).$$

It is easily seen that $R^*R = I$ and

$$RR^* = \bigoplus_{j \geqslant 0} \left(\bigoplus_{k=1}^{N_j - 1} Q_{j;k} \right)$$

where

$$Q_{j,k} = 0 \text{ if } k \geqslant 2 \text{ and } Q_{j,1} = V_{j,1}^{*}\,(P_{j+1,1} - P_{j,1})V_{j,1}.$$

Moreover

$$\|R_{j,1}\, \Phi^0(a_k) - \Psi_{j,1}(a_k)\, R_{j;1}\| =$$

$$= \|U_{j,1}^{*}(P_{j+1;0} - P_{j;0})\, \Phi^0(a_k) - (P_{j+1,1} - P_{j;1})\, \Phi^{1/N_j}(a_k)\,(P_{j+1,1} - P_{j,1})U_{j,1}\| \leqslant$$

$$\leqslant \|(P_{j+1,1} - P_{j,1})\,(U_{j,1}\, \Phi^0(a_k) - \Phi^{1/N_j}(a_k)\, U_{j,1}\| + \|[\Phi^{1/N_j}(a_k), (P_{j+1,1} - P_{j,1})]\| \leqslant$$

$$\leqslant \|[\Phi(a_k), (P_{j+1} - P_j)]\| + \|\Phi^0(a_k) - \Phi^{1/N_j}(a_k)\| + 2\|a_k\|\, \|U_{j,1} - 1\|.$$

Hence in view of (1), (3) and (4), it follows that

$$R\Phi^0(a_k) - \Psi(a_k)R \in C_n(X, K(H, H')).$$

Since $\{a_k\}_{k\in\mathbf{N}}$ is total in A it follows that

$$R\Phi^0(a) - \Psi(a)R \in C_n(X, K(H, H')) \text{ for all } a \in A.$$

Next $T \in C_{*s}(X, L(H', H))$ is defined by its components

$$T_{j,k} \in C_{*s}(X, L(H^{j,k}, H)).$$

These are

$$T_{j,k} = 0 \text{ if } k \leqslant N_j - 2 \text{ and } T_{j,N_j-1} = (P_{j+1,N_j} - P_{j,N_j})U_{j,N_j}V_{j,N_j-1}.$$

It is easily seen that T^* is constructed the same way as R after performing the symmetry $\alpha \mapsto 1-\alpha$ on the segment $[0, 1]$. So, the same kind of argument as for R, gives $TT^* = I$ and

$$T^*T = \bigoplus_{j\geqslant 0}\left(\bigoplus_{k=1}^{N_j-1} Q_{j,k}\right)$$

where $Q_{j,k} = 0$ for $k \leqslant N_j - 2$ and

$$Q_{j,N_j-1} = V^*_{j,N_j-1}(P_{j+1,N_j-1} - P_{j,N_j-1})V_{j,N_j-1}.$$

Moreover,

$$T\Psi(a) - \Phi^1(a)T \in C_n(X, K(H', H)) \text{ for all } a \in A.$$

Finally we construct S as the sum of two operators S_1, S_2. Here S_1 is

$$S_1 = \bigoplus_{j\geqslant 0}\left(\bigoplus_{k=1}^{N_j-1}(I - V^*_{j,k}(P_{j+1,k} - P_{j,k})V_{j,k})\right).$$

Clearly S_1 is a projection and $[S_1, \Psi(A)] = 0$. Next, S_2 will be such that

$$S_2(x)(H^{j,k}) \subset H^{j,k+1}, \quad (1 \leqslant k \leqslant N_j - 2) \text{ and } S_2(x)(H^{j,N_j-1}) = 0.$$

