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CLOSURE OF SIMILARITY ORBITS OF NILPOTENT OPERATORS I. FINITE RANK OPERATORS

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1. INTRODUCTION

Let $\mathscr{L}(\mathscr{X})$ be the algebra of all (continuous linear) operators acting on a (real or complex) Banach space $\mathscr{X}$. The similarity orbit of $T \in \mathscr{L}(\mathscr{X})$ is the set

$$\mathscr{S}(T) = \{WTW^{-1}: W \text{ is invertible in } \mathscr{L}(X)\}.$$

Let $\mathscr{S}(T)^{-}$ denote the *norm* closure of $\mathscr{S}(T)$. The main result of this article is the following one (This result completes the answer given in ref. [3]).

THEOREM 1.1. *Let $T \in \mathscr{L}(\mathscr{X})$ be a (necessarily algebraic) finite rank operator with minimal monic polynomial $p(z) = \prod_{j=1}^{m} (z - \lambda_j)^{k_j}$ (For $\mathscr{X}$ a real Banach space, it will be assumed that all the λ_j's are real); then*

$\mathscr{S}(T)^{-} = \{A \in \mathscr{L}(\mathscr{X})$: rank $q(A) \leqslant$ rank $q(T)$ *and* $\dim \operatorname{Ker} q(A) \geqslant \dim \operatorname{Ker} q(T)$ *for all* $q \mid p\}$,

where $q \mid p$ denotes a monic polynomial q dividing p.

Let $L \in \mathscr{L}(X)$, then $\mathscr{S}(L)^{-} = \mathscr{S}(T)^{-}$ if and only if rank $q(L) =$ rank $q(T)$ *for all $q \mid p$ if and only if L is similar to T.*

The real and complex case follow by the same proof, so we shall only consider the complex case. The second statement of the theorem is a trivial consequence of the first one and suggests the following

DEFINITION 1.1. *If A, $B \in \mathscr{L}(\mathscr{X})$* and $\mathscr{S}(A)^{-} = \mathscr{S}(B)^{-}$, A and B will be called *asymptotically similar* operators.

By $A \sim B$ ($A \# B$) it will be meant that A is similar (asymptotically similar, resp.) to B. Clearly, $\#$ is an equivalence relation in $\mathscr{L}(\mathscr{X})$. As in ref. [3], the equivalence class of A will be denoted by $[A]$. $(\mathscr{L}(\mathscr{X})/\#, <)$ is a partially ordered set (p.o.s.), where $<$ is the partial order induced by inclusion among the closures of similarity orbits.

Let $\mathscr{N}(\mathscr{X})$ denote the set of all nilpotent operators in $\mathscr{L}(\mathscr{X})$. Theorem 1.1 shows, in particular, that the section of the p.o.s. $(\mathscr{L}(\mathscr{X})/\#, <)$ corresponding to finite rank nilpotents is actually a lattice and, moreover, the structure of this lattice is independent of $\mathscr{X}$ provided $\mathscr{X}$ is an infinite dimensional space.

Theorem 1.1 has the following analog (In what follows $\mathscr{H}$ will always denote a complex *Hilbert space*).

THEOREM 1.2. *Let T be a compact nilpotent operator in Hilbert space $\mathscr{H}$; then*

$$\mathscr{S}(T)^{-} = \{A \in \mathscr{K}(\mathscr{H});\ \dim(A^j\mathscr{H})^{-} \leqq \dim(T^j\mathscr{H})^{-} \text{ for all } j\},$$

where $\mathscr{K}$ denotes the ideal of compact operators.

Further results for the case of a Hilbert space will be given in an oncoming paper [1]. The authors are deeply indebted to Professors Mischa Cotlar, Alain Etcheberry, Marta B. Pecuch and Lazaro Recht for several stimulating conversations, and to the referee for correcting several errors of the original version and providing an argument to simplify the proof of the main result.

2. SOME TECHNICAL RESULTS

This section contains the auxiliary mathematical tools to be used throughout the paper.

