J. OPERATOR THEORY
1(1979), 123–136

CONTRACTIONS WITH RICH SPECTRUM HAVE INVARIANT SUBSPACES

S. BROWN, B. CHEVREAU and C. PEARCY

1. INTRODUCTION

Let $\mathscr{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathscr{L}(\mathscr{H})$ denote the algebra of all bounded linear operators on $\mathscr{H}$. As usual, a subspace $\mathscr{M}$ of $\mathscr{H}$ is said to be a *nontrivial invariant subspace* for an operator A in $\mathscr{L}(\mathscr{H})$ if $(0) \neq \mathscr{M} \neq \mathscr{H}$ and $A\mathscr{M} \subset \mathscr{M}$. In [4] the first author solved the invariant subspace problem for subnormal operators in $\mathscr{L}(\mathscr{H})$ and in so doing originated a technique for constructing invariant subspaces that was amenable to wider application. In this paper we use the techniques and results of [4] to show (Theorem 4.1) that all contraction operators in $\mathscr{L}(\mathscr{H})$ with sufficiently rich spectrum have nontrivial invariant subspaces. The main new contributions to the ideas of [4] contained in the present paper are the use of the Sz.-Nagy-Foiaş functional calculus for contractions and Lemmas 4.5 and 4.6. For completeness, however, we have chosen to begin at the beginning, and thus we have included some preliminary material of a general nature (§2) as well as some material on the Sz.-Nagy-Foiaş functional calculus (§3). In addition, because of the different setting, many of the results from [4] appear in a slightly different form. Nevertheless, it must be said that the credit for Theorem 4.1 largely belongs to the first author.

Additional interesting results based on [4] and the present paper have been obtained by Agler [1] and Stampfli [12].

2. GENERAL PRELIMINARIES

For purposes of completeness, we include some preparatory material of a general nature.

PROPOSITION 2.1. *Let X be a complex Banach space with dual X^*, and let $\mathscr{L}$ be a weak* closed subspace of X^*. If ${}^a\mathscr{L}$ denotes the preannihilator of $\mathscr{L}$ in X, then the annihilator $({}^a\mathscr{L})^a$ of ${}^a\mathscr{L}$ in X^* is equal to $\mathscr{L}$, and the mapping α from $(X/{}^a\mathscr{L})^*$ onto $\mathscr{L}$ defined by setting $\alpha(\hat{f}) = \hat{f} \circ \pi$, $\hat{f} \in (X/{}^a\mathscr{L})^*$, where π is the quotient map of X onto $X/{}^a\mathscr{L}$, is an isomorphism of $(X/{}^a\mathscr{L})^*$ onto $\mathscr{L}$.*

Proof. That $({}^a\mathscr{L})^a = \mathscr{L}$ is Corollary 16.3 of [2], and that α has the desired properties is Proposition 16.6 of [2].

Throughout the paper, we will denote the Banach space of trace-class operators in $\mathscr{L}(\mathscr{H})$ with the trace norm $\| \ \|_\tau$ by (τc). Recall from [5, Theorem 8, p. 105] that setting

$$(1) \qquad \langle A, T\rangle = \mathrm{tr}(AT),\ A \in \mathscr{L}(\mathscr{H}), \quad T \in (\tau c),$$

induces a bilinear functional on $\mathscr{L}(\mathscr{H}) \times (\tau c)$ that allows us to identify $(\tau c)^*$ with $\mathscr{L}(\mathscr{H})$. This identification, which we use hereafter without further comment, has the further property that the weak* topology defined on $\mathscr{L}(\mathscr{H}) = (\tau c)^*$ coincides with the ultraweak topology on $\mathscr{L}(\mathscr{H})$. (For more information about the ultraweak topology, see [5, p. 35].)

COROLLARY 2.2. *If $\mathscr{A}$ is any ultraweakly closed subspace of $\mathscr{L}(\mathscr{H})$ and ${}^a\mathscr{A}$ is the preannihilator of A in (τc), then $((\tau c)/{}^a\mathscr{A})^*$ can be identified with $\mathscr{A}$.*

Proof. Apply Proposition 2.1 with $X = (\tau c)$ and $\mathscr{L} = \mathscr{A}$.

We write (X^*, w^*) for the topological linear space consisting of the dual of a Banach space X with its weak* topology.

THEOREM 2.3. *Let X and Y be complex Banach spaces with X separable. A linear mapping $S: (X^*, w^*) \to (Y^*, w^*)$ is continuous if and only if whenever a sequence $\{\varphi_n\}_{n=1}^{\infty}$ converges to 0 in (X^*, w^*) then so does the sequence $\{S\varphi_n\}_{n=1}^{\infty}$ in (Y^*, w^*).*

Before proving Theorem 2.3 we give a corollary needed later.

COROLLARY 2.4. *Let X be a complex Banach and let $\mathscr{M}$ be a weak* closed subspace of X^*. Then the relative topology induced on $\mathscr{M}$ by (X^*, w^*) coincides with the weak* topology $\mathscr{M}$ obtains as the dual of $X/{}^a\mathscr{M}$. Consequently, if X is separable, then the criterion for continuity obtained in Theorem 2.3 aslo applies to a linear mapping S from $\mathscr{M}$ equipped with its relative topology in (X^*, w^*) into a weak* closed subspace $\mathscr{N}$ of (Y^*, w^*) equipped with its relative topology.*

Proof. The equality of the two mentioned topologies is proved by inspection with nets. Since $X/{}^a\mathscr{M}$ inherits the separability of X and $\mathscr{M}$ and $\mathscr{N}$ can be identified with $(X/{}^a\mathscr{M})^*$ and $(Y/{}^a\mathscr{N})^*$, respectively, the last statement is just a rephrasing of Theorem 2.3.

In order to prove Theorem 2.3 we need two intermediate results.