The "matrix-element" of S_2 from $H^{j,k}$ to $H^{j,k+1}$ is given by

$$S_{2,j,k} = V^*_{j,k+1}U_{j,k+1}(P_{j+1,k} - P_{j,k})V_j.$$

Since

$$\|S_{2,j,k}\Psi_{j,k}(a_s)-\Psi_{j,k+1}(a_s)\,S_{2,j,k}\|=$$

$$=\|(P_{j+1,k+1}-P_{j,k+1})\,\Phi^{(k+1)/N_j}(a_s)\,(P_{j+1,k+1}-P_{j,k+1})\,U_{j,k+1}-$$

$$-U_{j,k+1}(P_{j+1,k}-P_{j,k})\,\Phi^{k/N_j}(a_s)\,(P_{j+1,k}-P_{j,k})\|=$$

$$=\|(P_{j+1,k+1}-P_{j,k+1})\,(\Phi^{-(k+1)/N_j}(a_s)\,U_{j,k+1}-U_{j,k+1}\,\Phi^{-k/N_j}(a_s))\times$$

$$\times(P_{j+1,k}-P_{j,k})\|\leqslant\|a_s\|\,\|U_{j,k+1}-I\|+\|\Phi^{(k+1)/N_j}(a_s)-\Phi^{k/N_j}(a_s)\|,$$

using (3) and (4) it is easily seen that

$$[S_2,\Psi(A)]\subset C_n(X,K(H')).$$

Also for $S=S_1+S_2$ it is immediate that

$$S^*S=I-RR^*\quad\text{and}\quad SS^*=I-T^*T.$$

The unitary $U\in C_{*s}(X,L(H\oplus H'))$ is now defined by the matrix $(U_{i,j})_{1\leqslant i,j\leqslant 2}$, where

$$U_{1,1}\in C_{*s}(X,L(H)),\qquad U_{1,1}=0;$$

$$U_{1,2}\in C_{*s}(X,L(H',H)),\qquad U_{1,2}=T;$$

$$U_{2,1}\in C_{*s}(X,L(H,H')),\qquad U_{2,1}=R;$$

$$U_{2,2}\in C_{*s}(X,L(H')),\qquad U_{2,2}=S.$$

We have

$$U(\Phi^0(a)\oplus\Psi(a))-(\Phi^1(a)\oplus\Psi(a))\,U\in C_n(X,K(H\oplus H'))$$

for all $a\in A$.

This gives

$$i_0^*[\tau]+[p\circ\Psi]=i_1^*[\tau]+[p\circ\Psi]$$

so that

$$i_0^*[\tau]=0\Rightarrow i_1^*[\tau]=0.\qquad\text{Q.E.D.}$$

For our next purposes, it will be useful to make the following working definition.

5.6. DEFINITION. *A nuclear separable unital C^*-algebra A is said to have the homotopy invariance property if for every finite-dimensional X and* $[\tau]\in\operatorname{Ext}(X\times[0,1],A)$ *we have*

$$i_0^*[\tau]=0\Rightarrow i_1^*[\tau]=0$$

(where $i_t\colon X\to X\times[0,1]$ is the injection $i_t(x)=(x,t)$).

Thus Proposition 5.5 means that nuclear quasidiagonal C^*-algebras have the homotopy-invariance property.

Endowing the space of $*$-monomorphisms $\tau: A \to C_{*s}(X, L(H))/C_{*s}(X, K(H))$ defining homogeneous X-extensions by A with the topology of point norm convergence, two such $*$-monomorphisms are called homotopic if they can be joined by a continuous curve in this space.

5.7. PROPOSITION. *Let A be a nuclear C^*-algebra which has the homotopy-invariance property, X, Y finite-dimensional compact metrizable spaces, $f, g: X \to Y$ continuous maps, and $[\tau_0]$, $[\tau_1] \in \mathrm{Ext}\,(X, A)$. Then we have:*

(i) if f and g are homotopic, then

$$f^*, g^*: \mathrm{Ext}\,(Y, A) \to \mathrm{Ext}\,(X, A)$$

are equal:

(ii) if τ_0 and τ_1 are homotopic then $[\tau_0] = [\tau_1]$.

Proof. (*i*) First let $[\tau] \in \mathrm{Ext}\,(X \times [0, 1], A)$, we shall prove that $i_0^*[\tau] = i_1^*[\tau]$. Indeed, by the symmetry $\alpha \mapsto 1 - \alpha$ of the segment $[0, 1]$ we infer that $i_0^*[\tau] = 0 \Leftrightarrow i_1^*[\tau] = 0$. Moreover, since i_0^* and i_1^* are surjective we infer that $i_0^*[\tau] \mapsto i_1^*[\tau]$ defines an automorphism of the group $\mathrm{Ext}\,(X, A)$. But since for every $[\sigma] \in \mathrm{Ext}\,(X, A)$ there is $[\tau] \in \mathrm{Ext}\,(X \times [0, 1], A)$ such that $i_0^*[\tau] = i_1^*[\tau] = [\sigma]$, we infer that $i_0^*[\tau] = i_1^*[\tau]$ always.