Let $\{e_1, e_2, ..., e_j\}$ be the canonical orthonormal basis of the Hilbert space $\mathbf{C}^j$, let q_j be the operator defined by $q_je_1 = 0$, $q_je_l = e_{l-1}$ for $l = 2, 3, ..., j$, and let

$$q_1(\alpha_1) \oplus q_2(\alpha_2) \oplus ... \oplus q_k(\alpha_k) = (\alpha_1, \alpha_2, ..., \alpha_k)$$

be the *orthogonal direct sum* of α_1 copies of q_1 (= the zero operator acting on $\mathbf{C}$), α_2 copies of $q_2, ..., a_k$ copies of q_k acting in the usual fashion on the Hilbert space $\mathscr{H} = \bigoplus_{j=1}^{k} (\mathbf{C}^j \oplus \mathbf{C}^j \oplus ... \oplus \mathbf{C}^j$ (α_j summands)). If $T \in \mathscr{N} \cap \mathscr{F}(\mathscr{X})$ (where $\mathscr{F}(\mathscr{X})$ denotes the ideal of finite rank operators), then $\mathscr{X}$ can be written as the algebraic direct sum of a finite dimensional subspace $\mathscr{X}_0$ and a subspace (= closed linear manifold) $\mathscr{X}_1$, both invariant under T, in such a way that $T|\mathscr{X}_0$ (the restriction of T to $\mathscr{X}_0$) is similar to $(0, \tau_2, \tau_3, ..., \tau_k)$ for a suitable finite sequence $\tau_2, \tau_3, ..., \tau_k$ of nonnegative integers with $\tau_k \neq 0$, and $T|\mathscr{X}_1 = 0$. The set of all nilpotents of order at most k will be denoted by $\mathscr{N}_k$ $(k = 1, 2, ...)$.

LEMMA 2.1. *If $1 \leqq l \leqq k-1$, then $q_{l+1} \oplus q_{k-1} \in \mathscr{S}(q_l \oplus q_k)^{-}$.*

Proof. Let $\{e_1, e_2, ..., e_l\}$ and $\{f_1, f_2, ..., f_k\}$ be the orthonormal basis of $\mathbf{C}^l$ and $\mathbf{C}^k$ related to q_l and q_k, resp., as above indicated ($q_le_1 = q_kf_1 = 0$, $q_l\,e_i = e_{i-1}$, $q_kf_j = f_{j-1}$ for $i, j > 1$) and let $W_\varepsilon \in \mathscr{L}(\mathbf{C}^l \oplus \mathbf{C}^i)$ $(0 < \varepsilon < 1)$ be the operator defined by $W_\varepsilon e_i = e_i - (1/\varepsilon)f_{i+1}$ for $i = 1, 2, ..., l$, $W_\varepsilon f_1 = \varepsilon f_1$, $W_\varepsilon f_j = f_j$ for $j = 2, 3, ..., k$. Then straightforward computations show that W_ε is invertible, $W_\varepsilon^{-1}e_i = e_i + (1/\varepsilon)f_{i+1}$ for $i = 1, 2, ..., l$, $W_\varepsilon^{-1}f_1 = (1/\varepsilon)f_1$, $W_\varepsilon^{-1}f_j = f_j$ for $j = 2, 3, ..., k$ and $Q_\varepsilon = W_\varepsilon(q_l \oplus q_k)W_\varepsilon^{-1}$ is defined by $Q_\varepsilon e_1 = f_1$, $Q_\varepsilon e_i = e_{i-1}$ for $i = 2, 3, ..., l$, $Q_\varepsilon f_1 = 0$, $Q_\varepsilon f_2 = \varepsilon f_1$, $Q_\varepsilon f_j = f_{j-1}$ for $j = 3, 4, ..., k$.