PROPOSITION 2.5. *Let X and Y be Banach spaces. A linear mapping $S: (X^*, w^*) \to (Y^*, w^*)$ is continuous if and only if there exists a map $T: Y \to X$*

such that

$$\langle Y, S\varphi\rangle = \langle Ty, \varphi\rangle, \quad y \in Y, \ \varphi \in X^*. \tag{2}$$

Furthermore, when this happens T is a bounded linear map and $S = T^*$.

Proof. Suppose first that there exists a map $T: Y \to X$ such that (2) holds. Let $\{\varphi_\lambda\}$ be a net converging to 0 in (X^*, w^*). For any y in Y we have $\langle y, S\varphi_\lambda\rangle = \langle Ty, \varphi_\lambda\rangle$, and therefore $\langle y, S\varphi_\lambda\rangle$ tends to 0 for all y. This proves the continuity of S (in the weak* topologies).

Conversely, suppose that $S: (X^*, w^*) \to (Y^*, w^*)$ is continous. Then for any y in Y, the mapping $\varphi \to \langle y, S\varphi\rangle$ is a continuous linear functional on (X^*, w^*), and therefore (cf. [2, Problem 15J]) can be identified with a unique element Ty of X satisfying $\langle y, S\varphi\rangle = \langle Ty, \varphi\rangle$. To prove the last statement, it is sufficient to show that if (2) holds, then T is linear and bounded, for in this case by definition we will have $S = T^*$. But linearity results from a standard uniqueness argument and boundedness from a straightforward application of the closed graph theorem.

The next proposition is contained in [2, Problem 16X].

PROPOSITION 2.6. *Let X be a Banach space. A linear submanifold of* X^* *is weak* closed if and only if its intersection with the unit ball of* X^* *is weak* closed.*

Proof of Theorem 2.3. Only the "if" part is not trivial. Let then S be a linear map from X^* to Y^* such that the sequence $\{S\varphi_n\}_{n=1}^\infty$ converges weak* to 0 in Y^* whenever $\{\varphi_n\}_{n=1}^\infty$ does so in X^*. For any y in Y the mapping $Ty: \varphi \to \langle y, S\varphi\rangle$ is a linear functional on X^*. Assume, for the moment, that Ty is weak* continuous for all y. Then Ty can be identified with a unique element of X (cf. [2, Problem 15J]), and the relation (2) holds. Thus, by Proposition 2.5, to complete the proof we only need to show that each mapping Ty is weak* continuous, or equivalently that it has weak* closed kernel (cf. [2, Problem 12U]). Let $E_y = \mathscr{B}^* \cap \operatorname{Ker}(Ty)$ where $\mathscr{B}^*$ denotes the unit ball of X^*. Since X is separable, $\mathscr{B}^*$ is metrizable and the weak* closure of E_y coincides with its weak* sequential closure. Let then $\{\varphi_n\}_{n=1}^\infty$ be a sequence in E_y converging to some φ in X^*. Since $\mathscr{B}^*$ is weak* closed, φ belongs to $\mathscr{B}^*$. By definition of E_y we have $\langle y, S\varphi_n\rangle = 0$ for all n, and by the hypothesis on S we get $\langle y, S\varphi\rangle = \lim \langle y, S\varphi_n\rangle = 0$, that is, φ belongs to $\operatorname{Ker}(Ty)$. Thus E_y is weak* closed and so is $\operatorname{Ker}(Ty)$ by Proposition 2.6. As observed before, this concludes the proof.

THEOREM 2.7. *Let X and Y be Banach spaces and let S be a continuous linear map from* (X^*, w^*) *into* (Y^*, w^*) *with trivial kernel and norm closed range. Then* $S(X^*)$ *is weak* closed and S is a weak* homeomorphism of* X^* *onto* $S(X^*)$.

Proof. By Proposition 2.5, $S = T^*$ with T a bounded linear operator from Y to X. Since $S(X^*)$ is norm closed, it follows from [2, p. 359] that $S(X^*)$ is also weak* closed and that the range of T is norm closed. By Proposition 2.1, there exists a Banach space Y_1 such that $Y_1^* = S(X^*)$, and by Corollary 2.4 there is no loss of generality in viewing S as a map from X^* onto Y_1^*. A repetition of the above arguments yields a bounded linear operator T_1 from Y_1 to X with norm closed range such that $S = T_1^*$. The operator T_1 has trivial kernel because S is onto and dense range because S is one-to-one. Therefore $T_1(Y_1) = X$ and T_1 is invertible. It follows that S is also invertible and the equality $S^{-1} = (T^{-1})^*$ implies (by Proposition 2.5) that $S^{-1}:(Y_1^*, w^*) \to (X^*, w^*)$ is continuous. Thus the proof is complete.

Recall that a set G in a vector space X is said to be *balanced* if $\lambda G \subset G$ for $|\lambda| \leqslant 1$. The *absolutely convex hull* of a set G is the smallest convex and balanced set containing G. Alternatively it is the collection of all linear combinations $\alpha_1 x_1 + \ldots + \alpha_n x_n$ of vectors $x_1, \ldots, x_n$ in G such that $|\alpha_1| + \ldots + |\alpha_n| \leqslant 1$. The following is the last proposition of a general nature that we shall need.

PROPOSITION 2.8. *Let X be a complex Banach space, and let E be a subset of the closed unit ball $\mathscr{B}$ of X such that for all φ in X^*, $\|\varphi\| = \sup_{x \in E} \langle x, \varphi \rangle$. Then the closure of the absolutely convex hull of E is the entire unit ball $\mathscr{B}$.*

Proof. Let $\mathscr{C}$ be the closed absolutely convex hull of E (that is, the closure of the absolutely convex hull of E). Clearly $\mathscr{C} \subset \mathscr{B}$. Suppose $\mathscr{B} \neq \mathscr{C}$ and let x_0 belong to $\mathscr{B} \setminus C$. By a standard consequence of the Hahn-Banach theorem (cf. [2, Prop. 14.15]), there exists a linear functional g in X^* and a real number c such that $\mathrm{Re}(g(x)) \leqslant c$ for all x in $\mathscr{C}$, while $\mathrm{Re}(g(x_0)) > c$. Since $0 \in \mathscr{C}$ the number c is non-negative. For any x in $\mathscr{C}$, write $|g(x)| = \lambda g(x)$ with $|\lambda| = 1$. We have $|g(x)| = \mathrm{Re}(\lambda g(x)) = \mathrm{Re}(g(\lambda x)) \leqslant c$. Thus $|g(x)| \leqslant c$ for all x in $\mathscr{C}$. But then, by the hypothesis on E, we have $\|g\| \leqslant c$, contradicting the fact that $\|g\| \geqslant |g(x_0)| \geqslant \mathrm{Re}\,(g(x_0)) > c$. Therefore $\mathscr{C} = \mathscr{B}$.