Now since f, g are homotopic, there is $F: X \times [0, 1] \to Y$ such that $F \circ i_0 = f$, $F \circ i_1 = g$, so that

$$f^*[\tau] = i_0^*(F^*[\tau]) = g^*[\tau]$$

for all $[\tau] \in \mathrm{Ext}\,(Y, A)$.

(*ii*) Since τ_0 and τ_1 are homotopic, there is *a* $*$-homomorphism

$$\sigma: A \to C_n([0, 1], C_{*s}(X, L(H))/C_n(X, K(H)))$$

such that each

$$A \ni a \mapsto (\sigma(a))\,(t) \in C_{*s}(X, L(H))/C_n(X, K(H))$$

defines a homogeneous X-extension by A and

$$(\sigma(a))\,(0) = \tau_0(a) \text{ and } (\sigma(a))\,(1) = \tau_1(a).$$

By the Bartle-Graves theorem ([33]),

$$C_n([0, 1],\; C_{*s}(X, L(H))/C_n(X, K(H)))$$

is isomorphic with

$$C_n([0, 1], C_{*s}(X, L(H)))/C_n([0, 1], C_n(X, K(H))).$$

But

$$C_n([0, 1], C_n(X, K(H))) \simeq C_n(X \times [0, 1], K(H))$$

and

$$C_n([0, 1], C_{*s}(X, L(H)))$$

is isomorphic with a C^*-subalgebra of

$$C_{*s}(X \times [0, 1], L(H)),$$

so we get a unital $*$-monomorphism

$$\tilde{\sigma}\colon A \to C_{*s}(X \times [0, 1], L(H))/C_n(X \times [0, 1], K(H)).$$

It is easily seen that $\tilde{\sigma}$ defines a homogeneous $(X \times [0, 1])$-extension by A and that $i_0^*[\tilde{\sigma}] = [\tau_0]$, $i_1^*[\tilde{\sigma}] = [\tau_1]$. Since i_0 and i_1 are homotopic, $[\tau_1] = [\tau_0]$ follows by (i). Q.E.D.

Recall that two unital $*$-homomorphisms $\rho_0, \rho_1\colon A \to B$ are called homotopic if there is a curve joining them in the space of unital $*$-homomorphisms endowed with the topology of point-norm convergence. Then Proposition 5.7. (ii) immediately yields the following corollary.

5.8. COROLLARY. *Let B be a nuclear unital separable C^*-algebra and A a nuclear C^*-algebra which has the homotopy-invariance property. Let further X be finite-dimensional and $\rho_0, \rho_1\colon A \to B$ be homotopic unital $*$-homomorphisms. Then*

$$\rho_{0*}, \rho_{1*}\colon \operatorname{Ext}(X, B) \to \operatorname{Ext}(X, A)$$

are equal.

Now we shall proceed to widen the class of C^*-algebras with the homotopy-invariance property.

For the next lemmas all ideals are closed two-sided and proper and for every ideal $J \subset A$, $\tilde{J}$ denotes the C^*-algebra, $\tilde{J} = \mathbf{C}\cdot 1 + J$.

5.9. LEMMA. *Let A be a unital nuclear separable C^*-algebra and $J \subset A$ an ideal. Then if A/J and $\tilde{J}$ have the homotopy-invariance property it follows that A has the homotopy-invariance property.*

Proof. Consider the diagram

$$\begin{array}{ccccc}
\operatorname{Ext}(X \times [0, 1], J) & \longleftarrow & \operatorname{Ext}(X \times [0, 1], A) & \longleftarrow & \operatorname{Ext}(X \times [0, 1], A/J) \\
\Big\downarrow i_0^* = i_1^* & & i_0^* \Big\downarrow \ \Big\downarrow i_1^* & & i_0^* = i_1^* \Big\downarrow \\
\operatorname{Ext}(X, \tilde{J}) & \longleftarrow & \operatorname{Ext}(X, A) & \longleftarrow & \operatorname{Ext}(X, A/J)
\end{array}$$

The horizontal rows are exact because of Thm. 4.1, also the vertical arrows at both ends are isomorphisms since $X \times \{0\}$ and $X \times \{1\}$ are clearly deformation retracts of $X \times [0, 1]$. Since the diagram is commutative with any of the two vertical arrows in the middle, we get that they must be equal. Q.E.D.