It readily follows that $Q = \text{(norm)-} \lim_{\varepsilon \to 0} Q_\varepsilon$ is unitarily equivalent to $q_{l+1} \oplus q_{k-1}$. Indeed, $Qf_1 = 0$, $Qe_1 = f_1$, $Q e_i = e_{i-1}$ for $i = 2, 3, \ldots, l$ and $Q f_2 = 0$, $Q f_j = f_{j-1}$ for $j = 3, 4, \ldots, k$. ▨

LEMMA 2.2. (*i*) *Let* $A \in \mathscr{L}(\mathscr{X})$ *be the weak limit of a net* $\{A_\nu\}_{\nu \in \Gamma}$ *with* rank $A_\nu < n < \infty$ *for all* $\nu \in \Gamma$; *then* rank $A < n$.

(*ii*) *Let* $\lim_{n\to\infty} \|A_n - A\| = 0$, $A, A_n \in \mathscr{L}(\mathscr{X})$; *then*

$$\dim (A\mathscr{X})^- \leqq \liminf_{n\to\infty} \dim (A_n\mathscr{X})^-.$$

(*iii*) *If* A_n *belongs to a proper* (*closed bilateral*) *ideal* $\mathscr{J}$ of $\mathscr{L}(\mathscr{X})$ *for all* n, *then* $A \in \mathscr{J}$.

Proof. (*i*) Assume that $A\mathscr{X}$ contains a subspace of dimension $n+1$. Then there exist $y_1, y_2, \ldots, y_{n+1} \in \mathscr{X}$ such that $\{Ay_j\}_{j=1}^{n+1}$ is a linearly independent set. Clearly, $\mathscr{Y} = \text{linear span } \{y_1, y_2, \ldots, y_{n+1}\}$ has dimension $n+1$ and therefore $\mathscr{Y} \cap \operatorname{Ker} A_\nu \neq \{0\}$ for all $\nu \in \Gamma$. It is easily seen that there exist $y \in \mathscr{Y}$, $\varphi \in \mathscr{X}^*$ and a cofinal set $\Sigma \subset \Gamma$ such that $\varphi(Ay) = 1$, but $\lim_{\nu \in \Sigma} \varphi(A_\nu y) = 0$, a contradiction. Therefore rank $A \leqq n$.

(*ii*) Passing if necessary to a subsequence, we can assume that $\alpha = \liminf_{n\to\infty} \dim (A_n\mathscr{X})^- = \lim_{n\to\infty} \dim(A_n\mathscr{X})^-$. If $\alpha < \infty$, then the result follows from (*i*). Otherwise, it is easily seen that $(A\mathscr{X})^- \subset \vee \{A_n\mathscr{X}\}_{n=m}^\infty$ (where $\vee$ denotes "the closed linear span of") for all $m \geqq 1$, therefore

$$\text{(topological) } \dim (A\mathscr{X})^- \leqq \lim_{m\to\infty} \sum_{n=m}^{\infty} \dim (A_n\mathscr{X})^- \leqq$$

$$\leqq \aleph_0 \lim_{n\to\infty} \dim(A_n \mathscr{X})^- = \lim_{n\to\infty} \dim(A_n \mathscr{X})^-.$$

The last statement is trivial. ▨

The following result is contained in ref. [4] for the Hilbert space case. The general case follows by the same proof.

LEMMA 2.3. (*i*) *Let* $A, B \in \mathscr{L}(\mathscr{X})$ *and let* $B \in \mathscr{S}(A)^-$. *Then* $p(B) \in \mathscr{S}(p(A))^-$ *for every polynomial* p.

(*ii*) *The analogous result holds for* $f(B)$, *for every function* f *analytic in a neighborhood of the spectrum* $\Lambda(B)$ *of* B.

As an immediate consequence of the last two lemmas, we have

COROLLARY 2.4. *Let* $T \in \mathscr{N}_k(\mathscr{X})$ *and let* $L \in \mathscr{S}(T)^-$; *then*

(*i*) $\mathscr{S}(T)^- \subset \mathscr{N}_k(\mathscr{X})$

(*ii*) $\dim (L^j\mathscr{X})^- \leqq \dim (T^j(\mathscr{X})^-$ *for* $j = 1, 2, \ldots, k$

(*iii*) *If* T^j *belongs to an ideal* $\mathscr{J}$, *for some* j, *then* $L^j \in \mathscr{J}$.

The following two results (see ref. [3]) will allow to extend every result about nilpotent operators to algebraic operators.