To see what the preceding results have to do with an arbitrary operator A in $\mathscr{L}(\mathscr{H})$, we need the following definition.

DEFINITION 2.9. Let $A \in \mathscr{L}(\mathscr{H})$. We denote by $\mathscr{A}_A$ the smallest ultraweakly closed subalgebra of $\mathscr{L}(\mathscr{H})$ containing A and 1, and we call $\mathscr{A}_A$ the *ultraweakly closed algebra generated* by A. (It is clear that $\mathscr{A}_A$ is just the closure in the ultraweak topology of the algebra of all polynomials $p(A)$.)

Thus $\mathscr{A}_A$ contains the norm closed algebra generated by A and is contained in the weakly closed algebra generated by A. Moreover, since $\mathscr{L}(\mathscr{H}) = (\tau c)^*$, it follows from Proposition 2.1 that $\mathscr{A}_A$ is the dual space of $(\tau c)/{}^a\mathscr{A}_A$, that is, $\mathscr{A}_A = ((\tau c)/{}^a\mathscr{A}_A)^*$, where the duality is given by the relation

$$\langle [T], B \rangle = \mathrm{tr}(BT), \qquad B \in \mathscr{A}_A,\ [T] \in (\tau c)/{}^a\mathscr{A}_A. \tag{3}$$

Thus $\mathscr{A}_A$ carries a weak* topology, and by virtue of Corollary 2.4, this topology on $\mathscr{A}_A$ coincides with the relative ultraweak topology $\mathscr{A}_A$ obtains as a subspace of

$\mathcal{L}(\mathcal{H})$. This topology on $\mathcal{A}_A$ will be called interchangeably the *ultraweak* or *weak** *topology* on $\mathcal{A}_A$.

3. FUNCTIONAL CALCULUS PRELIMINARIES

In this section we set forth for completeness some preliminary material on function spaces and the Sz.-Nagy-Foiaş functional calculus for contractions [8, p. 109] that will be needed in Section 4.

We write T for the unit circle in the complex plane and D for the open unit disc Int(T). For $1 \leqslant p < +\infty$, let $L_p(T)$ denote the Banach space of all functions f on T such that $|f|^p$ is integrable with respect to normalized Lebesgue measure on T, and let $H_2(T)$ denote the subspace of the Hilbert space $L_2(T)$ spanned by the functions $\{t^n\}_{n=0}^{\infty}$. Furthermore, let $L_\infty(T)$ denote the Banach algebra of all essentially bounded functions in $L_2(T)$ under the essential supremum norm, and let $H_\infty = H_\infty(T)$ be the closed subalgebra of $L_\infty(T)$ defined by $H_\infty = L_\infty(T) \cap H_2(T)$. Since $L_\infty(T) = L_1(T)^*$, $L_\infty(T)$ carries a weak* topology, and it is well known that H_∞ is also closed in this topology (cf. [6, p. 27]). By Corollary 2.4, the relative weak* topology on H_∞ is the same as the weak* topology that accrues to H_∞ by virtue of being the dual space of $L_1(T)/{}^{a}H_\infty$(Prop. 2.1). Henceforth in this paper we shall refer to this topology simply as the *weak* topology* on H_∞ without further explanation.

Let $\mathcal{A}_\infty$ denote the Banach algebra of all bounded holomorphic functions on D under the sup norm. There is an intimate and well-known connection between H_∞ and $\mathcal{A}_\infty$ that we shall need later (cf. [11], Th. 17.10]).

PROPOSITION 3.1. *There is a unique unital Banach algebra isomorphism* $\tilde{}$ *of* H_∞ *onto* $\mathcal{A}_\infty$ *such that* $\widetilde{p(t)} = p(\lambda)$ *for every polynomial p. If h is any function in* H_∞, *then the sequence* $\{\hat{h}(n)\}_{n=0}^{\infty}$ *of Fourier coefficients of h relative to the orthonormal basis* $\{t^n\}_{n=0}^{\infty}$ *for* $H_2(T)$ *is identical with the sequence of Taylor coefficients of* $\tilde{h}$. *Moreover* $h(t) = \lim_{r \uparrow 1} \tilde{h}(rt)$ *almost everywhere on T, and for every* λ_0 *in D,*

$$\tilde{h}(\lambda_0) = \frac{1}{2\pi i} \int_T \frac{h(t)}{t - \lambda_0} \, dt.$$

Consequently if $\lambda_0 \in D$ *and* E_{λ_0} *is the linear functional on* H_∞ *obtained by setting* $E_{\lambda_0}(h) = \tilde{h}(\lambda_0)$, $h \in H_\infty$, *then* E_{λ_0} *is weak* continuous. Furthermore, for every positive integer n, the linear functional* $h \to \hat{h}(n)$ *on* H_∞ *is weak* continuous. Finally, if* $\lambda_0 \in D$, *then any h in* H_∞ *can be decomposed as* $h(t) = \tilde{h}(\lambda_0) + (t - \lambda_0)g(t)$ *where* $g \in H_\infty$ *and*

$$\|g\|_\infty \leqslant (2/(1 - |\lambda_0|)) \, \|h\|_\infty .$$

We now recall some of the facts about the Sz.-Nagy-Foiaş functional calculus for contractions (i.e., for operators A satisfying $\|A\| \leqslant 1$) that we will need later. A contraction A is called *completely nonunitary* if there exists no reducing subspace $\mathcal{M} \neq (0)$ for A such that $A|\mathcal{M}$ is unitary.