5.10. LEMMA. *Let A be a nuclear C^*-algebra, $J \subset A$ an ideal, $q: A \to A/J$ the canonical homomorphism. Assume further that J has the homotopy-invariance property and let $[\tau] \in \mathrm{Ext}\,(X \times [0, 1], A)$ be such that $i_0^*[\tau] = 0$. Then there is $[\sigma] \in \mathrm{Ext}\,(X \times [0, 1], A/J)$ such that $q_*[\sigma] = [\tau]$ and $i_0^*[\sigma] = 0$.*

Proof. Consider $j: X \times [0, 1] \to X$ the projection $j(x, t) = x$ and let $i: \tilde{J} \to A$ be the natural inclusion. Since $i_0: X \to X \times [0, 1]$ is a homotopy-equivalence it follows that

$$i_0^*: \mathrm{Ext}\,(X \times [0, 1], \tilde{J}) \to \mathrm{Ext}\,(X, \tilde{J})$$

is an isomorphism. Hence $i_0^*(i_*[\tau]) = 0$ implies $i_*[\tau] = 0$. Thus using Thm. 4.1, there is $[\sigma'] \in \mathrm{Ext}\,(X \times [0, 1], A/J)$ such that $q_*[\sigma'] = [\tau]$. Then we may take $[\sigma] = [\sigma'] - j^* i_0^*[\sigma']$. Q.E.D.

5.11. LEMMA. *Let A be a nuclear C^*-algebra and $J_1 \subset J_2 \subset J_3 \subset \ldots$ an increasing sequence of ideals, such that $\overline{\bigcup_{k=1}^{\infty} \tilde{J}_k} = A$. Assume also that $\tilde{J}_{k+1}/J_k$ has the homotopy-invariance property for all $k \in \mathbf{N}$. Then A has the homotopy-invariance property.*

Proof. Let $[\tau] \in \mathrm{Ext}\,(X \times [0, 1], A)$ be such that $i_0^*[\tau] = 0$. We shall first prove the existence of $[\sigma_k] \in \mathrm{Ext}\,(X \times [0, 1], A/J_k)$ such that $q_{*k}[\sigma_{k+1}] = [\sigma_k]$, $[\sigma_0] = [\tau]$, $i_0^*[\sigma_k] = 0$, where $q_k: A/J_k \to A/J_{k+1}$, $(J_0 = 0)$, are the canonical homomorphisms.

Indeed, since $\tilde{J}_{k+1}/J_k$ have the homotopy-invariance property, the existence of the $[\sigma_k]$ with the above properties follows by using Lemma 5.10 recurrently.

We shall now prove that this implies $[\tau] = 0$. In view of the above, there are Hilbert spaces H_k, H_k', H_k'', $H_k = H_k' \oplus H_k''$, $(k \geqslant 0)$, $*$-monomorphisms σ_k'

$$\sigma_k': A/J_k \to C_{*s}(X \times [0, 1], L(H_k))/C_n(X \times [0, 1], K(H_k)),$$

$*$-homomorphisms ρ_k

$$\rho_k: A/J_k \to C_{*s}(X \times [0, 1], L(H_k')),$$

unital completely positive maps ψ_k

$$\psi_k: A/J_{k+1} \to C_{*s}(X \times [0, 1], L(H_k'')),$$

and unitaries

$$U_k \in C_{*s}(X \times [0, 1], L(H_{k+1}, H_k''))$$

such that

$$[\sigma_k'] = [\sigma_k], \qquad (k \geqslant 0);$$

$$p \circ (\rho_k \oplus (\psi_k \circ q_k)) = \sigma_k', \qquad (k \geqslant 0);$$

$$\tilde{\alpha}(U_k) \circ \sigma_{k+1}' = p \circ \psi_k, \qquad (k \geqslant 0).$$