LEMMA 2.5. *Let $A \in \mathcal{L}(\mathcal{H})$ and assume that $\Lambda(A)$ (the spectrum of A) is the disjoint union of finitely many clopen subsets $\Lambda_1, \Lambda_2, \ldots, \Lambda_m$. Let M_j be the invariant subspace of A associated with Λ_j via Riesz functional calculus* [8] *so that $\Lambda(A|M_j) = \Lambda_j$ $(j = 1, 2, \ldots, m)$ and $\mathcal{H} = \mathcal{M}_1 \dotplus \mathcal{M}_2 \dotplus \ldots \dotplus \mathcal{M}_m$ (algebraic direct sum). Then*

$$\mathcal{S}(A) = (B \sim B_1 \oplus B_2 \oplus \ldots \oplus B_m \colon B_j \in \mathcal{S}(A|\mathcal{M}_j),\ j = 1, 2, \ldots, m\}.$$

Furthermore, if $\Lambda(B_j) = \Lambda_j$ for all $B_j \in \mathcal{S}(A|\mathcal{M}_j)^-$ and for all $j = 1, 2, \ldots, m$, then

$$\mathcal{S}(A)^- = \{B \sim B_1 \oplus B_2 \oplus \ldots \oplus B_m;\ B_j \in \mathcal{S}(A|\mathcal{M}_j)^-,\ \ j = 1, 2, \ldots, m\}.$$

COROLLARY 2.6. *Let $A \in \mathcal{L}(\mathcal{H})$ be an algebraic operator with minimal monic polynomial $p(z) = \prod_{j=1}^{m} (z - \lambda_j)^{k_j} (\lambda_j \neq \lambda_i \text{ for } j \neq i)$ and let $\mathcal{H} = \mathcal{M}_1 \dotplus \mathcal{M}_2 \dotplus \ldots \dotplus \mathcal{M}_m$ be the decomposition of $\mathcal{H}$ associated with the spectrum $\Lambda(A) = \{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ of A via Riesz functional calculus. Then*

$$\mathcal{S}(A) = \{B \sim B_1 \oplus B_2 \oplus \ldots \oplus B_m \colon B_j \sim A|\mathcal{M}_j,\ j = 1, 2, \ldots, m\}$$

and

$$\mathcal{S}(A)^- = \{B \sim B_1 \oplus B_2 \oplus \ldots \oplus B_m \colon B_j \in \mathcal{S}(A|\mathcal{M}_j)^-,\ j = 1, 2, \ldots, m\}.$$

3. FINITE RANK OPERATORS

Let $A \in \mathcal{F}(\mathcal{X})$ for some Banach space $\mathcal{X}$. Then via Hahn-Banach's theorem A can be written as $A = \sum_{j=1}^{m} x_j \otimes \varphi_j$, where $x_j \in \mathcal{X}$, $\varphi_j \in \mathcal{X}^*$, and $x_j \otimes \varphi_j$ is defined by $x_j \otimes \varphi_j(y) = \varphi_j(y)x_j$ for $j = 1, 2, \ldots, m$. We can obviously assume that $\{x_j \otimes \varphi_j\}_{j=1}^{m}$ is a linearly independent subset of $\mathcal{L}(\mathcal{X})$; then there exist linearly independent vectors $z_1, z_2, \ldots, z_m \in \mathcal{X}$ such that $\varphi_j(z_j) \neq 0$ for all $j = 1, 2, \ldots, m$, and $\mathcal{X}$ can be written as the algebraic direct sum of the finite dimensional subspace $\mathcal{X}_0$ generated by $\{x_1, x_2, \ldots, x_m;\ z_1, z_2, \ldots, z_m\}$ and a subspace $\mathcal{X}_1 \subset \bigcap_{j=1}^{m} \operatorname{Ker} \varphi_j$, so that $A|\mathcal{X}_0$ is a finite dimensional operator and $A|\mathcal{X}_1 = 0$. Let $\mathcal{X} = \mathcal{X}_2 \dotplus \mathcal{X}_3$ be a second decomposition of $\mathcal{X}$ with $\dim \mathcal{X}_2 = \dim \mathcal{X}_0$; then there exists an invertible operator $W \in \mathcal{L}(\mathcal{X})$ of the form $I + \sum_{l=1}^{n} w_l \otimes \psi_l$ such that $W(\mathcal{X}_0) = \mathcal{X}_2$ and $W|\mathcal{X}_1 \cap \mathcal{X}_3$ is the identity on this subspace. Thus, by using Lemma 2.2 (*i*), it is easily seen than in order to prove Theorem 1.1 it will be enough to consider the case of a Hilbert space $\mathbf{C}^d$ for some $d < \infty$. Furthermore, by Corollary 2.5 we can restrict ourselves to nilpotent operators in $\mathcal{L}(\mathbf{C}^d)$. This is the content of the following