THEOREM 3.2. *Let A be a completely nonunitary contraction in $\mathcal{L}(\mathcal{H})$, and let $\mathcal{A}_A$ be the ultraweakly closed algebra generated by A* (Def. 2.9). *Then there is an algebra homomorphism $\varphi : h \to h(A)$ of H_∞ into $\mathcal{A}_A$ with the following properties:*

(a) *If $h(t) \equiv 1$, then $h(A) = 1$, and if $h(t) \equiv t$, then $h(A) = A$,*

(b) *If $h \in H_\infty$ and $\tilde{h}(\lambda) = \sum_{k=0}^{\infty} c_k \lambda^k$ then $h(A) = \lim_{r \to 1} \sum_{k=0}^{\infty} r^k c_k A^k$, where this limit exists in the strong operator topology,*

(c) *$\|h(A)\| \leqslant \|h\|_\infty$ for every h in H_∞,*

(d) *If $\{h_n\}$ is a bounded sequence in H_∞ and $\{\tilde{h}_n\}$ converges pointwise to 0 on D, then $\{h_n(A)\}$ converges to 0 in the weak* topology on $\mathcal{A}_A$; if the convergence o $\{\tilde{h}_n\}$ to 0 is uniform in D, then $\|h_n(A)\| \to 0$.*

(e) *If $\{h_n\}$ is a bounded sequence in H_∞ and $\{h_n(t)\}$ converges to zero almost everywhere on T, then $\{h_n(A)\}$ converges to 0 in the ultrastrong operator topology* (*cf.* [4, p. 36]),

(f) *The mapping φ is continuous if both H_∞ and $\mathcal{A}_A$ are given their weak* topologies,*

(g) *If φ is an isometry, then φ is a Banach algebra isomorphism of H_∞ onto $\mathcal{A}_A$ that is moreover a homeomorphism when H_∞ and $\mathcal{A}_A$ are given their weak* topologies.*

Proof. Most of this result is contained in Theorem III.1.2 of [8], but there are some few details to be checked. Since H_∞ is the weak* closure of the set of polynomials and (a) is valid, to verify that all operators $h(A)$ belong to $\mathcal{A}_A$, it suffices to prove (*f*). But according to Theorem 2.3 (which is applicable since $L_1(T)$ is separable), to prove (*f*) it suffices to show that if $\{h_n\}$ is a sequence in H_∞ that is converging weak* to 0, then the sequence $\{h_n(A)\}$ converges ultraweakly to zero. Such a sequence $\{h_n\}$ must be bounded by the uniform boundedness principle and must satisfy $\tilde{h}_n(\lambda) \to 0$ pointwise on D by Proposition 3.1. Thus the desired continuity follows from (d), which is essentially contained in Theorem III. 1.2. of [7], and the well-known fact that on bounded subsets of $\mathcal{L}(\mathcal{H})$ the weak and ultraweak operator topologies coincide, as do the strong and ultrastrong operator topologies [5, p. 36]. To prove (g) it suffices to note that if φ is an isometry, then the range of φ is norm closed, and thus by Theorem 2.7, range φ is weak* closed and φ is a weak* homeomorphism of H_∞ onto range φ. But the only ultraweakly closed subalgebra of $\mathcal{A}_A$ containing A and 1 is $\mathcal{A}_A$ itself, so range $\varphi = \mathcal{A}_A$.

4. THE MAIN THEOREM

In this section we finally prove the central theorem of the paper, which is the following.

THEOREM 4.1. *Let A be any operator in $\mathscr{L}(\mathscr{H})$ such that $\|A\| = 1$ and such that $\sigma(A) \cap D$ is sufficiently large that*

$$\sup_{\lambda \in \sigma(A) \cap D} |\tilde{h}(\lambda)| = \|h\|_\infty, \quad h \in H_\infty. \tag{4}$$

Then A has a nontrivial invariant subspace.

Obviously this theorem applies to any contraction whose spectrum is the closed unit disc, and, more generally, by the maximum modulus principle, to any contraction A such that $\sigma(A)$ contains some annulus $\{\lambda : 1 - \varepsilon \leqslant |\lambda| \leq 1\}$. Moreover, (4) will also be satisfied if $\sigma(A)$ simply contains the union of the unit circle and a spiral asymptotic to the unit circle. In fact, by [10, Prop. 4.15] there are operators A satisfying (4) such that $\sigma(A) \cap D$ is countable.

The proof of this theorem will be accomplished by making some preliminary reductions and then proving a sequence of lemmas. There are several assumptions that one can make about the operator A in question without loss of generality when looking for invariant subspaces. First, we may (and do) assume that A is completely nonunitary. (For if not, then the unitary part of A provides a supply of invariant subspaces.) Next, note that by virtue of (4), A is not a scalar. Thus by [8, Th. II.5.4] we may assume, by taking adjoints if necessary, that the sequence $\{A_n\}_{n=0}^{\infty}$ tends to 0 in the strong operator topology. (Note that taking adjoints does not affect the validity of (4).) Finally, we may suppose that the left essential spectrum $\sigma_{\ell e}(A)$ of A coincides with $\sigma(A)$, for otherwise either A or A^* has an eigenvalue (cf. [9, p. 47]).