Define also completely positive maps

$$\varphi_{j,k}: A/J_{k-j} \to C_{*s}(X \times [0, 1], L(H_{k-j}))$$

for $0 \leqslant j \leqslant k$, by taking

$$\varphi_{0,k} = \rho_k \oplus (\psi_k \circ q_k) \text{ and } \varphi_{j+1,k} = \rho_{k-j-1} \oplus (\alpha(U_{k-j-1}) \circ \varphi_{j,k} \circ q_{k-j-1})$$

Denoting by φ_k the completely positive map $\varphi_{k,k}$, it is easily seen that

$$p \circ \varphi_k = \sigma_0', \qquad (\forall)\ k \geqslant 0,$$

$$\varphi_{k+1}|\, \tilde{J}_k = \varphi_k|\, \tilde{J}_k, \qquad (\forall)\ k \geqslant 0,$$

and $\varphi_k|\, \tilde{J}_k$ is a $*$-homomorphism.

Since $\bigcup_{k=0}^{\infty} \tilde{J}_k$ is dense in A, it follows that the completely positive maps φ_k are point-norm convergent to some unital $*$-homomorphism $\varphi: A \to C_{*s}(X \times [0, 1], L(H_0))$. Since $\varphi|\, \tilde{J}_k = \varphi_k|\, \tilde{J}_k$ hence $(p \circ \varphi)(a) = (p \circ \varphi_k)(a) = \sigma_0'(a)$ for all $a \in \tilde{J}_k$. Again by the density of $\bigcup_{k=0}^{\infty} \tilde{J}_k$ in A, we infer $p \circ \varphi = \sigma_0'$. Thus $[\sigma_0'] = 0$ and since $[\sigma_0'] = [\sigma_0] = [\tau]$, the Lemma follows. Q.E.D.

The next theorem involves composition series for C^*-algebras, the definition of which can be found in (4.3.2, [21]).

5.12. THEOREM. *Let A be a separable nuclear unital C^*-algebra having a composition series $(J_\rho)_{0 \leqslant \rho \leqslant \alpha}$ such that $\tilde{J}_{\rho+1}/J_\rho$ are quasidiagonal. Then A has the homotopy-invariance property.*

Proof. We prove by transfinite induction that the $\tilde{J}_\rho$ have the homotopy-invariance property.

The step from $\tilde{J}_\rho$ to $\tilde{J}_{\rho+1}$ follows from Lemma 5.9.

In case $\beta \leqslant \alpha$ is a limit ordinal and $\tilde{J}_\rho$ have the homotopy-invariance property for all $\rho < \beta$ our assertion follows from Lemma 5.11 and the remark that A being separable, we can find a sequence $\rho_1 \leqslant \rho_2 \leqslant \ldots$ of ordinals, $\rho_j < \beta\ (\forall)\ j \in \mathbb{N}$ such that $\tilde{J}_\beta = \overline{\bigcup_{j=1}^{\infty} J_{\rho_j}}$. Q.E.D.

Since GCR-C^*-algebras have composition series with CCR quotients (see [21], 4.3.4) and since CCR-C^*-algebras are quasidiagonal ([44]), we have the following corollary.

5.13. COROLLARY. *The GCR separable unital C^*-algebras have the homotopy-invariance property.*

§ 6

In this section we establish a short exact sequence in the X-"variable" for $\operatorname{Ext}(X, x_0; A)$.

For the short exact sequence in the X-variable, some preparation is necessary.

6.1. LEMMA. *Let X be a finite-dimensional compact metrizable space and $Y \subset X$ a closed subset. Suppose e_n: $Y \to H$ are continuous functions with the property that $\{e_n(y)\}_{n \in \mathbf{N}}$ is an orthonormal basis of H for every $y \in Y$. Then there are continuous functions $\tilde{e}_n$: $X \to H$, $(n \in \mathbf{N})$, such that $\{\tilde{e}_n(x)\}_{n \in \mathbf{N}}$ is an orthonormal basis of H for every $x \in X$ and $\tilde{e}_n | Y = e_n$, $(n \in \mathbf{N})$.*