PROPOSITION 3.1. *Let $T \in \mathcal{N}(\mathbf{C}^d)$ be a nilpotent operator of order k; then*

$$\mathcal{S}(T)^- = \{A \in \mathcal{L}(\mathbf{C}^d): \text{ rank } A^j \leqq \text{rank } T^j \text{ for } j = 1,2,\dots,k\}.$$

If T and L are nilpotents, then $T \# L$ if and only if rank T^j = rank L^j *for all j if and only if $T \sim L$.*

Proof. The second statement follows immediately from the first one.

Let $A \in \mathcal{S}(T)^-$; then, by Corollary 2.4, $A^j \in \mathcal{S}(T^j)^-$ and rank $A^j \leqq$ rank T^j for $j = 1, 2, \dots, k$.

Conversely, assume that rank $A^j \leqq$ rank T^j for $j = 1, 2, \dots, k$.

Clearly, we can directly assume that both T and A are Jordan forms; i.e.,

$$T = q_{n_1}(\tau_1) \oplus q_{n_2}(\tau_2) \oplus \dots \oplus q_{n_k}(\tau_k)$$

and

$$A = q_{m_1}(\alpha_1) \oplus q_{m_2}(\alpha_2) \oplus \dots \oplus q_{m_h}(\alpha_h), \quad (\tau_j, \alpha_i > 1).$$

Put $m(T, A) = d + \sum_{j=1}^{\infty}$ (rank T^j-rank A^j) and proceed by induction on $m(T, A)$. The case $m(T, A) = 1$ is trivial. Suppose that $A_1 \in \mathcal{S}(T_1)^-$ if $m(T_1, A_1) \leqq n$ and let $A, T \in \mathcal{N}(\mathbf{C}^p)$ such that

$$\text{rank } A^j \leqq \text{rank } T^j \text{ for } j = 1, 2, 3, \dots$$

and

$$m(T, A) = n + 1.$$

If T and A have a common Jordan block q_r, then $T = q_r \oplus T_1$, $A = q_r \oplus A_1$, $m(T_1, A_1) \leqq n$, and by induction $A \in \mathcal{S}(T)^-$. If T and A have no common Jordan block and if $l \geqq 1$ is minimal such that $m_h < n_l$ then T is of the form

$$T = q_{n_{l-1}} \oplus q_{n_l} \oplus T'$$

where $n_0 = 0$ and q_0 acts on a $\{0\}$-space, if $l = 1$. Note now that we have

$$\text{rank } A^r < \text{rank } T^r \text{ when } n_{l-1} + 1 \leqq r \leqq m_n - 1.$$

Indeed, if rank A^r = rank T^r and if a_r (resp. t_r) denote the number of Jordan blocks of A (resp. T) with order of nilpotence greater than or equal to r, then obviously $a_r > t_r$ and this yields the contradiction

$$\text{rank } A^{r-1} = a_r + \text{rank } A^r > t_r + \text{rank } T^r = \text{rank } T^{r-1}.$$