Since A is completely nonunitary, we may consider the homomorphism $\varphi : H_\infty \to \mathscr{A}_A$ given by Theorem 3.2. By [7, Corollary 3.1], we have, for any h in H_∞, $\tilde{h}(\sigma(A) \subset D) \cap \sigma(h(A))$. Therefore, by (4),

$$\|h\|_\infty = \sup_{\lambda \in \sigma(A) \cap D} |\tilde{h}(\lambda)| \leq \|h(A)\|,$$

and by virtue of (c) of Theorem 3.2 we conclude that $\|h\|_\infty = \|h(A)\|$ for every h in H_∞, or, in other words, that φ is an isometry. Thus (g) of the same theorem tells us that the map $h \to h(A)$ is a Banach algebra isomorphism of H_∞ onto $\mathscr{A}_A$ that is, at the same time, a homeomorphism between H_∞ and $\mathscr{A}_A$ when these algebras are given their weak* topologies. This is an important ingredient in the proof of Theorem 4.1, and we use it first to obtain the following lemma. Throughout the remainder of the paper, we shall write Q for the quotient space $(\tau c)/^a \mathscr{A}_A$, which is the predual of $\mathscr{A}_A$. Elements of Q will be written as equivalence classes $[C]$, where $C \in (\tau c)$, and the quotient norm on Q will be denoted by $\| \quad \|_Q$.

LEMMA 4.2. *For every λ in D, there exists an element $[C_\lambda]$ in Q such that*

$$\langle [C_\lambda], h(A) \rangle = \operatorname{tr}(h(A)C_\lambda) = \tilde{h}(\lambda), \quad h \in H_\infty, \tag{5}$$

where the notation $\langle [C_\lambda], h(A) \rangle$ is as in (3) *and denotes the dual action between $\mathscr{A}_A$ and Q.*

Proof. We observed in Proposition 3.1 that for any fixed λ in D, the mapping $h \to \tilde{h}(\lambda)$ is a weak* continuous linear functional on H_∞. Thus, since the map $h(A) \to h$ is a weak* homeomorphism of $\mathscr{A}_A$ onto H_∞, the map $h(A) \to \tilde{h}(\lambda)$ is a weak* continuous linear functional on $\mathscr{A}_A$. But according to [2, Problem 15J], such linear functionals must arise from the dual action of Q on $\mathscr{A}_A$, and hence there exists an element $[C_\lambda]$ in Q such that (5) is valid.

We turn now to a brief outline of another key idea from [4]. As usual, if $x, y \in \mathscr{H}$, we denote by $x \otimes y$ the rank-one operator $u \to (u, y)x$ in $\mathscr{L}(\mathscr{H})$. It is well-known that $\|x \otimes y\| = \|x \otimes x\|_\tau = \|x\| \cdot \|y\|$ where $\| \quad \|_\tau$, as before, denotes the trace norm. Moreover, an easy computation shows that $\operatorname{tr}(x \otimes y) = (x, y)$ and that if $B \in \mathscr{L}(\mathscr{H})$, then $B(x \otimes y) = Bx \otimes y$. Suppose, for the moment, that it can be shown that there exist vectors x and y in $\mathscr{H}$ such that $[x \otimes y] = [C_0]$ in Q, where C_0 is as in Lemma 4.2. Then for every h in H_∞,

$$\tilde{h}(0) = \operatorname{tr}(h(A)(x \otimes y)) = \operatorname{tr}(h(A)x \otimes y) = (h(A)x, y).$$

In particular, taking $h = 1$, we see that $(x, y) = 1$, so that x and y are nonzero. Moreover, taking $h(t) = t^{n+1}$, we see that $(A^n(Ax), y) = 0$ for all positive integers n. If $Ax = 0$, then kernel A is a nontrivial invariant subspace for A, while if $Ax \neq 0$, then Ax is noncyclic for A, in which case $\bigvee_{n=0}^{\infty} A^n(Ax)$ is a nontrivial invariant subspace for A. Thus the proof of Theorem 4.1 can be completed by proving the existence of vectors x and y in $\mathscr{H}$ such that $[x \otimes y] = [C_0]$, and this is accomplished by establishing a sequence of lemmas. The first lemma shows that for any λ in $\sigma(A) \cap D$, $[C_\lambda]$ is at least a limit (in Q) of images of rank-one operators.

LEMMA 4.3. (cf. [4, Lemma 4.2]). *Let $\lambda \in \sigma(A) \cap D$. Then there exists an orthonormal sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ such that $\|(A - \lambda)x_i\| \to 0$, and for any such sequence, $\|[x_i \otimes x_i] - [C_\lambda]\|_Q \to 0$.*

Proof. Since $\lambda \in \sigma_{le}(A)$, one knows (cf. [9, Prop. 2.15]) that there exists an orthonormal sequence $\{x_i\}_{i=1}^{\infty}$ in $\mathscr{H}$ such that $\|(A - \lambda)x_i\| \to 0$. Since the element $[x_i \otimes x_i] - [C_\lambda]$ belongs to the Banach space Q whose dual is $\mathscr{A}_A$, it follows from a consequence of the Hahn-Banach theorem (cf. [2, Cor 14.11]) that for each i there exists an operator $B_i = h_i(A)$ in $\mathscr{A}_A$ such that $\|B_i\| = \|h_i\|_\infty = 1$ and

$$\begin{aligned}\|[x_i \otimes x_i] - [C_\lambda]\|_Q &= \langle [x_i \otimes x_i] - [C_\lambda], B_i \rangle \\ &= \operatorname{tr}(h_i(A)(x_i \otimes x_i - C_\lambda)).\end{aligned}$$

Let $\beta_i = \tilde{h}_i(\lambda)$ and write $h_i(t) = \beta_i + (t - \lambda)k_i(t)$. Then, by Proposition 3.1, $k_i \in H_\infty$ and $\|k_i\|_\infty \leq m = 2/(1 - |\lambda|)$. Thus for each i we have

$$\|[x_i \otimes x_i] - [C_\lambda]\|_Q = \mathrm{tr}(\{\beta_i + k_i(A)(A - \lambda)\}\{x_i \otimes x_i - C_\lambda\})$$

$$= \beta_i\{\mathrm{tr}(x_i \otimes x_i) - \mathrm{tr}(C_\lambda)\} + \mathrm{tr}(k_i(A)(A - \lambda)(x_i \otimes x_i))$$

$$= \mathrm{tr}(\{k_i(A)(A - \lambda)x_i\} \otimes x_i) = (k_i(A)(A - \lambda)x_i, x_i)$$

$$\leqslant \|k_i(A)\| \cdot \|(A - \lambda)x_i\|$$

since $\mathrm{tr}(x_i \otimes x_i) = 1$ and $\mathrm{tr}(C_\lambda) = \langle [C_\lambda], 1\rangle = 1$ also. From the facts that $\|k_i(A)\| = \|k_i\|_\infty \leq m$ and $\|(A - \lambda)x_i\| \to 0$ we deduce the desired result.