Proof. Let $\{h_n\}_{n \in \mathbf{N}}$ be a dense sequence of non-zero elements in H, each vector occuring an infinity of times. Let also $\{F_n\}_{n \in \mathbf{N}}$, be an increasing sequence of closed subsets of X, such that

$$\bigcup_{n \in \mathbf{N}} F_n = X \setminus Y.$$

We shall construct recurrently continuous maps $\tilde{e}_n$: $X \to H$ satisfying:

$$\tilde{e}_n | Y = e_n, \ m \leqslant n \Rightarrow \langle \tilde{e}_n(x), \tilde{e}_m(x) \rangle = \delta_{m,n}, \quad (\forall) \ x \in X$$

and

$$\left\| h_n - \sum_{k=1}^{n} \langle h_n, \tilde{e}_k(x) \rangle \tilde{e}_k(x) \right\| \leqslant 1/n \text{ for all } x \in F_n.$$

Clearly the constructed $\tilde{e}_n$ will then satisfy the requirements of the lemma.

Suppose $\tilde{e}_k$ have been constructed for $k < n$ (if $n = 1$, the set of $k < n$ is void). Consider for each $x \in X$, the set $S_x \subset H$, which is the set of all vectors of length 1 in H which are orthogonal to $\{\tilde{e}_k(x);\ 1 \leqslant k < n\}$. It is easily seen that the set-valued function $X \ni x \mapsto S_x \subset H$ is lower-semicontinuous in the sense appearing in Michael's theorem ([33]). Also if $\varepsilon \leqslant 1/2$, then if

$$Q = \{f \in H;\ \|f - h\| < \varepsilon\} \cap S_x \neq \emptyset$$

for some $h \in H$ and $x \in X$, then Q is contractible, as can be easily seen using the map

$$F(t, f) = \|(1-t)h_0 + tf\|^{-1}((1-t)h_0 + tf), \quad h_0 \in Q.$$

Also by (10.8.2 in [21]) each S_x is contractible. Thus the set-valued map $X \ni x \mapsto S_x \subset H$ satisfies the conditions of Michael's theorem.

Defining $\zeta: X \to H$ by

$$\zeta(x) = h_n - \sum_{k=1}^{n-1} \langle h_n, \tilde{e}_k(x) \rangle \tilde{e}_k(x)$$

and considering $M \subset F_n$ the closed subset of F_n on which $\|\zeta(x)\| \geqslant 1/n$, let $g: M \cup Y \to H$ be the continuous map which is equal to $\zeta(x)/\|\zeta(x)\|$ for $x \in M$ and equal to $e_n(x)$ for $x \in Y$. Then $g(x) \in S_x$ for each $x \in M \cup Y$. Hence by

Michael's theorem ([33]), there is a continuous map $\tilde{e}_n: X \to H$ such that $\tilde{e}_n(x) \in S_x$ for all $x \in X$ and $\tilde{e}_n | (M \cup Y) = g$. Clearly

$$\langle \tilde{e}_n(x), \tilde{e}_m(x) \rangle = \delta_{n,m}$$

for all $m \leqslant n$ and $x \in X$. Also since

$$\left\| h_n - \sum_{k=1}^{n} \langle h_n, \tilde{e}_k(x) \rangle \tilde{e}_k(x) \right\| = \|\zeta(x) - \langle \zeta(x), \tilde{e}_n(x) \rangle \tilde{e}_n(x)\|,$$

we infer that

$$\left\| h_n - \sum_{k=1}^{n} \langle h_n, \tilde{e}_k(x) \rangle \tilde{e}_k(x) \right\|$$

is $< 1/n$ on $F_n \setminus M$ and is $= 0$ on M. Q.E.D.

6.2. COROLLARY. *Let X be a finite-dimensional compact metrizable space and $Y \subset X$ a closed subset. Let further $U: Y \to L(H)$ be a $*$-strongly continuous map such that $U(x)$ is unitary for each $x \in Y$. Then there is a $*$-strongly continuous map $\tilde{U}: Y \to L(H)$, such that $\tilde{U}(x)$ is unitary for every $x \in X$ and $\tilde{U} | Y = U$.*

Proof. Let $\{f_n\}_{n \in \mathbf{N}} \subset H$ be an orthonormal basis of H. Let further $e_n: Y \to H$ be defined by $e_n(y) = U(y) f_n$. Consider then the $\tilde{e}_n: X \to H$ provided by Lemma 6.1, and define

$$\tilde{U}: X \to L(H) \text{ by } \tilde{U}(x) f_n = \tilde{e}_n(x).$$

Then $X \ni x \mapsto \tilde{U}(x) \in L(H)$ is clearly unitary-valued and strongly-continuous. Since $\tilde{U}$ is unitary valued and strongly-continuous, it follows that it is also $*$-strongly-continuous. Q.E.D.