Finally, setting $T_1 = q_{n_{l-1}+1} \oplus q_{n_l-1} \oplus T'$, we can check that

$$\text{rank } A^j \leqq \text{rank } T_1^j \text{ for } j = 1, 2, 3, \dots$$

and

$$m(T_1, A) \leq n;$$

consequently $A \in \mathscr{S}(T_1)^-$. To conclude the proof we observe that $T_1 \in \mathscr{S}(T)^-$ by [3] if $l = 1$ or by Lemma 2.1 if $l > 1$. ▨

COROLLARY 3.2. *$(\mathscr{N} \cap \mathscr{F}(\mathscr{X}) / \#, <)$ is a lattice.*

Proof. Consider first the case when $\mathscr{X} = \mathbf{C}^d$, for some $d < \infty$. Let $A = (\alpha_1, \alpha_2, ..., \alpha_k)$ and $B = (\beta_1, \beta_2, ..., \beta_l)$ $(\alpha_k > 0, \beta_l > 0)$ be two Jordan forms in $\mathscr{L}(\mathbf{C}^d)$. Define $r_j = \max\{\text{rank } A^j, \text{rank } B^j\}$ for $j = 0, 1, ..., m$ with $m = \max\{k, l\}$. Let $\gamma_j = r_{j-1} - 2r_j + r_{j+1}$ for $j = 1, 2, ..., m$. Then $\gamma_j \geq \text{rank } A^{j-1} - 2 \text{ rank } A^j + \text{rank } A^{j+1}$ or $\gamma_j \geq \text{rank } B^{j-1} - 2 \text{ rank } B^j + \text{rank } B^{j+1}$. But $\text{rank } A^j = \alpha_{j+1} + 2\alpha_{j+2} + ... + (k-j)\alpha_k$ implies that $\text{rank } A^{j-1} - 2 \text{ rank } A^j + \text{rank } A^{j+1} = \alpha_j$ for $j = 1, 2, ..., k$. Therefore, $\gamma_j \geq 0$ for $j = 1, 2, ..., m$. Let $S = (\gamma_1, \gamma_2, ..., \gamma_m)$. Then $\text{rank } S^j = \gamma_{j+1} + 2\gamma_{j+2} + ... + (m-j)\gamma_m = r_j$ for $j = 0,1, ..., m$. Therefore, $\text{rank } S^j = \max\{\text{rank } A^j, \text{rank } B^j\}$ for $j = 0, 1, ..., m$. In particular, $S \in \mathscr{L}(\mathbf{C}^d)$. From Proposition 3.1 it follows that $[A] \vee [B] = [S]$.

The set of elements $[R]$ such that $[R] < [A]$ and $[R] < [B]$ is a finite set $[R_1]$, $[R_2]$, ..., $[R_p]$. From the proof about the supremum, there exists R in $\mathscr{L}(\mathbf{C}^d)$ such that $\text{rank } R^j = \max\{\text{rank } R_i^j: i = 1, 2, ..., p\}$. From Proposition 3.1 it follows that $[A] \wedge [B] = [R]$.

This completes the proof of the corollary when $\mathscr{X}$ is finite dimensional. Now the general case follows from the observations at the beginning of this section. ▨

4. COMPACT NILPOTENT OPERATORS IN HILBERT SPACES

Let T be a compact nilpotent of order k and let $A \in \mathscr{S}(T)^-$. By Corollary 2.4, A is a compact nilpotent such that

$$\dim (A^j\mathscr{H})^- \leqq \dim (T^j\mathscr{H})^- \tag{4.1}$$

for all $j = 1, 2, ..., k$.

Now assume that A is a compact nilpotent satisfying (4.1). Since $\dim (T\mathscr{H})^- \leqq \aleph_0$ and $\dim (A\mathscr{H})^- \leqq \aleph_0$, there exists an infinite dimensional separable subspace $\mathscr{H}_0$ reducing both T and A such that $T \mid \mathscr{H}_0^\perp = A \mid \mathscr{H}_0^\perp = 0$, so that we can directly assume that $\mathscr{H}$ itself is separable.