LEMMA 4.4 (cf. [4, Lemma 4.3]). *Let $\lambda \in \sigma(A) \cap D$ and let $\{x_i\}_{i=1}^\infty$ be any orthonormal sequence such that $\|(A - \lambda)x_i\| \to 0$. Then for any fixed w in $\mathcal{H}$, $\|[x_i \otimes w]\|_Q \to 0$.*

Proof. Just as in the previous lemma, we conclude that there exists a sequence $\{h_i\}_{i=1}^\infty$ of functions in H_∞ such that for all i, $\|h_i\|_\infty = 1$ and

$$\|[x_i \otimes w]\|_Q = \mathrm{tr}(h_i(A)(x_i \otimes w)) = (h_i(A)x_i, w).$$

Writing as before $h_i(t) = \tilde{h}_i(\lambda) + (t - \lambda)k_i(t)$, we have

$$\|[x_i \otimes w]\|_Q = \tilde{h}_i(\lambda)(x_i, w) + (k_i(A)(A - \lambda)x_i, w).$$

The first summand on the right tends to zero since the $\tilde{h}_i(\lambda)$ are bounded and $\{x_i\}$ is an orthonormal sequence, and the second summand tends to zero as before; hence the result.

The next lemma is almost a symmetrical version of Lemma 4.4, but it is much harder to prove.

LEMMA 4.5. *Let $\{x_i\}_{i=1}^\infty$ be any orthonormal sequence in $\mathcal{H}$. Then for every fixed w in $\mathcal{H}$, $\|[w \otimes x_i]\|_Q \to 0$.*

Proof. Once again, let $\{h_i\}$ be a sequence of functions in the unit sphere of H_∞ such that

$$\|[w \otimes x_i]\|_Q = \langle [w \otimes x_i], h_i(A)\rangle =$$

$$= \mathrm{tr}(h_i(A)(w \otimes x_i)) = \mathrm{tr}(\{h_i(A)w\} \otimes x_i) =$$

$$= (h_i(A)w, x_i).$$

Suppose now that this sequence does not converge to zero. Then there exist a positive number δ and a subsequence $\{i_j\}$ of positive integers such that $(h_{i_j}(A)w, x_{i_j}) > \delta$.

Since the closed unit ball of H_∞ is weak* compact, there exists a subsequence $\{h_{i_{j_k}}\}$ of h_{i_j} converging (in the weak* topology) to some g in the unit ball of H_∞. For convenience of notation we set $g_k = h_{i_{j_k}} - g$ and $y_k = x_{i_{j_k}}$. Since $(g(A)w, y_k) \to 0$, it follows that for k sufficiently large, $|(g_k(A)w, y_k)| > \delta$, and it is clear from the definition that $\|g_k\|_\infty \le 2$ and that $\{g_k\}$ converges weak* to 0. Using the fact that $\{A^n\}$ converges strongly to 0, we fix a positive integer N such that $\|A^N w\| \le \delta/6$. We write

$$g_k(t) = p_k(t) + t^N m_k(t)$$

where p_k is a polynomial of degree at most $N-1$, the coefficients of p_k are the corresponding Fourier coefficients of g_k, and $m_k \in H_\infty$.

By Proposition 3.1, if n is any fixed positive integer, then the sequence $\{\hat{g}_k(n)\}$ of n^{th} Fourier coefficients of the sequence $\{g_k\}$ must tend to zero. Thus, since $\|p_k\|_\infty$ is bounded by the sum of the moduli of the N coefficients of p_k, we have $\|p_k\|_\infty \to 0$. Furthermore, since $\|g_k\|_\infty \le 2$ for all k, it follows that $\|m_k\|_\infty \le 3$ for k sufficiently large. Thus the inequality

$$\delta < |(g_k(A)w, y_k)| = |(p_k(A)w, y_k) + (m_k(A)A^N w, y_k)| \le$$

$$\le \|p_k(A)\| \cdot \|w\| + \|m_k(A)\| \cdot \|A^N w\| \le \|p_k\|_\infty \cdot \|w\| + 3(\delta/6)$$

is both valid and untenable for k sufficiently large, and we have reached a contradiction.

LEMMA 4.6. *Let $\lambda_1, \dots, \lambda_n$ be any finite sequence of (not necessarily distinct) numbers from $\sigma(A) \cap D$. Then there exists a corresponding family $\{x_i^1\}, \dots, \{x_i^n\}$ of mutually orthogonal orthonormal sequences such that $\lim_{i\to\infty} \|(A - \lambda_j)x_i^j\| = 0$ for $1 \le j \le n$ and $\lim_{i\to\infty} \|[x_i^j \otimes x_i^k]\|_Q = 0$ for all $1 \le j, k \le n$ with $j \ne k$. Furthermore if $\delta_1, \dots, \delta_n$ is any sequence of n scalars and $u_i = \sum_{j=1}^n \delta_j x_i^j$, $v_i = \sum_{j=1}^n \bar{\delta}_j x_i^j$, then*

$$\lim_{i\to\infty} \|[u_i \otimes v_i] - \sum_{j=1}^n \delta_j^2 [C_{\lambda_j}]\|_Q = 0.$$

Proof. Consider the operator $\lambda_1 \oplus \dots \oplus \lambda_n$, where each λ_j acts on an infinite dimensional space. Since the set $\{\lambda_1, \dots, \lambda_n\} \subset \sigma_{le}(A)$, one can apply [3, Theorem A] with $\lambda_1 \oplus \dots \oplus \lambda_n = N_\varepsilon$ in the notation of that theorem, and it follows easily that there exists a family $\{x_i^1\}, \dots, \{x_i^n\}$ of mutually orthogonal orthonormal sequences (i.e., $(x_i^j, x_l^k) = \delta_{jk}\delta_{il}$) such that $\lim_{i\to\infty} \|(A - \lambda_j)x_i^j\| = 0$ for all $1 \leqslant j \leqslant n$.