If Y is a closed subspace of X, then considering X/Y endowed with the base-point Y/Y, we shall write $\operatorname{Ext}(X, Y; A)$ instead of $\operatorname{Ext}(X/Y, Y/Y; A)$.

6.3. PROPOSITION. *Let Y be a closed subspace of the finite-dimensional metrizable compact space X and let $i: Y \to X$ and $j: X \to X/Y$ be the natural maps. Then assuming A is nuclear, we have the following exact sequence:*

$$\operatorname{Ext}(X, Y; A) \xrightarrow{j^*} \operatorname{Ext}(X, A) \xrightarrow{i^*} \operatorname{Ext}(Y, A).$$

Proof. Clearly $i^* \circ j^* = 0$, so it will be sufficient to prove that $\operatorname{Im} j^* \supset \operatorname{Ker} i^*$. Thus let $[\tau] \in \operatorname{Ext}(X, A)$ be such that $i^*[\tau] = 0$; we shall prove the existence of $[\sigma] \in \operatorname{Ext}(X, Y; A)$ such that $j^*[\sigma] = [\tau]$.

Since $i^*[\tau] = [i^*(\tau)] = 0$, there is a unitary $U \in C_{*s}(Y, L(H))$ implementing the equivalence of $i^*(\tau)$ and of some *constant* trivial homogeneous Y-extension by A. Thus there is a $*$-monomorphism $\mu_0: A \to L(H)$, $\mu_0(A) \cap K(H) = 0$, such that defining

$$\mu: A \to C_{*s}(Y, L(H)) \text{ by } (\mu(a))(y) = \mu_0(a), \ (\forall)\ y \in Y,$$

we have

$$\mu(a) - U(f \mid Y)\, U^* \in C_n(Y, K(H)) \text{ for } f \in p^{-1}(\tau(a)).$$

By Corollary 6.2, there is a unitary $\tilde{U} \in C_{*s}(X, L(H))$ such that $\tilde{U} \mid Y = U$. Then using the theorem of Dugundji for

$$\mu(a) - U(f \mid Y)\, U^* \in C_n(Y, K(H)),$$

we obtain that for every $a \in A$, there is $g \in C_{*s}(X/Y, L(H))$ such that $g \circ j \in \tilde{U}(p^{-1}(\tau(a)))\, \tilde{U}^*$ and $g(Y/Y) - \mu_0(a) \in K(H)$. Also, clearly two such g's differ only by an element of $C_n(X/Y, K(H))$. Thus defining

$$\sigma\colon A \to C_{*s}(X/Y, L(H))/C_n(X/Y, K(H))$$

by $\sigma(a) = p(g)$, where g is such that

$$g \circ j \in \tilde{U}(p^{-1}(\tau(a)))\, \tilde{U}^*,$$

we have

$$j^*(\sigma) = \tilde{\alpha}(\tilde{U}) \circ \tau$$

and $[\sigma] \in \operatorname{Ext}(X, Y; A)$. Q.E.D.

The following consequence of the preceding proposition is immediate:

6.4. COROLLARY. *Let X be a finite-dimensional compact metrizable space, $Y \subset X$ a closed subset and $x_0 \in Y \subset X$. Denoting by $i\colon Y \to X$ and $j\colon X \to X/Y$ the natural maps, for nuclear A we have the exact sequence:*

$$\operatorname{Ext}(X, Y; A) \xrightarrow{j^*} \operatorname{Ext}(X, x_0; A) \xrightarrow{i^*} \operatorname{Ext}(Y, x_0; A).$$

6.5. REMARK. *In case A is not nuclear, the proof of Proposition 6.3 still shows that $i^* \circ j^* = 0$ and $\operatorname{Ker} i^* \subset \operatorname{Im} j^*$.*

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Added in Proof: The authors have learned that the result contained in corollary 6.2. is not new, it can be found in a paper by M. J. Dupré in J. Functional Analysis, **22**, *3*, (1976), 295–322.