Either $\dim (T^{k-1}\mathscr{H})^- = \aleph_0$ or there exists an s, $1 \leq s < k$, such that $\dim (T^j\mathscr{H})^- = \aleph_0$ for $j=0, 1, ..., s-1$ and $\text{rank } T^j < \infty$ for $j=s, s+1, ..., k-1$ $(T^{k-1} \neq 0$

by hypothesis). It is a standard fact that T can be written as an upper triangular matrix with zero diagonal entries

$$(4.2)\qquad T=\begin{pmatrix} 0 & t_{12} & t_{13} & \cdots & \cdots & t_{1,n-1} & t_{1,n} & \cdots \\ & 0 & t_{23} & \cdots & \cdots & t_{2,n-1} & t_{2,n} & \cdots \\ & & 0 & \cdots & \cdots & t_{3,n-1} & t_{3,n} & \cdots \\ & & & \ddots & & \vdots & \vdots & \\ & & & & \ddots & \vdots & \vdots & \\ & & & & & 0 & t_{n-1,n} & \cdots \\ & & & & & & 0 & \cdots \\ & & & & & & & \ddots \end{pmatrix}$$

with respect to some suitably chosen orthonormal basis $\{e_n\}_{n=1}^{\infty}$ of $\mathscr{H}$.

Let P_n be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}_n = \vee\{e_1, e_2, ..., e_n\}$ ($n = 1, 2, ...$) and let Q_n be similarly defined with T replaced by A.

Since $T, A \in \mathscr{K}$, it readily follows that $\|P_nT^jP_n - T^j\| \to 0$ ($n \to \infty$) and $\|Q_n A^j Q_n - A^j\| \to 0$ ($n \to \infty$) for every j. Since $\mathscr{M}_n$ is invariant under T, we have $(P_n TP_n)^j = P_nT^jP_n$ and rank $(P_nTP_n)^j \leqq \dim(T^j\mathscr{H})^-$ for all j (and similarly for Q_nAQ_n). It is clear that P_nTP_n and Q_nAQ_n are finite rank nilpotent operators for every n and, moreover, that rank $(P_nTP_n)^j =$ rank T^j and rank $(Q_nAQ_n)^j =$ rank A^j for $j=s, s+1, ..., k$, for all $n \geqq n_0$. Furthermore, rank $(P_nTP_n)^j \to \infty (n \to \infty)$ for $j = 1, 2, ..., s-1$.

Hence, for a given m, there exists $n_1 \geqq n_0$ such that rank$(Q_m A Q_m)^j \leqq$ rank $(P_nTP_n)^j$ for $j = 1, 2, ..., k$ and for $n \geqq n_1$. Thus, by Proposition 3.1, $Q_mAQ_m \in \mathscr{S}(P_nTP_n)^-$ for all $n \geqq n_1$.

Let

$$T=\begin{pmatrix} T_1 & * \\ 0 & T_2 \end{pmatrix}$$

be the matrix of T with respect to the decomposition $\mathscr{H} = \mathscr{M}_n \oplus \mathscr{M}_n^{\perp}$. It follows by standard arguments (see, e.g., refs. [2; 3; 7]) that $P_nTP_n = T_1 \oplus 0 \in \mathscr{S}(T)^-$. Hence $Q_mAQ_m \in \mathscr{S}(T)^-$ for all $m = 1, 2, ...,$ and, a fortiori, $A \in \mathscr{S}(T)^-$.

The proof of Theorem 1.2 is complete now.

Let T be an arbitrary nilpotent of order k with upper triangular matrix (4.2). Then $\{P_nT^jP_n\}_{n=1}^{\infty}$ is a sequence of finite rank nilpotents converging *strongly* to T^j, $j = 1, 2, ..., k$. Minor modifications of the above proof yield the following

COROLLARY 4.1. *Let $T \in \mathscr{N}_k \setminus \mathscr{N}_{k-1}$ and assume that* $\dim(T^j\mathscr{H})^-$ *is finite for $j = s, s+1, ..., k$, but not for $j = 0, 1, ..., s-1$. Then $\mathscr{S}(T)^-$ contains every compact nilpotent operator A satisfying the conditions* (4.1).