Suppose now that $1 \leqslant j, k \leqslant n$ with $j \ne k$. For clarity of notation we set $\lambda = \lambda_j$, $\mu = \lambda_k$, $x_i = x_i^j$, and $y_i = x_i^k$. As before, there exists a sequence $\{h_i\}_{i=1}^\infty$

in H_∞ such that $\|h_i\|_\infty = 1$ and $\|[x_i \otimes y_i]\|_Q = (h_i(A)x_i, y_i)$ for each i. Using the decomposition $h_i(t) = \tilde{h}_i(\lambda) + (t-\lambda)g_i(t)$, we obtain

$$\|[x_i \otimes y_i]\|_Q = \tilde{h}_i(\lambda)(x_i, y_i) + (g_i(A)(A-\lambda)\,x_i, y_i) = (g_i(A)(A-\lambda)\,x_i, y_i),$$

and the right-hand side of this equation tends to zero as before.

To prove the last assertion in the statement of the lemma, we write

$$[u_i \otimes v_i] = \sum_{j=1}^{n} \delta_j^2 [x_i^j \otimes x_i^j] + \sum_{1 \leqslant j \neq k \leqslant n} \delta_j \delta_k [x_i^j \otimes x_i^k].$$

By what has already been proved, the second summand on the right converges to 0 (in $\|\ \|_Q$), and by Lemma 4.3 the first summand converges to $\sum_{j=1}^{n} \delta_j^2 [C_{\lambda_j}]$.

LEMMA 4.7 ([4, Lemma 4.4]). *Let $S \subset Q$ be the closed absolutely convex hull of the set $\{[C_\lambda]: \lambda \in \sigma(A) \cap D\}$. Then S is equal to the closed unit ball of Q.*

Proof. For any $h(A)$ in $\mathscr{A}_A$ we have

$$\|h(A)\| = \|h\|_\infty = \sup_{\lambda \in \sigma(A) \cap D} |\tilde{h}(\lambda)| = \sup_{\lambda \in \sigma(A) \cap D} |\langle [C_\lambda], h(A) \rangle|,$$

and Proposition 2.8 gives the desired conclusion.

LEMMA 4.8 (cf. [4, Lemma 4.5]). *Let $[L] \in Q$ and suppose there exist vectors s and s' in $\mathscr{H}$ such that $\|[s \otimes s'] - [L]\|_Q < \varepsilon < 1$. Then there exist vectors t and t' in $\mathscr{H}$ such that $\|s - t\| < \sqrt{\varepsilon}$, $\|s' - t'\| < \sqrt{\varepsilon}$, and $\|[t \otimes t'] - [L]\|_Q < \varepsilon/4$.*

Proof. Let $[K] = [L] - [s \otimes s']$. If $[K] = 0$, set $t = s$ and $t' = s'$. If $d = \|[K]\|_Q \neq 0$, then by Lemma 4.7 there exist $\lambda_1, \ldots, \lambda_m$ in D (not necessarily distinct) and scalars $\alpha_1, \ldots, \alpha_m$ such that

$$\left\| [K/d] - \sum_{j=1}^{m} \alpha_j [C_{\lambda_j}] \right\|_Q < \varepsilon/8d, \quad \sum_{j=1}^{m} |\alpha_j| \leqslant 1.$$

For each j choose δ_j so that $\delta_j^2 = d\alpha_j$. Then we have

$$\left\| [K] - \sum_{j=1}^{m} \delta_j^2 [C_{\lambda_j}] \right\|_Q < \varepsilon/8. \tag{6}$$

By Lemma 4.6 there exist m mutually orthogonal, orthonormal sequences $\{x_i^j\}_{i=1}^{\infty}$, $1 \leqslant j \leqslant m$, such that $\lim_{i\to\infty} \|(A-\lambda_j)x_i^j\| = 0$ for $1 \leqslant j \leqslant m$ and such that, if we set $u_i = \sum_{j=1}^{m} \delta_j x_i^j$ and $v_i = \sum_{j=1}^{m} \bar{\delta}_j x_i^j$, then

$$\lim_i \|[u_i \otimes v_i] - \sum_{j=1}^{m} \delta_j^2 [C_{\lambda_j}]\|_Q = 0.$$

Thus, by virtue of (6), we know that for all i sufficiently large,

$$\|[K] - [u_i \otimes v_i]\|_Q < \varepsilon/8. \tag{7}$$

Define $s_i = s + u_i$ and $s_i' = s' + v_i$. We show that we can choose $t = s_{i_0}$ and $t' = s_{i_0}'$ for some integer i_0 sufficiently large. Note first that for each i,

$$\|u_i\|^2 = \|v_i\|^2 = \sum_{j=1}^{m} |\delta_j|^2 \leqslant d = \|[K]\|_2 < \varepsilon.$$

Thus for any choice of i_0, if t and t' are chosen as indicated, we have $\|s - t\| \leqslant \sqrt{\varepsilon}$ and $\|s' - t'\| < \sqrt{\varepsilon}$. On the other hand,

$$[s_i \otimes s_i'] - [L] = [s \otimes v_i] + [u_i \otimes s'] + [u_i \otimes v_i] + [s \otimes s'] - [L]$$

$$= [s \otimes v_i] + [u_i \otimes s'] + [u_i \otimes v_i] - [K].$$

Thus, by (7), for i sufficiently large we have

$$\|[s_i \otimes s_i'] - [L]\|_Q < \|[s \otimes v_i]\|_Q + \|[u_i \otimes s']\|_Q + \varepsilon/8.$$

Since $[s \otimes v_i] = \sum_{j=1}^{m} \delta_j [s \otimes x_i^j]$ and similarly for $[u_i \otimes s']$, both $\|[s \otimes v_i]\|_Q$ and $\|[u_i \otimes s']\|_Q$ tend to zero by Lemmas 4.4 and 4.5, so that we may choose i_0 as desired.