By reducing the algebraic case to the nilpotent case, we obtain

COROLLARY 4.2. *Let T be an algebraic operator in a Hilbert space $\mathscr{H}$ of infinite dimension h, with minimal monic polynomial p and assume that $p_0(T) \in \mathscr{K}$ for some polynomial p_0 with simple roots; then $\mathscr{S}(T)^-$ is the set of all $A \in \mathscr{L}(\mathscr{H})$ such that $\Lambda(A) = \Lambda(T)$, $p_0(A) \in \mathscr{K}$, $\dim (q(A)\mathscr{H})^- \leqq \dim (q(T)\mathscr{H})^-$ for all $q|p\}$ and the spectral subspaces corresponding to T and A associated with every point of the spectrum have the same dimension.*

If $p_0(T) \notin \mathscr{F}$ then there exists an operator $L \# T$ such that L is not similar to T.

Proof. The characterization of $\mathscr{S}(T)^-$ follows from Theorem 1.2 and Corollary 2.5. The details are left to the reader.

It follows, in particular, that $L \# T$ if and only if $\Lambda(L) = \Lambda(T)$, $p_0(L) \in \mathscr{K}$ and $\dim (q(L)\mathscr{H})^- = \dim (q(T)\mathscr{H})^-$ for all $q|p$.

Assume that $p_0(T) \notin \mathscr{F}$; then it is not difficult to construct, by using the results of ref. [9], an operator L satisfying all our requirements such that $p_0(L)$ and $p_0(T)$ do not belong to the same *not closed* bilateral subideals of $\mathscr{K}$. Since $A \sim T$ implies $p_0(A) \sim p_0(T)$, it readily follows that L and T cannot be similar.

REMARK. The complete description of $\mathscr{S}(T)^-$ for an arbitrary $T \in \mathscr{N}(\mathscr{H})$ will be given in ref. [1]. The analogous problem for an arbitrary Banach space is much more difficult, as the following example shows:

EXAMPLE 5.1. Let $\mathscr{X} = l^1 \oplus l^2 \oplus l^3$, let $T = \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \oplus 0$ and $A = 0 \oplus \begin{pmatrix} 0 & 0 \\ G & 0 \end{pmatrix}$, where $Fe_n = f_n$ and $Gf_n = g_n$ ($\{e_n\}$, $\{f_n\}$, $\{g_n\}$ the canonical bases of l^1, l^2, l^3, resp.). Then $AX = XT$ an $YA = TY$, where $Xe_n = f_n$, $Xf_n = g_n$ and $Xg_n = (1/n)e_n$, and $Ye_n = g_n$, $Yf_n = (1/n)e_n$ and $Yg_n = (1/n)f_n$, whence it readily follows that A and T are quasi-similar operators in the sense of B. Sz.-Nagy and C. Foiaş [10]; furthermore, A and T are algebraic operators with minimal monic polynomial $p(z) = z^2$. However, $A \notin \mathscr{S}(T)^-$ and $T \notin \mathscr{S}(A)^-$.

Proof. Let W be an invertible element of $\mathscr{L}(\mathscr{X})$. Since every continuous linear map $K: l^p \to l^q$ with $1 \leqq q < p \leqq \infty$ is necessarily compact [6], the matrices of W and W^{-1} must have the form

$$W = \begin{pmatrix} W_{11} & W_{12} & W_{13} \\ W_{21} & W_{22} & W_{23} \\ W_{31} & W_{32} & W_{33} \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{pmatrix},$$

where $W_{12}, S_{12}: l^2 \to l^1$, $W_{13}, S_{13}: l^3 \to l^1$ and $W_{23}, S_{23}: l^3 \to l^2$ are compact. Then

$$\|A - WTW^{-1}\| = \left\| \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & G & 0 \end{pmatrix} - \begin{pmatrix} * & * & * \\ * & * & * \\ * & W_{32}FS_{12} & * \end{pmatrix} \right\| \geqq$$

$$\geqq \|G - W_{32}FS_{12}\| \geqq 1,$$

because $W_{32}FS_{12}$ is compact. Hence, dist $[A, \mathscr{S}(T)] = 1$ and, similarly, dist $[T, \mathscr{S}(A)] = 1$. ▨

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