The next and last lemma concludes the proof of Theorem 4.1.

LEMMA 4.9 (cf. [4, Theorem 4.6]). *If $[L]$ is an arbitrary element of Q, then there exist vectors x and y in $\mathscr{H}$ such that $[L] = [x \otimes y]$. In particular, this is true for $[L] = [C_0]$.*

Proof. It suffices to prove the lemma for all $[L]$ in Q such that $\|[L]\|_Q \leqslant 1$. Applying Lemma 4.7 to such an $[L]$, we obtain the existence of a finite sequence $\lambda_1, \ldots, \lambda_m$ of (not necessarily distinct) numbers from $\sigma(A) \cap D$ and a corresponding sequence $\alpha_1, \ldots, \alpha_m$ of scalars such that $\|[L] - \sum_{j=1}^{m} \alpha_j [C_{\lambda_j}]\|_Q < 1/8$. Moreover, if we set $\delta_j^2 = \alpha_j$ for each j, then by Lemma 4.6 there exist vectors u and v such that $\|[u \otimes v] - \sum_{j=1}^{m} \alpha_j [C_{\lambda_j}]\|_Q < 1/8$. Thus $\|[L] - [u \otimes v]\| < 1/4$, and we set $u_0 = u_1 = u$,

$v_0 = v_1 = v$. Suppose now, by induction, that u_k and v_k have been chosen for $1 \leqslant k \leqslant n$ so as to satisfy

$$\|[L] - [u_k \otimes v_k]\|_Q < 1/2^{2k}, \ \|u_k - u_{k-1}\| < 1/2^{k-1}, \ \|v_k - v_{k-1}\| < 1/2^{k-1}.$$

Applying Lemma 4.8 with $\varepsilon = 1/2^{2n}$, $s = u_n$, and $s' = v_n$, we obtain $u_{n+1} = t$ and $v_{n+1} = t'$ satisfying the above inequalities for $k = n + 1$. We have thus constructed by induction two sequences $\{u_n\}$ and $\{v_n\}$ such that $\|[u_n \otimes v_n] - [L]\|_Q \to 0$. But the inequalities $\|u_{n+1} - u_n\| < 1/2^n$ and $\|v_{n+1} - v_n\| < 1/2^n$ clearly imply that these two sequences are Cauchy in $\mathscr{H}$. Let $x = \lim u_n$ and $y = \lim v_n$. Since

$$\begin{aligned} \|[u_n \otimes v_n] - [x \otimes y]\|_Q &\leqslant \|u_n \otimes v_n - x \otimes y\|_\tau \\ &\leqslant \|u_n \otimes (v_n - y)\|_\tau + \|(u_n - x) \otimes y\|_\tau \\ &= \|u_n\| \cdot \|v_n - y\| + \|u_n - x\| \cdot \|y\| \to 0, \end{aligned}$$

we have $[x \otimes y] = [L]$. Thus the proof of Theorem 4.1 is complete.

Remark. It seems likely that some modification of the techniques employed herein and in [4] should permit one to prove that every operator A in $\mathscr{L}(\mathscr{H})$ such that $\|A\| = 1$ and $\sigma(A)$ contains the unit circle has a nontrivial invariant subspace, but the present authors have been unable to prove this.

REFERENCES

1. Agler, J., An invariant subspace theorem, *J. Functional Analysis,* to appear.
2. Brown, A.; Pearcy, C., *Introduction to operator theory I: Elements of functional analysis,* Springer-Verlag, New York, 1977.
3. Brown, A.; Pearcy, C., Jordan loops and decompositions of operators, *Canad.J.Math.,* **29**(1977), 1112–1119.
4. Brown, S., Some invariant subspaces for subnormal operators, *Journal of Integral Equations and Operator Theory,* to appear.
5. Dixmier, J., *Les algèbres d'opérateurs dans l'espace hilbertien,* Gauthier-Villars, Paris, 1957.
6. Douglas, R., G., *Banach algebra techniques in operator theory,* Academic Press, New York, 1972.
7. Foiaş, C.; Mlak, W., The extended spectrum of completely non-unitary contractions and the spectral mapping theorem, *Studia Math.,* **26** (1966), 239–245.
8. Sz.-Nagy, B.; Foiaş, C., *Harmonic analysis of operators on Hilbert space,* Akademiai Kiadó, Budapest, 1970.
9. Pearcy, C., *Some recent developments in operator theory,* CBMS-NSF Lecture Notes No. 36, A.M.S., Providence, 1978.

10. Rubel, L. A.; Shields, A. L., The space of bounded analytic functions on a region, *Ann. Inst. Fourier (Grenoble)*, **1** (1966), 235–277.
11. Rudin, W., *Real and complex analysis*, McGraw-Hill, New York, 1966.
12. Stampfli, J., An extension of Scott Brown's invariant subspace theorem: K-spectral sets, preprint, 1978.

S. BROWN, University of California, at Santa Barbara U.S.A.

B. CHEVREAU, University of Michigan and University of Idaho U.S.A.

C. PEARCY, University of California at Santa Barbara and University of Michigan U.S.A.

(Received June 29, 1978; revised November 30, 1978.)

Added in Proof. After this paper was written, the authors learned that Dan Voiculescu has independently proved Theorem 4.1. above. His proof was presented to the Colloquium in Functional Analysis in Timişoara in May, 1978.